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L^p–dimension free boundedness for Riesz transforms associated to Hermite functions⋆

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Abstract. Riesz transforms associated to Hermite functions were introduced by S. Thangavelu, who proved that they are bounded operators on $L^p(\mathbb{R}^d)$, $1 < p < \infty$. In this paper we give a different proof that allows us to show that the L^p-norms of these operators are bounded by a constant not depending on the dimension d. Moreover, we define Riesz transforms of higher order and free dimensional estimates of the L^p-bounds of these operators are obtained. In order to prove the mentioned results we give an extension of the Littlewood-Paley theory that we believe of independent interest.

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1. Introduction

Let $\{H_k\}_{k\geq 0}$ be the family of 1−dimensional Hermite polynomials

$$H_k(x) = (-1)^k \frac{d^k}{dx^k}(e^{-x^2})e^{x^2}.$$

We have,

$$H_k^{'}(x) = 2kH_{k-1}(x) \qquad \text{and} \qquad H_{k+1}(x) = -H_k^{'}(x) + 2xH_k(x),$$

see [Th] and [Sz]. We define the 1−dimensional Hermite functions

$$h_k(x) = (2^k k!\pi^{1/2})^{-1/2} H_k(x) e^{-x^2/2},\ k = 0, 1, \dots .$$

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This is a complete orthonormal system in $L^2(\mathbb{R})$. Then follows that

$$\left(-\frac{d}{dx}+x\right)h_k=\sqrt{2(k+1)}h_{k+1} \text{ and } \left(\frac{d}{dx}+x\right)h_k=\sqrt{2k}h_{k-1}. \quad (1.1)$$

We observe that the operator

$$L=-\frac{d^2}{dx^2}+|x|^2$$

can be factorized as

$$L=\frac{1}{2}\left[\left(\frac{d}{dx}+x\right)\left(-\frac{d}{dx}+x\right)+\left(-\frac{d}{dx}+x\right)\left(\frac{d}{dx}+x\right)\right].$$

It follows from (1.1) that

$$Lh_k=(2k+1)h_k. \quad (1.2)$$

We define the $d-$dimensional Hermite functions $h_\alpha(x)$ with $\alpha=(\alpha_1,\dots,\alpha_d)$ and $x=(x_1,\dots,x_d)$, as the product $h_\alpha(x)=\Pi_{i=1}^d h_{\alpha_i}(x_i)$. This is a complete orthonormal system in $L^2(\mathbb{R}^d)$. Let L be the differential operator given by

$$L=-\Delta+|x|^2. \quad (1.3)$$

A consequence of (1.2) is that

$$Lh_\alpha=(2|\alpha|+d)h_\alpha, \qquad \text{where} \quad |\alpha|=\alpha_1+\cdots+\alpha_d. \quad (1.4)$$

The operator L is positive, self-adjoint with respect to the Lebesgue measure in $\mathbb{R}^d$, and admits the spectral decomposition

$$Lf=\sum_{n=0}^{\infty}(2n+d)\sum_{|\alpha|=n}c_\alpha h_\alpha, \qquad f\in Dom(L),$$

where $c_\alpha=<f,h_\alpha>=\int f(x)h_\alpha(x)dx$.

By $(L)^{-1/2}$ we mean the operator defined over the Hermite functions as

$$(L)^{-1/2}h_\alpha=\frac{1}{\sqrt{2|\alpha|+d}}h_\alpha$$

and extended to the space of linear combinations of Hermite functions.

Let us denote by $\mathcal{A}_i$, $1\le|i|\le d$, the $2d$ differential operators given by

$$\mathcal{A}_i=\frac{\partial}{\partial x_i}+x_i, \quad \mathcal{A}_{-i}=-\frac{\partial}{\partial x_i}+x_i\ , \quad \text{for} \quad 1\le i\le d. \quad (1.5)$$

Then, we define the "Riesz" transforms

$$R_i=\mathcal{A}_iL^{-1/2}\ , \quad \text{for} \quad 1\le|i|\le d.$$

These Riesz transforms associated to the differential operator L just defined, were introduced by S. Thangavelu, see [Th]. He showed that they are bounded from $L^p(\mathbb{R}^d)$ into $L^p(\mathbb{R}^d)$ for p in the range $1 < p < \infty$ and also of weak type $(1, 1)$. In dimension one conjugate expansions were studied in [GoSt]. In this paper we give a new proof of Thangavelu's result that has the advantage of showing that the L^p-boundedness, for $1 < p < \infty$, is a dimension free phenomenon. More precisely, we shall prove the following theorem:

Theorem A. *Let p be in the range $1 < p < \infty$, then*

$$\left\| \left(\sum_{1 \le |i| \le d} |R_i f|^2 \right)^{1/2} \right\|_{L^p(\mathbb{R}^d)} \le C_p \|f\|_{L^p(\mathbb{R}^d)},$$

where C_p is a positive constant which depends only on p (and not on the dimension d).

The corresponding result for the classical Euclidean Riesz transforms associated to the differential operator Δ was proved by E. Stein in 1983, see [S3], by using Littlewood-Paley $g-$functions. By transference methods J. Duoandikoetxea and J. L. Rubio de Francia, see [DR], gave an alternative proof of Stein's result.

The result is also known for the Riesz transforms associated to the differential operator in $\mathbb{R}^d$ given by $\Delta - 2x.\nabla$. In this case the natural measure is the gaussian measure and the eigenvalues of the operator turn out to be the Hermite polynomials. Proofs of these results were given by G. Pisier, see [P] and C. Gutiérrez, who used an extension of the Littlewood-Paley $g-$functions, see [Gu]. In this case, since the measure is finite, R. Gundy and P. A. Meyer obtained dimension free results using probabilistic methods, see [Gn] and [Me]. Analogous results were proved in [CMZ] for the case of the Heisenberg group.

The proof of Theorem A will follow some ideas introduced by E. Stein in [S3], [S2] and [Gu]. In the way of proving our result we shall introduce in section 2, new Littlewood-Paley $g-$type functions. We believe that the boundedness properties of the $g-$type functions stated in Theorem 1 are of independent interest.

Theorem A can be extended to the Riesz transforms of higher order. More precisely, let $L^{-m/2}$ be the operator defined by

$$L^{-m/2} h_\alpha = (2|\alpha| + d)^{-m/2} h_\alpha, \tag{1.6}$$

and defined the Riesz transforms of order m as

$$R_{i_1 i_2 \ldots i_m} = \mathcal{A}_{i_1} \mathcal{A}_{i_2} \ldots \mathcal{A}_{i_m} L^{-m/2}, \tag{1.7}$$

where $1 \le |i_j| \le d$, for every $1 \le j \le m$. In order to deal with these Riesz transforms we introduce higher order Littlewood-Paley $g-$type functions in section 3 whose boundedness properties are stated in Theorem 4 and Theorem 5. The method used in proving these theorems follows ideas contained in [GuSeT]. For the Riesz transforms of order m we have the following theorem:

Theorem B. *Let p be in the range $1 < p < \infty$, there exist constants $C_{p,m}$ not depending on the dimension d, such that*

$$\left\| \left(\sum_{1\leq |i_1|,\dots,|i_m|\leq d} |R_{i_1 i_2 \dots i_m} f|^2 \right)^{1/2} \right\|_{L^p(\mathbb{R}^d)} \leq C_{p,m} \|f\|_{L^p(\mathbb{R}^d)}. \tag{1.8}$$

2. Littlewood-Paley $g-$functions of order one

The well known Gauss-Weierstrass kernel on $\mathbb{R}^d$ is defined as

$$W_t(x) = (4\pi t)^{-d/2} e^{-|x|^2/(4t)}. \tag{2.1}$$

It is easy to check that

$$W_t(x) > 0, \quad \int_{\mathbb{R}^d} W_t(y)dy = 1, \quad \text{and} \quad W_t * W_s(x) = W_{t+s}(x)$$

hold for every $x \in \mathbb{R}^d$ and $t, s > 0$. The Gauss-Weierstrass integral of a function $f(x)$ is given by

$$W_t(f)(x) = \int_{\mathbb{R}^d} W_t(x-y) f(y) dy.$$

For every $f \in L^p(\mathbb{R}^d)$, $1 \leq p < \infty$, we have that $\lim_{t\to 0^+} W_t(f) = f$ in $L^p(\mathbb{R}^d)$ and almost everywhere. Thus, W_t defines a contraction semigroup on $L^p(\mathbb{R}^d)$, $1 \leq p < \infty$.

We denote the maximal function f^* associated to this kernel, of a function $f(x)$, as

$$f^*(x) = \sup_{t>0} |W_t(f)(x)|.$$

As it is well known (see [S2], page 73), f^* satisfies the inequality

$$\|f^*\|_{L^p(\mathbb{R}^d)} \leq C_p \|f\|_{L^p(\mathbb{R}^d)}, \tag{2.2}$$

for every $1 < p < \infty$ where the constant C_p does not depend on the dimension d.

For every $t > 0$ we consider the kernel G_t defined on $\mathbb{R}^d \times \mathbb{R}^d$ by the expression

$$G_t(x, y) = r^{d/2} M_r(x, y), \tag{2.3}$$

where $r = e^{-2t}$ and

$$M_r(x, y) = \sum_{n=0}^{\infty} r^n \sum_{|\alpha|=n} h_\alpha(x) h_\alpha(y). \tag{2.4}$$

By the $d-$dimensional version of the Mehler's formula (see [Th], page 6), we have that

$$M_r(x,y) = \frac{1}{\pi^{d/2}} \frac{1}{\left(1-r^2\right)^{d/2}} \times \exp\left\{-\frac{1}{4}\left[\left(\frac{1-r}{1+r}\right)|x+y|^2 + \left(\frac{1+r}{1-r}\right)|x-y|^2\right]\right\}.$$

Then, if $u = \frac{1-r}{1+r}$, we can write

$$\begin{aligned} G_t(x,y) &= \left[\frac{1-u^2}{4\pi u}\right]^{d/2} \exp\left\{-\frac{1}{4}\left[u\,|x+y|^2 + \frac{1}{u}\,|x-y|^2\right]\right\} \\ &\le \left[\frac{1-u^2}{4\pi u}\right]^{d/2} e^{-\frac{1}{4u}|x-y|^2}. \end{aligned} \tag{2.5}$$

From (2.1) it follows that

$$G_t(x,y) \le W_u(x-y) \qquad \text{with} \quad u = \frac{1-e^{-2t}}{1+e^{-2t}}. \tag{2.6}$$

Moreover, the following properties of the kernel G_t hold:

(i) $\int_{\mathbb{R}^d} G_t(x,y) G_s(y,z) dy = G_{t+s}(x,z)$,

(ii) $\int_{\mathbb{R}^d} G_t(x,y) dy = \left(\frac{1-u^2}{1+u^2}\right)^{d/2} e^{-\frac{u}{1+u^2}|x|^2} = (\cosh 2t)^{-d/2} e^{-\frac{\tanh 2t}{2}|x|^2}$, and

(iii) from property (ii) above and (2.6) it follows that for every $f \in L^p(\mathbb{R}^d)$, $1 \le p < \infty$, $\lim_{t\to 0^+} \int_{\mathbb{R}^d} G_t(x,y) f(y) dy = f(x)$ in $L^p(\mathbb{R}^d)$ and almost everywhere.

Thus, by (2.6), (i) and (iii) we see that

$$G_t(f)(x) = \int_{\mathbb{R}^d} G_t(x,y) f(y) dy$$

defines a contraction semigroup on $L^p(\mathbb{R}^d)$, $1 \le p < \infty$. It can be shown that the infinitesimal generator of this semigroup is the differential operator $\Delta - |x|^2 = -L$, see (1.3).

Let b be an integer. We define

$$G_t^b(x,y) = e^{bt} G_t(x,y).$$

Then,

$$G_t^b(f)(x) = \int_{\mathbb{R}^d} G_t^b(x,y) f(y) dy$$

defines a semigroup on $L^p(\mathbb{R}^d)$, $1 \le p < \infty$, and $\|G_t^b 00(f)\|_{L^p(\mathbb{R}^d)} \le e^{bt} \|f\|_{L^p(\mathbb{R}^d)}$. Obviously, the infinitesimal generator of this semigroup is $-L + b$.

Recalling that $u = \frac{1-e^{-2t}}{1+e^{-2t}}$, it turns out that $0 \le u < 1$.
If $d \ge b+1$, then

$$e^{bt}(1-u^2)^{d/2} = e^{bt}\left[\frac{4e^{-2t}}{(1+e^{-2t})^2}\right]^{\frac{b+1}{2}}(1-u^2)^{\frac{d}{2}-\frac{b+1}{2}}$$
$$\le C_b e^{-t},$$

holds, where the constant C_b is equal to 2^{b+1} if $b+1 \ge 0$ and $C_b = 1$ if $b+1 < 0$. Thus, from (2.5) it follows that

$$G_t^b(x, y) \le C_b e^{-t} W_u(x-y).$$

Using the subordination formula (see [S1])

$$e^{-t} = \frac{1}{\sqrt{\pi}}\int_0^\infty e^{-s}s^{-1/2}e^{-\frac{t^2}{4s}}ds = \frac{t}{\sqrt{4\pi}}\int_0^\infty e^{-s}s^{-3/2}e^{-\frac{t^2}{4s}}ds, \tag{2.7}$$

we have that for $d \ge b+1$

$$\frac{t}{\sqrt{4\pi}}\int_0^\infty \int_{\mathbb{R}^d} G_s^b(x, y)|f(y)|dys^{-3/2}e^{-\frac{t^2}{4s}}ds \le C_b e^{-t}|f|^*(x). \tag{2.8}$$

In particular, in the case of $f(x) \equiv 1$, (2.8) implies that

$$\frac{t}{\sqrt{4\pi}}\int_{\mathbb{R}^d}\int_0^\infty G_s^b(x, y)s^{-3/2}e^{-\frac{t^2}{4s}}dsdy \le C_b e^{-t}. \tag{2.9}$$

Taking into account these estimations, we define for $d \ge b+1$ the "Poisson" kernel $P_t^b(x, y)$ as

$$P_t^b(x, y) = \frac{t}{\sqrt{4\pi}}\int_0^\infty G_s^b(x, y)s^{-3/2}e^{-\frac{t^2}{4s}}ds.$$

This kernel is positive and symmetric in the variables x and y. Moreover, by (2.9), it follows that

$$\int_{\mathbb{R}^d} P_t^b(x, y)dy \le C_b e^{-t}. \tag{2.10}$$

By (2.8) we can define the "Poisson" integral $u_b(x, t) = P_t^b(f)(x)$ of a function $f(x)$ as

$$u_b(x, t) = P_t^b(f)(x) = \int_{\mathbb{R}^d} P_t^b(x, y)f(y)dy$$
$$= \frac{t}{\sqrt{4\pi}}\int_{\mathbb{R}^d}\int_0^\infty G_s^b(x, y)f(y)s^{-3/2}e^{-\frac{t^2}{4s}}dsdy. \tag{2.11}$$

From (2.8) it follows that, if $d \ge b+1$

$$|u_b(x, t)| \le C_b e^{-t}|f|^*(x) \tag{2.12}$$

holds. The Poisson integral $P_t^b(f)$ defines a semigroup on $L^p(\mathbb{R}^d)$, $1 < p < \infty$. Using (2.3) and (2.4), it is not difficult to check that

$$u_b(x,t) = \sum_{n=0}^{\infty} e^{-(2n+d-b)^{1/2}t} \sum_{|\alpha|=n} c_\alpha h_\alpha(x), \tag{2.13}$$

where

$$c_\alpha =< f, h_\alpha >= \int_{\mathbb{R}^d} f(y) h_\alpha(y) dy.$$

In order to deal with the case $b+1 > d$, we shall need an estimate of $\left(\frac{d}{dr}\right)^k M_r(x,y)$. It is easy to see that,

$$\left(\frac{d}{dr}\right)^k M_r(x,y) = Q_r(|x+y|^2, |x-y|^2) \exp -\frac{1}{4}\left\{\left(\frac{1-r}{1+r}\right)|x+y|^2 + \left(\frac{1+r}{1-r}\right)|x-y|^2\right\},$$

where $Q_r(u,v)$ is a polynomial of k degree whose coefficients are functions depending on

$$\frac{1}{\left(1-r^2\right)^{d/2}} \;,\; \frac{1-r}{1+r} \;\text{ and }\; \frac{1+r}{1-r}$$

and their derivatives up to the order k. Since these functions are analytic on $|r| < 1$, it follows that for $0 \le r \le 1/2$

$$\left|Q_r(|x+y|^2, |x-y|^2)\right| \le C_{k,d}\left(1+|x+y|^2+|x-y|^2\right)^k.$$

Moreover, if $0 \le r \le 1/2$ the inequalities $\frac{1-r}{1+r} \ge 1/4$ and $\frac{1+r}{1-r} \ge 1$ hold. Then,

$$\left|\left(\frac{d}{dr}\right)^k M_r(x,y)\right| \le C_{k,d}\left(1+|x+y|^2+|x-y|^2\right)^k e^{-\frac{1}{16}\left(|x+y|^2+|x-y|^2\right)} \le C_{k,d} e^{-\frac{\left(|x+y|^2+|x-y|^2\right)}{32}} \le C_{k,d} e^{-\frac{|x-y|^2}{32}}.$$

Thus, by Taylor's formula, we have the estimate

$$\Big|\sum_{n=k}^{\infty} r^n \sum_{|\alpha|=n} h_\alpha(x) h_\alpha(y)\Big| \le r^k C_{k,d} e^{-\frac{|x-y|^2}{32}}, \quad \text{for } 0 < r \le 1/2. \tag{2.14}$$

In the case $b+1 > d$, we shall show that the "Poisson" integral of $f(x)$ defined as before by

$$u_b(x,t) = P_t^b(f)(x) = \frac{t}{\sqrt{4\pi}} \int_0^\infty \left(\int_{\mathbb{R}^d} G_s^b(x,y) f(y) dy\right) s^{-3/2} e^{-\frac{t^2}{4s}} ds, \tag{2.15}$$

is well defined if we restrict the functions $f(x)$ to satisfy that $c_\alpha =< f, h_\alpha >= 0$ for every α, $|\alpha| \le k-1$, where k is a positive integer such that $2k \ge b+2-d$.

In fact, since for $s \ge 1$, $r = e^{-2s} \le 1/2$, by (2.3), (2.4) and (2.14) we obtain

$$\begin{aligned}\left|\int_{\mathbb{R}^d} G_s^b(x,y)f(y)dy\right| &\le e^{(b-d)s}\int_{\mathbb{R}^d}\left|\sum_{n=k}^{\infty} e^{-2ns}\sum_{|\alpha|=n} h_\alpha(x)h_\alpha(y)\right| |f(y)|\,dy\\ &\le C_{k,d}e^{-(2k+d-b)s}\int_{\mathbb{R}^d} e^{-\frac{|x-y|^2}{32}}\,|f(y)|\,dy.\end{aligned}$$

Then, by the subordination formula (2.7), we have

$$\begin{aligned}&\left|\frac{t}{\sqrt{4\pi}}\int_0^\infty\left(\int_{\mathbb{R}^d} G_s^b(x,y)f(y)dy\right)s^{-3/2}e^{-\frac{t^2}{4s}}ds\right| \le \frac{t}{\sqrt{4\pi}}\left|\left(\int_0^1+\int_1^\infty\right)\right|\\ &\le \frac{te^{b+1}}{\sqrt{4\pi}}\int_0^1\left(\int_{\mathbb{R}^d} G_s(x,y)\,|f(y)|\,dy\right)e^{-s}s^{-3/2}e^{-\frac{t^2}{4s}}ds\\ &\quad +C_{k,d}\frac{t}{\sqrt{4\pi}}\int_{\mathbb{R}^d}\int_1^\infty e^{-(2k+d-b)s}s^{-3/2}e^{-\frac{t^2}{4s}}e^{-\frac{|x-y|^2}{32s}}\,|f(y)|\,dsdy\\ &\le e^{b+1}e^{-t}\,|f|^*(x)+C_{k,d}\frac{t}{\sqrt{4\pi}}e^{-(2k+d-b-2)}\\ &\quad\times\int_1^\infty(e^{-s}s^{\frac{d}{2}})e^{-s}s^{-\frac{3}{2}}e^{-\frac{t^2}{4s}}s^{-\frac{d}{2}}\int_{\mathbb{R}^d} e^{-\frac{|x-y|^2}{32s}}\,|f(y)|\,dyds\\ &= C_{b,k,d}e^{-t}\,|f|^*(x).\end{aligned}$$

Thus, the Poisson integral $u_b(x,t)$ of $f(x)$ as given in (2.15) is well defined and

$$|u_b(x,t)| \le C_{b,k,d}e^{-t}|f|^*(x) \tag{2.16}$$

holds, where $2k \ge b+2-d$.

If

$$f(x) = \sum_{n=0}^{\infty}\sum_{|\alpha|=n} c_\alpha h_\alpha(x),$$

where $c_\alpha = 0$, for every α, $|\alpha| \le k-1$, $d < b+1$ and $2k \ge b+2-d$ we have

$$\begin{aligned}u_b(x,t) &= \frac{t}{\sqrt{4\pi}}\int_0^\infty\left(\int_{\mathbb{R}^d} G_s^b(x,y)f(y)dy\right)s^{-3/2}e^{-\frac{t^2}{4s}}ds\\ &= \frac{t}{\sqrt{4\pi}}\int_0^\infty\left(\sum_{n=k}^{\infty} e^{-(2n+d-b)s}\sum_{|\alpha|=n} c_\alpha h_\alpha(x)\right)s^{-3/2}e^{-\frac{t^2}{4s}}ds\\ &= \sum_{n=0}^{\infty} e^{-(2n+d-b)^{1/2}t}\sum_{|\alpha|=n} c_\alpha h_\alpha(x),\end{aligned} \tag{2.17}$$

as in the case $d \ge b+1$, see (2.13).

Collecting our results we have that the Poisson integral defines a semigroup on $L^p(\mathbb{R}^d)$, $1 < p < \infty$, in the case $d \geq b+1$, and in the case $d < b+1$ defines a semigroup on the subspace of $L^p(\mathbb{R}^d)$ consisting of all functions $f(x)$ that satisfy the condition $c_\alpha =< f, h_\alpha >= 0$ for every α, $|\alpha| \leq k-1$, where k is a positive integer such that $2k \geq b+2-d$.

Let us define the differential operator

$$L_b = L - bI = -\Delta + |x|^2 - b.$$

Obviously, for $b = 0$, this operator coincides with the operator given in (1.3). We shall also consider the operator $\mathcal{L}_b$ defined on $\mathbb{R}^d \times (0, \infty)$ by

$$\mathcal{L}_b = -\frac{\partial^2}{\partial t^2} + L_b = -\frac{\partial^2}{\partial t^2} - \Delta + |x|^2 - b.$$

Then, by (1.4) we get

$$L_b h_\alpha = (L - bI)\, h_\alpha = (2\,|\alpha| + d - b)\, h_\alpha.$$

From (2.13) and (2.17), it follows that

$$\mathcal{L}_b u_b(x, t) = 0, \tag{2.18}$$

holds for every integer b.

Now, we define

$$\mathcal{A} = (\mathcal{A}_{-d}, \cdots, \mathcal{A}_{-1}, \mathcal{A}_0, \mathcal{A}_1, \cdots, \mathcal{A}_d)$$

where as in (1.5)

$$\mathcal{A}_i = \frac{\partial}{\partial x_i} + x_i \quad \text{and} \quad \mathcal{A}_{-i} = -\frac{\partial}{\partial x_i} + x_i \quad \text{for } 1 \leq i \leq d,$$

and

$$\mathcal{A}_0 = \frac{\partial}{\partial t}.$$

The following identities for $1 \leq i \leq d$,

$$(\mathcal{A}_i u_b)^2 = \left[\left(\tfrac{\partial}{\partial x_i} + x_i\right) u_b\right]^2 = \left(\frac{\partial u_b}{\partial x_i}\right)^2 + 2x_i u_b \frac{\partial u_b}{\partial x_i} + (x_i u_b)^2,$$
$$(\mathcal{A}_{-i} u_b)^2 = \left[\left(-\tfrac{\partial}{\partial x_i} + x_i\right) u_b\right]^2 = \left(\frac{\partial u_b}{\partial x_i}\right)^2 - 2x_i u_b \frac{\partial u_b}{\partial x_i} + (x_i u_b)^2$$

and

$$(\mathcal{A}_0 u_b)^2 = \left(\frac{\partial}{\partial t} u_b\right)^2,$$

imply that

$$|\nabla_{(x,t)} u_b|^2 \le |\mathcal{A} u_b|^2 \le 2\left(|\nabla_{(x,t)} u_b|^2 + |x|^2 u_b^2\right). \tag{2.19}$$

Moreover, by (2.18)

$$\begin{aligned}
&\frac{\partial^2 u_b^2}{\partial t^2} + \sum_{i=1}^{d} \frac{\partial^2 u_b^2}{\partial x_i^2} + 2bu_b^2 \\
&= 2\left[\left(\frac{\partial u_b}{\partial t}\right)^2 + \sum_{i=1}^{d}\left(\frac{\partial u_b}{\partial x_i}\right)^2\right] + 2u_b\left[\frac{\partial^2 u_b}{\partial t^2} + \sum_{i=1}^{d}\frac{\partial^2 u_b}{\partial x_i^2} + bu_b\right] \\
&= 2\left[\left(\frac{\partial u_b}{\partial t}\right)^2 + \sum_{i=1}^{d}\left(\frac{\partial u_b}{\partial x_i}\right)^2 + |x|^2 u_b^2\right] \ge |\mathcal{A} u_b|^2 .
\end{aligned} \tag{2.20}$$

Lemma 1. *Let* $\epsilon > 0$. *Then, for* p *in the range* $1 < p \le 2$, *we have that*

$$\begin{aligned}
&\frac{\partial^2 \left(u_b^2 + \varepsilon^2\right)^{\frac{p}{2}}}{\partial t^2} + \Delta\left(u_b^2 + \varepsilon^2\right)^{\frac{p}{2}} + p\,|b|\left(u_b^2 + \varepsilon^2\right)^{\frac{p}{2}} \\
&\quad \ge \frac{p\,(p-1)}{2}\left(u_b^2 + \varepsilon^2\right)^{\frac{p-2}{2}} |\mathcal{A} u_b|^2
\end{aligned} \tag{2.21}$$

holds.

Proof. Simple calculations show that

$$\begin{aligned}
&\frac{\partial^2 \left(u_b^2 + \varepsilon^2\right)^{p/2}}{\partial t^2} + \Delta\left(u_b^2 + \varepsilon^2\right)^{p/2} \\
&= \frac{p(p-2)}{4}\left(u_b^2 + \varepsilon^2\right)^{(p-4)/2}\left[\left(\frac{\partial u_b^2}{\partial t}\right)^2 + \sum_{i=1}^{d}\left(\frac{\partial u_b^2}{\partial x_i}\right)^2\right] \\
&\quad + \frac{p}{2}\left(u_b^2 + \varepsilon^2\right)^{(p-2)/2}\left[\frac{\partial^2 u_b^2}{\partial t^2} + \sum_{i=1}^{d}\frac{\partial^2 u_b^2}{\partial x_i^2}\right] \\
&= \frac{p}{2}\left(u_b^2 + \varepsilon^2\right)^{(p-2)/2}\left[\frac{\partial^2 u_b^2}{\partial t^2} + \sum_{i=1}^{d}\frac{\partial^2 u_b^2}{\partial x_i^2} + 2bu_b^2\right] - pb\left(u_b^2 + \varepsilon^2\right)^{(p-2)/2} u_b^2 \\
&\quad + \frac{p(p-2)}{4}\left(u_b^2 + \varepsilon^2\right)^{(p-4)/2}\left[\left(\frac{\partial u_b^2}{\partial t}\right)^2 + \sum_{i=1}^{d}\left(\frac{\partial u_b^2}{\partial x_i}\right)^2 + 4\,|x|^2 u_b^4\right] \\
&\quad - p(p-2)\,|x|^2\left(u_b^2 + \varepsilon^2\right)^{(p-4)/2} u_b^4.
\end{aligned}$$

Then, by (2.20) and (2.19) we see that this expression is greater than or equal to

$$\begin{aligned}
&\frac{p}{2}\left(u_b^2 + \varepsilon^2\right)^{(p-2)/2} |\mathcal{A} u_b|^2 + \frac{p(p-2)}{2}\left(u_b^2 + \varepsilon^2\right)^{(p-4)/2} u_b^2\, |\mathcal{A} u_b|^2 \\
&\quad - pb\left(u_b^2 + \varepsilon^2\right)^{(p-2)/2} u_b^2 - p(p-2)\left(u_b^2 + \varepsilon^2\right)^{(p-4)/2} u_b^4\, |x|^2 .
\end{aligned}$$

Consequently,

$$\begin{aligned}
&\frac{\partial^2 \left(u_b^2+\varepsilon^2\right)^{p/2}}{\partial t^2} + \Delta \left(u_b^2+\varepsilon^2\right)^{p/2} \\
&\quad \geq \frac{p}{2}\left(u_b^2+\varepsilon^2\right)^{(p-2)/2} |\mathcal{A}u_b|^2 + \frac{p(p-2)}{2}\left(u_b^2+\varepsilon^2\right)^{(p-4)/2} u_b^2\, |\mathcal{A}u_b|^2 \\
&\quad\quad - pb\left(u_b^2+\varepsilon^2\right)^{(p-2)/2} u_b^2 - p(p-2)\left(u_b^2+\varepsilon^2\right)^{(p-4)/2} u_b^4\, |x|^2 .
\end{aligned}$$

Since we are assuming that $1 < p \leq 2$, the former inequality implies

$$\begin{aligned}
&\frac{\partial^2 \left(u_b^2+\varepsilon^2\right)^{p/2}}{\partial t^2} + \Delta \left(u_b^2+\varepsilon^2\right)^{p/2} \\
&\quad \geq \frac{p\,(p-1)}{4}\left(u_b^2+\varepsilon^2\right)^{(p-2)/2} |\mathcal{A}u_b|^2 - p|b|\left(u_b^2+\varepsilon^2\right)^{p/2},
\end{aligned}$$

which completes the proof of Lemma 1 □

Definition 1. *Given a function $f \in L^p(\mathbb{R}^d)$, with p in the range $1 < p < \infty$, we define the square functions $g_b(f)$ and $g_b^1(f)$ as*

$$g_b(f)(x) = \left(\int_0^\infty |t\mathcal{A}u_b(x,t)|^2 \frac{dt}{t}\right)^{1/2},$$

and

$$g_b^1(f)(x) = \left(\int_0^\infty \left|t\frac{\partial u_b}{\partial t}(x,t)\right|^2 \frac{dt}{t}\right)^{1/2},$$

where $u_b(x,t) = P_t^b f(x)$, is the "Poisson" integral of the function $f(x)$, defined in (2.11) and (2.15).

Lemma 2. *Let us assume that p is in the range $1 < p \leq 2$. Then*

$$\|g_b(f)\|_{L^p(\mathbb{R}^d)} \leq C\|f\|_{L^p(\mathbb{R}^d)},$$

holds under the following conditions:

(i) if $d \geq b+1$ the function $f(x)$ belongs to $L^p(\mathbb{R}^d)$ and the constant C depends on p, b and not on the dimension d,
and
(ii) if $d \leq b$ the function $f(x)$ belongs to $L^p(\mathbb{R}^d)$ and $c_\alpha =< f, h_\alpha >= 0$ for every $|\alpha| \leq k-1$, where the non negative integer k satisfies $2k \geq b-d+2$, and the constant C depends on p, k, d and b.

Proof. We can assume that $f(x)$ is a finite linear combination of the Hermite functions h_α, since these functions are dense in $L^p(\mathbb{R}^d)$. For each $N > 0$, applying Lemma 1 we have that

$$\int_0^N |t\mathcal{A}u_b|^2 \frac{dt}{t} \le \frac{2}{p(p-1)} \int_0^N (u_b^2+\varepsilon^2)^{\frac{2-p}{2}} \times \left(\frac{\partial^2 (u_b^2+\varepsilon^2)^{\frac{p}{2}}}{\partial t^2} + \Delta (u_b^2+\varepsilon^2)^{\frac{p}{2}} + p\,|b|\,(u_b^2+\varepsilon^2)^{\frac{p}{2}} \right) t dt. \tag{2.22}$$

By (2.21) the sum $\left(\frac{\partial^2 (u_b^2+\varepsilon^2)^{p/2}}{\partial t^2} + \Delta (u_b^2+\varepsilon^2)^{\frac{p}{2}} + p\,|b|\,(u_b^2+\varepsilon^2)^{\frac{p}{2}} \right)$ is non-negative, then the former integral is bounded by

$$\frac{2}{p(p-1)} \left(\sup_{0<t\le N} u_b(x,t)^2 + \varepsilon^2 \right)^{\frac{2-p}{2}} \times \left(\int_0^N t\Delta_{(x,t)} (u_b^2+\varepsilon^2)^{\frac{p}{2}} dt + p|b| \int_0^\infty t(u_b^2+\varepsilon^2)^{\frac{p}{2}} dt \right).$$

Applying Beppo Levi's theorem, we get

$$\int_{\mathbb{R}^d} g_b(f)(x)^p dx = \lim_{N\to\infty} \int_{|x|\le N} \left(\int_0^N |t\mathcal{A}u_b(x,t)|^2 \frac{dt}{t} \right)^{p/2} dx.$$

In virtue of (2.22), we have that

$$\begin{aligned} &\int_{|x|\le N} \left(\int_0^N |t\mathcal{A}u_b(x,t)|^2 \frac{dt}{t} \right)^{p/2} dx \\ &\le C_p \int_{|x|\le N} \left(\sup_{0<t\le N} u_b(x,t)^2 + \varepsilon^2 \right)^{\frac{p(2-p)}{4}} \\ &\quad \times \left(\int_0^N t\Delta_{(x,t)} \left(u_b^2+\varepsilon^2\right)^{\frac{p}{2}} dt + p\,|b| \int_0^\infty t\left(u_b^2+\varepsilon^2\right)^{\frac{p}{2}} dt \right)^{\frac{p}{2}} dx. \end{aligned}$$

By Hölder's inequality, this expression is bounded by the constant C_p times the product of

$$\left(\int_{|x|\le N} \left(\sup_{t>0} u_b(x,t)^2 + \varepsilon^2 \right)^{p/2} dx \right)^{(2-p)/2}$$

and

$$\left(\int_{|x|\le N} \left(\int_0^N t\Delta_{(x,t)} \left(u_b^2+\varepsilon^2\right)^{p/2} dt + p\,|b| \int_0^N t\left(u_b^2+\varepsilon^2\right)^{p/2} dt \right) dx \right)^{p/2}.$$

Taking into account (2.12), (2.16) and (2.2) we have that

$$\begin{aligned}
&\lim_{\varepsilon\to 0}\left(\int_{|x|\le N}\left(\sup_{t>0} u_b(x,t)^2+\varepsilon^2\right)^{p/2} dx\right)^{(2-p)/2}\\
&=\left(\int_{|x|\le N}\left(\sup_{t>0}|u_b(x,t)|\right)^{p} dx\right)^{(2-p)/2}\le C\left(\int_{\mathbb{R}^d}|f|^*(x)^p dx\right)^{(2-p)/2}\\
&\le C\left(\int_{\mathbb{R}^d}|f(x)|^p\,dx\right)^{(2-p)/2},
\end{aligned}$$

where the constant C depends only on p if $d\ge b+1$, and depends on p, k, d and b in the case $d\le b$.

Let us compute

$$\lim_{\varepsilon\to 0}\left(\int_{|x|\le N}\left(\int_0^N t\Delta_{(x,t)}\left(u_b^2+\varepsilon^2\right)^{p/2}dt+p\,|b|\int_0^N t\left(u_b^2+\varepsilon^2\right)^{p/2}dt\right)dx\right). \tag{2.23}$$

If ∂Q_N denotes the boundary of $Q_N=\{(x,t):|x|\le N, 0\le t\le N\}$, then by Green's formula, we obtain that

$$\begin{aligned}
&\int_{|x|\le N}\int_0^N t\Delta_{(x,t)}\left(u_b^2+\varepsilon^2\right)^{p/2}dtdx\\
&=\int_{\partial Q_N}\left(t\frac{\partial}{\partial\eta}\left(u_b^2+\varepsilon^2\right)^{p/2}-\left(u_b^2+\varepsilon^2\right)^{p/2}\frac{\partial t}{\partial\eta}\right)d\sigma(x,t)\\
&=\int_{\partial Q_N}\left(tp\left(u_b^2+\varepsilon^2\right)^{(p/2)-1}u_b\frac{\partial u_b}{\partial\eta}-\left(u_b^2+\varepsilon^2\right)^{p/2}\frac{\partial t}{\partial\eta}\right)d\sigma(x,t)\\
&\le\int_{\partial Q_N}\left(tp\left(u_b^2+\varepsilon^2\right)^{(p-1)/2}\left|\frac{\partial u_b}{\partial\eta}\right|-\left(u_b^2+\varepsilon^2\right)^{p/2}\frac{\partial t}{\partial\eta}\right)d\sigma(x,t).
\end{aligned}$$

Then, the limit in (2.23) is less than or equal to

$$\int_{\partial Q_N}\left(tp|u_b|^{p-1}\left|\frac{\partial u_b}{\partial\eta}\right|-|u_b|^p\frac{\partial t}{\partial\eta}\right)d\sigma(x,t)+p|b|\int_{|x|\le N}\int_0^N t|u_b|^p dtdx. \tag{2.24}$$

Recalling that we are assuming that u_b is a finite linear combination of functions of the type $e^{-(2|\alpha|+d-b)^{1/2}t}h_\alpha(x)$, taking the limit for N tending to infinite, (2.12), (2.16) and (2.2), we have that (2.24) is majorized by

$$\int_{\mathbb{R}^d}|f(x)|^p\,dx+C\int_{\mathbb{R}^d}|f|^*(x)^p dx\int_0^\infty te^{-pt}dt\le C\int_{\mathbb{R}^d}|f(x)|^p\,dx.$$

Therefore, if $1 < p \le 2$ it follows that

$$\|g(f)\|_{L^p(\mathbb{R}^d)} \le C\|f\|_{L^p(\mathbb{R}^d)},$$

where if $d \ge b+1$, the constant C depends on p, b and not on the dimension d, and if $d \le b$ the constant C depends on p, k, d and b, where k satisfies the condition $2k \ge b - d + 2$ □

Lemma 3. *The following inequalities hold:*

(i) If $d \ge b+3$ and $1 \le i \le d$

$$|\mathcal{A}_{-i}u_b(x,t)|^2 \le C_b e^{-t/2} \int_{\mathbb{R}^d} P_{t/2}^{b+2}(x,y)\, |\mathcal{A}_{-i}u_b(y,t/2)|^2\, dy,$$

$$|\mathcal{A}_{i}u_b(x,t)|^2 \le C_b e^{-t/2} \int_{\mathbb{R}^d} P_{t/2}^{b-2}(x,y)\, |\mathcal{A}_{i}u_b(y,t/2)|^2\, dy,$$

and

$$|\mathcal{A}_{0}u_b(x,t)|^2 \le C_b e^{-t/2} \int_{\mathbb{R}^d} P_{t/2}^{b}(x,y)\, |\mathcal{A}_{0}u_b(y,t/2)|^2\, dy$$

where the constant C_b does not depend on the dimension d.

(ii) If $d \le b+2$ and $2k \ge b-d+3$, let $f(x)$ be a function with Fourier coefficients $c_\alpha = 0$ for every $|\alpha| \le k-1$, then

$$|\mathcal{A}_{i}u_b(x,t)|^2 \le C_{k,d,b}\Big(\int_{\mathbb{R}^d} P^0_{t/2}(x,y)|\mathcal{A}_i u_b(y,t/2)|^2 dy + \int_{\mathbb{R}^d} \frac{t}{(|x-y|^2+t^2)^{\frac{(d+1)}{2}}} |\mathcal{A}_i u_b(y,t/2)|^2 dy \Big)$$

for every $-d \le i \le d$.

Proof. Let us consider (i). For $1 \le i \le d$, we have

$$\begin{aligned}\mathcal{A}_{-i}u_b(x,t) &= -\frac{\partial u_b}{\partial x_i}(x,t) + x_i u_b(x,t) \\ &= \sum_{n=0}^{\infty} e^{-(2n+d-b)^{1/2}t/2} \sum_{|\alpha|=n} e^{-(2n+d-b)^{1/2}t/2} c_\alpha \sqrt{2(\alpha_i+1)}\, h_{\alpha+e_i}(x).\end{aligned}$$

Since $d \ge b+3$,

$$\mathcal{A}_{-i}u_b(x,t) = P_{t/2}^{b+2}\left(\mathcal{A}_{-i}u_b(\cdot,t/2)\right)(x).$$

Therefore, by Schwarz's inequality we get

$$\begin{aligned}|\mathcal{A}_{-i}u_b(x,t)|^2 &= \left| P_{t/2}^{b+2}\left(\mathcal{A}_{-i}u_b(\cdot,t/2)\right)(x)\right|^2 \\ &\le \left(\int_{\mathbb{R}^d} P_{t/2}^{b+2}(x,y)dy\right)\left(\int_{\mathbb{R}^d} P_{t/2}^{b+2}(x,y)\, |\mathcal{A}_{-i}u_b(y,t/2)|^2\, dy\right),\end{aligned}$$

which by (2.10) implies

$$|\mathcal{A}_{-i}u_b(x,t)|^2 \le C_b e^{-t/2} \int_{\mathbb{R}^d} P_{t/2}^{b+2}(x,y)\, |\mathcal{A}_{-i}u_b(y,t/2)|^2\, dy.$$

Analogously, we can obtain the others inequalities in (i).

Now, let us consider case (ii). For $-d \le i \le d$, by (2.17) we have

$$\begin{aligned}|\mathcal{A}_i u_b(x,t)| &\le \Bigg| \frac{t}{4\sqrt{\pi}} \int_0^\infty \Big(\int_{\mathbb{R}^d} e^{(b-2sg(i)-d)s} \Big[\sum_{n=k}^\infty e^{-2ns} \sum_{|\alpha|=n} h_\alpha(x) h_\alpha(y) \Big] \\ &\qquad \times \mathcal{A}_i u_b(y,t/2) dy \Big) s^{-\frac{3}{2}} e^{-\frac{t^2}{16s}} ds \Bigg| \\ &\le \frac{t}{4\sqrt{\pi}} \left(\left| \int_0^1 \right| + \left| \int_1^\infty \right| \right) = A + B.\end{aligned}$$

We observe that this is valid also taking $k = 0$. Let us estimate A. By (2.9) and Schwarz's inequality we have

$$\begin{aligned}A &\le \frac{t e^{|b|+2}}{4\sqrt{\pi}} \int_0^1 \int_{\mathbb{R}^d} G_s(x,y)\, |\mathcal{A}_i u_b(y,t/2)|\, dy s^{-3/2} e^{-\frac{t^2}{16s}} ds \\ &\le C_b e^{-t/4} \left(\int_{\mathbb{R}^d} P_{t/2}^0(x,y)\, |\mathcal{A}_i u_b(y,t/2)|^2\, dy \right)^{1/2}.\end{aligned}$$

As for B, it is bounded by

$$\frac{t}{4\sqrt{\pi}} \int_{\mathbb{R}^d} \int_1^\infty e^{(b+2-d)s} \Big| \sum_{n=k}^\infty e^{-2ns} \sum_{|\alpha|=n} h_\alpha(x) h_\alpha(y) \Big| |\mathcal{A}_i u_b(y,t/2)| dy s^{-\frac{3}{2}} e^{-\frac{t^2}{16s}} ds$$

which, in virtue of (2.14), is less than or equal to

$$\frac{C_{k,d}\, t}{4\sqrt{\pi}} \int_{\mathbb{R}^d} \left(\int_1^\infty e^{-(2k+d-b-3)s} (e^{-s} s^{\frac{d}{2}}) s^{-\frac{d}{2}} e^{-\frac{|x-y|^2}{32s}} s^{-\frac{3}{2}} e^{-\frac{t^2}{16s}} ds \right) |\mathcal{A}_i u_b(y,t/2)| dy.$$

This expression is bounded by

$$\begin{aligned}&C_{k,d,b} \frac{t}{4\sqrt{\pi}} \int_{\mathbb{R}^d} \left(\int_1^\infty s^{-d/2} e^{-\frac{|x-y|^2}{32s}} s^{-3/2} e^{-\frac{t^2}{32s}} ds \right) |\mathcal{A}_i u_b(y,t/2)| dy \\ &\le C_{k,d,b} \int_{\mathbb{R}^d} \frac{t}{(|x-y|^2+t^2)^{(d+1)/2}} |\mathcal{A}_i u_b(y,t/2)| dy \\ &\le C_{k,d,b} \left(\int_{\mathbb{R}^d} \frac{t}{(|x-y|^2+t^2)^{(d+1)/2}} |\mathcal{A}_i u_b(y,t/2)|^2 dy \right)^{1/2}\end{aligned}$$

as we wanted to prove. □

Lemma 4. *Let us assume that* p *is in the range* $2 < p < \infty$. *Then*

$$\|g_b(f)\|_{L^p(\mathbb{R}^d)} \le C\|f\|_{L^p(\mathbb{R}^d)},$$

holds under the following conditions:

(i) if $d \ge b+3$ *the function* $f(x)$ *belongs to* $L^p(\mathbb{R}^d)$ *and the constant* C *depends on* p, b *and not on the dimension* d,
and

(ii) if $d \le b+2$ *the function* $f(x)$ *belongs to* $L^p(\mathbb{R}^d)$ *and* $c_\alpha =< f, h_\alpha >= 0$ *for every* $|\alpha| \le k-1$, *where the non negative integer* k *satisfies* $2k \ge b-d+3$, *and the constant* C *depends on* p, k, d *and* b.

Proof. Let us assume first that $p > 4$ and $\phi(x)$ be a non-negative function. We can take $f(x)$ a finite linear combination of the Hermite functions h_α, since these functions are dense in $L^p(\mathbb{R}^d)$. Let us consider (i). Since P_t^a for $a = b-2, b$ or $b+2$ are self-adjoint operators, then by Lemma 3 part (i) we have

$$\begin{aligned}
&\int_{\mathbb{R}^d} [g_b(f)(x)]^2 \phi(x)dx \\
&= \int_{\mathbb{R}^d}\int_0^\infty t|\mathcal{A}u_b(x,t)|^2\phi(x)dxdt \\
&\le C_b \int_{\mathbb{R}^d}\int_0^\infty t(P_t^b + P_t^{b-2} + P_t^{b+2})(|\mathcal{A}u_b|^2)(x,t)\phi(x)dxdt \\
&= C_b \int_{\mathbb{R}^d}\int_0^\infty t|\mathcal{A}u_b(x,t)|^2(P_t^b + P_t^{b-2} + P_t^{b+2})\phi(x)dxdt.
\end{aligned}$$

Applying Lemma 1 with $p = 2$, the last expression is bounded by

$$\begin{aligned}
&C_b \int_{\mathbb{R}^d}\int_0^\infty t\left(\frac{\partial^2 u_b^2}{\partial t^2} + \Delta u_b^2 + 2|b|u_b^2\right)\left(P_t^b + P_t^{b-2} + P_t^{b+2}\right)\phi(x)dxdt \\
&= C_b \int_{\mathbb{R}^d}\int_0^\infty t\left(\frac{\partial^2 u_b^2}{\partial t^2} + \Delta u_b^2\right)\left(P_t^b + P_t^{b-2} + P_t^{b+2}\right)\phi(x)dxdt \\
&\quad + C_b \int_{\mathbb{R}^d}\int_0^\infty 2t|b|u_b^2\left(P_t^b + P_t^{b-2} + P_t^{b+2}\right)\phi(x)dxdt = A + B.
\end{aligned}$$

Taking into account (2.12), applying Hölder's inequality with exponents $\frac{2}{p}+\frac{1}{q} = 1$ and (2.2) we have that

$$\begin{aligned}
B &= C_b \int_{\mathbb{R}^d}\int_0^\infty tu_b^2\left(P_t^b + P_t^{b-2} + P_t^{b+2}\right)\phi\,(x)\,dxdt \\
&\le C_b \int_{\mathbb{R}^d}\left(\int_0^\infty te^{-t}dt\right)\left(\sup_{t>0}|u_b(x,t)|\right)^2\phi^*(x)dx \\
&\le C_b \int_{\mathbb{R}^d} |f|^*(x)^2\phi^*(x)dx \le C_{b,p}\,\|f\|_p^2\,\|\phi\|_q\,.
\end{aligned}$$

On the other hand and since $\mathcal{L}_a P_t^a \phi = 0$ it follows that

$$\begin{aligned}\Delta_{(x,t)}\left(u_b^2 P_t^a \phi\right) &= P_t^a \phi \Delta_{(x,t)} u_b^2 + 4u_b\left(\nabla_{(x,t)} u_b \cdot \nabla_{(x,t)} P_t^a \phi\right) + u_b^2 \Delta_{(x,t)} P_t^a \phi \\ &= P_t^a \phi \Delta_{(x,t)} u_b^2 + 4u_b\left(\nabla_{(x,t)} u_b \cdot \nabla_{(x,t)} P_t^a \phi\right) + u_b^2\left(|x|^2 P_t^a \phi - a P_t^a \phi\right) \\ &\geq P_t^a \phi \Delta_{(x,t)} u_b^2 + 4u_b\left(\nabla_{(x,t)} u_b \cdot \nabla_{(x,t)} P_t^a \phi\right) - a u_b^2 P_t^a \phi.\end{aligned}$$

In consequence,

$$P_t^a \phi \Delta_{(x,t)} u_b^2 \leq \Delta_{(x,t)}\left(u_b^2 P_t^a \phi\right) - 4u_b\left(\nabla_{(x,t)} u_b \cdot \nabla_{(x,t)} P_t^a \phi\right) + a u_b^2 P_t^a \phi. \tag{2.25}$$

Then, we have

$$\begin{aligned}A &= C_b \int_{\mathbb{R}^d}\int_0^\infty t\left(\frac{\partial^2 u_b^2}{\partial t^2} + \Delta u_b^2\right)\left(P_t^b + P_t^{b-2} + P_t^{b+2}\right)\phi(x)\,dxdt \\ &\leq C_b \sum\{I_a : a = b-2, b, b+2\},\end{aligned}$$

where

$$\begin{aligned}I_a &= \int_{\mathbb{R}^d}\int_0^\infty t\left(\Delta_{(x,t)}(u_b^2 P_t^a \phi) - 4u_b(\nabla_{(x,t)} u_b \cdot \nabla_{(x,t)} P_t^a \phi) + a u_b^2 P_t^a \phi\right) dxdt \\ &\leq \int_{\mathbb{R}^d}\int_0^\infty t\Delta_{(x,t)}(u_b^2 P_t^a \phi)dxdt + 4\int_{\mathbb{R}^d}\int_0^\infty t|u_b||\nabla_{(x,t)} u_b||\nabla_{(x,t)} P_t^a \phi|dxdt \\ &\quad + |a|\int_{\mathbb{R}^d}\int_0^\infty t u_b^2 P_t^a \phi dxdt.\end{aligned}$$

By Green's formula, (2.19) and (2.12) it follows that I_a is bounded by

$$\begin{aligned}&\int_{\mathbb{R}^d} f(x)^2 \phi(x)dx + C_b \int_{\mathbb{R}^d} |f|^*(x) g_b(f)(x) g_a(\phi)(x)dx \\ &\quad + C_b|a|\int_{\mathbb{R}^d}\left(\int_0^\infty t e^{-t} dt\right)|f|^*(x)^2 \phi^*(x)dx.\end{aligned}$$

Since $q < 2$ for $p > 4$, applying Hölder's inequality and (2.2) this expression is majorized by

$$C_{b,p}\left(\|f\|_p^2 \|\phi\|_q + \|g_a(\phi)\|_q \|g_b(f)\|_p \|f\|_p\right).$$

By Lemma 2, this is bounded by

$$C_{b,p}\left(\|f\|_p^2 \|\phi\|_q + \|\phi\|_q \|g_b(f)\|_p \|f\|_p\right)$$

and if we suppose that $\|\phi\|_q = 1$, it follows that

$$\|g_b(f)\|_p^2 \leq A + B \leq C_{b,p}\left(\|f\|_p^2 + \|g_b(f)\|_p \|f\|_p\right),$$

which implies that

$$\|g_b(f)\|_p \leq \left(2C_{b,p}+1\right) \|f\|_p\,,$$

holds for $d \geq b+3$ and $p > 4$, with a constant not depending on the dimension d.

Now, let us consider (ii). We have that

$$\int_{\mathbb{R}^d} [g_b(f)(x)]^2\, \phi(x)dx = \int_{\mathbb{R}^d}\int_0^\infty t\, |\mathcal{A}u_b|^2\, \phi(x)dxdt$$

which, in virtue of Lemma 3 part (ii), it is bounded by

$$\begin{aligned} C_{k,d,b} \int_{\mathbb{R}^d}\int_0^\infty t \int_{\mathbb{R}^d} P^0_{t/2}\,(x,y)\, |\mathcal{A}u_b\,(y,t/2)|^2\, dy\phi(x)dtdx \\ + C_{k,d,b} \int_{\mathbb{R}^d}\int_0^\infty t \int_{\mathbb{R}^d} \frac{t}{\left(|x-y|^2+t^2\right)^{(d+1)/2}}\, |\mathcal{A}u_b\,(y,t/2)|^2\, dy\phi(x)dxdt. \end{aligned}$$

Applying Fubini's theorem we see that this expression is less than or equal to

$$\begin{aligned} C_{k,d,b} \int_0^\infty t \int_{\mathbb{R}^d} P^0_{t/2}\,(\phi)\,(y)\, |\mathcal{A}u_b\,(y,t/2)|^2\, dydt \\ + C_{k,d,b} \int_0^\infty t \int_{\mathbb{R}^d} \phi_t\,(y)\, |\mathcal{A}u_b\,(y,t/2)|^2\, dydt, \end{aligned}$$

where $\phi_t(y)$ denotes the harmonic function

$$\phi_t(y) = \frac{\Gamma\left(\frac{d+1}{2}\right)}{\pi^{\frac{d+1}{2}}} \int_{\mathbb{R}^d} \frac{t}{\left(|x-y|^2+t^2\right)^{(d+1)/2}} \phi\,(x)\, dx.$$

As it is well known, (see [S1]), $\lim_{t\to 0} \phi_t(y) = \phi(y)$ almost everywhere. Therefore, by Lemma 1, we have

$$\begin{aligned} &\int_{\mathbb{R}^d} [g_b(f)(x)]^2\, \phi(x)dx \\ &\quad \leq C_{k,d,b} \int_{\mathbb{R}^d}\int_0^\infty t \left(\frac{\partial^2 u_b^2}{\partial t^2} + \Delta u_b^2 + 2bu_b^2\right) \left(P^0_t(\phi)(x) + \phi_t(x)\right) dxdt. \end{aligned}$$

Moreover, by (2.25) with $a = 0$

$$P^0_t \phi \Delta_{(x,t)} u_b^2 \leq \Delta_{(x,t)} \left(u_b^2 P^0_t \phi\right) - 4u_b \left(\nabla_{(x,t)} u_b \cdot \nabla_{(x,t)} P^0_t \phi\right),$$

and since $\Delta_{(x,t)}\phi_t = 0$ we get

$$\phi_t \Delta_{(x,t)} u_b^2 = \Delta_{(x,t)} \left(u_b^2 \phi_t\right) - 4u_b \left(\nabla_{(x,t)} u_b \cdot \nabla_{(x,t)} \phi_t\right).$$

Then,

$$\begin{aligned}
&\int_{\mathbb{R}^d} [g_b(f)(x)]^2 \phi(x)dx \\
&\leq C_{k,d,b} \left[\int_{\mathbb{R}^d} \int_0^\infty t\Delta_{(x,t)}(u_b^2 P_t^0 \phi)dxdt \right. \\
&\quad + \int_{\mathbb{R}^d} \int_0^\infty t|u_b||\nabla_{(x,t)} u_b||\nabla_{(x,t)} P_t^0 \phi|dxdt \int_{\mathbb{R}^d} \int_0^\infty t u_b^2 P_t^0(\phi)dxdt \\
&\quad + \int_{\mathbb{R}^d} \int_0^\infty t\Delta_{(x,t)}(u_b^2 \phi_t)dxdt + \int_{\mathbb{R}^d} \int_0^\infty t|u_b||\nabla_{(x,t)} u_b||\nabla_{(x,t)} \phi_t|dxdt \\
&\quad \left. + \int_{\mathbb{R}^d} \int_0^\infty t u_b^2 \phi dxdt \right].
\end{aligned}$$

Considering that the classical $g-$function for harmonic functions is bounded on $L^q(\mathbb{R}^d)$ (see [S1]), and proceeding as in the proof of (i) above, we get that

$$\|g_b(f)\|_p \leq \left(2C_{k,d,b,p} + 1\right) \|f\|_p$$

holds whenever $d \leq b+2$, $2k \geq b-d+3$, with a constant depending on the dimension d in this case □

Lemmas 2 and 4 imply the following theorem:

Theorem 1. *The functions g_b and g_b^1 introduced in Definition 1 satisfy*

$$\|g_b^1(f)\|_{L^p(\mathbb{R}^d)} \leq \|g_b(f)\|_{L^p(\mathbb{R}^d)} \leq C\|f\|_{L^p(\mathbb{R}^d)},$$

for $1 < p < \infty$, under the following conditions:

(i) if $d \geq b+3$ the function $f(x)$ belongs to $L^p(\mathbb{R}^d)$ and the constant C depends on p, b and not on the dimension d,
and
(ii) if $d \leq b+2$ the function $f(x)$ belongs to $L^p(\mathbb{R}^d)$ and $c_\alpha =< f, h_\alpha >= 0$ for every $|\alpha| \leq k-1$, where the non negative integer k satisfies $2k \geq b-d+3$, and the constant C depends on p, k, d and b.

Proof. The first inequality is obvious. The second inequality is a consequence of Lemma 2 for $1 < p \leq 2$ and of Lemma 4 for $4 \leq p < \infty$. The case $2 < p < 4$ follows by interpolation □

3. Littlewood-Paley $g-$functions of higher order

Let H be a Hilbert space and $f(x)$ an $H-$valued strongly measurable function defined on $\mathbb{R}^d$. As usual, we say that the function $f \in L^p_H(\mathbb{R}^d)$ if and only if $\|f(x)\|_H \in L^p(\mathbb{R}^d)$. Let us denote the norm in $L^p_H(\mathbb{R}^d)$ by

$$\|f\|_{L^p_H(\mathbb{R}^d)} = \left(\int_{\mathbb{R}^d} \|f(x)\|_H^p \, dx\right)^{1/p}.$$

If $S \subseteq \mathbb{R}^d$ is a measurable set, $\int_S f(x)dx$ will stand for the integral of $f(x)$ in the sense of Bochner (see [Y], page 132). Let us consider the kernel $G_t(x, y)$ defined in (2.3). Given $f \in L^p_H(\mathbb{R}^d)$ and b an integer, we define the "Poisson" integral of $f(x)$ as in (2.11)

$$P_t^b f(x) = \frac{t}{\sqrt{4\pi}} \int_0^\infty \left(\int_{\mathbb{R}^d} e^{bs} G_s(x, y) f(y) dy \right) s^{-3/2} e^{-\frac{t^2}{4s}} ds,$$

if $d \geq b + 1$. In the case when $d \leq b$, we define $P_t^b f(x)$ by the same formula but we ask $f(x)$ to satisfy the condition $c_\alpha = \int_{\mathbb{R}^d} f(y) h_\alpha(y) dy = 0$ for every α, $|\alpha| \leq k - 1$, where k is a positive integer such that $2k \geq b - d + 2$. As before, we have the expansion

$$P_t^b f(x) = \sum_{n=0}^{\infty} e^{-(2n+d-b)^{1/2} t} \sum_{|\alpha|=n} c_\alpha h_\alpha(x), \tag{3.1}$$

where $c_\alpha = \int_{\mathbb{R}^d} f(y) h_\alpha(y) dy \in H$.

It is easy to check that for $1 \leq |i| \leq d$

$$\mathcal{A}_i h_\alpha(x) = \sqrt{2\alpha_{|i|} + 1 - sg(i)}\; h_{\alpha - sg(i) e_{|i|}}(x)$$

and as before,

$$\mathcal{A}_0 = \frac{\partial}{\partial t}.$$

Using the expansion (3.1) it is easy to verify that for $t > 0$ we can define $\mathcal{A}_i P_t^b f(x) \in H$, $-d \leq i \leq d$, in such a way that

$$< \mathcal{A}_i P_t^b f(x), v >_H = \mathcal{A}_i P_t^b (< f(\cdot), v >_H)(x),$$

holds for every $v \in H$ and every $x \in \mathbb{R}^d$.

Let $\mathcal{S}_{H,b}$ be the linear application from $L^p_H(\mathbb{R}^d)$ into $H^{2d+1}-$valued function defined on $\mathbb{R}^d$ given by

$$\mathcal{S}_{H,b} f(x) = \left\{ \mathcal{S}_{H,b,j} f(x) \right\}_{j=-d}^{d} = \left\{ t \mathcal{A}_j P_t^b f(x) \right\}_{j=-d}^{d}.$$

Then, we define the Littlewood-Paley function $g_{H,b}$ as

$$g_{H,b}(f)(x) = \left(\int_0^\infty \sum_{j=-d}^{d} \left\| \mathcal{S}_{H,b,j} f(x) \right\|_H^2 \frac{dt}{t} \right)^{1/2}.$$

Remark 1. For $H = \mathbb{R}$ we have that $g_{\mathbb{R},b}(f)$ is equal to the function $g_b(f)$ given in Definition 1. By Theorem 1 we have that

$$\left\| g_{\mathbb{R},b}(f) \right\|_{L^p(\mathbb{R}^d)} = \left\| \left(\int_0^\infty \sum_{j=-d}^{d} \left| \mathcal{S}_{\mathbb{R},b,j} f(x) \right|^2 \frac{dt}{t} \right)^{1/2} \right\|_{L^p(\mathbb{R}^d)} \leq C \, \| f \|_{L^p(\mathbb{R}^d)} \,, \tag{3.2}$$

holds for $1 < p < \infty$ and $d \geq b + 3$, with a constant C not depending on the dimension d. For the case $d \leq b + 2$, the inequality (3.2) holds provided that $\int f(y) h_\alpha(y) dy = 0$ for every α such that $|\alpha| \leq k - 1$, with k an integer satisfying $2k \geq b - d + 3$, but this time the constant C depends on the dimension d.

The following theorem due to J. L. Krivine allows us to extend (3.2) to $H-$valued functions.

Theorem 2. *Given two Banach lattices A and B and a bounded linear operator $T : A \to B$, we have that for any sequence $f_1, f_2, \cdots, f_n, \cdots \in A$,*

$$\left\| \left(\sum_{j=1}^{\infty} \left| T f_j \right|^2 \right)^{1/2} \right\|_B \leq G \, \| T \| \left\| \left(\sum_{j=1}^{\infty} \left| f_j \right|^2 \right)^{1/2} \right\|_A \,,$$

where G is the Grothendieck's universal constant, which satisfies $1 < G < 2$.

For a proof of this theorem see [Kr] or [LT].

The following theorem generalizes (3.2) for $H-$valued functions.

Theorem 3. *Let H be a Hilbert space. Then, for $1 < p < \infty$ it follows that*

$$\left\| g_{H,b}(f) \right\|_{L^p(\mathbb{R}^d)} = \left\| \left(\int_0^\infty \sum_{j=-d}^{d} \left\| \mathcal{S}_{H,b,j} f(x) \right\|_H^2 \frac{dt}{t} \right)^{1/2} \right\|_{L^p(\mathbb{R}^d)} \leq GC \, \| f \|_{L^p_H(\mathbb{R}^d)}$$

with the same restrictions on d, b and k as in the case when $H = \mathbb{R}$, C the constant of that case (see Remark 1) and G the Grothendieck's universal constant. We observe that GC does not depend on the Hilbert space H.

Proof. Let $\{v_n\}$ be an orthonormal basis of the Hilbert space H. Then,

$$f(x) = \sum_n f_n(x) v_n \quad \text{and} \quad \| f(x) \|_H^2 = \sum_n | f_n(x) |^2 \,.$$

Thus,

$$\left\| \left(\int_0^\infty \sum_{j=-d}^{d} \left\| \mathcal{S}_{H,b,j} f(x) \right\|_H^2 \frac{dt}{t} \right)^{1/2} \right\|_{L^p(\mathbb{R}^d)}$$

$$= \left\| \left(\int_0^\infty \sum_{j=-d}^{d} \sum_n \left| \mathcal{S}_{\mathbb{R},b,j} f_n(x) \right|^2 \frac{dt}{t} \right)^{1/2} \right\|_{L^p(\mathbb{R}^d)}$$

$$= \left\| \left(\sum_n \int_0^\infty \left\| \mathcal{S}_{\mathbb{R},b} f_n(x) \right\|^2_{\mathbb{R}^{2d+1}} \frac{dt}{t} \right)^{1/2} \right\|_{L^p(\mathbb{R}^d)} .$$

Since by Theorem 1, $\mathcal{S}_{\mathbb{R},b}$ is a linear operator from the Banach lattice $A = L^p(\mathbb{R}^d)$ into the Banach lattice $B = L^p_{L^2_{\mathbb{R}^{2d+1}}((0,\infty),dt/t)}(\mathbb{R}^d)$ we can apply Theorem 2 obtaining

$$\left\| \left(\sum_n \int_0^\infty \left\| \mathcal{S}_{\mathbb{R},b} f_n(x) \right\|^2_{\mathbb{R}^{2d+1}} \frac{dt}{t} \right)^{1/2} \right\|_{L^p(\mathbb{R}^d)}$$

$$\leq GC_{p,b} \left\| \left(\sum_n |f_n(x)|^2 \right)^{1/2} \right\|_{L^p(\mathbb{R}^d)}$$

$$= GC_{p,b} \, \|f\|_{L^p_H(\mathbb{R}^d)}$$

□

Given a positive integer m and $f \in L^p_H(\mathbb{R}^d)$, we introduce

$$\mathcal{S}_{m,H,b} f(x) = \left\{ t^m \mathcal{A}_{i_1} \mathcal{A}_{i_2} \ldots \mathcal{A}_{i_m} P_t^b f(x) \right\}_{-d \leq i_1, i_2, \ldots, i_m \leq d}$$

and the Littlewood-Paley $g-$functions $g_{m,H,b}$ y $g^1_{m,H,b}$ given by

$$g_{m,H,b}(f)(x)^2 = \int_0^\infty \sum_{-d \leq i_1, i_2, \cdots, i_m \leq d} \left\| t^m \mathcal{A}_{i_1} \mathcal{A}_{i_2} \ldots \mathcal{A}_{i_m} P_t^b f(x) \right\|^2_H \frac{dt}{t},$$

and

$$g^1_{m,H,b}(f)(x)^2 = \int_0^\infty \left\| t^m \mathcal{A}_0^m P_t^b f(x) \right\|^2_H \frac{dt}{t} = \int_0^\infty \left\| t^m \frac{\partial^m P_t^b f}{\partial t^m}(x) \right\|^2_H \frac{dt}{t}.$$

We observe that if $m = 1$, $g_{1,H,b}(f)$ coincides with the $g_{H,b}(f)$ already defined.

For the function $g_{m,H,b}$ we have the following theorem:

Theorem 4. *Let $1 < p < \infty$, b an integer, m a positive integer and H a Hilbert space. Then, the inequality*

$$\left\| g_{m,H,b}(f) \right\|_{L^p(\mathbb{R}^d)} \leq C \, \|f\|_{L^p_H(\mathbb{R}^d)} \tag{3.3}$$

holds, for $d \geq b+2m+1$, with a constant C depending on p, b and m, but not on the dimension d. For the case $d \leq b+2m$, if $f(y)$ satisfies that $\int f(y) h_\alpha(y) dy = 0$ for every α, $|\alpha| \leq k-1$, where k is an integer satisfying $2k \geq b+2m-d+1$, the inequality (3.3) holds but the constant C also depends on the dimension d.

Proof. For $m = 1$ the inequality (3.3) holds by Theorem 3.

Let us assume that the inequality (3.3) is valid for m then, we shall show that it holds por $m + 1$.

Given a Hilbert space H let K the Hilbert space defined as

$$\begin{aligned} &h \in K \qquad \text{if and only if} \\ &h = \left\{h_{i_1,i_2,\dots,i_m}(t)\right\}\ , \\ &\quad -d \le i_1, i_2, \dots, i_m \le d\ ,\ 0 < t < \infty \text{ and } h_{i_1,i_2,\dots,i_m}(t) \in H \end{aligned}$$

with the norm

$$\|h\|_K^2 = \int_0^\infty \left(\sum_{-d\le i_1,i_2,\dots,i_m \le d} \left\|h_{i_1,i_2,\dots,i_m}(t)\right\|_H^2 \right) \frac{dt}{t} < \infty.$$

The Hilbert space K we have just defined is usually denoted as $K = L^2_{H^{(2d+1)m}}\left((0,\infty), \frac{dt}{t}\right)$.

By Theorem 3, we have that for every a

$$\begin{aligned} \left\|g_{1,K,a}(h)\right\|_{L^p(\mathbb{R}^d)} &= \left\| \left(\int_0^\infty \sum_{j=-d}^{d} \left\|\mathcal{S}_{K,a,j} h(x)\right\|_K^2 \frac{ds}{s} \right)^{1/2} \right\|_{L^p(\mathbb{R}^d)} \\ &\le GC_{p,a} \|h\|_{L^p_K(\mathbb{R}^d)} \end{aligned}$$

holds, where the constant $C_{p,a}$ does not depend on the Hilbert space H.

Now, we consider the $K-$valued function $h(x)$, defined on $\mathbb{R}^d$, as

$$h(x) = \mathcal{S}_{m,H,b} f(x) = \left\{t^m \mathcal{A}_{i_1}\mathcal{A}_{i_2}\dots\mathcal{A}_{i_m} P_t^b f(x)\right\}_{-d\le i_1,i_2,\dots,i_m\le d} \in K.$$

Then, we obtain

$$\begin{aligned} \mathcal{S}_{1,K,a} h(x) &= \left\{\mathcal{S}_{1,K,a,j} h(x)\right\}_{j=-d}^{d} \\ &= \left\{s\mathcal{A}_{i_1} P_s^a t^m \mathcal{A}_{i_2}\dots\mathcal{A}_{i_{m+1}} P_t^b f(x)\right\}_{-d\le i_1,\dots,i_{m+1}\le d}, \end{aligned}$$

and therefore

$$\begin{aligned} g_{1,K,a}(h)(x)^2 &= \int_0^\infty \sum_{j=-d}^{d} \left\|\mathcal{S}_{1,K,a,j} h(x)\right\|_K^2 \frac{ds}{s} \\ &= \int_0^\infty \int_0^\infty \left(\sum_{-d\le i_1,i_2,\dots,i_{m+1}\le d} \left\|s\mathcal{A}_{i_1} P_s^a t^m \mathcal{A}_{i_2}\dots\mathcal{A}_{i_{m+1}} P_t^b f(x)\right\|_H^2 \right) \frac{dt}{t}\frac{ds}{s}. \end{aligned}$$

Let $e_1, e_2, \ldots, e_r$ be the standard basis of $\mathbb{R}^r$ and $e_0 = 0$. For each multi-index $(i_2, \ldots, i_{m+1})$ with $|i_j| \le d,\ 2 \le j \le m+1$ let $\ell = \sum_{n=2}^{m+1} sg(i_n)$. Obviously, $|\ell| \le m$. Then, it follows that

$$\begin{aligned} &t^m \mathcal{A}_{i_2} \ldots \mathcal{A}_{i_{m+1}} P_t^b h_\alpha(x) \\ &\quad = t^m c\left(i_2, \ldots, i_{m+1,} \alpha\right) e^{-(2|\alpha|+d-b)^{1/2} t} h_{\left(\alpha - \sum_{n=2}^{m+1} sg(i_n) e_{|i_n|}\right)}(x), \end{aligned}$$

with $c\,(i_2, \ldots, i_{m+1}, \alpha) = 0$ in the case when any component of $\alpha - \sum_{n=2}^{m+1} sg(i_n)$ $e_{|i_n|}$ turn out to be negative. Consequently,

$$\begin{aligned} &s\mathcal{A}_{i_1} P_s^{-2\ell+b} t^m \mathcal{A}_{i_2} \ldots \mathcal{A}_{i_{m+1}} P_t^b h_\alpha(x) \\ &\quad = st^m \mathcal{A}_{i_1} e^{-\left(2\left|\alpha - \sum_{n=2}^{m+1} sg(i_n) e_{|i_n|}\right| + d + 2\ell - b\right)^{1/2} s} \\ &\qquad \times c\,(i_2, \ldots, i_{m+1}, \alpha)\, e^{-(2|\alpha|+d-b)^{1/2} t} h_{\left(\alpha - \sum_{n=2}^{m+1} sg(i_n) e_{|i_n|}\right)}(x). \end{aligned}$$

Since $\left|\alpha - \sum_{n=2}^{m+1} sg(i_n) e_{|i_n|}\right| = |\alpha| - \sum_{n=2}^{m+1} sg(i_n) = |\alpha| - \ell$, then

$$\begin{aligned} &s\mathcal{A}_{i_1} P_s^{-2\ell+b} t^m \mathcal{A}_{i_2} \ldots \mathcal{A}_{i_{m+1}} P_t^b h_\alpha(x) \\ &\quad = st^m \mathcal{A}_{i_1} e^{-(2(|\alpha|-\ell)+d+2\ell-b)^{1/2} s} c(i_2, \ldots, i_{m+1}, \alpha) \\ &\qquad \times e^{-(2|\alpha|+d-b)^{1/2} t} h_{\left(\alpha - \sum_{n=2}^{m+1} sg(i_n) e_{|i_n|}\right)}(x) \\ &\quad = st^m \mathcal{A}_{i_1} c(i_2, \ldots, i_{m+1,} \alpha) e^{-(2|\alpha|+d-b)^{1/2}(t+s)} h_{\left(\alpha - \sum_{n=2}^{m+1} sg(i_n) e_{|i_n|}\right)}(x) \\ &\quad = st^m c(i_{1,} i_2, \ldots, i_{m+1}, \alpha) e^{-(2|\alpha|+d-b)^{1/2}(t+s)} h_{\left(\alpha - \sum_{n=1}^{m+1} sg(i_n) e_{|i_n|}\right)}(x) \\ &\quad = st^m \mathcal{A}_{i_1} \mathcal{A}_{i_2} \ldots \mathcal{A}_{i_{m+1}} P_{t+s}^b h_\alpha(x). \end{aligned}$$

Therefore

$$\begin{aligned} &s\mathcal{A}_{i_1} P_s^{-2\ell+b} t^m \mathcal{A}_{i_2} \ldots \mathcal{A}_{i_{m+1}} P_t^b h_\alpha(x) \\ &\quad = st^m \mathcal{A}_{i_1} \mathcal{A}_{i_2} \ldots \mathcal{A}_{i_{m+1}} \left(e^{-(2|\alpha|+d-b)^{1/2}(t+s)} h_\alpha(x)\right). \end{aligned}$$

This implies that if $\ell = \sum_{n=2}^{m+1} sg\,(i_n)$, then

$$st^m \mathcal{A}_{i_1} \mathcal{A}_{i_2} \ldots \mathcal{A}_{i_{m+1}} \left(P_{t+s}^b f(x)\right) = s\mathcal{A}_{i_1} P_s^{-2\ell+b} t^m \mathcal{A}_{i_2} \ldots \mathcal{A}_{i_{m+1}} P_t^b f(x). \quad (3.4)$$

Thus, if $D_\ell = \left\{(i_2, \ldots, i_{m+1}) : \sum_{n=2}^{m+1} sg\,(i_n) = \ell\right\}$, by (3.4) we get

$$\sum_{D_\ell} \int_0^\infty \int_0^\infty \left\| st^m \mathcal{A}_{i_1} \mathcal{A}_{i_2} \ldots \mathcal{A}_{i_{m+1}} \left(P_{t+s}^b f(x)\right)\right\|_H^2 \frac{ds}{s} \frac{dt}{t} \le g_{1,K,-2\ell+b}(h)(x)^2.$$

By a change of variables, it is easy to check that

$$\int_0^\infty \int_0^\infty \left\| st^m \mathcal{A}_{i_1}\mathcal{A}_{i_2}\ldots\mathcal{A}_{i_{m+1}}\left(P^b_{t+s}f(x)\right)\right\|_H^2 \frac{ds}{s}\frac{dt}{t}$$
$$= \frac{1}{2m\,(2m+1)}\int_0^\infty \left\| t^{m+1}\mathcal{A}_{i_1}\mathcal{A}_{i_2}\ldots\mathcal{A}_{i_{m+1}}\left(P^b_t f(x)\right)\right\|_H^2 \frac{dt}{t}.$$

In consequence, we obtain

$$g_{m+1,H,b}f(x)^2 \le 2m\,(2m+1)\sum_{\ell=-m}^{m} g_{1,K,b+2\ell}h(x)^2.$$

Then, by Theorem 3

$$\left\| g_{m+1,H,b}f(x)\right\|_{L^p(\mathbb{R}^d)} \le C\,\|h\|_{L^p_K(\mathbb{R}^d)} = C\left\| g_{m,H,b}f(x)\right\|_{L^p_K(\mathbb{R}^d)}, \tag{3.5}$$

holds for $1 < p < \infty$ and $d \ge b+2m+3$, with a constant C not depending on the dimension d. For the case $d \le b+2(m+1)$, the inequality (3.5) holds provided that $\int f(y)h_\alpha(y)dy = 0$ for every α such that $|\alpha| \le k-1$, with k an integer satisfying $2k \ge b+2(m+1)-d+1$, but this time the constant C depends on the dimension d.

Since the conditions for case $m+1$ imply the conditions for m, by the inductive hypothesis, we get that the theorem holds for $m+1$ □

Theorem 5. *Let H be a Hilbert space, $1 < p < \infty$ and m a positive integer. Then,*

$$\|f\|_{L^p_H(\mathbb{R}^d)} \le C\left\| g^1_{m,H,b}(f)\right\|_{L^p(\mathbb{R}^d)} \tag{3.6}$$

holds, with a constant C depending on p, b and m, but not on the dimension d in the case when $d \ge b+2m+1$. For the case $d \le b+2m$, if $f(x)$ satisfies the conditions $\int f(y)h_\alpha(y)dy = 0$ for every α, $|\alpha| \le k-1$, with k an integer such that $2k \ge b+2m-d+1$, then (3.6) holds but the constant C also depends on the dimension d. We observe that the constant C does not depend on the Hilbert space H.

Proof. Let $f \in L^p_H(\mathbb{R}^d)\cap L^2_H(\mathbb{R}^d)$. We have

$$\int_{\mathbb{R}^d} g^1_{m,H,b}(f)(x)^2dx = \int_{\mathbb{R}^d}\int_0^\infty \left\| t^m \frac{\partial^m P^b_t f}{\partial t^m}(x)\right\|_H^2 \frac{dt}{t}dx$$
$$= \int_{\mathbb{R}^d}\int_0^\infty t^{2m-1}\left\| \sum_{n=0}^\infty (-1)^m\,(2n+d-b)^{m/2}\,e^{-(2n+d-b)^{1/2}t}\sum_{|\alpha|=n} c_\alpha h_\alpha\right\|_H^2 dt\,dx$$

$$
\begin{aligned}
&= \sum_\beta \int_{\mathbb{R}^d} \int_0^\infty t^{2m-1} \left(\sum_{n=0}^\infty (2n+d-b)^{m/2}\, e^{-(2n+d-b)^{1/2}t} \sum_{|\alpha|=n} c_{\alpha,\beta} h_\alpha \right)^2 dt dx \\
&= \sum_\beta \int_0^\infty t^{2m-1} \left(\sum_{n=0}^\infty (2n+d-b)^{m}\, e^{-2(2n+d-b)^{1/2}t} \sum_{|\alpha|=n} \left| c_{\alpha,\beta} \right|^2 \right) dt \\
&= \frac{(2m-1)!}{2^{2m}} \sum_\beta \sum_{n=0} \sum_{|\alpha|=n} \left| c_{\alpha,\beta} \right|^2 = \frac{(2m-1)!}{2^{2m}} \int_{\mathbb{R}^d} \sum_\beta \left| f_\beta(x) \right|^2 dx \\
&= \frac{(2m-1)!}{2^{2m}} \int_{\mathbb{R}^d} \| f(x) \|_H^2 \, dx.
\end{aligned}
$$

Let

$$
P_k f = \sum_{|\alpha| \le k-1} \langle f, h_\alpha \rangle\, h_\alpha,
$$

be the projection of f onto the finite-dimensional space generated by the functions $h_\alpha,\ |\alpha| \le k-1$. We have

$$
\begin{aligned}
\| P_k f \|_{L^p_H(\mathbb{R}^d)} &\le \sum_{|\alpha| \le k-1} \| \langle f, h_\alpha \rangle \|_H \| h_\alpha \|_{L^p(\mathbb{R}^d)} \\
&\le \left(\sum_{|\alpha| \le k-1} \| h_\alpha \|_{L^{p'}(\mathbb{R}^d)} \| h_\alpha \|_{L^p(\mathbb{R}^d)} \right) \| f \|_{L^p_H(\mathbb{R}^d)} \le C \| f \|_{L^p_H(\mathbb{R}^d)}.
\end{aligned} \tag{3.7}
$$

In consequence,

$$
\| (I - P_k) f \|_{L^p_H(\mathbb{R}^d)} \le (1 + C) \| f \|_{L^p_H(\mathbb{R}^d)}. \tag{3.8}
$$

If $d \ge 2m+1$ we define $Pf = 0$ and if $d \le 2m$ and k satisfies $2k \ge 2m - d + 1$ we define $Pf = P_k f$. Then, it follows that

$$
\| (I - P) f \|_{L^p_H(\mathbb{R}^d)} \le C \| f \|_{L^p_H(\mathbb{R}^d)}. \tag{3.9}
$$

By hyphotesis we assume that $f_2 = (I-P) f_2$. Then, by polarization it follows that

$$
\begin{aligned}
&\frac{(2m-1)!}{2^{2m}} \int_{\mathbb{R}^d} < f_1(x),\, f_2(x) >_H dx \\
&\quad = \frac{(2m-1)!}{2^{2m}} \int_{\mathbb{R}^d} < f_1(x),\, (I-P) f_2(x) >_H dx \\
&\quad = \int_{\mathbb{R}^d} \int_0^\infty \left\langle t^m \frac{\partial^m P_t^b f_1}{\partial t^m}(x),\, t^m \frac{\partial^m P_t^b (I-P) f_2}{\partial t^m}(x) \right\rangle_H \frac{dt}{t} dx \\
&\quad \le \int_{\mathbb{R}^d} g^1_{m,H,b}(f_1)(x) g^1_{m,H,b}((I-P) f_2)(x) dx.
\end{aligned}
$$

By Hölder's inequality and Theorem 4, we get

$$\|g^1_{m,H,b}(f_1)\|_{L^p(\mathbb{R}^d)}\,\|g^1_{m,H,b}((I-P)f_2)\|_{L^{p'}(\mathbb{R}^d)}$$
$$\le C\|g^1_{m,H,b}(f_1)\|_{L^p(\mathbb{R}^d)}\|(I-P)f_2\|_{L^{p'}_H(\mathbb{R}^d)}.$$

Then, by (3.9), we get

$$\|g^1_{m,H,b}(f_1)\|_{L^p(\mathbb{R}^d)}\,\|g^1_{m,H,b}((I-P)f_2)\|_{L^{p'}(\mathbb{R}^d)}$$
$$\le C\|g^1_{m,H,b}(f_1)\|_{L^p(\mathbb{R}^d)}\,\|f_2\|_{L^{p'}_H(\mathbb{R}^d)},$$

which implies the theorem □

4. Riesz transforms

In this section we apply the results obtained on the Littlewood-Paley $g-$functions to prove the Theorems A and B. Obviously Theorem A is a particular case of Theorem B. Thus we go directly to the proof of Theorem B.

Proof of the Theorem B. We recall that $L = -\Delta + |x|^2$, and for every positive integer m by (1.6) we have that

$$L^{-m/2}h_\alpha(x) = (2|\alpha| + d)^{-m/2}h_\alpha(x).$$

By (1.7), if $1 \le |i_1|, |i_2|, \dots, |i_m| \le d$,

$$R_{i_1,i_2,\dots,i_m}f = \mathcal{A}_{i_1}\mathcal{A}_{i_2}\dots\mathcal{A}_{i_m}L^{-m/2}f.$$

Therefore

$$\begin{aligned} R_{i_1,i_2,\dots,i_m}h_\alpha &= \mathcal{A}_{i_1}\mathcal{A}_{i_2}\dots\mathcal{A}_{i_m}(2\,|\alpha| + d)^{-m/2}h_\alpha \\ &= c\,(i_1, i_2, \dots, i_m, \alpha)\,(2\,|\alpha| + d)^{-m/2}h_{\left(\alpha-\sum_{n=1}^m sg(i_n)e_{|i_n|}\right)}. \end{aligned}$$

Classifying the family of the indexes $(i_1, i_2, \dots, i_m)$ with $1 \le |i_1|, |i_2|, \dots, |i_m| \le d$, as the union of the sets $D_\ell = \left\{(i_1, i_2, \dots, i_m) : \sum_{n=1}^m sg(i_n) = \ell\right\}$, for $-m \le \ell \le m$, it follows that if $(i_1, i_2, \dots, i_m) \in D_\ell$

$$\begin{aligned} \frac{\partial^m}{\partial t^m}P_t^{-2\ell}\left(R_{i_1,i_2,\dots,i_m}h_\alpha\right) &= \frac{\partial^m}{\partial t^m}\left(e^{-(2|\alpha|+d)^{1/2}t}\mathcal{A}_{i_1}\mathcal{A}_{i_2}\dots\mathcal{A}_{i_m}(2\,|\alpha| + d)^{-m/2}h_\alpha\right) \\ &= (-1)^m\mathcal{A}_{i_1}\mathcal{A}_{i_2}\dots\mathcal{A}_{i_m}e^{-(2|\alpha|+d)^{1/2}t}h_\alpha \\ &= (-1)^m\mathcal{A}_{i_1}\mathcal{A}_{i_2}\dots\mathcal{A}_{i_m}P_t^0h_\alpha \end{aligned}$$

holds. Then, we also have

$$\frac{\partial^m}{\partial t^m}P_t^{-2\ell}\left(R_{i_1,i_2,\dots,i_m}f\right) = (-1)^m\mathcal{A}_{i_1}\mathcal{A}_{i_2}\dots\mathcal{A}_{i_m}P_t^0f. \tag{4.1}$$

By Minkowski's inequality

$$\left\| \left(\sum_{1\le |i_1|,|i_2|,\ldots,|i_m|\le d} \left| R_{i_1,i_2,\ldots,i_m} f \right|^2 \right)^{1/2} \right\|_{L^p(\mathbb{R}^d)} \le \sum_{\ell=-m}^{m} \left\| \left(\sum_{D_\ell} \left| R_{i_1,i_2,\ldots,i_m} f \right|^2 \right)^{1/2} \right\|_{L^p(\mathbb{R}^d)}. \tag{4.2}$$

By Theorem 5, with $b \le 2m$, the last expression is less than or equal to a constant C times

$$\sum_{-m\le \ell\le m} \left\| \left(\sum_{D_\ell} \int_0^\infty \left| t^m \frac{\partial^m}{\partial t^m} P_t^{-2\ell} \left(R_{i_1,i_2,\ldots,i_m} f \right) \right|^2 \frac{dt}{t} \right)^{1/2} \right\|_{L^p(\mathbb{R}^d)}$$

where the constant C depends on p, b and m, but not on the dimension d in the case $d \ge 4m+1$, and for the case $d \le 4m$, if $f(y)$ satisfies the condition $\int f(y)h_\alpha(y)dy = 0$ for every α, $|\alpha| \le k-1$, with k a positive integer such that $2k \ge 4m - d + 1$, the constant C also depends on the dimension d.

Using (4.1) and Theorem 4 with $H = \mathbb{R}$ and $b = 0$, we obtain that (4.2) is bounded by

$$C \left\| \left(\int_0^\infty \sum_{-d\le i_1,i_2,\ldots,i_m\le d} \left(t^m \mathcal{A}_{i_1}\mathcal{A}_{i_2}\ldots\mathcal{A}_{i_m} P_t^0 f \right)^2 \frac{dt}{t} \right)^{1/2} \right\|_{L^p(\mathbb{R}^d)} \le C \left\| g_{m,\mathbb{R},0}(f) \right\|_{L^p(\mathbb{R}^d)} \le C \left\| f \right\|_{L^p(\mathbb{R}^d)}.$$

Thus, we have shown that

$$\left\| \left(\sum_{1\le |i_1|,|i_2|,\ldots,|i_m|\le d} \left| R_{i_1,i_2,\ldots,i_m} f \right|^2 \right)^{1/2} \right\|_{L^p(\mathbb{R}^d)} \le C \left\| f \right\|_{L^p(\mathbb{R}^d)} \tag{4.3}$$

where the constant C depends on p, b and m, but not on the dimension d in the case $d \ge 4m+1$. For the case $d \le 4m$, if $f(y)$ satisfies the condition $\int f(y)h_\alpha(y)dy = 0$ for every α, $|\alpha| \le k-1$, with k a positive integer such that $2k \ge 4m - d + 1$, (4.3) holds but the constant C also depends on the dimension d.

For the case when $d \ge 4m+1$, (4.3) proves that Theorem B holds. Now, suppose that $d \le 4m$ and choose k such that $2k \ge 4m - d + 1$. Let

$$P_k f = \sum_{|\alpha|\le k-1} \langle f, h_\alpha \rangle h_\alpha,$$

be the projection of f onto the finite-dimensional space generated by the functions $h_\alpha,\ |\alpha| \le k-1$. Then,

$$\begin{aligned}
&\left\| R_{i_1,i_2,\ldots,i_m} P_k f \right\|_{L^p(\mathbb{R}^d)} \\
&\quad \le \sum_{|\alpha|\le k-1} |\langle f, h_\alpha\rangle| \left\| \mathcal{A}_{i_1}\mathcal{A}_{i_2}\ldots\mathcal{A}_{i_m} L^{-m/2} h_\alpha \right\|_{L^p(\mathbb{R}^d)} \\
&\quad \le \left(\sum_{|\alpha|\le k-1} \|h_\alpha\|_{L^{p'}(\mathbb{R}^d)} \left\| \mathcal{A}_{i_1}\mathcal{A}_{i_2}\ldots\mathcal{A}_{i_m} L^{-m/2} h_\alpha \right\|_{L^p(\mathbb{R}^d)} \right) \|f\|_{L^p(\mathbb{R}^d)} \\
&\quad = C' \, \|f\|_{L^p(\mathbb{R}^d)} \, .
\end{aligned} \tag{4.4}$$

By Minkowski's inequality we get

$$\begin{aligned}
&\left\| \left(\sum_{1\le|i_1|,|i_2|,\ldots,|i_m|\le d} \left| R_{i_1,i_2,\ldots,i_m} f \right|^2 \right)^{1/2} \right\|_{L^p(\mathbb{R}^d)} \\
&\quad \le \left\| \left(\sum_{1\le|i_1|,|i_2|,\ldots,|i_m|\le d} \left| R_{i_1,i_2\ldots,i_m} P_k f \right|^2 \right)^{1/2} \right\|_{L^p(\mathbb{R}^d)} \\
&\qquad + \left\| \left(\sum_{1\le|i_1|,|i_2|,\ldots,|i_m|\le d} \left| R_{i_1,i_2,\ldots,i_m} \left(I - P_k\right) f \right|^2 \right)^{1/2} \right\|_{L^p(\mathbb{R}^d)}
\end{aligned}$$

which by (4.4), (4.3), (3.7) and (3.8) it is less than or equal to

$$C \left(\|P_k f\|_{L^p(\mathbb{R}^d)} + \|(I-P_k)\, f\|_{L^p(\mathbb{R}^d)} \right) \le C \, \|f\|_{L^p(\mathbb{R}^d)} \, .$$

This shows that also in the case $d \le 4m$ the inequality (1.8) holds with a constant $C_{p,m}$ depending on the dimension d. Taking into account that for a given value of m there are only a finite number of dimensions less than or equal to $4m$ the constant $C_{p,m}$ can be found in such a way that Theorem B holds □

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