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Research Article

A Liouville theorem for some Bessel generalized operators

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ABSTRACT
In this paper we establish a Liouville theorem in $\mathcal{H}'_\mu$ for a wider class of operators in $(0,\infty)^n$ that generalizes the n-dimensional Bessel operator. We will present two different proofs, based in two representation theorems for certain distributions 'supported in zero'.

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1. Introduction

Liouville type theorems have been studied in many works under different contexts. In analytic theory, Liouville theorems stated that a bounded entire function reduces to a constant. A first version of Liouville theorem in distributional theory is due to L. Schwartz [1], and assert that any bounded harmonic function in $\mathbb{R}^n$ is a constant.

Currently, this result has been generalized in many directions. A well known generalization states that:

> Let $L = \sum_{|\alpha|\le m} a_\alpha D^\alpha$ be a linear differential operator with constant coefficients such that $\sum_{|\alpha|\le m} a_\alpha (2\pi i\xi)^\alpha \neq 0$ for all $\xi \in \mathbb{R}^n - \{0\}$. If a tempered distribution u, solves $Lu = 0$, then u is a polynomial function. In particular, if u is bounded then it reduces to a constant.

In this work, we established a Liouville type theorem for a large class of operators in $(0,\infty)^n$, that are lineal combinations of operators

$$S^k = S^{k_1}_{\mu_1} \circ \ldots \circ S^{k_n}_{\mu_n}, \tag{1.1}$$

where k is a multi-index, $k = (k_1, \ldots, k_n)$, $\mu_i \in \mathbb{R}$, $\mu_i \geq -1/2$ and

$$S_{\mu_i} = \frac{\partial^2}{\partial x_i^2} - \frac{4\mu_i - 1}{4x_i^2}. \tag{1.2}$$

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The operators given by linear combination of (1.1) contain as a particular case the n-dimensional operator defined in [2] and given by:

$$S_\mu = \Delta - \sum_{i=1}^{n} \frac{4\mu_i^2 - 1}{4x_i^2}, \tag{1.3}$$

where $\mu = (\mu_1, \ldots, \mu_n)$ and S_μ is a n-dimensional version of the well know Bessel operator

$$S_\alpha = \frac{\mathrm{d}^2}{\mathrm{d}x^2} - \frac{4\alpha^2 - 1}{4x^2}. \tag{1.4}$$

This operators were introduced in relation to the Hankel transform given by

$$h_\alpha f(y) = \int_0^\infty f(x)\sqrt{xy}J_\alpha(xy)\,\mathrm{d}x \tag{1.5}$$

with $\alpha \geq -1/2$, for 1-dimensional case and the n-dimensional case

$$(h_\mu \phi)(y) = \int_{(0,\infty)^n} \phi(x_1, \ldots, x_n) \prod_{i=1}^{n} \{\sqrt{x_i y_i} J_{\mu_i}(x_i y_i)\}\,\mathrm{d}x_1 \ldots \mathrm{d}x_n \tag{1.6}$$

with $\mu = (\mu_1, \ldots, \mu_n)$, $\mu_i \geq -1/2$, $i = 1, \ldots, n$. And J_ν represents the Bessel functions of the first kind and order ν.

Bessel operators (1.3) and (1.4) and Hankel Transforms (1.5) and (1.6) were studied on Zemanian spaces $\mathcal{H}_\mu$ and $\mathcal{H}'_\mu$ in [2–4].

The space $\mathcal{H}_\mu$ is a space of functions $\phi \in C^\infty((0,\infty)^n)$ such that for all $m \in \mathbb{N}_0, k \in \mathbb{N}_0^n$ verifies

$$\gamma^\mu_{m,k}(\phi) = \sup_{x \in (0,\infty)^n} |(1 + \|x\|^2)^m T^k \{x^{-\mu-1/2}\phi(x)\}| < \infty, \tag{1.7}$$

where $-\mu - 1/2 = (-\mu_1 - 1/2, \ldots, -\mu_n - 1/2)$ and the operators T^k are given by $T^k = T_n^{k_n} \circ T_{n-1}^{k_{n-1}} \circ \ldots \circ T_1^{k_1}$, where $T_i = x_i^{-1}(\partial/\partial x_i)$. Thus $\mathcal{H}_\mu$ is Frèchet space. The dual space of $\mathcal{H}_\mu$ is denoted by $\mathcal{H}'_\mu$.

In [2] the authors proved that S_{μ_i} are continuous from $\mathcal{H}_\mu$ into itself for all $i = 1, \ldots, n$ and self-adjoint lineal mappings. This fact also implies that the operator $S^k = S_{\mu_n}^{k_n} \ldots S_{\mu_1}^{k_1}$ is continuous from $\mathcal{H}_\mu$ into itself. Then, since they are self-adjoints the generalized operators can be extended to $\mathcal{H}'_\mu$ by

$$(S_{\mu_i} f, \phi) = (f, S_{\mu_i}\phi) \quad \text{and} \quad (S^k f, \phi) = (f, S^k \phi), \quad f \in \mathcal{H}'_\mu \quad \phi \in \mathcal{H}_\mu. \tag{1.8}$$

The generalized Hankel transformation $h_\mu f$ of $f \in \mathcal{H}'_\mu$ is defined by

$$(h_\mu f, \phi) = (f, h_\mu \phi), \quad f \in \mathcal{H}'_\mu, \quad \phi \in \mathcal{H}_\mu$$

for $\mu \in [-1/2, \infty)^n$. Then h_μ is an automorphism onto $\mathcal{H}_\mu$ and $\mathcal{H}'_\mu$ and $h_\mu = (h_\mu)^{-1}$.

The Hankel transform and Bessel operator are related by $h_\mu(S_\mu) = -\|y^2\| h_\mu$ in $\mathcal{H}_\mu$ and $\mathcal{H}'_\mu$.

Now we shall describe the main result of this work.

Theorem 1.1: *Let $P[x]$ be a polynomial in n-variables such that $\sum_{|\alpha|\le N} a_\alpha x^\alpha \neq 0$ for all $x \in \mathbb{R}^n - \{0\}$ and all its coefficients have the same sign. Let L be the operator $L = \sum_{|\alpha|\le N}(-1)^{|\alpha|} a_\alpha S^\alpha$. If $f \in \mathcal{H}'_\mu$ and*

$$Lf = 0, \tag{1.9}$$

then there exists a polynomial in n-variables Q such that $f(x) = x^{\mu+1/2} Q[x_1^2, \dots, x_n^2]$.

Corollary 1.2: *If f is a classical solution of (1.9) of slow growth then there exists a polynomial in n-variables Q such that $f(x) = x^{\mu+1/2} Q[x_1^2, \dots, x_n^2]$. In particular if f is bounded then f is a constant.*

Remark 1.1: The cases $\mu = (\mu_1, \dots, \mu_n) = (1/2, \dots, 1/2)$ or $(-1/2, \dots, -1/2)$ produce in (1.3) the Laplacian operator in $(0, \infty)^n$.

This paper is organized as follows. In Section 2, we present some notational conventions that will allow us to simplify the presentation of our results. In Section 3 we propose a characterization of a certain family of functions on the multiplier space $\mathcal{O}$ of the n-dimensional space $\mathcal{H}_\mu$ that extends the result proved by Zemanian in [4]. In Sections 4 and 5 we give two different proofs of Theorem 1.1.

2. Preliminaries and notations

In this section we summarize without proof the relevant material on Hankel transforms and the Zemanian spaces studied in [2,3,5].

We now present some notational conventions that will allow us to simplify the presentation of our results. We denote by $x = (x_1, \dots, x_n)$ and $y = (y_1, \dots, y_n)$ elements of $(0, \infty)^n$ or $\mathbb{R}^n$. Let $\mathbb{N}$ be the set $\{1, 2, 3, \dots\}$ and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, $\|x\| = (x_1^2 + \cdots + x_n^2)^{1/2}$. The notations $x < y$ and $x \le y$ mean, respectively, $x_i < y_i$ and $x_i \le y_i$ for $i = 1, \dots, n$. Moreover, $x = a$ for $x \in \mathbb{R}^n$, $a \in \mathbb{R}$ means $x_1 = x_2 = \dots = x_n = a$, $x^m = x_1^{m_1} \dots x_n^{m_n}$ and e_j for $j = 1, \dots, n$, denotes the members of the canonical basis of $\mathbb{R}^n$. An element $k = (k_1, \dots, k_n) \in \mathbb{N}_0^n = \mathbb{N}_0 \times \mathbb{N}_0 \times \cdots \times \mathbb{N}_0$ is called multi-index. For k,m multi-index we set $|k| = k_1 + \cdots + k_n$ the length of the multi-index.

Also we will note

$$k! = k_1! \dots k_n!, \quad \binom{k}{m} = \binom{k_1}{m_1} \dots \binom{k_n}{m_n} \quad \text{for } k, m \in \mathbb{N}_0^n.$$

Remark 2.1: Let k be a multi-index and θ, φ diferenciable functions up to order $|k|$, the following equality is valid

$$T^k\{\theta.\varphi\} = \sum_{j=0}^{k} \binom{k}{j} T^{k-j}\theta . T^j \varphi, \tag{2.1}$$

where '.' denote the usual product of functions, $\binom{k}{j}$ and $\sum_{j=0}^{k}$ must be interpreted as in the previous section for $j = 0 = (0, \dots, 0)$.

Remark 2.2: If e_i is an element of the canonical base of $\mathbb{R}^n$, since $S^{e_i} = S^0_{\mu_n} \ldots \circ S^1_{\mu_i} \ldots S^0_{\mu_1} = S_{\mu_i}$, then $\sum_{i=1}^n S^{e_i} = \sum_{i=1}^n S_{\mu_i} = S_\mu$.

In [6] was defined the generalized function δ_α, as

$$(\delta_\alpha, \phi) = C_\alpha \lim_{x \to 0^+} x^{-\alpha - 1/2} \phi(x), \tag{2.2}$$

where $C_\alpha = 2^\alpha \Gamma(\alpha + 1)$. The distribution given by (2.2) can be extended in the same way to the n-dimensional case. Moreover we can consider the following distribution

$$(T^k \delta_\mu, \phi) = C_\mu \lim_{\substack{x \to 0 \\ x_i > 0}} T^k \{x^{-\mu - 1/2} \phi(x)\}, \tag{2.3}$$

where k is a multi-index, $\mu \in \mathbb{R}^n$ and C_μ is a constant depending on μ given by $C_\mu = \prod_{i=1}^n 2^{\mu_i} \Gamma(\mu_i + 1)$. The generalized function (2.3) is well defined as it can be seen in the proof of Lemma 3.1. Let $\phi \in \mathcal{H}_\mu$, since

$$|(T^k \delta_\mu, \phi)| = \left| C_\mu \lim_{\substack{x \to 0 \\ x_i > 0}} T^k \{x^{-\mu - 1/2} \phi(x)\} \right| \leq C_\mu \sup_{x \in (0,\infty)^n} |T^k \{x^{-\mu - 1/2} \phi(x)\}| = C_\mu \gamma^\mu_{0,k}(\phi),$$

then $T^k \delta_\mu$ lies in $\mathcal{H}'_\mu$. Moreover,

$$h_\mu T^k \delta_\mu = C^\mu_k \, t^{\mu + 2k + 1/2} \quad \text{in } \mathcal{H}'_\mu, \tag{2.4}$$

where $C^\mu_k = (-1)^{|k|} (C_\mu / C_{\mu + k})$. Indeed, since the well known formula $(\mathrm{d}/\mathrm{d}z)(z^{-\alpha} J_\alpha) = -z^{-\alpha} J_{\alpha+1}$ is valid for $\alpha \neq -1, -2, \ldots$, if we consider $k = e_j$, then

$$\begin{aligned}
(h_\mu T_j \delta_\mu, \phi) &= (T_j \delta_\mu, h_\mu \phi) = C_\mu \lim_{\substack{x \to 0 \\ x_i > 0}} T_j \{x^{-\mu - 1/2} h_\mu \phi(x)\} \\
&= C_\mu \lim_{\substack{x \to 0 \\ x_i > 0}} x_j^{-1} \partial / \partial x_j \left\{ \int_{(0,\infty)^n} t^{\mu + 1/2} \phi(t_1, \ldots, t_n) \prod_{i=1}^n \{(x_i t_i)^{-\mu_i} J_{\mu_i}(x_i t_i)\} \, \mathrm{d}t_1 \ldots \mathrm{d}t_n \right\} \\
&= C_\mu \lim_{\substack{x \to 0 \\ x_i > 0}} x_j^{-1} \left\{ \int_{(0,\infty)^n} t^{\mu + 1/2} \phi(t_1, \ldots, t_n) \, \partial / \partial x_j \left\{ \prod_{i=1}^n \{(x_i t_i)^{-\mu_i} J_{\mu_i}(x_i t_i)\} \right\} \mathrm{d}t_1 \ldots \mathrm{d}t_n \right\} \\
&= -C_\mu \lim_{\substack{x \to 0 \\ x_i > 0}} \int_{(0,\infty)^n} t^{\mu + 2e_j + 1/2} \phi(t_1, \ldots, t_n) [(x_j t_j)^{-(\mu_j + 1)} J_{\mu_j + 1}(x_j t_j)] \\
&\quad \times \prod_{\substack{i=1 \\ i \neq j}}^n \{(x_i t_i)^{-\mu_i} J_{\mu_i}(x_i t_i)\} \, \mathrm{d}t_1 \ldots \mathrm{d}t_n \\
&= -C_\mu \left\{ C_{\mu_j + 1} \prod_{\substack{i=1 \\ i \neq j}}^n C_{\mu_i} \right\}^{-1} \int_{(0,\infty)^n} t^{\mu + 2e_j + 1/2} \phi(t_1, \ldots, t_n) \, \mathrm{d}t_1 \ldots \mathrm{d}t_n \\
&= \left(-\frac{C_{\mu_j}}{C_{\mu_j + 1}} t^{\mu + 2e_j + 1/2}, \phi \right).
\end{aligned}$$

Therefore the assertion is true for $k = e_j$. The general case follows in a similar way. Indeed, let $r \in \mathbb{N}_0$ and let us observe that

$$\begin{aligned} T_j^r &\left\{ t^{\mu+1/2} \prod_{i=1}^n \{(x_i t_i)^{-\mu_i} J_{\mu_i}(x_i t_i)\} \right\} \\ &= (x_j^{-1}\partial/\partial x_j)^r \left\{ t^{\mu+1/2} \prod_{i=1}^n \{(x_i t_i)^{-\mu_i} J_{\mu_i}(x_i t_i)\} \right\} \\ &= (-1)^r t^{\mu+2r\,e_j+1/2}(x_j t_j)^{-(\mu_j+r)} J_{\mu_j+r}(x_j t_j) \prod_{\substack{i=1 \\ i\neq j}}^n \{(x_i t_i)^{-\mu_i} J_{\mu_i}(x_i t_i)\}, \end{aligned} \tag{2.5}$$

then (2.5) yields

$$\begin{aligned} (h_\mu T_j^r \delta_\mu, \phi) &= (T_j^r \delta_\mu, h_\mu \phi) = C_\mu \lim_{\substack{x\to 0 \\ x_i>0}} T_j^r \{x^{-\mu-1/2} h_\mu \phi(x)\} \\ &= C_\mu \lim_{\substack{x\to 0 \\ x_i>0}} (x_j^{-1}\partial/\partial x_j)^r \left\{ x^{-\mu-1/2} \int_{(0,\infty)^n} \phi(t_1,\ldots,t_n) \prod_{i=1}^n \{\sqrt{x_i t_i} J_{\mu_i}(x_i t_i)\}\, \mathrm{d}t_1 \ldots \mathrm{d}t_n \right\} \\ &= C_\mu \lim_{\substack{x\to 0 \\ x_i>0}} (x_j^{-1}\partial/\partial x_j)^r \left\{ \int_{(0,\infty)^n} t^{\mu+1/2} \phi(t_1,\ldots,t_n) \prod_{i=1}^n \{(x_i t_i)^{-\mu_i} J_{\mu_i}(x_i t_i)\}\, \mathrm{d}t_1 \ldots \mathrm{d}t_n \right\} \\ &= (-1)^r C_\mu \lim_{\substack{x\to 0 \\ x_i>0}} \int_{(0,\infty)^n} t^{\mu+2r\,e_j+1/2} \phi(t_1,\ldots,t_n) (x_j t_j)^{-(\mu_j+r)} J_{\mu_j+r}(x_j t_j) \\ &\quad \times \prod_{\substack{i=1 \\ i\neq j}}^n \{(x_i t_i)^{-\mu_i} J_{\mu_i}(x_i t_i)\}\, \mathrm{d}t_1 \ldots \mathrm{d}t_n \\ &= (-1)^r C_\mu \left\{ C_{\mu_j+r} \prod_{\substack{i=1 \\ i\neq j}}^n C_{\mu_i} \right\}^{-1} \int_{(0,\infty)^n} t^{\mu+2re_j+1/2} \phi(t_1,\ldots,t_n)\, \mathrm{d}t_1 \ldots \mathrm{d}t_n \\ &= \left((-1)^r \frac{C_{\mu_j}}{C_{\mu_j+r}} t^{\mu+2re_j+1/2}, \phi \right). \end{aligned}$$

For the general case, if we compute for $j \neq k \in \{1,\ldots,n\}$ and $r, m \in \mathbb{N}_0$ then we obtain that

$$(h_\mu T_j^r T_k^m \delta_\mu, \phi) = (T_j^r T_k^m \delta_\mu, h_\mu \phi) = \left((-1)^{r+m} \frac{C_{\mu_j} C_{\mu_k}}{C_{\mu_j+r} C_{\mu_k+m}} t^{\mu+2re_j+2me_k+1/2}, \phi \right),$$

and the result follows.

3. Some results about Taylor's expansions and a special family of multipliers in $\mathcal{H}_\mu$

In this section we extend the characterization obtained by Zemanian in [4] related to Taylor's expansions of functions in $\mathcal{H}_\mu$. Moreover we give a result which improve Lemma 3.2 in [5].

Lemma 3.1: *Let $\mu \in \mathbb{R}^n$. Then ϕ is a member of $\mathcal{H}_\mu$ if and only if it satisfies the following three conditions:*

(i) *$\phi(x)$ is a smooth complex valued function on $(0,\infty)^n$.*

(ii) *For each $r \in \mathbb{N}_0$*

$$x^{-\mu-1/2}\phi(x) = a_0 + \sum_{|k_1|=1} a_{2k_1}x^{2k_1} + \sum_{|k_2|=2} a_{2k_2}x^{2k_2} + \cdots + \sum_{|k_r|=r} a_{2k_r}x^{2k_r} + R_{2r}(x), \tag{3.1}$$

where

$$a_{2k_r} = \frac{1}{2^r k_r!} \lim_{\substack{x\to 0\\ x_i>0}} T^{k_r}\{x^{-\mu-1/2}\phi(x)\}, \tag{3.2}$$

and the remainder term $R_{2r}(x)$ satisfies

$$T^k R_{2r}(x) = o(1) \qquad \underset{x_i>0}{x \to 0} \tag{3.3}$$

for k multi-index such that $|k| = r$.

(iii) *For each multi-index k_r, $D^{k_r}\phi(x)$ is of rapid descent as $|x| \to \infty$.*

Proof: Since $\phi(x) \in \mathcal{H}_\mu$ condition (i) is satisfied by definition. For a multi-index k let us consider the smooth function in $(0,\infty)^n$ given by

$$\psi(x) = \psi(x_1,\ldots,x_n) = T^k\{x^{-\mu-1/2}\phi(x)\}. \tag{3.4}$$

Let us see that the coefficients given by (3.2) are well defined, that is,

$$\lim_{\substack{(x_1,\ldots,x_n)\to(0,\ldots,0)\\ x_i>0}} \psi(x_1,\ldots,x_n) < \infty. \tag{3.5}$$

Since

$$\left|\frac{\partial^n}{\partial x_n\ldots\partial x_1}\psi(x)\right| \le M|x_1x_2\ldots x_n|, \tag{3.6}$$

if $(a_1,\ldots,a_n)$ in $[0,\infty)^n$ such that there exist $1 \le j \le n$ and $a_j = 0$ then

$$\lim_{\substack{(x_1,\ldots,x_n)\to(a_1,\ldots,a_n)\\ x_i>0}} \frac{\partial^n}{\partial x_n\ldots\partial x_1}\psi(x) = 0,$$

then $(\partial^n/(\partial x_n\ldots\partial x_1))\psi(x)$ is C^∞ in $(0,\infty)^n$, continuous in $[0,\infty)^n$ and consequently integrable in $[0,1]^n$.

Moreover,

$$\int_1^{x_1}\cdots\int_1^{x_n}\frac{\partial^n}{\partial y_n\ldots\partial y_1}\psi(y_1,\ldots,y_n)\,\mathrm{d}y_n\ldots\mathrm{d}y_1=\psi(x_1,\ldots,x_n)+\sum_\lambda a_\lambda\psi(b_\lambda) \tag{3.7}$$

with $a_\lambda=1$ or -1 and $b_\lambda=(b_{\lambda_1},\ldots,b_{\lambda_n})$ with $b_{\lambda_i}=x_i$ or $b_{\lambda_i}=1$.

Let us see now $\lim_{\substack{(x_1,\ldots,x_n)\to(0,\ldots,0)\\ x_i>0}}\psi(b_\lambda)<\infty$. First, let us consider b_λ such that $b_{\lambda_j}=1$ if $j\neq i$ and $b_{\lambda_i}=x_i$. Since $|(\partial/\partial y_i)\psi(y_1,\ldots,y_n)|\leq M|y_i|$,

$$\lim_{x_i\to 0}\int_1^{x_i}\frac{\partial}{\partial y_i}\psi(1,\ldots,y_i,\ldots,1)\,\mathrm{d}y_i<\infty.$$

So, $\lim_{x\to 0}\psi(1,\ldots,x_i,\ldots,1)<\infty$.

Now let us see that $\lim_{x\to 0}\psi(1,\ldots,x_i,1,\ldots,1,x_j\ldots,1)<\infty$. In fact $|(\partial/\partial x_i\partial x_j)\psi(1,\ldots,1,x_i,1,\ldots,1,x_j,\ldots,1)|\leq M|x_ix_j|$.

Then $(\partial/\partial x_i\partial x_j)\psi(1,\ldots,1,x_i,1,\ldots,1,x_j,\ldots,1)$ is integrable in $[0,1]^2$ and

$$\begin{aligned}&\int_1^{x_j}\int_1^{x_i}\frac{\partial}{\partial y_i\partial y_j}\psi(1,\ldots,1,y_i,1,\ldots,1,y_j,\ldots,1)\,\mathrm{d}y_i\,\mathrm{d}y_j\\&\quad=\psi(1,\ldots,x_i,\ldots,x_j,\ldots,1)-\psi(1,\ldots,x_i,\ldots,1,\ldots,1)\\&\quad-\psi(1,\ldots,1,\ldots,x_j,\ldots,1)+\psi(1,\ldots,1).\end{aligned}$$

Then, taking limit when $x\to 0$ to both sides of the previous formula we obtain that $\lim_{x\to 0}\psi(1,\ldots,x_i,\ldots,x_j,\ldots,1)<\infty$.

If we continuous this process recursively, in the $(n-1)$ step then we obtain that $\lim_{x\to 0}\psi(b)$ is finite if $b=(1,x_2,\ldots x_n)$, or $(x_1,1,\ldots,x_n)$, etc. Finally from (3.7) we deduce (3.5).

Now let us make the following observation. If $r,p\in\mathbb{N}_0$

$$T_i^r x_i^{2p}=\begin{cases}2^r r! & \text{if } r=p,\\ 2^r\dfrac{p!}{(p-r-1)!}x_i^{2(p-r)} & \text{if } r<p,\\ 0 & \text{if } r>p.\end{cases} \tag{3.8}$$

Let m and k be multi-index such as $|m|=|k|=r$, then

$$T^m\{x^{2k}\}=\begin{cases}2^r k! & \text{if } m=k,\\ 0 & \text{if } m\neq k.\end{cases} \tag{3.9}$$

Upon choosing a_{2k_r} according to (3.2) and observing that

$$\begin{aligned}\lim_{\substack{x\to 0\\ x_i>0}}T^{k_r}R_{2r}(x)&=\lim_{\substack{x\to 0\\ x_i>0}}T^{k_r}\left\{x^{-\mu-1/2}\phi(x)-\sum_{j=0}^{r}\sum_{|k_j|=j}a_{k_j}x^{2k_j}\right\}\\&=\lim_{\substack{x\to 0\\ x_i>0}}T^{k_r}\{x^{-\mu-1/2}\phi(x)\}-a_{k_r}2^r k_r!=0,\end{aligned}$$

we obtain (3.3). Condition (iii) was already proved in [5, Lemma 2.1]. Conversely, if conditions (i) and (ii) hold, then $\sup_{x\in(0,1]^n} |(1+\|x\|^2)T^k\{x^{-\mu-1/2}\phi(x)\}| < \infty$.

From (2.1) it can be deduce the formula

$$T^k\{x^{-\mu-1/2}\phi(x)\} = x^{-\mu-1/2}\left\{\sum_{j=0}^{k} b_{k,j}\frac{D^j\phi}{x^{2k-j}}\right\},$$

which implies $\sup_{x\in(1,\infty)^n} |(1+\|x\|^2)T^k\{x^{-\mu-1/2}\phi(x)\}| < \infty$ since the conditions (i) and (iii) hold. Therefore $\gamma^{\mu}_{m,k}(\phi)$ are finite for all $m \in \mathbb{N}_0$ and $k \in \mathbb{N}_0^n$ which completes the theorem. ∎

Let $\mathcal{O}$ be the space of functions $\theta \in C^\infty((0,\infty)^n)$ with the property that for every $k \in \mathbb{N}_0^n$ there exists $n_k \in \mathbb{Z}$ and $C > 0$ such that, $|(1+\|x\|^2)^{n_k}T^k\theta| < C$, for all $x \in (0,\infty)^n$.

For the next Lemma, we will consider polynomials of n-variables, $P[x] = P[x_1,\dots,x_n] = \sum_{|\alpha|\le N} a_\alpha x^\alpha$, with $a_\alpha \in \mathbb{R}$.

Lemma 3.2: *Let $P[x]$ and $Q[x]$ be polynomials of n-variables such that $Q[x] = \sum_{|\alpha|\le N} b_\alpha x^\alpha \neq 0$ for all $x \in [0,\infty)^n$ and all its coefficients have the same sign then $P[x_1^2,\dots,x_n^2]/Q[x_1^2,\dots,x_n^2] \in \mathcal{O}$.*

Proof: Let us show that $P[x_1^2,\dots,x_n^2] \in \mathcal{O}$. We want to see that for all $k \in \mathbb{N}_0^n$ there exists $n_k \in \mathbb{Z}$ such that

$$|(1+\|x\|^2)^{n_k}T^kP[x_1^2,\dots,x_n^2]| < \infty. \tag{3.10}$$

If $k = e_i$,

$$T^{e_i}P[x_1^2,\dots,x_n^2] = x_i^{-1}\sum_{|\zeta|\le N'} 2\zeta_i\, a_\zeta\, x_1^{2\zeta_1}\dots x_i^{2\zeta_i-1}\dots x_n^{2\zeta_n} = \tilde{P}[x_1^2,\dots,x_n^2].$$

Any polynomial of the form $\sum_{|\beta|\le N} c_\beta\, x_1^{2\beta_1}\dots x_n^{2\beta_n}$ can be bounded in the following way

$$\left|\sum_{|\beta|\le N} c_\beta\, x_1^{2\beta_1}\dots x_n^{2\beta_n}\right| \le \sum_{|\beta|\le N} |c_\beta||x_1^{2\beta_1}|\dots|x_n^{2\beta_n}| < C(1+\|x\|^2)^{|\gamma|},$$

for suitables $C > 0$ and a multi-index γ. So $|(1+\|x\|^2)^{-|\gamma'|}\tilde{P}[x_1^2,\dots,x_n^2]| < C$, for some multi-index γ'.

Now let us see that $1/Q[x_1^2,\dots,x_n^2]$ is also in $\mathcal{O}$. Let $Q[x] = \sum_{|\alpha|\le N} b_\alpha x_1^{\alpha_1}\dots x_n^{\alpha_n}$ and without loss of generality we assume that $b_\alpha \ge 0$, for all $\alpha : |\alpha| \le N$, then

$$T^{e_i}(Q[x_1^2,\dots,x_n^2])^{-1} = (Q[x_1^2,\dots,x_n^2])^{-2}\tilde{Q}[x_1^2,\dots,x_n^2], \tag{3.11}$$

since $Q[x]$ does not have any zeros in $[0,\infty)^n$ then $b_0 \neq 0$, so

$$Q[x_1^2,\dots,x_n^2] = b_0 + \sum_{0<|\alpha|\le N} b_\alpha x_1^{2\alpha_1}\dots x_n^{2\alpha_n} \ge b_0,$$

therefore

$$(Q[x_1^2,\ldots,x_n^2])^{-2} \leq \frac{1}{b_0^2} < \infty. \tag{3.12}$$

From (3.11) and (3.12), it follows (3.10) for $k = e_i$. The general case follows in a similar way. ∎

4. Proofs of Liouville type theorem in $\mathcal{H}'_\mu$

The following is a representation theorem for distributions 'supported in zero' in $\mathcal{H}'_\mu$.

Theorem 4.1: *Let $T \in \mathcal{H}'_\mu$ satisfying $(T,\phi) = 0$ for all $\phi \in \mathcal{H}_\mu$ with* $\mathbf{supp}(\phi) \subset \{x \in (0,\infty)^n : \|x\| \geq a\}$ *for some $a \in \mathbb{R}$, $a > 0$. Then there exist $N \in \mathbb{N}_0$ and scalars c_k, $|k| \leq N$ such that*

$$T = \sum_{|k|\leq N} c_k S^k \delta_\mu,$$

where δ_μ is given by (2.3) for $k = 0$.

Proof: The proof will follow directly from [7, Lemma 1.4.1] if we can show that there exists N_0 such that if $\phi \in \mathcal{H}_\mu$ satisfies $(S^k\delta_\mu, \phi) = 0$ for $|k| \leq N_0$, then $(T,\phi) = 0$.

Consider the family of seminorms $\{\lambda^\mu_{m,k}\}$ defined by (A.1) which generate the same topology in $\mathcal{H}_\mu$ as the family $\{\gamma^\mu_{m,k}\}$ (see Appendix) and let

$$\rho^\mu_R(\phi) = \sum_{\substack{m\leq R\\ |k|\leq R}} \lambda^\mu_{m,k}(\phi).$$

This family of seminorms result to be an increasing and equivalent to $\{\lambda^\mu_{m,k}\}$. So, given $T \in \mathcal{H}'_\mu$, there exist $c > 0$ and $N \in \mathbb{N}_0$ such that $|(T,\phi)| \leq C\rho^\mu_N(\phi)$, $\phi \in \mathcal{H}_\mu$.

Now, let $\phi \in \mathcal{H}_\mu$ satisfying $(S^k\delta_\mu, \phi) = 0$, for all $|k| \leq N_0$, where $N_0 = 2N$ then:

$$\lim_{\substack{x\to 0\\ x_i>0}} x^{-\mu-1/2} S^k\phi(x) = 0.$$

Given $\varepsilon > 0$ there exists $\eta_k > 0$ such as $|x^{-\mu-1/2}S^k\phi(x)| < \varepsilon$, for all $x \in (0,\infty)^n$, $\|x\| < \eta_k$ for all k such that $|k| < N_0$.

Set $\eta = \min_{|k|\leq N_0}\{\eta_k\}$ and $\eta < 1$, then $|x^{-\mu-1/2}S^k\phi(x)| < \varepsilon$, for all $x \in (0,\infty)^n$, $\|x\| < \eta$.

Fix η^* satisfying $0 < \eta^* < \eta < 1$ and define a smooth function ψ on $(0,\infty)^n$ by $\psi(x) = 1$ for $\{x \in (0,\infty)^n : \|x\| < \eta^*\}$ and $\psi(x) = 0$ for $\{x \in (0,\infty)^n : \|x\| \geq \eta\}$.

We claim that $\psi \in \mathcal{O}$. In fact, since $\psi \in C^\infty((0,\infty)^n)$ there exist $M_k > 0$ such that $|T^k\psi(x)| \leq M_k$ then there exist $n_k \in \mathbb{N}$ such that $|(1+\|x\|^2)^{-n_k}T^k\psi(x)| < \infty$.

Since $\mathbf{supp}((1-\psi)\phi) \subset \{x \in (0,\infty)^n : \|x\| \geq \eta^*\}$, then for the hypothesis

$$((1-\psi)T, \phi) = (T, (1-\psi)\phi) = 0 \quad \forall \phi \in \mathcal{H}_\mu.$$

From the above it follows that $T = \psi T$, then

$$\begin{aligned} |(T,\phi)| &= |(\psi T, \phi)| = |(T, \psi\phi)| \leq C\rho_N^\mu(\psi\phi) \\ &= C \sum_{\substack{m\leq N \\ |k|\leq N}} \sup_{x\in(0,\infty)^n} |(1+\|x\|^2)^m x^{-\mu-1/2} S^k(\psi\phi)(x)|. \end{aligned} \tag{4.1}$$

Since $\mathbf{supp}\, \psi \subset \{x \in (0,\infty)^n : \|x\| \leq \eta\}$, then

$$\begin{aligned} &\sup_{x\in(0,\infty)^n} |(1+\|x\|^2)^m x^{-\mu-1/2} S^k(\psi\phi)(x) \\ &\quad \leq \sup_{\|x\|<\eta^*} |(1+\|x\|^2)^m x^{-\mu-1/2} S^k\phi(x)| \\ &\quad\quad + \sup_{\eta^*\leq\|x\|<\eta} |(1+\|x\|^2)^m x^{-\mu-1/2} S^k(\psi\phi)(x)|. \end{aligned} \tag{4.2}$$

If we consider $\|x\| < \eta^*$, then

$$\sup_{\|x\|\leq\eta^*} |(1+\|x\|^2)^m x^{-\mu-1/2} S^k\phi(x)| \leq 2^{|m|}\varepsilon. \tag{4.3}$$

Now we consider $\eta^* \leq \|x\| < \eta$. Applying (A.3) and (2.1) we obtain that

$$\begin{aligned} x^{-\mu-1/2} S^k(\psi\phi)(x) &= \sum_{l=0}^{k} b_{l,k} x^{2l} T^{k+l}\{x^{-\mu-1/2}(\psi\phi)(x)\} \\ &= \sum_{l=0}^{k} b_{l,k} x^{2l} \sum_{r=0}^{k+l} \binom{k+l}{r} T^{k+l-r}\psi(x) T^r\{x^{-\mu-1/2}\phi(x)\}. \end{aligned} \tag{4.4}$$

Since $\psi \in C^\infty((0,\infty)^n)$, there exist positive constants such that

$$|T^{k+l-r}\psi(x)| \leq M_{k,l,r}, \tag{4.5}$$

in $\eta^* \le \|x\| < \eta$. Accordingly to (4.4) and (4.5) we now have that

$$
\begin{aligned}
&|(1+\|x\|^2)^m x^{-\mu-1/2} S^k(\psi\phi)(x)| \\
&\le (1+\|x\|^2)^m \sum_{l=0}^{k}\sum_{r=0}^{k+l} |b_{l,k}| \binom{k+l}{r} M_{k,l,r}\, |x^{2l} T^r\{x^{-\mu-1/2}\phi(x)\}| \\
&= \sum_{l=0}^{k}\sum_{r=0}^{k+l} M^*_{k,l,r}\, (1+\|x\|^2)^m x^{2l} |T^r\{x^{-\mu-1/2}\phi(x)\}| \\
&\le \sum_{l=0}^{k}\sum_{r=0}^{k+l} M^*_{k,l,r}\, (1+\|x\|^2)^{m+l} |T^r\{x^{-\mu-1/2}\phi(x)\}| \\
&\le \sum_{l=0}^{k}\sum_{r=0}^{k+l} B_{k,l,r} \sup_{x\in(0,\infty)^n} |(1+\|x\|^2)^{m+l} x^{-\mu-1/2} S^r\phi(x)|.
\end{aligned} \tag{4.6}
$$

Since $|r| \le |2k| \le 2N = N_0$ then

$$
|(1+\|x\|^2)^{m+l} x^{-\mu-1/2} S^r\phi(x)| \le 2^{|m+l|} |x^{-\mu-1/2} S^r\phi(x)| \le 2^{|m+l|}\varepsilon. \tag{4.7}
$$

From (4.1), (4.2), (4.3), (4.6) and (4.7) then:

$$
\begin{aligned}
|(T,\phi)| &\le C \sum_{\substack{m\le N\\ |k|\le N}} \sup_{x\in(0,\infty)^n} |(1+\|x\|^2)^m x^{-\mu-1/2} S^k(\psi\phi)(x)| \\
&\le C \sum_{\substack{m\le N\\ |k|\le N}} \left(2^{|m|}\varepsilon + \sum_{l=0}^{k}\sum_{r=0}^{k+l} B_{k,l,r} 2^{|m+l|}\varepsilon \right) = C'\varepsilon
\end{aligned}
$$

with $C' = C\sum_{\substack{m\le N\\ |k|\le N}} (2^{|m|} + \sum_{l=0}^{k}\sum_{r=0}^{k+l} B_{k,l,r} 2^{|m+l|})$. Hence $(T,\phi) = 0$ since $\varepsilon > 0$ was arbitrarily chosen. ■

Lemma 4.2: *Let $\psi \in C^\infty((0,\infty)^n)$ such that $\psi(x) = 1$ if $x_1 + \cdots + x_n \ge a^2$, $\psi(x) = 0$ if $x_1 + \cdots + x_n \le b^2$ with $0 < b^2 \le a^2$ and $0 \le \psi \le 1$. And let $P[x] = \sum_{|\alpha|\le N} a_\alpha x^\alpha \neq 0$ for all $x \in \mathbb{R}^n - \{0\}$ and all its coefficients have the same sign, therefore $P[x_1^2,\dots,x_n^2]^{-1}\psi(x_1^2,\dots,x_n^2) \in \mathcal{O}$.*

Proof: Let $P[x_1,\dots,x_n] = \sum_{|\alpha|\le N} a_\alpha x_1^{\alpha_1}\dots x_n^{\alpha_n}$.

The aim of this proof is to verify that for all $k \in \mathbb{N}_0^n$ there exists $n_k \in \mathbb{Z}$ such that

$$
|(1+\|x\|^2)^{n_k} T^k\{P[x_1^2,\dots,x_n^2]^{-1}\psi(x_1^2,\dots,x_n^2)\}| \le C \quad \forall x \in (0,\infty)^n.
$$

For $b \leq \|x\| \leq a$ it turns out that

$$\begin{aligned} T^{e_i}\{P[x_1^2,\dots,x_n^2]^{-1}\psi(x_1^2,\dots,x_n^2)\} &= x_i^{-1}\frac{\partial}{\partial x_i}\{P[x_1^2,\dots,x_n^2]^{-1}\psi(x_1^2,\dots,x_n^2)\} \\ &= P[x_1^2,\dots,x_n^2]^{-2}\tilde{P}[x_1^2,\dots,x_n^2]\psi(x_1^2,\dots,x_n^2) \\ &\quad + 2P[x_1^2,\dots,x_n^2]^{-1}\frac{\partial\psi}{\partial x_i}(x_1^2,\dots,x_n^2). \end{aligned} \tag{4.8}$$

Since all the functions involved, ψ and its derivatives are all continuous in $b \leq \|x\| \leq a$, it is clear that (4.8) is bounded. On the other hand, if $\|x\| \geq a$, since $\psi(x) = 1$ then

$$T^{e_i}\{P[x_1^2,\dots,x_n^2]^{-1}\} = x_i^{-1}\frac{\partial}{\partial x_i}\{P[x_1^2,\dots,x_n^2]^{-1}\} = P[x_1^2,\dots,x_n^2]^{-2}\tilde{P}[x_1^2,\dots,x_n^2].$$

We already shown that $\tilde{P}$ is in $\mathcal{O}$, so, there exist $r \in \mathbb{Z}$ such that $|\tilde{P}[x_1^2,\dots,x_n^2]| \leq C(1 + \|x\|^2)^r$. Without loss of generality suppose that all a_α are positives and let us first consider $a_0 \neq 0$, then $P[x_1^2,\dots,x_n^2]^{-2}$ is bounded as in (3.11).

If now we consider $a_0 = 0$, since $P[x_1^2,\dots,x_n^2] > 0$ for $(x_1,\dots,x_n) \neq (0,\dots,0)$ then P must attain a minimum in S^{n-1}. Let δ be such that

$$\delta < P\left[\frac{x_1^2}{\|x\|^2},\dots,\frac{x_n^2}{\|x\|^2}\right] = \sum_{1\leq|\alpha|\leq N} a_\alpha \frac{x_1^{2\alpha_1}\dots x_n^{2\alpha_n}}{\|x\|^{2|\alpha|}}. \tag{4.9}$$

Since $\|x\| \geq a$ and $|\alpha| \geq 1$ then

$$\|x\|^{2|\alpha|} > a^{2|\alpha|} \tag{4.10}$$

From (4.9) and (4.10) we obtain that

$$\delta < C \sum_{1\leq|\alpha|\leq N} a_\alpha x_1^{2\alpha_1}\dots x_n^{2\alpha_n} \tag{4.11}$$

with $C = \max_{1\leq|\alpha|\leq N} a^{-2|\alpha|}$, then $P[x_1^2,\dots,x_n^2]^{-2} \leq C^2\delta^{-2}$.

Then,

$$\sup_{\|x\|\geq a} |T^{e_i}\{P[x_1^2,\dots,x_n^2]^{-1}| \leq C'(1 + \|x\|^2)^r. \tag{4.12}$$

From equations (4.8) and (4.12) the Lemma follows for $k = e_i$. The general case follows in a similar way. ∎

Now we are ready for the proof of Theorem 1.1.

Proof of Theorem 1.1.: If $L(f) = 0$ this means that $\sum_{|\alpha|\leq N}(-1)^{|\alpha|}a_\alpha S^\alpha f = 0$.

Since $h_\mu(S_{\mu_i}f) = -y_i^2 h_\mu f$ (see [2]), applying Hankel transform to both sides, we have

$$h_\mu\left(\sum_{|\alpha|\le N}(-1)^{|\alpha|}a_\alpha S^\alpha f\right) = \sum_{|\alpha|\le N}(-1)^{|\alpha|}a_\alpha(-1)^{|\alpha|}y_1^{2\alpha_1}\dots y_n^{2\alpha_n}h_\mu f$$
$$= P[y_1^2,\dots,y_n^2]h_\mu f = 0. \tag{4.13}$$

Let ψ being as in the previous Lemma. Then $[P[y_1^2,\dots,y_n^2]]^{-1}\psi(y_1^2,\dots,y_n^2) \in \mathcal{O}$. Then multiplying in (4.13) we obtain that

$$\psi(y_1^2,\dots,y_n^2).h_\mu f = 0. \tag{4.14}$$

Let $\phi \in \mathcal{H}_\mu$ with $\mathbf{supp}\,\phi \subset \{x \in (0,\infty)^n : \|x\| \ge a\}$ and let us see that $(h_\mu f, \phi) = 0$.

Since $\psi(x_1^2,\dots,x_n^2).\phi(x_1,\dots,x_n) = \phi(x_1,\dots,x_n)$ in $(0,\infty)^n$, then

$$(h_\mu f,\phi) = (h_\mu f,\psi\phi) = (\psi h_\mu f,\phi) = 0, \tag{4.15}$$

where we have used (4.14). Consequently $h_\mu f$ is zero for all ϕ such that $\mathbf{supp}\,\phi \subset \{x \in (0,\infty)^n : \|x\| \ge a\}$. For Theorem 4.1 there exist $N_1 \in \mathbb{N}_0$ and constants c_k, $|k| \le N_1$ such that

$$h_\mu f = \sum_{|k|\le N_1} c_k S^k\delta_\mu. \tag{4.16}$$

Therefore, applying the Hankel transform h_μ to both sides of (4.16) and since $h_\mu = (h_\mu)^{-1}$ we obtain that

$$f = h_\mu(h_\mu f) = \sum_{|k|\le N_1} c_k h_\mu(S^k\delta_\mu) =$$
$$= \sum_{|k|\le N_1} c_k(-1)^{|k|}y_1^{2k_1}\dots y_1^{2k_n}h_\mu\delta_\mu$$
$$= \sum_{|k|\le N_1} c_k(-1)^{|k|}y_1^{2k_1}\dots y_1^{2k_n}y^{\mu+1/2},$$

which completes the proof. ■

5. Another proof of Theorem 1.1

We establish a different representation theorem from the one proved in the previous section.

Theorem 5.1: *Let $f \in \mathcal{H}'_\mu$ satisfying $(f,\phi) = 0$ for all $\phi \in \mathcal{H}_\mu$ with $\mathbf{supp}(\phi) \subset \{x \in (0,\infty)^n : \|x\| \ge a\}$ for some $a \in \mathbb{R}$, $a > 0$. Then there exist $N \in \mathbb{N}_0$ and scalars c_k, $|k| \le N$ such that*

$$f = \sum_{|k|\le N} c_k T^k\delta_\mu,$$

where $T^k\delta_\mu$ given by (2.3).

Proof: Let $f \in \mathcal{H}'_\mu$, such that f verifies the hypothesis of the theorem and $c > 0$, $N \in \mathbb{N}_0$ such that

$$|(f,\phi)| \le C \sum_{\substack{m\le N\\ |k|\le N}} \gamma^{\mu}_{m,k}(\phi), \quad \phi \in \mathcal{H}_\mu. \tag{5.1}$$

By the Taylor formula and (2.3), if $\phi \in \mathcal{H}_\mu$

$$\phi(x) = \frac{x^{\mu+1/2}}{C_\mu}\left\{(\delta_\mu,\phi) + \sum_{|k_1|=1}(T^{k_1}\delta_\mu,\phi)\frac{x^{2k_1}}{2k_1!} + \cdots \right.$$
$$\left. + \sum_{|k_N|=N}(T^{k_N}\delta_\mu,\phi)\frac{x^{2k_N}}{2^N k_N!} + C_\mu R_{2N}(x)\right\}, \tag{5.2}$$

where the remain term satisfies $\lim_{\substack{x\to 0\\ x_i>0}} T^k R_{2N}(x) = 0$ for all k multi-index such that $|k| \le N$. Then, given $\varepsilon > 0$ there exist $\eta_k > 0$ such that $|T^k R_{2N}(x)| < \varepsilon$ for $x \in (0,\infty)^n$ such that $\|x\| < \eta_k$. Set $\eta = \min_{|k|\le N}\{\eta_k\}$ and $\eta < 1$, then $|T^k R_{2N}(x)| < \varepsilon$ for all $x \in (0,\infty)^n$ such that $\|x\| < \eta$ and $|k| \le N$.

Let $a \in \mathbb{R}$ such that $0 < a < \eta$ and define ψ a smooth function on $(0,\infty)^n$ by $\psi(x) = 1$ for $\{x \in (0,\infty)^n : \|x\| < a/2\}$ and $\psi(x) = 0$ for $\{x \in (0,\infty)^n : \|x\| \ge a\}$ and therefore $(f,(1-\psi(x))\phi(x)) = 0$ for any $\phi \in \mathcal{H}_\mu$. Hence

$$(f,\phi) = (f,\psi\phi). \tag{5.3}$$

Therefore

$$(f,\phi) = \sum_{|k|\le N} c_k(T^k\delta_\mu,\phi) + (f, x^{\mu+1/2}\psi(x)R_{2N}(x)), \tag{5.4}$$

where $c_k = (1/C_\mu 2^{|k|}k!)(f, x^{\mu+1/2}x^{2k}\psi(x))$.

Applying the estimate (5.1) to $x^{\mu+1/2}\psi(x)R_{2N}(x)$, we get

$$|(f, x^{\mu+1/2}\psi(x)R_{2N}(x))| \le C \sum_{\substack{m\le N\\ |k|\le N}} \gamma^{\mu}_{m,k}(x^{\mu+1/2}\psi(x)R_{2N}(x)).$$

Then

$$\begin{aligned}
&|(1+\|x\|^2)^m T^k\{x^{-\mu-1/2}x^{\mu+1/2}\psi(x)R_{2N}(x)\}| \\
&\quad \le \sup_{\|x\|<a/2} |(1+\|x\|^2)^m T^k\{R_{2N}(x)\}| + \sup_{a/2\le\|x\|<a} |(1+\|x\|^2)^m T^k\{\psi(x)R_{2N}(x)\}| \\
&\quad \le \sup_{\|x\|<a/2} |(1+\|x\|^2)^m T^k\{R_{2N}(x)\}| + \sup_{a/2\le\|x\|<a} |(1+\|x\|^2)^m \sum_{j=0}^{k} T^{k-j}\psi(x)T^j R_{2N}(x)| \\
&\quad \le \sup_{\|x\|<a/2} |(1+\|x\|^2)^m T^k\{R_{2N}(x)\}| + \sup_{a/2\le\|x\|<a} \sum_{j=0}^{k} M_{j,k}|(1+\|x\|^2)^m T^j R_{2N}(x)|.
\end{aligned}$$

For $\|x\| < \eta$ result that

$$|(f, x^{\mu+1/2}\psi(x)R_{2N}(x))| \leq C \sum_{\substack{m\leq N\\|k|\leq N}} 2^m \left(1 + \sum_{j=0}^{k} M_{j,k}\right) \varepsilon = C'\varepsilon.$$

Thus $(f, x^{\mu+1/2}\psi(x)R_{2N}(x)) = 0$ since ε was arbitrarily chosen. Therefore

$$(f, \phi) = \sum_{|k|\leq N} c_k(T^k\delta_\mu, \phi).$$

■

Now we can sketch a different proof for Theorem 1.1.

Another proof of Theorem 1.1.: If $L(f) = 0$, then we obtain as in (4.15) that $h_\mu f$ is zero for all ϕ such that **supp** $\phi \subset \{x \in (0,\infty)^n : \|x\| \geq a\}$ with $a > 0$, $a \in \mathbb{R}$. Then, since Theorem 5.1 holds, there exist $N_2 \in \mathbb{N}_0$ and constants c_k, $|k| \leq N_2$ such that

$$h_\mu f = \sum_{|k|\leq N_2} c_k T^k \delta_\mu. \tag{5.5}$$

Therefore, applying the Hankel transform h_μ to both sides of (5.5) and since $h_\mu = (h_\mu)^{-1}$ we obtain that

$$f = h_\mu(h_\mu f) = \sum_{|k|\leq N_2} c_k h_\mu(T^k\delta_\mu) = \sum_{|k|\leq N_2} c_k M_k^\mu y_1^{2k_1} \dots y_n^{2k_n} y^{\mu+1/2},$$

where we have used (2.4). The proof is this complete. ■

Disclosure statement

No potential conflict of interest was reported by the authors.

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Appendix. Equivalence of the seminorms $\gamma^{\mu}_{m,k}$ and $\lambda^{\mu}_{m,k}$

The main result of this paper needs of the existence of another family of seminorms, different from the family $\gamma^{\mu}_{m,k}$, which is defined as

$$\lambda^{\mu}_{m,k}(\phi) = \sup_{x\in(0,\infty)^n} |(1+\|x\|^2)^m x^{-\mu-1/2} S^k \phi(x)|, \quad \phi \in \mathcal{H}_\mu. \tag{A1}$$

The construction of the family $\{\lambda^{\mu}_{m,k}\}_{m\in\mathbb{N}_0, k\in\mathbb{N}_0^n}$ was motivated by the works of Marrero and Betancor [6], Sánchez [8] and Koh and Zemanian [9]. This multinorm is important because generates on $\mathcal{H}_\mu$ the same topology as the family $\{\gamma^{\mu}_{m,k}\}$.

Remark A.1: Let k be a multi-index, the following equality is valid

$$x^{-\mu-1/2} S^k \phi(x) = \sum_{l=0}^{k} b_{l,k} x^{2l} T^{k+l}\{x^{-\mu-1/2}\phi(x)\}. \tag{A2}$$

This formula can be derived from the equation

$$x_i^{-\mu_i-1/2} S^{k_i}_{\mu_i} \phi(x) = \sum_{l=0}^{k_i} b_{l,k_i} x_i^{2l} T_i^{k_i+l}\{x_i^{-\mu_i-1/2}\phi(x)\}, \tag{A3}$$

where the constants b_{j,k_i}, $j = 0, \dots, k_i$, are suitable real constants, only depending on μ_i. The formula (A.3) is due to Koh and Zemanian (see [9, p.948]) and is valid for every $k_i \in \mathbb{N}_0$. Indeed, if $k \in \mathbb{N}_0^n$, $k = (k_1, \dots, k_n)$ then,

$$(x_i^{-\mu_i-1/2} S^{k_i}_{\mu_i})(x_j^{-\mu_j-1/2} S^{k_j}_{\mu_j})\phi(x) = \sum_{l_i=0}^{k_i}\sum_{l_j=0}^{k_j} b_{l_i,k_i} b_{l_j,k_j} x_i^{2l_i} x_j^{2l_j} T_i^{k_i+l_i} T_j^{k_j+l_j}\{x_i^{-\mu_i-1/2} x_j^{-\mu_j-1/2}\phi(x)\}.$$

Repeating this process we obtain that

$$x^{-\mu-1/2} S^k \phi(x) = (x_1^{-\mu_1-1/2} S^{k_1}_{\mu_1}) \dots (x_n^{-\mu_n-1/2} S^{k_n}_{\mu_n})\phi(x) = \sum_{l=0}^{k} b_{l,k} x^{2l} T^{k+l}\{x^{-\mu-1/2}\phi(x)\},$$

where $l = (l_1, \dots, l_n)$ and $k = (k_1, \dots, k_n)$.

On the other hand, from [8, Propositions IV.2.2 and IV.2.4] we have that for all $k_i \in \mathbb{N}_0$, $i = 1, \dots, n$ result that $|T_i^{k_i}\{x_i^{-\mu_i-1/2}\phi(x)\}| \le C_i \sup_{x_i\in(0,\infty)} |x_i^{-\mu_i-1/2} S^{k_i}_{\mu_i}\phi(x)|$.

So, we can generalize this inequality and obtain the following result

Remark A.2: Let k be a multi-index, the following inequality is valid

$$|T^k\{x^{-\mu-1/2}\phi(x)\}| \le C \sup_{x\in(0,\infty)^n} |x^{-\mu-1/2} S^k \phi(x)|. \tag{A4}$$

Set $i, j \in \{1, \dots, n\}$, $i \ne j$ and computing

$$\begin{aligned}
&|T_i^{k_i} T_j^{k_j}\{x^{-\mu-1/2}\phi(x)\}| \\
&\quad \le C_i \sup_{x_i\in(0,\infty)} |x_i^{-\mu_i-1/2} S^{k_i}_{\mu_i}\{T_j^{k_j}\{x_j^{-\mu_j-1/2} x^{-\mu-1/2+(\mu_i e_i+1/2)+(\mu_j e_j+1/2)}\phi(x)\}\}| \\
&\quad = C_i \sup_{x_i\in(0,\infty)} |T_j^{k_j}\{x_j^{-\mu_j-1/2}(x^{-\mu-1/2+(\mu_j e_j+1/2)} S^{k_i}_{\mu_i})\phi(x)\}|
\end{aligned}$$

$$\leq C_i C_j \sup_{x_j, x_i \in (0,\infty)} |x_j^{-\mu_j - 1/2} S_{\mu_j}^{k_j} \{x^{-\mu-1/2+(\mu_j e_j + 1/2)} S_{\mu_i}^{k_i} \phi(x)\}|$$

$$= C_i C_j \sup_{x_j, x_i \in (0,\infty)} |x^{-\mu-1/2} S_{\mu_i}^{k_i} S_{\mu_j}^{k_j} \phi(x)|.$$

The general case follows from an inductive argument.

From (A2) and (A4) we obtain that the families of seminorms $\gamma_{m,k}^{\mu}$ and $\lambda_{m,k}^{\mu}$ are equivalents.

$$|(1 + \|x\|^2)^m T^k \{x^{-\mu-1/2} \phi(x)\}| \leq C \sup_{x \in (0,\infty)^n} |(1 + \|x\|^2)^m x^{-\mu-1/2} S^k \phi(x)| = C \lambda_{m,k}^{\mu}(\phi),$$

therefore $\gamma_{m,k}^{\mu}(\phi) \leq C \lambda_{m,k}^{\mu}(\phi)$. On the other hand, (A.2) imply that

$$|(1 + \|x\|^2)^m x^{-\mu-1/2} S^k \phi(x)| \leq \sum_{l=0}^{k} |b_{l,k}| \, |(1 + \|x\|^2)^{m+|l|} T^{k+l} \{x^{-\mu-1/2} \phi(x)\}|$$

$$\leq \sum_{l=0}^{k} |b_{l,k}| \gamma_{m+|l|,k+l}^{\mu}(\phi),$$

which leads to $\lambda_{m,k}^{\mu}(\phi) \leq \sum_{l=0}^{k} |b_{l,k}| \gamma_{m+|l|,k+l}^{\mu}(\phi)$.