Discrete Applied Mathematics 245 (2018) 16–27

Contents lists available at ScienceDirect

Discrete Applied Mathematics

journal homepage: www.elsevier.com/locate/dam

Polyhedra associated with identifying codes in graphs

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ARTICLE INFO

Article history:
Received 23 December 2015
Received in revised form 2 June 2017
Accepted 7 June 2017
Available online 10 July 2017

Keywords:
Identifying code polyhedron
Identifying code clutter
Odd hypercycles

ABSTRACT

The identifying code problem is a newly emerging search problem, challenging both from a theoretical and a computational point of view, even for special graphs like bipartite graphs. Hence, a typical line of attack for this problem is to determine minimum identifying codes of special graphs or to provide bounds for their size.

In this work we study the associated polyhedra and present some general results on their combinatorial structure. We demonstrate how the polyhedral approach can be applied to find minimum identifying codes for special graphs, and discuss further lines of research in order to obtain strong lower bounds stemming from linear relaxations of the identifying code polyhedron, enhanced by suitable cutting planes to be used in a B&C framework.

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1. Introduction

Many search problems as, e.g., fault detection in networks, fire detection in buildings, or performing group tests, can be modeled by so-called identifying codes in graphs [18].

Consider a graph $G = (V, E)$ with a set of nodes $V = \{1, \dots, |V|\}$ and a set of edges E. Given a node $i \in V$ let denote by $N[i] = \{i\} \cup N(i)$ the closed neighborhood of i, i.e., the node i together with all its neighbors. A subset $C \subseteq V$ is *dominating* (resp. *separating*) if $N[i] \cap C$ are non-empty (resp. distinct) sets for all $i \in V$. An *identifying code* of G is a node subset which is dominating and separating, see Fig. 1 for illustration.

Not every graph G admits an identifying code or is *identifiable*: this holds if and only if there are no true twins in G, i.e., there is no pair of distinct nodes $i, j \in V$ with $N[i] = N[j]$ [18]. On the other hand, for every identifiable graph, its whole node set trivially forms an identifying code.

The *identifying code number* $\gamma^{ID}(G)$ of a graph G is the minimum cardinality of any identifying code of G. Determining $\gamma^{ID}(G)$ is in general NP-hard [11]. From a combinatorial point of view, the problem has been actively studied during the last decade. Typical lines of attack are to determine minimum identifying codes of special graphs or to provide bounds for their size. Closed formulas for the exact value of $\gamma^{ID}(G)$ have been found so far only for restricted graph families (e.g. for paths and cycles by [10] and for stars by [17]). A linear time algorithm to determine $\gamma^{ID}(G)$ if G is a tree was provided by [7], but for many other graph classes where several other in general hard problems are easy to solve, it turned out that the identifying code problem remains hard. This includes bipartite graphs [11] and two classes of chordal graphs, namely split graphs and interval

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1 This work was supported by an ECOS-MINCyT cooperation France–Argentina, A12E01.

http://dx.doi.org/10.1016/j.dam.2017.06.005
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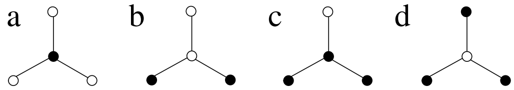

Fig. 1. A graph, where the subset of black nodes forms (a) a dominating (but not separating) set, (b) a separating (but not dominating) set, and (c), (d) minimum identifying codes.

graphs [13]. This motivates the study of bounds for $\gamma^{ID}(G)$. For instance, a canonical lower bound is $\lceil\log(n+1)\rceil \le \gamma^{ID}(G)$ for general graphs G of order n by [18]. The trivial upper bound $\gamma^{ID}(G) \le n$ has been improved for connected graphs G to $\gamma^{ID}(G) \le n-1$ by [17] (with stars being examples where this bound is tight) and for line graphs to $\gamma^{ID}(L(G)) \le 2|V(G)| - 5$ by [14].

As polyhedral methods have been already proved to be successful for several NP-hard combinatorial optimization problems, our aim is to apply such techniques to the identifying code problem. For that, a reformulation as set covering problem is in order. For a 0/1-matrix M with n columns, the set covering polyhedron is $Q^*(M) = \text{conv}\left\{x \in \mathbf{Z}^n_+ : Mx \ge \mathbf{1}\right\}$ and $Q(M) = \left\{x \in \mathbf{R}^n_+ : Mx \ge \mathbf{1}\right\}$ is its linear relaxation. A *cover* of M is a 0/1-vector x such that $Mx \ge \mathbf{1}$, and the *covering number* $\tau(M)$ equals $\min \mathbf{1}^T x$, $x \in Q^*(M)$ (see Section 2 for more details).

We obtain such a constraint system $Mx \ge \mathbf{1}$ for the identifying code problem as follows. Domination clearly requires that any identifying code C intersects the closed neighborhood $N[i]$ of each node $i \in V$. Separation means that no two intersections $C \cap N[i]$ and $C \cap N[j]$ are equal or, equivalently, that C intersects each symmetric difference $N[i] \bigtriangleup N[j]$ for distinct nodes $i, j \in V$.

From now on given $S \subseteq \{1, \dots, n\}$ by $x(S)$ we mean $\sum_{i \in S} x_i$. Hence, the following constraints encode the domination and separation requirements:

$$\begin{array}{rcll} \min \mathbf{1}^T x & & & \\ x(N[j]) & \ge & 1 \ \ \forall j \in V & \text{(domination)} \\ x(N[j] \bigtriangleup N[k]) & \ge & 1 \ \ \forall j,k \in V, j \ne k & \text{(separation)} \\ x & \in & \{0,1\}^{|V|}. & \end{array}$$

Let $M_{ID}(G)$ be the resulting *identifying code matrix* of G, i.e., the matrix having as rows the incidence vectors of the closed neighborhoods of the nodes of G and their pairwise symmetric differences. Accordingly, we define the *identifying code polyhedron of* G as

$$P_{ID}(G) = Q^*(M_{ID}(G)) = \text{conv}\left\{x \in \mathbf{Z}^{|V|}_+ : M_{ID}(G)\, x \ge \mathbf{1}\right\}.$$

It is clear from the definition that a graph is identifiable if and only if none of the symmetric differences results in a zero-row of $M_{ID}(G)$, and that $\gamma^{ID}(G)$ equals the covering number $\tau(M_{ID}(G)) = \min \mathbf{1}^T x$, $x \in P_{ID}(G)$.

Our aim is to apply the polyhedral approach to find minimum identifying codes.

We first provide some definitions and results related to covering polyhedra (Section 2), then we focus on general properties of the identifying code polyhedron $P_{ID}(G)$ and introduce the canonical linear relaxation (Section 3). Afterwards, we discuss several lines to apply polyhedral techniques. In Section 4, we present cases where $M_{ID}(G)$ falls into a class of matrices for which the set covering polyhedron is known and we, thus, immediately can obtain a complete description of $P_{ID}(G)$ and the exact value of $\gamma^{ID}(G)$.

Furthermore, we present cases where a complete description of $P_{ID}(G)$ involves many and complicated facets, but where we can identify facet-defining substructures (related to minors of $M_{ID}(G)$) that allow us to derive the full rank inequality $x(V) \ge \tau(M_{ID}(G)) = \gamma^{ID}(G)$ and, thus, the exact value of $\gamma^{ID}(G)$ (Section 5).

This demonstrates how polyhedral techniques can be applied in this context. We close with a discussion on future lines of research, including how the here obtained results can be extended to other classes of graphs.

Some of the results in this contribution appeared without proofs in [2–4,6].

2. Properties of set covering polyhedra

We introduce definitions and basic concepts related to set covering polyhedra and provide results which are crucial for the proofs in the subsequent sections.

2.1. Preliminaries

Given two vectors $x, y \in \mathbb{R}^n$, we say that $x \le y$ if $x_i \le y_i$ for all $i \in \{1, \dots, n\}$.

Let $M \in \{0,1\}^{n \times m}$. If x and y are two rows of M and $x \le y$, we say that y is *redundant*.

Remind that a *cover* of a matrix M is a vector $x \in \{0,1\}^n$ such that $Mx \ge \mathbf{1}$. A cover x of M is *minimal* if there is no other cover y of M such that $y \le x$. The blocker of M, denoted by $b(M)$, is the matrix whose rows are the minimal covers of M. It is known that $b(b(M)) = M$ and, thus, we can refer to $Q^*(M)$ and $Q(b(M))$ as a blocking pair of polyhedra. Moreover, $\mathbf{a}$ is an extreme point of $Q(b(M))$ if and only if $\mathbf{a}^T x \geq 1$ is a facet defining inequality of $Q^*(M)$ (see [15]). In the sequel we will refer to this property as *blocking duality*.

Given a matrix M and $j \in \{1, \ldots, n\}$, we introduce two matrix operations: the contraction of j, denoted by M/j, means that column j is removed from M as well as the resulting redundant rows and hence, corresponds to setting $x_j = 0$ in the constraints $Mx \geq \mathbf{1}$. The deletion of j, denoted by $M \setminus j$ means that column j is removed from M as well as all the rows with a 1 in column j and this corresponds to setting $x_j = 1$ in the constraints $Mx \geq \mathbf{1}$.

The contraction of a set V_1 of columns from M is the matrix M/V_1 obtained by contracting all the columns $j \in V_1$ and the deletion of a set V_2 of columns from M is the matrix $M \setminus V_2$ obtained by deleting columns $j \in V_2$.

Then, given M and $V_1, V_2 \subseteq \{1, \ldots, n\}$ disjoint, we will say that $M/V_1 \setminus V_2$ is a *minor* of M and this minor does not depend on the order of operations or elements in $\{1, \ldots, n\}$. It is clear that M is always a minor of itself and we will say that a minor $M/V_1 \setminus V_2$ is *proper* if $V_1 \cup V_2 \neq \emptyset$. It is not hard to see that $b(M/j) = b(M) \setminus j$ and $b(M \setminus j) = b(M)/j$ for every $j \in \{1, \ldots, n\}$.

Let $U \subseteq \{1, \ldots, n\}$ be a subset of columns of M and $\overline{U} = \{1, \ldots, n\} - U$ its complement. A *rank inequality* associated with a minor $M' = M \setminus U$ is

$$\sum_{i \in \overline{U}} x_i \geq \tau(M'). \tag{1}$$

Remark 1. In [1], it is shown the following. If (1) is a facet of $Q^*(M')$, then it is also a facet of $Q^*(M)$. In addition, if the rank constraint associated with some minor induces a facet-defining inequality of $Q^*(M)$ then this inequality is also induced by a minor obtained by deletion only.

2.2. Set covering polyhedra associated with q-roses

Let $\mathcal{H} = (V, \mathcal{E})$ be a hypergraph with $\mathcal{E} \subseteq 2^V$ and let $M(\mathcal{H})$ denote its incidence matrix, i.e., $M(\mathcal{H})$ encodes row-wise the incidence vectors of the hyperedges in $\mathcal{E}$. Given $n > q \geq 2$, let $\mathcal{R}_n^q = (V, \mathcal{E})$ be the hypergraph where $V = \{1, \ldots, n\}$ and $\mathcal{E}$ contains all q-element subsets of V. Nobili and Sassano [19] called the incidence matrix of $\mathcal{R}_n^q$ the *complete q-rose of order n* and we denote it by $M(\mathcal{R}_n^q)$. In [20] it is proved the following result.

Theorem 2 ([20]). *For $n > q \geq 2$, the inequality*

$$\sum_{i=1}^{n} x_i \geq \tau(M(\mathcal{R}_n^q)) = n - q + 1$$

is a facet defining inequality for $Q^*(M(\mathcal{R}_n^q))$.

For the sake of completeness, we here present the unpublished proofs of the results in [6] describing the set covering polyhedron of q-roses of order n. We start with the study of minors of $M(\mathcal{R}_n^q)$. It can be easily observed that the following holds.

Remark 3. For $n > q \geq 2$ and $i \in \{1, \ldots, n\}$

1. $M(\mathcal{R}_n^q) \setminus i = M(\mathcal{R}_{n-1}^q)$.
2. $M(\mathcal{R}_n^q)\,/i = M(\mathcal{R}_{n-1}^{q-1})$.

In addition, the next result proves that the blocker of a complete q-rose is a complete $n - q + 1$-rose.

Lemma 4. *Let $n > q \geq 2$, then $b(M(\mathcal{R}_n^q)) = M(\mathcal{R}_n^{n-q+1})$.*

Proof. From Theorem 2, $\tau(M(\mathcal{R}_n^q)) = n - q + 1$. Let d be a 0/1-vector with $n - q + 1$ entries at value one. By definition every row of $M(\mathcal{R}_n^q)$ has $n - q$ entries at value zero. It is easy to check that $M(\mathcal{R}_n^q)d \geq \mathbf{1}$; i.e., d is a minimum cover of $M(\mathcal{R}_n^q)$. It follows that $M(\mathcal{R}_n^{n-q+1})$ is a row submatrix of $b(M(\mathcal{R}_n^q))$. Let d' be a 0/1-vector with more than $n - q + 1$ entries at value one. It is clear that there is a row r in $M(\mathcal{R}_n^{n-q+1})$ such that $r \leq d'$. Hence d' is not a minimal cover of $M(\mathcal{R}_n^q)$. Then, the rows in $M(\mathcal{R}_n^{n-q+1})$ are the only minimal covers of $M(\mathcal{R}_n^q)$, i.e., $b(M(\mathcal{R}_n^q)) = M(\mathcal{R}_n^{n-q+1})$. $\square$

Theorem 5. *Let $n > q \geq 2$. The point $\bar{x}$ is a fractional extreme point of $Q(M(\mathcal{R}_n^q))$ if and only if*

$$\bar{x}_i = \begin{cases} \dfrac{1}{q-s} & \text{if } i \notin C_s, \\ 0 & \text{if } i \in C_s, \end{cases} \tag{2}$$

where $s \in \{0, \ldots, q-2\}$ and $C_s \subseteq \{1, \ldots, n\}$, with $|C_s| = s$.

Proof. Firstly consider $\bar{x} = \frac{1}{q}\mathbf{1}$, then $s = 0$ and $C_s = \emptyset$. It is immediate that $M(\mathcal{R}_n^q)\bar{x} = \mathbf{1}$. In order to prove that $\bar{x}$ is an extreme point we need to find n linearly independent constraints of the system $M(\mathcal{R}_n^q)x \geq \mathbf{1}$.

Then, for every $i = 1, \dots, q+1$, we select a row f_i such that its jth entry equals

$$(f_i)_j = \begin{cases} 0 & \text{if } j = q+2, \dots, n \ \text{ or } j = i \\ 1 & \text{otherwise,} \end{cases}$$

and, for every $i = q+2, \dots, n$, we select a row f_i of $M(\mathcal{R}_n^q)$ such that its jth entry equals

$$(f_i)_j = \begin{cases} 1 & \text{if } j = i \\ 0 & \text{for every } j \geq q+2 \text{ but } i \neq j. \end{cases}$$

The considered rows can be reordered in a matrix M in the following way

$$M = \left(\begin{array}{c|c} J & 0 \\ \hline A & I \end{array} \right),$$

where J is the square matrix of order $q+1$ with all its entries at value one except for its main diagonal that has all 0's, I is the identity matrix of order $n-q-1$ and A is a $(n-q-1) \times (q+1)$ matrix with $q-1$ ones per row.

Trivially J has rank $q+1$. It follows that M has rank n and since $\bar{x}$ satisfies the system $M\bar{x} = \mathbf{1}$ it is an extreme point of $Q(M(\mathcal{R}_n^q))$.

It remains to prove that if $s = 0$ then $\bar{x} = \frac{1}{q}\mathbf{1}$ is the only fractional extreme point of $Q(M(\mathcal{R}_n^q))$ with no zero entries.

Let $\bar{y}$ be a fractional extreme point of $Q(M(\mathcal{R}_n^q))$ with $s = 0$. Then the n linearly independent facet inducing inequalities that $\bar{y}$ satisfies at equality are associated with a square row submatrix M' of $M(\mathcal{R}_n^q)$ with rank n. Observe that if $M'x = \mathbf{1}$ has a unique solution then $\bar{y} = \bar{x}$.

Now, consider $s \in \{1, \dots, q-2\}$ and an extreme point $\bar{x}$ defined by (2). Observe that $\bar{x} \in Q(M(\mathcal{R}_n^q)) \cap \{x : x_i = 0 \text{ for all } i \in C_s\}$. From Remark 3.2 we have that $M(\mathcal{R}_n^q)\ /C_s = M(\mathcal{R}_{n-s}^{q-s})$, hence $\bar{x}$ can be written as $(\bar{z}, 0)$ where $\bar{z} = \frac{1}{q-s}\mathbf{1}$ with $\bar{z} \in Q(M(\mathcal{R}_{n-s}^{q-s}))$. As a consequence of the case $s = 0$ already proved, $\bar{z}$ is an extreme point of $Q(M(\mathcal{R}_{n-s}^{q-s}))$ and then $\bar{x}$ is an extreme point of $Q(M(\mathcal{R}_n^q))$.

Conversely, let $\bar{x}$ be an extreme point of $Q(M(\mathcal{R}_n^q))$ and suppose it has zero components. Let $C_s = \{i : \ \bar{x}_i = 0\}$. Then, the point $\bar{z} \in \mathbb{R}^{n-|C_s|}$ such that $\bar{z}_i = \bar{x}_i, i \in \{1, \dots, n\} - C_s$ is an extreme point of $Q(M(\mathcal{R}_n^q)/C_s)$. From Remark 3.2, if $s = |C_s|$ we have that $\bar{z}$ is an extreme point of $Q(M(\mathcal{R}_{n-s}^{q-s}))$ with no zero components. Hence, $\bar{z} = \frac{1}{q-|C_s|}\mathbf{1}$. Then we have,

$$\bar{x}_i = \begin{cases} \dfrac{1}{q-s} & \text{if } i \notin C_s \\ 0 & \text{if } i \in C_s. \quad \square \end{cases} \tag{3}$$

As a consequence we have:

Theorem 6. *Let $n > q \geq 2$. An inequality $\sum_{i=1}^n a_i x_i \geq 1$ with $a_j \notin \{0, 1\}$ for some $j \in \{1, \dots, n\}$ is a facet-defining inequality for $Q^*(b(M(\mathcal{R}_n^q))) = Q^*(M(\mathcal{R}_n^{n-q+1}))$ if and only if $\sum_{i=1}^n a_i x_i \geq 1$ can be written as $x(A_s) \geq q-s$ for some $A_s \subseteq \{1, \dots, n\}$ where $s \in \{0, \dots, q-2\}$ and $|A_s| = n-s$.*

Proof. Consider a facet-defining inequality of $Q^*(M(\mathcal{R}_n^{n-q+1}))$ of the form $\sum_{i=1}^n a_i x_i \geq 1$ with $a_j \notin \{0, 1\}$ for some $j \in \{1, \dots, n\}$. From Lemma 4, $M(\mathcal{R}_n^{n-q+1}) = b(M(\mathcal{R}_n^q))$ and using blocking duality it holds that the vector $a \in \mathcal{R}^n$ is a fractional extreme point of $Q(M(\mathcal{R}_n^q))$. From Theorem 5 it follows that $a_i = \frac{1}{q-s}$ if $i \notin C_s \subseteq \{1, \dots, n\}$ and $a_i = 0$ otherwise. Using the same results the converse is straightforward. $\square$

3. General properties of identifying code polyhedra

In this section, we examine general properties of identifying code polyhedra concerning their dimension and study which of the constraints defining the canonical linear relaxation define facets. From the set covering formulation, it is clear that the inequalities

$$x_i \geq 0 \quad \text{for } i \in V, \tag{4}$$

$$x(N[i]) \geq 1 \quad \text{for } i \in V, \tag{5}$$

$$x(N[i] \triangle N[j]) \geq 1 \quad \text{for } i, j \in V,\ j \neq i \tag{6}$$

are always valid for $P_{ID}(G)$. The inequalities (4) are called *trivial*, we refer to the inequalities (5) as *closed neighborhood inequalities* and to the inequalities (6) as *symmetric difference inequalities*.

Accordingly, the identifying code matrix is composed by

$$M_{ID}(G) = \begin{pmatrix} N[G] \\ \triangle[G] \end{pmatrix},$$

encoding row-wise the closed neighborhoods of the nodes of G (in $N[G]$) and their pairwise symmetric differences (in $\triangle[G]$).

A graph G is identifiable if and only if $P_{ID}(G)$ is non-empty. As $N[G]$ has no zero-row, G is identifiable if and only if $\triangle[G]$ has no zero-row (i.e. if and only if G has no true twins [18]).

We first address the question when $P_{ID}(G)$ is full-dimensional. It is known from Balas and Ng [8] that a polyhedron $Q^*(M)$ is full-dimensional if and only if the matrix M has at least two ones per row.

For $P_{ID}(G)$, this means that G must not have isolated nodes (to ensure $|N[i]| \geq 2$ for all $i \in V(G)$) and that there are no two adjacent nodes i and j with $N[i] = N[j] \cup \{k\}$ for some node k (to ensure $|N[i] \triangle N[j]| \geq 2$ for all distinct $i, j \in V(G)$).

Let $V_1(G)$ be the set of nodes $k \in V(G)$ such that $\{k\} = N[i] \triangle N[j]$ for two different nodes i and j in $V(G)$. We immediately obtain:

Corollary 7. *Let G be a graph without isolated nodes. Then, we have:*

1. *$P_{ID}(G)$ is full-dimensional if and only if $V_1(G) = \emptyset$.*
2. *The constraint $x_i \geq 0$ defines a facet of $P_{ID}(G)$ if and only if $i \notin V_1(G)$.*

In addition, $M_{ID}(G)$ may contain rows which are redundant. We, therefore, define the corresponding clutter matrix, the *identifying code clutter matrix* $C_{ID}(G)$ of a graph G, obtained by removing redundant rows from $M_{ID}(G)$. We clearly have $P_{ID}(G) = \text{conv}\{x \in \mathbf{Z}_+^{|V|} : C_{ID}(G)\, x \geq \mathbf{1}\}$. Moreover, in [8] it is proved that the only facet-defining inequalities of a set covering polyhedron $Q^*(A)$ with integer coefficients and right hand side equal to 1 are those of the system $Ax \geq \mathbf{1}$. Hence we have:

Theorem 8. *All constraints from $C_{ID}(G)\, x \geq \mathbf{1}$ define facets of $P_{ID}(G)$.*

We obtain a linear relaxation, the *fractional identifying code polyhedron* $Q_{ID}(G)$ of G, by considering all vectors satisfying the above inequalities:

$$Q_{ID}(G) = \left\{ x \in \mathbf{R}_+^{|V|} : C_{ID}(G)\, x \geq \mathbf{1} \right\}.$$

We, therefore, propose to firstly determine the identifying code clutter matrix $C_{ID}(G)$ and then to study which further constraints have to be added to $Q_{ID}(G)$ to obtain $P_{ID}(G)$.

In order to discuss which rows from $M_{ID}(G)$ remain in $C_{ID}(G)$ it is convenient to consider the hypergraph associated with $C_{ID}(G)$.

We define the *identifying code hypergraph* $H_{ID}(G)$ to be the hypergraph whose incidence matrix $M(H_{ID}(G))$ equals $C_{ID}(G)$. Clearly, every hyperedge of $H_{ID}(G)$ corresponds to the closed neighborhood of a node in G or the symmetric difference of two nodes in G. But, since $C_{ID}(G)$ is a clutter matrix, there is no hyperedge in $H_{ID}(G)$ that contains another hyperedge. Therefore there is no hyperedge containing a node from $V_1(G)$. In addition, we observe that if i and j are neither adjacent nor have a common neighbor, then $N[i]$ and $N[j]$ are disjoint, hence $N[i] \triangle N[j] = N[i] \cup N[j]$ follows and its characteristic vector is redundant in $M_{ID}(G)$. This implies a symmetric difference $N[i] \triangle N[j]$ is a hyperedge of $H_{ID}(G)$ only if i and j are adjacent or have a common neighbor.

4. Identifying code polyhedra of complete *p*-partite graphs

In this section, we consider complete p-partite graphs and establish a connection to complete 2-roses of order n, $\mathcal{R}_n^2$, already mentioned in Section 2.2.

4.1. Complete bipartite graphs

First we consider complete bipartite graphs $K_{m,n}$ with bipartition $A = \{1, \ldots, m\}$ and $B = \{m+1, \ldots, m+n\}$. We begin with the case of stars $K_{1,n}$, i.e., $A = \{1\}$ and $n \geq 2$. Note that $K_{1,2} = P_3$ and it is easy to see that $V_1 = B$ is the unique minimum identifying code.

Lemma 9. *For a star $K_{1,n}$ with $n \geq 3$, we have $H_{ID}(K_{1,n}) = K_{1+n}$ and $C_{ID}(K_{1,n}) = M(\mathcal{R}_{n+1}^2)$.*

Proof. For a star $K_{1,n}$ with $n \geq 3$, we have that

- $N[1] = \{1\} \cup B$,
- $N[i] = \{1, i\}$ for all $i \in B$,
- $N[1] \triangle N[i] = B - \{i\}$ for all $i \in B$,
- $N[j] \triangle N[k] = \{j, k\}$ for distinct $j, k \in B$.

This shows that $V_1(K_{1,n}) = \emptyset$. After removing those sets whose characteristic vectors are redundant, namely $N[1] = \{1\} \cup B$ and $N[1] \triangle N[i] = B - \{i\}$ for all $i \in B$, we obtain that $H_{ID}(K_{1,n})$ exactly contains all 2-element subsets of $A \cup B$ and, thus, it induces a clique K_{1+n} and $C_{ID}(K_{1,n}) = M(\mathcal{R}_{n+1}^2)$ follows. □

Then we deduce from Theorem 6:

Corollary 10. *$P_{ID}(K_{1,n})$ with $n \geq 3$ is described by the inequalities $x(C) \geq |C| - 1$ for all nonempty subsets $C \subseteq \{1, \ldots, n+1\}$.*

The above inequalities yield, for $|C| = 1$, the trivial inequalities $x_i \geq 0$ and, for $|C| = 2$, the closed neighborhood and symmetric difference inequalities $x_i + x_j \geq 1$ with $i \neq j$ describing $Q_{ID}(K_{1,n})$. On the other hand, $C = V$ yields the full rank facet which immediately implies $\gamma^{ID}(K_{1,n}) = |V| - 1$ (and provides an alternative proof for the result given in [17]).

Observe that for $K_{2,2}$, it is easy to see that $C_{ID}(K_{2,2}) = M(\mathcal{R}_4^2)$. Therefore, Corollary 10 also applies to $P_{ID}(K_{2,2})$. For general complete bipartite graphs $K_{m,n}$ with $m \geq 2$, $n \geq 3$, we obtain:

Lemma 11. *For a complete bipartite graph $K_{m,n}$ with $m \geq 2$, $n \geq 3$, we have $H_{ID}(K_{m,n}) = K_m \cup K_n$ and*

$$C_{ID}(K_{m,n}) = \begin{pmatrix} M(\mathcal{R}_m^2) & 0 \\ 0 & M(\mathcal{R}_n^2) \end{pmatrix}.$$

Proof. Let $k, l \in A$. Clearly, $N[k] \triangle N[l] = \{k, l\}$. Hence the rows corresponding to the sets $N[i] = \{i\} \cup B$, for $i \in A$, and $N[i] \triangle N[j] = (A \cup B) - \{i, j\}$ for $i \in A$ and $j \in B$ are redundant. Symmetric considerations show that only symmetric differences $N[i] \triangle N[j]$ remain where i, j come either both from A or both from B. Thus, $H_{ID}(K_{m,n})$ exactly contains all 2-element subsets of A and all 2-element subsets of B. □

Remark 12. It is known that the set of facet-defining inequalities of $Q^*(M)$ when M is a block matrix of the form

$$M = \begin{pmatrix} M_1 & \mathbf{0} \\ \mathbf{0} & M_2 \end{pmatrix}$$

is the union of the sets of facet-defining inequalities for $Q^*(M_1)$ and $Q^*(M_2)$.

As a consequence of Theorem 6, Lemma 11 and the above remark we conclude the following:

Corollary 13. *$P_{ID}(K_{m,n})$ is given by the inequalities*

1. *$x(C) \geq |C| - 1$ for all nonempty $C \subseteq A$,*
2. *$x(C) \geq |C| - 1$ for all nonempty $C \subseteq B$.*

Moreover, $\gamma^{ID}(K_{m,n}) = |V| - 2$.

4.2. Complete p-partite graphs

The results above can be further generalized for complete p-partite graphs. Consider $K_{n_1,\ldots,n_p} = (U_1, \ldots, U_p, E)$ where each $U_i = \{v_{i1}, \ldots, v_{in_i}\}$ induces a nonempty stable set and all edges between U_i and U_j , $i \neq j$ are present. We use $|U_i| = n_i$ for $i = 1, \ldots, p$, $|V| = n$ and assume $n_1 \leq n_2 \leq \cdots \leq n_p$ as well as $p \geq 3$.

Firstly note that $K_{n_1,\ldots,n_p}$ is not identifiable if $n_2 = 1$ (because in this case, $U_1 = \{v_{11}\}$ and $U_2 = \{v_{21}\}$ holds and v_{11} and v_{21} become true twins).

For illustration in Fig. 2, complete 3-partite and 4-partite graphs are depicted and the black dots in each of them correspond to their minimum identifying codes.

Lemma 14. *Let $K_{n_1,n_2,\ldots,n_p}$ be a complete p-partite graph with $n_1 = 1$. Let $r = |\{i : n_i = 2\}|$ then*

- *if $r = 0$ we have:*

$$C_{ID}(K_{1,n_2,\ldots,n_p}) = \begin{pmatrix} 0 & M(\mathcal{R}_{n_2}^2) & & \\ \vdots & \ddots & \ddots & \vdots \\ 0 & & 0 & M(\mathcal{R}_{n_p}^2) \end{pmatrix}$$

- *if $1 \leq r < p - 1$ we have:*

$$C_{ID}(K_{1,n_2,\ldots,n_p}) = \begin{pmatrix} 0 & I_{2r} & 0 & \ldots & 0 \\ 0 & 0 & M(\mathcal{R}_{n_{r+2}}^2) & & \\ \vdots & & \ddots & \ddots & \vdots \\ 0 & \ldots & & 0 & M(\mathcal{R}_{n_p}^2) \end{pmatrix}$$

- *if $r = p - 1$, $C_{ID}(K_{1,n_2,\ldots,n_p}) = (0 \ I_{2r})$.*

Proof. Let $G = K_{n_1,n_2,\ldots,n_p}$ be a complete p-partite graph with $n_1 = 1$. If $r = |\{i : n_i = 2\}|$ and $1 \leq r < p - 1$, we have the following closed neighborhoods:

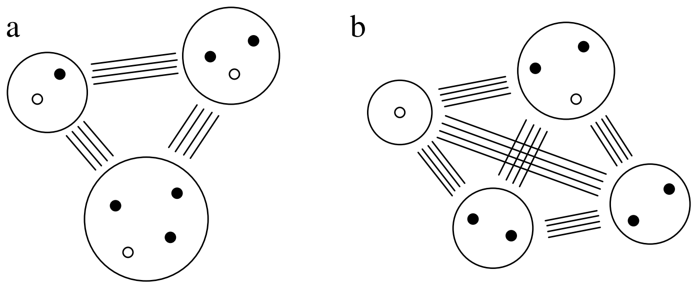

Fig. 2. (a) A complete 3-partite graph with $n_1 = 2$, $n_2 = 3$ and $n_3 = 4$, (b) A complete 4-partite graph with $n_1 = 1$, $n_2 = n_3 = 2$ and $n_4 = 3$.

- $N[v_{11}] = V$,
- $N[v_{i1}] = V - \{v_{i2}\}$ and $N[v_{i2}] = V - \{v_{i1}\}$ for $i = 2, \ldots, r+1$,
- $N[v_{ij}] = (V - U_i) \cup \{v_{ij}\}$ for $i = r+2, \ldots, p$

Hence, $N[v_{11}] \triangle N[v_{i1}] = \{v_{i2}\}$ and $N[v_{11}] \triangle N[v_{i2}] = \{v_{i1}\}$ for all $i = 2, \ldots, r+1$ shows that $U_2 \cup \cdots \cup U_r \subseteq V_1(G)$. All closed neighborhoods contain at least one node from $V_1(G)$ and, thus, all its characteristic vectors are redundant. Moreover, all the characteristic vectors associated with the symmetric differences distinct from $N[v_{ij}] \triangle N[v_{ik}] = \{v_{ij}, v_{ik}\}$ for $i = r+2, \ldots, p$ are redundant:

- $N[v_{11}] \triangle N[v_{ij}] = U_i - \{v_{ij}\}$ for $i = r+2, \ldots, p$ contains $N[v_{ij}] \triangle N[v_{ik}]$ (by $n_i \geq 3$),
- $N[v_{i1}] \triangle N[v_{i2}]$ and $N[v_{il}] \triangle N[v_{jk}]$ for all $i, j = 2, \ldots, r+1$, as well as $N[v_{il}] \triangle N[v_{jk}]$ for $i = 2, \ldots, r+1, j = r+2, \ldots, p$ intersect $V_1(G)$.

Thus, there is no hyperedge in $H_{ID}(G)$ containing v_{11}. The nodes from $U_2 \cup \cdots \cup U_r$ form $V_1(G)$ (leading to an identity matrix in $C_{ID}(G)$), and each U_i with $i = r+2, \ldots, p$ induces a 2-rose of order n_i in $C_{ID}(G)$.

The proofs of the remaining cases are particular situations of the proof above (if $r = 0$, then the submatrix I_{2r} disappears, if $r = p-1$, then all the submatrices $M(\mathcal{R}^2_{n_{r+i}})$ disappear) and the lemma follows. □

As a consequence of Theorem 6, Remark 12 and Lemma 14, we obtain:

Theorem 15. *Let $G = K_{n_1,n_2,\ldots,n_p}$ be a complete p-partite graph with $n_1 = 1$. Let $r = |\{i : n_i = 2\}|$ and consider the following inequalities:*

(1) *$x(v_{11}) \geq 0$ and $x(v_{ij}) \geq 0$ for all $v_{ij} \in U_i$, $i = r+2, \ldots, p$,*
(2) *$x(v_{ij}) \geq 1$ for all $v_{ij} \in U_i$, $i = 2, \ldots, r+1$,*
(3) *$x(V') \geq |V'| - 1$ for all nonempty subsets $V' \subseteq U_i$ for $i = r+2, \ldots, p$.*

Then $P_{ID}(G)$ is given by the inequalities

- *(1) and (3) if $r = 0$,*
- *(1), (2) and (3) if $1 \leq r < p-1$,*
- *(1) and (2) if $r = p-1$.*

Moreover $\gamma^{ID}(G) = n - p + r$.

Using similar arguments as in the proof of Lemma 14, we obtain:

Lemma 16. *For a complete p-partite graph $K_{n_1,n_2,\ldots,n_p}$ with $n_i = 2$ for $i = 1, \ldots, r$ and $n_i \geq 3$ for $i = r+1, \ldots, p$, we have:*

$$C_{ID}(K_{n_1,n_2,\ldots,n_p}) = \begin{pmatrix} M(\mathcal{R}^2_{2r}) & 0 & \ldots & 0 \\ 0 & M(\mathcal{R}^2_{n_{r+1}}) & 0 & \\ \vdots & & \ddots & \vdots \\ 0 & \ldots & 0 & M(\mathcal{R}^2_{n_p}) \end{pmatrix}.$$

Theorem 6, Remark 12 and Lemma 16 imply:

Theorem 17. *For a complete p-partite graph $G = K_{n_1,n_2,\ldots,n_p}$ with $n_i = 2$ for all $i = 1, \ldots, r$, and $n_i \geq 3$ for $i = r+1, \ldots, p$, $P_{ID}(G)$ is given by the inequalities*

1. *$x(v) \geq 0$ for all $v \in V$,*

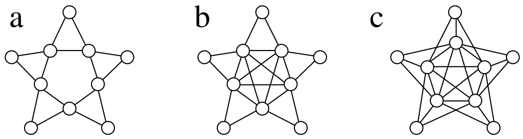

Fig. 3. Three examples of suns (a) the 5-sun M_5, (b) the complete sun S_5 and (c) its complement, the co-sun $\overline{S}_5$.

2. *$x(V') \geq |V'| - 1$ for all nonempty subsets $V' \subseteq U_1 \cup \cdots \cup U_r$,*
3. *$x(V') \geq |V'| - 1$ for all nonempty subsets $V' \subseteq U_i$ for $i = r+1, \ldots, p$.*

Moreover $\gamma^{ID}(G) = n - p + r - 1$.

Remark 18. Note that any 2-rose minor in $C_{ID}(G)$ corresponds to a clique in $H_{ID}(G)$. Lemma 9 shows that the identifying code hypergraph of stars is a clique, and Corollary 10 implies that cliques form facet-defining substructures. In particular, every set of pairwise false twins in a graph gives rise to a clique in $H_{ID}(G)$ since for non-adjacent nodes i and j with $N(i) = N(j)$, we have $N[i] \triangle N[j] = \{i, j\}$ (see complete multi-partite graphs for examples). Hence, each set V' of pairwise false twins in a graph G leads to a facet $x(V') \geq |V'| - 1$ of $P_{ID}(G)$.

5. Identifying code polyhedra of suns

In this section, we discuss hypercycles as further relevant substructures in $H_{ID}(G)$ that can lead to valid or facet-defining inequalities of $P_{ID}(G)$.

Let $\mathcal{H} = (V, \mathcal{E})$ be a hypergraph with $\mathcal{E} \subseteq 2^V$. A hypercycle $\mathcal{C} = (V', \mathcal{E}')$ of length m is a hypergraph defined by an alternating sequence $i_1E_1i_2 \ldots i_mE_mi_1$ of m nodes and m hyperedges with $\{i_j, i_{j+1}\} \in E_i$, $i_{m+1} = i_1$. It is an *induced hypergraph* of $\mathcal{H}$ if $M(\mathcal{C})$ is a deletion minor of $M(\mathcal{H})$, i.e., if it is obtained by removing the columns outside V' and the rows with a 1-entry outside V'. The result in Remark 1 can be restated as follows:

Lemma 19. *Let $\mathcal{H}' = (V', \mathcal{E}')$ be an induced hypergraph of $\mathcal{H} = (V, \mathcal{E})$. The inequality $x(V') \geq \tau(M(\mathcal{H}'))$ is valid for $Q^*(M(\mathcal{H}))$. Moreover if it is a facet of $Q^*(M(\mathcal{H}'))$ then it is also a facet of $Q^*(M(\mathcal{H}))$.*

In the sequel, we consider three families of suns and study hypercycles in their identifying code hypergraphs in order to determine minimum identifying codes.

It will turn out that the corresponding identifying code clutters are related to different circulant matrices. A *circulant matrix* is a square matrix where each row vector is shifted one element to the right relative to the preceding row. We denote by C_n^k the circulant matrix in $\{0, 1\}^{n\times n}$ having as first row the vector starting with k 1-entries and having 0-entries otherwise. In contrary to the case of q-roses, the covering polyhedron of general circulant matrices has not yet been described, except for some special cases (see [1] and [12] for further references).

A *sun* is a graph $G = (C \cup S, E)$ whose node set can be partitioned into S and C, where $S = \{s_1, \ldots, s_n\}$ is a stable set and $C = \{c_1, \ldots, c_n\}$ is a (not necessarily chordless) cycle.

Here, we focus our consideration on three cases:

- n-suns M_n where C induces a hole and s_i is adjacent to exactly c_i and c_{i+1} for all $1 \leq i \leq n$,
- complete suns S_n where C induces a clique and s_i is adjacent to exactly c_i and c_{i+1} for all $1 \leq i \leq n$ and
- co-suns $\overline{S}_n$ (the complements of complete suns S_n)

(indices are taken modulo n), see Fig. 3 for examples. By definition, we immediately see that all such suns with $n \geq 3$ are identifiable.

5.1. *n*-suns

We start our considerations with n-suns. Note that $\gamma_{ID}(M_3) = 3$ is easy to see.

Theorem 20. *For an n-sun $M_n = (C \cup S, E)$ with $n \geq 4$, we have*

$$C_{ID}(M_n) = \begin{pmatrix} I & I \\ C_w & I \end{pmatrix}$$

where C_w is the circulant matrix whose first row is $(0, 1, 0, \ldots, 0)$. Moreover, $H_{ID}(M_n) = C_{2n}$ and $C_{ID}(M_n) = C_{2n}^2$.

Proof. The neighborhood matrix of M_n can be written as

$$N[M_n] = \begin{pmatrix} C_n^2 & I \\ C_n^3 & C_n^2 \end{pmatrix}$$

because we have that $N[s_i] = \{s_i, c_i, c_{i+1}\}$ and $N[c_i] = \{c_{i-1}, c_i, c_{i+1}, s_{i-1}, s_i\}$. Clearly, $N[c_i]$ contains $N[s_i]$ for all $i \le n$. To find $\triangle[M_n]$, we consider the following cases:

- We have $N[s_i] \triangle N[c_i] = \{s_i, c_i, c_{i+1}\} \triangle \{c_{i-1}, c_i, c_{i+1}, s_{i-1}, s_i\} = \{c_{i-1}, s_{i-1}\}$ and $N[s_{i-1}] \triangle N[c_i] = \{c_{i+1}, s_i\}$, which is clearly contained in $N[u]$ for every $u \in M_n$.
- Consider $c_i, c_j \in C$. If c_i and c_j are adjacent nodes, say, if $j = i + 1$ holds, then since $n \ge 4$, $N[c_i] \triangle N[c_{i+1}] = \{c_{i-1}, c_i, c_{i+1}, s_{i-1}, s_i\} \triangle \{c_i, c_{i+1}, c_{i+2}, s_i, s_{i+1}\} = \{c_{i-1}, c_{i+2}, s_{i-1}, s_{i+1}\}$ follows. As $N[s_i] \triangle N[c_i] = \{c_{i-1}, s_{i-1}\} \subseteq N[c_i] \triangle N[c_{i+1}]$ the characteristic vector of $N[c_i] \triangle N[c_{i+1}]$ is redundant. If c_i and c_j are not adjacent nodes then $N[c_i] \triangle N[c_j]$ is redundant since it contains $\{s_i, c_i\} = N[s_{i+1}] \triangle N[c_{i+1}]$.
- Let $s_i, s_j \in S$. If s_i and s_j have a common neighbor, say, c_{i+1} and $j = i + 1$ holds, then $N[s_i] \triangle N[s_{i+1}] = \{s_i, c_i, c_{i+1}\} \triangle \{s_{i+1}, c_{i+1}, c_{i+2}\} = \{s_i, s_{i+1}, c_i, c_{i+2}\}$. Due to $N[s_{i+1}] \triangle N[c_{i+1}] = \{s_i, c_i\} \subseteq N[s_i] \triangle N[s_{i+1}]$, the characteristic vector of $N[s_i] \triangle N[s_{i+1}]$ is redundant. If s_i and s_j have no common neighbor then $N[s_i] \triangle N[s_j]$ is redundant since $N[s_i]$ and $N[s_j]$ are disjoint sets.

Since all rows of $C_{ID}(M_n)$ have exactly two 1-entries, it is clear that $H_{ID}(M_n)$ is a graph. It is a cycle since $N[s_i] \triangle N[c_i] = \{c_{i-1}, s_{i-1}\}$ and $N[s_{i-2}] \triangle N[c_{i-1}] = \{c_i, s_{i-1}\}$ share the node s_{i-1}, and $N[s_{i-2}] \triangle N[c_{i-1}]$ and $N[s_{i+1}] \triangle N[c_{i+1}] = \{s_i, c_i\}$ share node c_i. Accordingly, its incidence matrix $C_{ID}(M_n)$ can be re-arranged as C_{2n}^2 (by ordering the columns as $c_1, s_1, c_2, s_2, \dots, c_n, s_n$ and the rows as $N[c_2] \triangle N[s_2], N[c_1] \triangle N[s_n], N[c_3] \triangle N[s_3], N[c_2] \triangle N[s_1], \dots, N[c_1] \triangle N[s_1], N[c_n] \triangle N[s_{n-1}]$). □

Hence, $H_{ID}(M_n)$ is an even (hyper)cycle and $\tau(M_n) = n$ clearly holds. In addition, C_{2n}^2 is one of the few circulant matrices where $Q^*(C_{2n}^2)$ is known [12], and we conclude:

Corollary 21. *For $M_n = (C \cup S, E)$ with $n \ge 4$, $P_{ID}(M_n)$ coincides with its linear relaxation $Q(C_{ID}(M_n))$ and $\gamma_{ID}(M_n) = n$.*

5.2. Complete suns

Let us now consider a complete sun $S_n = (C \cup S, E)$ with $n \ge 4$. In contrary to n-suns, the identifying code clutters of complete suns have a much more complex structure [5], involving different combinations of circulant matrices, where some submatrices occur for all $n \ge 4$, others not (depending on the parity of n and the size of the graph). Accordingly, the description of $P_{ID}(S_n)$ requires many and complex facets. However, an analysis of $C_{ID}(S_n)$ shows that S is an identifying code and $\gamma^{ID}(S_n) \le |S| = n$. In [5] we conjectured that this bound is tight. In order to prove the conjecture, we rely on the following result:

Lemma 22. *Let $S_n = (C \cup S, E)$ be a complete sun with $n \ge 4$. The hyperedges $N[s_i]$, $N[s_{i+1}]$, and $N[s_i] \triangle N[s_{i+1}]$ form a hypercycle in $H_{ID}(S_n)$ that induces a rank facet $x(\{c_i, c_{i+1}, c_{i+2}, s_i, s_{i+1}\}) \ge 2$ of $P_{ID}(S_n)$.*

Proof. Consider the following hyperedges from $H_{ID}(S_n)$: the neighborhoods $N[s_i] = \{c_i, c_{i+1}, s_i\}$, $N[s_{i+1}] = \{c_{i+1}, c_{i+2}, s_{i+1}\}$ and their symmetric difference $N[s_i] \triangle N[s_{i+1}] = \{c_i, c_{i+2}, s_i, s_{i+1}\}$.

They form, for all $n \ge 4$, a hypercycle of length 3 with support $\{c_i, c_{i+1}, c_{i+2}, s_i, s_{i+1}\}$. It is clear that this hypercycle is obtained by deletion of the columns in $V - \{c_i, c_{i+1}, c_{i+2}, s_i, s_{i+1}\}$ in $C_{ID}(S_n)$. As $x(\{c_i, c_{i+1}, c_{i+2}, s_i, s_{i+1}\}) \ge 2$ for $n \ge 4$ is a facet of this deletion minor, it is also a facet of $P_{ID}(S_n)$ by Lemma 19. □

Theorem 23. *For a complete sun $S_n = (C \cup S, E)$ with $n \ge 4$, the stable set S is a minimum identifying code and, thus, $\gamma^{ID}(S_n) = n$.*

Proof. Let us firstly observe that the stable set S is an identifying code in S_n: indeed, $N[s_i] \cap S = \{s_i\}$ and $N[c_i] \cap S = \{s_{i-1}, s_i\}$ holds for $i = 1, \dots, n$, thus each node is dominated and separated.

In order to show that S is a minimum identifying code in S_n, we consider an arbitrary identifying code I in S_n with $I \ne S$ and show that $|I| \ge n$.

Observe that I contains nodes from both C and S: we can neither have $I \subset S$ (otherwise I does not dominate any node in $S - I$) nor we can have $I \subseteq C$ (otherwise I does not separate any two nodes in C).

In order to show $|I| \ge n$, we provide arguments implying $|I \cap C| \ge |S - I|$. Note that $S - I$ cannot contain 3 consecutive nodes s_{i-1}, s_i, s_{i+1} (otherwise $I \cap N[c_i] = I \cap C = I \cap N[c_{i+1}]$ holds). Hence, $S - I$ can be partitioned into blocks containing either a single node or two consecutive nodes from S, where no two blocks are consecutive. Suppose that $S - I$ consists of p blocks A_j of cardinality 1 and q blocks B_k with two consecutive nodes each and consequently $p + 2q = |S - I|$.

For each block $A_j = \{s_i\}$, it clearly follows $|I \cap \{c_{i+1}, c_i\}| \ge 1$ from the inequality $x(\{s_i, c_{i+1}, c_i\}) \ge 1$ associated to the hyperedge $N[s_i]$ in $H_{ID}(S_n)$.

For each block $B_k = \{s_i, s_{i+1}\}$, we have by Lemma 22 that $N[s_i]$, $N[s_{i+1}]$, and $N[s_i] \triangle N[s_{i+1}]$ form a hypercycle with rank facet $x(\{c_i, c_{i+1}, c_{i+2}, s_i, s_{i+1}\}) \geq 2$, which clearly implies $|I \cap \{c_i, c_{i+1}, c_{i+2}\}| \geq 2$.

In addition, if s_i and s_j belong to different blocks of $S - I$, then the sets of their neighbors in C are disjoint. This finally shows $|I \cap C| \geq p + 2q$ and implies $|I| \geq n$. □

5.3. Co-suns

Finally, let us consider co-suns $\overline{S_n} = (C \cup S, E)$, where C is a clique and S is a stable set. Note that $\overline{S}_3 = M_3$ and $\overline{S}_4 = S_4$ holds. Also the identifying code clutters of co-suns have a complex structure [3], involving different combinations of circulant matrices, where some submatrices occur for all $n \geq 4$, others not (depending on the parity of n and the size of the graph). Accordingly, the description of $P_{ID}(\overline{S}_n)$ requires many and complex facets, too. An analysis of $C_{ID}(\overline{S}_n)$ shows that S is an identifying code and $\gamma^{ID}(\overline{S}_n) \leq |S| = n$ holds. However, this bound is tight only for $n = 5, 6$. Hence, in the sequel, we will consider the cases when $n \geq 7$.

Remark 24. From the definition of $\overline{S}_n$, we obtain the following hyperedges of $H_{ID}(\overline{S}_n)$:

(1) $N[s_i] = (C - \{c_i, c_{i-1}\}) \cup \{s_i\}$,
(2) $N[s_i] \triangle N[s_j] = \{c_{i-1}, c_i, c_{j-1}, c_j, s_i, s_j\}$, in particular $N[s_i] \triangle N[s_{i+1}] = \{c_{i-1}, c_{i+1}, s_i, s_{i+1}\}$,
(3) $N[c_i] \triangle N[c_j] = \{s_i, s_{i+1}, s_j, s_{j+1}\}$, in particular $N[c_i] \triangle N[c_{i+1}] = \{s_i, s_{i+2}\}$.

Theorem 25. *The identifying code number of $\overline{S}_n$ with $n \geq 7$ is $n - 1$.*

Proof. Let us show that $I^* = \{c_1, c_3, s_2, s_3\} \cup \bigcup_{i=5}^{n-1} s_i$ is an identifying code. Indeed, all nodes in $C \cup S$ are separated and dominated since we have that:
$N[c_i] \cap I^* = I^* - \{s_{i+1}\}$ with $i = 1$ or $i = 4$.
$N[c_i] \cap I^* = I^* - \{s_i\}$ with $i = 3$ or $i = n - 1$.
$N[c_i] \cap I^* = I^* - \{s_i, s_{i+1}\}$ with $i = 2$ or $i \in \{5, \ldots, n - 2\}$.
$N[c_n] \cap I^* = I^*$.
$N[s_1] \cap I^* = \{c_3\}$.
$N[s_2] \cap I^* = \{s_2, c_3\}$.
$N[s_3] \cap I^* = \{s_3, c_1\}$.
$N[s_4] \cap I^* = \{c_1\}$.
$N[s_i] \cap I^* = \{s_i, c_1, c_3\}$ with $i = 5, 6, \ldots, n - 1$.
$N[s_n] \cap I^* = \{c_1, c_3\}$.

Also we can observe that $|I^*| = n - 1$, hence $\gamma^{ID}(\overline{S}_n) \leq n - 1$ follows. Let I be an identifying code of $\overline{S}_n$, we will show that $|I| \geq n - 1$.

Claim 1. *There cannot be 3 consecutive nodes in $S - I$. Suppose that $s_i \notin I$ holds. Then we have by Remark 24(3) that $\{s_{i-2}, s_{i+2}\} \subseteq I$ and $\{s_{i-1}, s_{i+1}\} \cap I \neq \emptyset$.* ◇

Claim 2. *There is at most one pair of consecutive nodes in $S - I$. Suppose that $\{s_i, s_{i+1}, s_j, s_{j+1}\} \subset S - I$ with $|j - i - 1| \geq 2$. Then, according to Remark 24(3) I cannot be an identifying code.* ◇

As a consequence of Claims 1 and 2, the set $S - I$ can be partitioned as $S - I = B \cup A_1 \cup \cdots \cup A_t$ with $t < n$ where B is a block of either none or 2 consecutive nodes of S and each A_i is a block having exactly one node of S. We next study the possible gaps between two consecutive blocks A_j and A_{j+1}.

Claim 3. *Between A_j and A_{j+1}, there are at least two nodes from $I \cap S$, for all $1 \leq j < t$. Suppose to the contrary that we have $A_j = \{s_i\}$, $s_{i+1} \in I \cap S$ and $A_{j+1} = \{s_{i+2}\}$ for some j with $1 \leq j < t$. By Remark 24(3), the symmetric difference $N[c_i] \triangle N[c_{i+1}] = \{s_i, s_{i+2}\}$ is a hyperedge of $H_{ID}(\overline{S}_n)$ and thus $|\{s_i, s_{i+2}\} \cap I| \geq 1$ must hold, a contradiction to the assumption that $A_j \cup A_{j+1} = \{s_i, s_{i+2}\} \subseteq S - I$.* ◇

Thus, for the set $S_A = A_1 \cup \cdots \cup A_t$ we have that if $A_j = \{s_i\}$ then $A_{j+1} = \{s_k\}$ with $|i - k| \geq 3$ for all $1 \leq j < t$ (indices for s_i are taken modulo n).

Claim 4. *From $|S_A| = t$ we obtain $|I \cap C| \geq t - 1$. Let $S_A = \{z_1, z_2, \ldots, z_t\}$. W.l.o.g. assume that $z_1 = s_1$ and $z_j = s_m$ for some $m \geq 4$. According to Remark 24(2) there is $w_1 \in \{c_1, c_{m-1}, c_m, c_n\} \cap I$ that separates z_1 from z_j. If $w_1 \in \{c_1, c_n\}$ then w_1 separates z_1 from z_i for all $i \in \{2, \ldots, t\}$ (an analogous conclusion follows if $w_1 \in \{c_m, c_{m-1}\}$). Since $w_1 \in N[z_i]$ for all $i \in \{2, \ldots, t\}$ then there is another node in I that separates z_i from z_j for $i, j \neq 1$. Let us call $w_2 \in (C - \{w_1\}) \cap I$ the node that separates z_2 from z_i for all $i \in \{3, \ldots, t\}$. Applying this reasoning it is clear that I has $t - 1$ different nodes in C that separate the nodes in S_A. Hence $|I \cap C| \geq t - 1$.* ◇

If $B = \emptyset$ then $|I \cap S| = n - t$. From Claim 4 it holds that $|I \cap C| \geq t - 1$ and then $|I| \geq n - 1$.

It is left to treat the case if $B \neq \emptyset$ and thus $|S \cap I| = n - t - 2$ holds. W.l.o.g. assume that $B = \{s_1, s_2\}$. By Remark 24(2), $\{c_2, c_n\} \cap I \neq \emptyset$. Let us assume that $c_2 \in I$ (the same argument can be applied to $c_n \in I$). As $c_2 \in N[x]$ for every $x \in \{s_1\} \cup S_A$ then I has other nodes in C that separate them. Using Claim 4 with $S'_A = \{s_1\} \cup S_A$ we obtain that $|I| = |I \cap S| + |I \cap (C - \{c_2\})| + 1 \geq n - t - 2 + t + 1 = n - 1$. □

6. Concluding remarks

The identifying code problem is hard in general and challenging both from a theoretical and a computational point of view, even for special graphs like bipartite graphs [11] and split graphs [13]. Hence, a typical line of attack is to determine minimum identifying codes of special graphs (as paths [9,16], stars [17] and cycles [9,16]), or to provide lower and upper bounds [14,17,18].

In this paper, we demonstrated how polyhedral techniques can help to find identifying codes of minimum size. For that, we rely on a reformulation of the identifying code problem in terms of a set covering problem in a suitable hypergraph $H_{ID}(G)$ and study the identifying code polyhedron $P_{ID}(G) = Q^*(C_{ID}(G))$ as covering polyhedron associated with its incidence matrix $C_{ID}(G)$.

We provided some general properties of the identifying code polyhedron $P_{ID}(G)$ and its canonical linear relaxation (Section 3). Afterwards, we discussed several lines to apply polyhedral techniques to the identifying code problem. In any case, the first step is to determine $H_{ID}(G)$ and its incidence matrix $C_{ID}(G)$.

If $C_{ID}(G)$ falls into a class of matrices M for which the set covering polyhedron $Q^*(M)$ is already known, then we immediately obtain a complete description of $P_{ID}(G)$ and can deduce the exact value of $\gamma^{ID}(G)$. This turned out to be the case for stars $K_{1,n}$ (where $C_{ID}(K_{1,n})$ equals a 2-rose R^2_{n+1}) and for general complete multipartite graphs G (where $C_{ID}(G)$ is composed by blocks of 2-roses). Moreover, the identifying code clutter of n-suns M_n turned out to equal the circulant matrix C^2_{2n} which implied $P_{ID}(M_n) = Q_{ID}(M_n)$. In all these cases, we obtained a complete description of $P_{ID}(G)$ and a closed formula for the exact value of $\gamma^{ID}(G)$.

A matrix M is *ideal* if $Q^*(M) = Q(M)$. Hence, we can conclude from our result on n-suns:

Corollary 26. *The identifying code clutters of n-suns M_n are ideal for all $n \geq 3$.*

A way to evaluate how far a nonideal matrix is from being ideal consists in classifying the inequalities that have to be added to $Q(M)$ in order to obtain $Q^*(M)$. In [1], a matrix M is called *rank-ideal* if $Q^*(M)$ is described by rank constraints only. Thus, the results in Section 4 imply:

Corollary 27. *The identifying code clutters of complete multipartite graphs G are rank-ideal since rank constraints associated with cliques in $H_{ID}(G)$ suffice to describe $P_{ID}(G)$.*

In general, we cannot expect identifying code clutters to be (rank-)ideal. Complete suns S_n and their complements are examples of graphs G where $C_{ID}(G)$ is far from being rank-ideal. However, an analysis of $C_{ID}(G)$ implies $\gamma^{ID}(S_n) \leq n$ and raised a conjecture in [5] that this bound is tight. Here, we were able to verify this conjecture by combining polyhedral and combinatorial arguments. Finally, we provided a purely combinatorial proof for $\gamma^{ID}(\overline{S}_n) \leq n - 1$ for all complements of complete suns with $n \geq 7$.

Note that the arguments and techniques applied to complete suns are rather general and have the potential to be applied to all graphs G, even if their identifying code clutters are matrices with a complex structure and a complete description of $P_{ID}(G)$ involves many and complicated facets. In all such cases, an analysis of $C_{ID}(G)$ can provide, on the one hand, upper bounds for $\gamma^{ID}(G)$ and, on the other hand, minors of $C_{ID}(G)$ (e.g. associated with cliques or odd hypercycles in $H_{ID}(G)$) whose rank constraints strengthen the linear relaxation $Q_{ID}(G)$ and can be used to obtain lower bounds for $\gamma^{ID}(G)$.

Future lines of our research include to identify more facet-defining substructures in $H_{ID}(G)$ (related to minors of $C_{ID}(G)$) that allow us to strengthen the linear relaxation $Q_{ID}(G)$. Thereby, our goal is to obtain either the identifying code of minimum size or strong lower bounds stemming from linear relaxations of the identifying code polyhedron, enhanced by suitable cutting planes. Recall that facets associated with deletion minors of $C_{ID}(G)$ remain facets in $P_{ID}(G)$, so according facets identified for special graphs are relevant for every graph having such subgraphs.

Acknowledgments

We would like to thank the anonymous referees for their helpful comments that improved the final presentation of these results.

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