

Gauge invariant fluctuations of the metric during inflation from a new scalar-tensor Weyl-integrable gravity model

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We investigate gauge invariant scalar fluctuations of the metric during inflation in a nonperturbative formalism in the framework of a recently formulated scalar-tensor theory of gravity, in which the geometry of space-time is that of a Weyl integrable manifold. We show that in this scenario the Weyl scalar field can play the role of the inflaton field. As an application of the theory, we examine the case of a power-law inflation. In this case, the quasi-scale invariance of the spectrum for scalar fluctuations of the metric is achieved for determined values of the parameter ω of the scalar-tensor theory. We stress the fact that in our formalism the physical inflaton field has a purely geometrical origin.

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I. INTRODUCTION

The existence of an inflationary stage [1–4] of the early Universe is now supported by much observational evidence, in particular, by the discovery of temperature anisotropies present in the cosmic microwave background [5,6]. In fact, in recent years, there has been an extraordinary development on observational tests of inflationary models [7]. On the theoretical side, among the most popular and pioneering models of inflation, we would like to mention the supercooled chaotic inflation model [8]. In this proposal, as we know, the expansion of the Universe is driven by a scalar field known as the inflaton field.

In the current state of affairs, one would say that an inflationary model is considered viable when, among other features, it provides a mechanism for the creation of primordial density fluctuations, needed to explain the subsequent structure formation in the Universe [9–11]. According to some authors, fluctuations of the inflaton field induce scalar fluctuations of the metric around a Friedmann-Robertson-Walker (FRW) geometrical background. On the other hand, scalar fluctuations of the metric on cosmological scales can be studied in a nonperturbative formalism, describing not only small fluctuations but the larger ones [12]. In this kind of model, it is assumed that the inflaton field exists at the beginning of the inflationary

stage [13]. In view of this, it seems rather appealing, and perhaps closer to the spirit of general relativity, to look for a model that explains the origin of the inflaton scalar field at a purely geometrical level.

We all know that in the so-called scalar-tensor theories of gravity, in addition to the space-time metric, a scalar field is required to describe gravity. Historically, an early motivation for this new degree of freedom came from the attempt to incorporate Mach's principle in a relativistic theory of gravity [14,15]. Later, new classes of scalar-tensor theories of which the main motivation has a geometric character have appeared [16]. Among these, there has been an increasing interest in gravitational theories defined in a Weyl integrable space-time geometry [17]. These may be regarded as scalar-tensor theories in which the scalar field plays a clear geometric role.¹ In the present paper, we shall consider a more recent geometrical approach to scalar-tensor theory [20]. This approach starts by considering the action of Brans-Dicke theory and introduces the space-time geometry from first principles, which is done by applying the Palatini formalism. Cosmological models in this new geometrical scalar-tensor theory have been studied, and cosmological scenarios have been found, which seems to

¹Incidentally, it should be noted that general relativity can also be formulated in the language of Weyl-integrable space-time [18]. In cosmology, it was shown that this new formulation of general relativity in terms of the Weyl-integrable geometry admits solutions capable of explaining the present accelerating expansion of the Universe, as a natural consequence of the existence of the Weyl scalar field [19].

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indicate the presence of a geometric phase transition of the Universe [21].

In this paper, we consider gauge invariant scalar fluctuations of the metric during inflation, employing a nonperturbative formalism in the context of the Weyl scalar-tensor theory of gravity. Here, the physical inflaton field has a geometrical origin since it is part of the affine structure of the space-time manifold. As an application of this idea, we study a scenario in which the early Universe underwent an expansion phase given by power-law inflation. Our approach is different than one worked earlier [12], in which the inflaton field has a physical origin. In this work, we demonstrate that the expansion of the Universe can be driven by a geometrical field (in a Weyl frame), which can be interpreted as the inflaton field when we use the Einstein frame. The paper is organized as follows. In Sec. II, we present the general formalism that leads to Weyl geometrical scalar-tensor theory. We proceed to Sec. III, where we work out the formalism of gauge invariant scalar fluctuations of our model in a nonperturbative way. The particular case of the dynamical equations for the small scalar fluctuations of the metric are studied using the linearized case. The power spectrum and the mean square scalar fluctuations of the metric in a more general way are calculated at the end of this section. In Sec. IV, we present the example in which the Universe describes a power-law inflationary expansion. Finally, in Sec. V, we conclude with some comments.

II. FORMALISM

To begin with, let us now consider a scalar-tensor theory of gravity of which the action is given by

$$S = \frac{1}{16\pi} \int d^4x \sqrt{-g} \left\{ \Phi R + \frac{\tilde{\omega}(\Phi)}{\Phi} g^{\mu\nu} \Phi_{,\mu} \Phi_{,\nu} - \tilde{V}(\Phi) \right\}, \quad (1)$$

where R denotes the Ricci scalar, $\tilde{\omega}(\Phi)$ is a function of the scalar field, and $\tilde{V}(\Phi)$ is a scalar potential.

It is easy to see that, in terms of the new variable $\phi = -\ln(G\Phi)$, the action (1) can be rewritten as

$$S = \int d^4x \sqrt{-g} \left\{ e^{-\phi} \left[\frac{R}{16\pi G} + \omega(\phi) g^{\mu\nu} \phi_{,\mu} \phi_{,\nu} \right] - V(\phi) \right\}, \quad (2)$$

where we have defined $\omega(\phi) = (16\pi G)^{-1} \tilde{\omega}[\Phi(\phi)]$ and $V(\phi) = \tilde{V}(\Phi(\phi))$. Adopting the Palatini procedure, it can be shown that the variation of the action (2) with respect to the affine connection yields [20]

$$\nabla_\alpha g_{\mu\nu} = \phi_{,\alpha} g_{\mu\nu}, \quad (3)$$

which corresponds to the nonmetricity condition characterizing a Weyl-integrable space-time, where ϕ is interpreted as the Weyl scalar field. We thus see that if we adopt the Palatini variational principle we are naturally led to the geometry of the Weyl-integral space-time. To get the complete set of field equations, we next perform the variation of the action (2) with respect to the metric $g_{\alpha\beta}$ and the scalar field ϕ , which then gives

$$G_{\mu\nu} = 8\pi G \left[\omega(\phi) \left(\frac{1}{2} g_{\mu\nu} \phi_{,\alpha} \phi^{,\alpha} - \phi_{,\mu} \phi_{,\nu} \right) - \frac{1}{2} e^{\phi} g_{\mu\nu} V(\phi) \right], \quad (4)$$

$$\square\phi = - \left(1 + \frac{1}{2\omega(\phi)} \frac{d\omega(\phi)}{d\phi} \right) \phi_{,\mu} \phi^{,\mu} - \frac{e^{\phi}}{\omega(\phi)} \left(\frac{1}{2} \frac{dV}{d\phi} + V \right), \quad (5)$$

where $G_{\mu\nu} = R_{\mu\nu} - (1/2)Rg_{\mu\nu}$ is calculated in terms of the affine connection $\Gamma_{\mu\nu}^\alpha$, given by

$$\Gamma_{\mu\nu}^\alpha = \left\{ \begin{matrix} \alpha \\ \mu\nu \end{matrix} \right\} - \frac{1}{2} g^{\alpha\beta} [g_{\beta\mu} \phi_{,\nu} + g_{\beta\nu} \phi_{,\mu} - g_{\mu\nu} \phi_{,\beta}], \quad (6)$$

$\left\{ \begin{matrix} \alpha \\ \mu\nu \end{matrix} \right\} = (1/2)g^{\alpha\beta}(g_{\beta\mu,\nu} + g_{\beta\nu,\mu} - g_{\mu\nu,\beta})$ denoting the Levi-Civita connection.

An important fact to be noted here is that the nonmetricity condition (3) is invariant under the Weyl transformations

$$\bar{g}_{\mu\nu} = e^f g_{\mu\nu}, \quad (7)$$

$$\bar{\phi} = \phi + f, \quad (8)$$

where f is an arbitrary scalar function of the coordinates. It is usually said in the literature that the transformations (7)–(8) lead from one *frame* (M, g, ϕ) to another frame $(M, \bar{g}, \bar{\phi})$. For the particular choice $f = -\phi$, we have $\bar{g}_{\mu\nu} = e^{-\phi} g_{\mu\nu}$ and $\bar{\phi} = 0$, and in this case, the condition (3) reduces to the Riemannian metricity condition. Due to this fact the frame $(\bar{M}, \bar{g}, \bar{\phi} = 0)$ will be referred as the *Einstein frame*. However, the terminology ‘‘Einstein frame’’ used here is different from that traditionally employed in Jordan-Brans-Dicke (JBD) scalar-tensor theories. This is because in the JBD theories the frame transformations do not preserve the compatibility of the metric and the affine connection, generating geodesics in the traditional Einstein frame with an extra acceleration term. Here, the *Riemann* or *Einstein frame* is a geometric object constructed as a result of the invariance under Weyl transformations of the nonmetricity condition (3), and thus the geodesics are preserved in all Weyl frames, including what this paper calls the Einstein frame, which is defined by means of the

effective metric $\bar{g}_{\mu\nu}$. Now, it is not difficult to verify that in the Einstein frame the action (2) takes the form

$$S^{(R)} = \int d^4x \sqrt{-\bar{g}} \left\{ \frac{\bar{R}}{16\pi G} + \omega(\phi) \bar{g}^{\mu\nu} \phi_{,\mu} \phi_{,\nu} - e^{2\phi} V(\phi) \right\}, \quad (9)$$

with \bar{R} denoting the transformed Riemannian Ricci scalar. The field equations derived from the action (9) will be given by

$$\bar{G}_{\mu\nu} = 8\pi G \left[\omega(\phi) \left(\frac{1}{2} \phi_{,\alpha} \phi^{,\alpha} \bar{g}_{\mu\nu} - \phi_{,\mu} \phi_{,\nu} \right) - \frac{e^{2\phi}}{2} \bar{g}_{\mu\nu} V(\phi) \right], \quad (10)$$

$$\bar{\square} \phi = -\frac{1}{2\omega} \frac{d\omega}{d\phi} \phi_{,\alpha} \phi^{,\alpha} - \frac{e^{2\phi}}{\omega} \left(V + \frac{1}{2} \frac{dV}{d\phi} \right), \quad (11)$$

where $\bar{G}_{\mu\nu}$ and $\bar{\square}$ denote the Einstein tensor and the D'Alembertian operator, respectively, both calculated with the affine connection in the Einstein frame.²

To study scalar fluctuations of the metric during inflation, we shall consider the simplest scalar-tensor theory derived from Eqs. (10) and (11). This is achieved when we choose the parameter $\omega(\phi)$ to be a constant. In this case, the field equations (10)–(11) reduce to

$$\bar{G}_{\mu\nu} = 8\pi G \left[\omega \left(\frac{1}{2} \phi_{,\alpha} \phi^{,\alpha} \bar{g}_{\mu\nu} - \phi_{,\mu} \phi_{,\nu} \right) - \frac{e^{2\phi}}{2} \bar{g}_{\mu\nu} V(\phi) \right], \quad (12)$$

$$\bar{\square} \phi = -\frac{e^{2\phi}}{\omega} \left(V + \frac{1}{2} \frac{dV}{d\phi} \right). \quad (13)$$

One interesting feature of this framework is that, in contrast to what happens in a general Weyl frame, in the Einstein frame the scalar field ϕ is no longer a geometric field and should be regarded as a physical field. In this way, we shall consider ϕ in (12) and (13) the inflaton field, the scalar field that drives the expansion of the Universe during inflation. From this point of view, the term between brackets in the right side of Eq. (12) can be interpreted as an induced energy-momentum tensor in the Einstein frame,

$$\bar{T}_{\mu\nu} = \omega \left(\frac{1}{2} \phi_{,\alpha} \phi^{,\alpha} \bar{g}_{\mu\nu} - \phi_{,\mu} \phi_{,\nu} \right) - \frac{e^{2\phi}}{2} \bar{g}_{\mu\nu} V(\phi). \quad (14)$$

²Here, the energy momentum-tensor $T_{\mu\nu}$ is defined according the prescription adopted in Ref. [20].

For convenience, we shall work in the Einstein frame, although the results hold in a general frame.³

III. NONPERTURBATIVE GAUGE INVARIANT SCALAR FLUCTUATIONS OF THE METRIC

To study gauge invariant scalar fluctuations of the metric, let us just by generality start by following the nonperturbative formalism introduced in Ref. [12]. Thus, the perturbed line element can be written in the form

$$ds^2 = e^{2\psi} dt^2 - a^2(t) e^{-2\psi} (dx^2 + dy^2 + dz^2), \quad (15)$$

where $a(t)$ is the cosmological scale factor and $\psi(t, x, y, z)$ is a metric function describing gauge invariant scalar fluctuations of the metric in a nonperturbative way. The Ricci scalar calculated with the metric in (15) is given by

$$\bar{R} = 6e^{-2\psi} \left[\frac{\ddot{a}}{a} + H^2 - \ddot{\psi} - 5H\dot{\psi} + 3\dot{\psi}^2 + \frac{e^{4\psi}}{3a^2} (\nabla^2 \psi - (\nabla \psi)^2) \right], \quad (16)$$

with $H = \dot{a}/a$ denoting the Hubble parameter. In the absence of matter, the perturbed field equations (12) reduce to

$$e^{-2\psi} (3H^2 - 6H\dot{\psi} + 3\dot{\psi}^2) + \frac{1}{a^2} [2\nabla^2 \psi - (\nabla \psi)^2] e^{2\psi} = 8\pi G \left[\frac{\omega}{2} \left(e^{-2\psi} \dot{\phi}^2 + \frac{e^{2\psi}}{a^2} (\nabla \phi)^2 \right) + \frac{1}{2} e^{2\phi} V(\phi) \right], \quad (17)$$

$$\left(-2\ddot{\psi} + 5\dot{\psi}^2 - 8H\dot{\psi} + \frac{2\ddot{a}}{a} + H^2 \right) e^{-2\psi} - \frac{1}{3a^2} (\nabla \psi)^2 e^{2\psi} = 8\pi G \left[-\frac{\omega}{2} \left(e^{-2\psi} \dot{\phi}^2 - \frac{1}{3a^2} e^{2\psi} (\nabla \phi)^2 \right) + \frac{1}{2} e^{2\phi} V(\phi) \right], \quad (18)$$

$$\frac{1}{a} \frac{\partial}{\partial x^i} \left[\frac{\partial}{\partial t} (a\psi) \right] - \frac{\partial \psi}{\partial t} \frac{\partial \psi}{\partial x^i} = 8\pi G \frac{\omega}{2} \frac{\partial \phi}{\partial t} \frac{\partial \phi}{\partial x^i}, \quad (19)$$

$$\frac{\partial \psi}{\partial x^i} \frac{\partial \psi}{\partial x^j} = -8\pi G \frac{\omega}{2} \frac{\partial \phi}{\partial x^i} \frac{\partial \phi}{\partial x^j}, \quad (20)$$

while the dynamics of the inflaton field ϕ is given by

³It is important to note that in Weyl geometrical scalar-tensor theory all geometrical objects, such as geodesics, curvature, and length of a curve, are constructed in an invariant way with respect to Weyl transformations. This is done with the help of the “effective” metric $\gamma_{\mu\nu} = e^{-\phi} g_{\mu\nu}$, which is a fundamental invariant of the equivalence class of Weyl manifolds. It turns out that in the Einstein frame $\gamma_{\mu\nu} = \bar{g}_{\mu\nu}$. See, for instance, Ref. [20].

$$\ddot{\phi} + (3H - 4\dot{\psi})\dot{\phi} - \frac{e^{4\psi}}{a^2}\nabla^2\phi + \frac{1}{\omega}e^{2(\psi+\phi)}\left[V(\phi) + \frac{1}{2}V'(\phi)\right] = 0, \quad (21)$$

where the prime mark denotes a derivative with respect to ϕ . It is not difficult to verify that after some algebraic manipulations Eqs. (17) and (18) yield

$$(\ddot{\psi} - 4\dot{\psi}^2 + 7H\dot{\psi})e^{-2\psi} + \frac{2}{3a^2}(\nabla\psi)^2e^{2\psi} - \frac{1}{a^2}(\nabla^2\psi)e^{2\psi} = 8\pi G\left[-\frac{\omega}{3a^2}e^{2\psi}(\nabla\phi)^2 - \frac{1}{2}e^{2\phi}V(\phi)\right], \quad (22)$$

where we have eliminated background contributions. This equation determines the dynamics of ψ , the function that describes the scalar fluctuations of the metric with arbitrary amplitude.⁴

A. Linear approximation

In general, finding solutions for the dynamics of ψ from Eq. (22) is not an easy task. However, we can obtain some solutions in the weak field limit, i.e., solutions corresponding to small amplitudes of the scalar fluctuations. In this limit, a linear approximation of the scalar fluctuations of the metric is useful, and the gauge invariance is preserved. To implement this limit, we use $e^{\pm n\psi(x^\sigma)} \simeq 1 \pm n\psi(x^\sigma)$. In this regime, a semiclassical approximation for the inflaton field is also valid. Thus, we write $\phi(t, x^i) = \phi_b(t) + \delta\phi(t, x^i)$, where the background classical field $\phi_b = \langle E|\phi|E\rangle$ is the expectation value of ϕ , with $|E\rangle$ denoting a physical quantum state given by the Bunch-Davies vacuum [22], and $\delta\phi$ describes the quantum fluctuations of the field ϕ . The line element (15) in the weak field limit becomes

$$ds^2 = (1 + 2\psi)dt^2 - a^2(t)(1 - 2\psi)(dx^2 + dy^2 + dz^2). \quad (23)$$

Here, the metric $\bar{g}_{\mu\nu}^{(0)} = \text{diag}[1, -a^2(t), -a^2(t), -a^2(t)]$ describes the background metric in the Einstein frame, which is supposed to be isotropic and homogenous. Notice that the linear approximation (23) agrees with the longitudinal gauge in the standard approach to perturbation theory [23,24]. The linearization of Eq. (22) leads to

⁴We would like to point out that the nonperturbative method adopted here is an extension of the standard approach. The main difference lies in the fact that in the nonperturbative approach cosmological curvature fluctuations of the metric with large amplitude can be also treated, whereas in the standard approach, the fluctuations must be small compared to the metric background. A more complete description of the nonperturbative method can be found in Ref. [7].

$$\ddot{\psi} + 7H\dot{\psi} - \frac{1}{a^2}\nabla^2\psi = -4\pi Ge^{2\phi_b}V'(\phi_b)\delta\phi. \quad (24)$$

From Eqs. (19) and (20), Eq. (24) can be put in the form

$$\ddot{\psi} + \alpha(t)\dot{\psi} - \frac{1}{a^2}\nabla^2\psi + \beta(t)\psi = 0, \quad (25)$$

where

$$\alpha(t) = 7H + \frac{e^{2\phi_b}}{\omega\phi_b}V'(\phi_b), \quad (26)$$

$$\beta(t) = \frac{e^{2\phi_b}}{\omega}\left(V(\phi_b) + \frac{V'(\phi_b)}{\phi_b}H\right). \quad (27)$$

On the other hand, with respect to the background part (i.e., on cosmological scales), the linearization of Eq. (21) gives

$$\ddot{\phi}_b + 3H_c\dot{\phi}_b + \frac{e^{2\phi_b}}{\omega}\left[V(\phi_b) + \frac{1}{2}V'(\phi_b)\right] = 0, \quad (28)$$

whereas, on small quantum scales, the dynamics of $\delta\phi$ and ψ is given by

$$\begin{aligned} \ddot{\delta\phi} + 3H_c\dot{\delta\phi} - \frac{1}{a^2}\nabla^2\delta\phi + \frac{e^{2\phi_b}}{\omega}\left[V'(\phi_b) + \frac{1}{2}V''(\phi_b)\right]\delta\phi \\ = 4\dot{\phi}_b\dot{\psi} - \frac{2e^{2\phi_b}}{\omega}\left[V(\phi_b) + \frac{1}{2}V'(\phi_b)\right]\psi. \end{aligned} \quad (29)$$

At the same time, the Friedmann equation for the background metric is

$$3H_c^2 = 4\pi G[\omega\dot{\phi}_b^2 + e^{2\phi_b}V(\phi_b)]. \quad (30)$$

Finally, from Eqs. (28) and (30), we find that the background inflaton field satisfies the equation

$$\dot{\phi}_b^2 = -\frac{\dot{H}_c}{4\pi G\omega}. \quad (31)$$

Now, in order to study the quantum dynamics of the inflaton field ϕ and of the scalar fluctuations of the metric ψ , we shall follow the canonical quantization procedure.

B. Quantization and spectrum for scalar fluctuations of the metric

Following the canonical quantization procedure of quantum field theory, we start by imposing the commutation relation

$$[\psi(\bar{x}), \Pi_{(\psi)}^0(t, \bar{x}')] = i\delta^{(3)}(\bar{x} - \bar{x}'), \quad (32)$$

where the quantity $\Pi_{(\psi)}^0 = \partial L/\partial\dot{\psi}$ is the canonical conjugate momentum to ψ and L is the Lagrangian. The equal times quantization condition (32) implies that

$$[\psi(t, \bar{x}), \dot{\psi}(t, \bar{x}')] = \frac{i}{t_0} e^{-\int^t \alpha(t) dt} \delta^{(3)}(\bar{x} - \bar{x}'). \quad (33)$$

According to the action (9), now L takes the form

$$L = \sqrt{-\bar{g}} \left[\frac{\bar{R}}{8\pi G} - \omega \bar{g}^{\mu\nu} \phi_{,\mu} \phi_{,\nu} - e^{2\phi} V(\phi) \right], \quad (34)$$

where \bar{R} is given by the expression (16).

To simplify the structure of Eq. (25), we can introduce the auxiliary field

$$\psi(\bar{x}, t) = e^{-\frac{1}{2} \int^t \alpha(t) dt} \chi(\bar{x}, t) \quad (35)$$

so that Eq. (25) can be written in terms of χ

$$\ddot{\chi} - \frac{1}{a^2} \nabla^2 \chi + \left[\beta - \left(\frac{\alpha^2}{4} + \frac{\dot{\alpha}}{2} \right) \right] \chi = 0. \quad (36)$$

The auxiliary field $\chi(\bar{x}, t)$ can be expanded in Fourier modes as

$$\chi(\bar{x}, t) = \frac{1}{(2\pi)^{3/2}} \int d^3 k [a_k e^{i\bar{k}\cdot\bar{x}} \xi_k(t) + a_k^\dagger e^{-i\bar{k}\cdot\bar{x}} \xi_k^*(t)] \quad (37)$$

with the asterisk mark denoting a complex conjugate and a_k^\dagger and a_k denoting the creation and annihilation operators, which satisfy the commutation relations

$$[a_k, a_{k'}^\dagger] = \delta^{(3)}(\bar{k} - \bar{k}'), [a_k, a_{k'}] = [a_k^\dagger, a_{k'}^\dagger] = 0. \quad (38)$$

From Eq. (37), we can see that Eq. (36) leads to

$$\ddot{\xi}_k + \left[\frac{k^2}{a^2} - \left(\frac{\alpha^2}{4} + \frac{\dot{\alpha}}{2} - \beta \right) \right] \xi_k = 0, \quad (39)$$

which determines the dynamics of the quantum modes ξ_k for the scalar fluctuations of the metric. The squared quantum fluctuations of ψ in the IR sector (cosmological scales) are given by the expression

$$\langle \psi^2 \rangle_{\text{IR}} = \frac{e^{-\int^t \alpha(t) dt}}{2\pi^2} \int_0^{\epsilon k_H} \frac{dk}{k} k^3 [\xi_k(t) \xi_k^*(t)]|_{\text{IR}}, \quad (40)$$

where $\epsilon = k_{\text{max}}^{\text{IR}}/k_p \ll 1$ is a dimensionless parameter, $k_{\text{max}}^{\text{IR}} = k_H(t_r)$ being the wave number related to the Hubble radius at time t_r , when the modes reenter the horizon, while k_p is the Planckian wave number. For the Hubble parameter $H = 0.5 \times 10^{-9} M_p$, the values of ϵ range between 10^{-5} and 10^{-8} , and this corresponds to a number of e -foldings at the end of inflation $N_e = 63$.

IV. EXAMPLE: A POWER-LAW INFLATION

As an application of the ideas developed previously, we now obtain the spectrum of squared scalar quantum fluctuations of the metric ψ in the framework of the Weyl geometric scalar-tensor theory in the case of a power-law inflationary expansion of the Universe.

Considering the power-law expansion $a(t) = a_0(t/t_0)^p$, Eq. (31) leads to the classical solution

$$\phi_b(t) = \phi_0 \left[1 + n_0 \ln \left(\frac{t}{t_e} \right) \right], \quad (41)$$

where $\phi_0 = \phi_b(t_0)$, t_0 being the time when inflation starts. For the potential, if we substitute Eq. (31) in Eq. (30), we then get

$$e^{2\phi_b/\phi_0} V(t) = \frac{3H_c^2 + \dot{H}_c}{4\pi G}, \quad (42)$$

which, for $H_c = p/t$, reduces to

$$V(t) = \frac{1}{4\pi G} \frac{p(3p-1)}{t_e^{-2n_0}} t^{-2(1+n_0)}, \quad (43)$$

where

$$n_0 = -\sqrt{p/(4\pi G \phi_0^2 \omega)} \quad (44)$$

is a negative dimensionless parameter. Taking into account Eqs. (26) and (27), the expression (39) can be written in the form

$$\ddot{\xi}_k + \left(\frac{\kappa^2}{t^{2p}} - \frac{\gamma_0^2}{t^2} \right) \xi_k = 0, \quad (45)$$

with $\gamma_0^2 = \alpha_0^2/4 - \alpha_0/2 - \beta_0$ and $\kappa^2 = (k^2 t_0^{2p})/a_0^2$. The parameters α_0 and β_0 are given by

$$\alpha_0 = 7p - \frac{2p(3p-1)(1+n_0)}{4\pi G \omega n_0^2},$$

$$\beta_0 = \frac{p(3p-1)}{4\pi G \omega} \left[1 - \frac{2(1+n_0)}{n_0^2} e^{2\phi_0} \right]. \quad (46)$$

The general solution of Eq. (45) reads

$$\xi_k(t) = A_1 \sqrt{t} \mathcal{H}_\nu^{(1)}[Z(t)] + A_2 \sqrt{t} \mathcal{H}_\nu^{(2)}[Z(t)], \quad (47)$$

with $\nu = \sqrt{4\gamma_0^2 + 1}/(2p-2)$, $Z(t) = \kappa t^{1-p}/(p-1)$, and A_1 and A_2 being integration constants. The functions $\mathcal{H}_\nu^{(1,2)}$ denote the first and second kinds Hankel functions, respectively. The expression (33) in terms of the χ field becomes

$$[\chi(\bar{x}, t), \dot{\chi}(\bar{x}', t)] = \frac{i}{t_0} \delta^{(3)}(\bar{x} - \bar{x}'), \quad (48)$$

and thus the normalization condition for the modes will be given by

$$\dot{\xi}_k^* \xi_k - \xi_k^* \dot{\xi}_k = \frac{i}{t_0}. \quad (49)$$

From Eq. (49) and by choosing the Bunch-Davies condition, the normalized solution for the modes ξ_k is

$$\xi_k(t) = i \sqrt{\frac{\pi}{4t_0}} \left(\frac{1}{p-1} \right)^{\frac{1-p}{2}} \sqrt{t} \mathcal{H}_\nu^{(2)}[Z(t)]. \quad (50)$$

The mean square fluctuations for ψ on the IR sector according to Eq. (40) are then given by

$$\begin{aligned} \langle \psi^2 \rangle_{\text{IR}} &= \frac{(-2a_0)^{2\nu}}{8\pi^3} \frac{\Gamma^2(\nu)}{(p-1)^{1-p-2\nu}} \frac{t^{1-2\nu(1-p)-\alpha_0}}{t_0^{1-2p(3+\nu)}} \\ &\times \frac{\epsilon^{3-2\nu}}{3-2\nu} \beta_0^2 t_r^{2(p-1)(3-2\nu)}, \end{aligned} \quad (51)$$

where we have used that $k_H = (\gamma_0^2/t_0^{2p})t_r^{2(p-1)}$. The corresponding power spectrum for ψ has the form

$$\mathcal{P}_k(\psi) = \frac{(-2a_0)^{2\nu}}{8\pi^3(3-2\nu)} \frac{\Gamma^2(\nu)}{(p-1)^{1-p-2\nu}} \frac{t^{1-2\nu(1-p)-\alpha_0}}{t^{1+2\nu p}} k^{3-2\nu}. \quad (52)$$

We then see that the quasi-scale invariance for the spectrum of scalar fluctuations of the metric $\mathcal{P}_k(\psi)$ is achieved for the values of $P/\rho \gtrsim -1$. The spectral index is given by $n_s - 1 = 3 - 2\nu$, so that once n_s is known, we can determine the parameter p of the power-law expansion of the Universe, which then will be given by

$$p = 1 + \frac{2}{1 - n_s}. \quad (53)$$

For $n_s \simeq 0.96$ [25], we obtain $p \simeq 51$. For this spectral index, the equation of state $P = -(\frac{2\dot{H}}{3H^2} + 1)\rho$ will lead to

$$P/\rho = \frac{2-3p}{3p} \simeq -0.9869, \quad (54)$$

where P is the pressure and ρ is the energy density. This is in good agreement with the data obtained by the WMAP9- ω CDM(flat) observations [26]. This is very interesting because, since $P/\rho = -\frac{p^2}{4\pi G\phi_0^2}$, for $n_0 = -1$ we obtain from Eq. (41) that ϕ_0 is

$$\phi_0 \simeq \frac{3.57}{\sqrt{\pi\omega}} G^{-1/2}, \quad (55)$$

where we have taken into account the value $p = 51$. Hence, this means that ϕ_0 takes sub-Planckian values for $\omega > 4.06$, which solves the problem of the trans-Planckian values in models of standard inflation. Furthermore, the choice $n_0 = -1$ corresponds to a constant $V(t)$ and $\gamma_0^2 = -\beta_0 = \frac{p(1-3p)}{4\pi G\omega}$. With this choice for n_0 , and taking into account the expression (42), it is possible to set the following correspondence with standard models of inflation:

$$\mathcal{V}(\phi_b) = 2V(t)e^{2\phi_b/\phi_0}. \quad (56)$$

In other words, the effective potential $\mathcal{V}(\phi_b)$ can be interpreted potential in standard models of inflation for $n_0 = -1$. This is very interesting because now it is possible to define the slow-roll parameters of inflation evaluated at $k = aH$: $\epsilon = \frac{1}{16\pi G} (\frac{\mathcal{V}'}{\mathcal{V}})^2$, $\eta = \frac{1}{8\pi G} \frac{\mathcal{V}''}{\mathcal{V}}$. In our case, we obtain that

$$2\epsilon = \eta = \frac{1}{2\pi G\phi_0^2} = 0.03887. \quad (57)$$

With this value, one can automatically calculate the scalar and tensor spectral indices: $n_s = 1 - 6\epsilon + 2\eta = 1 - \eta = 0.9612$, $n_t \simeq -2\epsilon = -\eta = -0.0387$. Hence, it follows immediately that the scalar to tensor index r is given by: $r = n_t/n_s = -\eta/(1-\eta) = -0.04$. This value is in agreement with observations [25].

V. FINAL COMMENTS

In this paper, we have considered gauge invariant scalar fluctuations of the cosmological metric during the inflationary phase of the Universe in the framework of the Weyl geometrical scalar-tensor theory of gravity and using a nonperturbative formalism. By investigating the limit of the field equations in the case of small perturbations, we have obtained the spectrum of scalar fluctuations at the end of inflation. An important feature of this model is that the inflaton field is modeled by a geometrical scalar field, which is part of the affine structure of the background geometry. As far as inflationary models are concerned, it is perhaps tempting to take the view that a geometrical origin of the inflaton might be regarded as more natural than its introduction *a priori* without a clear theoretical justification.

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