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Complexity of k-tuple total and total $\{k\}$-dominations for some subclasses of bipartite graphs ☆

G. Argiroffo [a], V. Leoni [b], P. Torres [a,b,*]

[a] Depto. de Matemática, Facultad de Ciencias Exactas, Ingeniería y Agrimensura, Universidad Nacional de Rosario, Argentina
[b] Consejo Nacional de Investigaciones Científicas y Técnicas, Argentina

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ABSTRACT

We consider two variations of graph total domination, namely, k-tuple total domination and total $\{k\}$-domination (for a fixed positive integer k). Their related decision problems are both NP-complete even for bipartite graphs. In this work, we study some subclasses of bipartite graphs. We prove the NP-completeness of both problems (for every fixed k) for bipartite planar graphs and we provide an APX-hardness result for the total domination problem for bipartite subcubic graphs. In addition, we introduce a more general variation of total domination (total (r, m)-domination) that allows us to design a specific linear time algorithm for bipartite distance-hereditary graphs. In particular, it returns a minimum weight total $\{k\}$-dominating function for bipartite distance-hereditary graphs.

1. Introduction and preliminaries

All the graphs in this paper are finite, simple and without isolated vertices. Given a graph $G = (V(G), E(G))$, $V(G)$ and $E(G)$ denote its vertex and edge sets, respectively. For any $v \in V(G)$, $N(v)$ is the *open neighborhood* of v in G, i.e. the set of vertices adjacent to v in G and $N[v] = N(v) \cup \{v\}$ is the *closed neighborhood* of v in G. Two vertices $u, v \in V(G)$ are false (true) twins if $N(u) = N(v)$ (resp. $N[u] = N[v]$). For a graph G and $v \in V(G)$, $G - v$ denotes the graph induced by $V(G) - \{v\}$. A *pendant* vertex in G is a vertex of degree one in G. Given a function f, a graph G and $S \subseteq V(G)$, $f(S) = \sum_{v \in S} f(v)$ denotes the *weight* of f on S, if $S = V(G)$ we just say the *weight* of f. A function $f : V(G) \mapsto \{0, 1\}$ is a *total dominating function* of G if $f(N(v)) \geq 1$ for all $v \in V(G)$. The *total domination number* of G is the minimum weight of a total dominating function of G, and it is denoted by $\gamma_t(G)$ [4]. Total domination in graphs is now well studied in graph theory. The literature on the subject has been surveyed and detailed in the book [10].

In [9] Henning and A. Kazemi defined a generalization of total domination as follows: let k be a positive integer, a function $f : V(G) \mapsto \{0, 1\}$ is a *k-tuple total dominating function* of G if $f(N(v)) \geq k$ for all $v \in V(G)$. It is clear that a graph has a k-tuple total dominating function if its minimum degree is at least k. The minimum possible weight of a k-tuple total dominating function of G is called the *k-tuple total domination number* of G and denoted by $\gamma_{\times k,t}(G)$. Another generalization (defined by N. Li and X. Hou in [13]) is the following: a function $f : V(G) \mapsto \{0, 1, \ldots, k\}$ is a *total $\{k\}$-dominating function* of G if $f(N(v)) \geq k$ for all $v \in V(G)$. The minimum possible weight of a total $\{k\}$-dominating function of G is called the *total $\{k\}$-domination number* of G and denoted by $\gamma_{\{k\},t}(G)$.

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* Corresponding author at: Depto. de Matemática, Facultad de Ciencias Exactas, Ingeniería y Agrimensura, Universidad Nacional de Rosario, Argentina.

E-mail addresses: garua@fceia.unr.edu.ar (G. Argiroffo), valeoni@fceia.unr.edu.ar (V. Leoni), ptorres@fceia.unr.edu.ar (P. Torres).

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As usual, these definitions induce the study of the following decision problems for a positive fixed integer k:

k-TUPLE TOTAL DOMINATION PROBLEM (k-DOM-T)
Inst.: $G = (V(G), E(G))$, $j \in \mathbb{N}$
Quest.: Does G have a k-tuple total dominating function f with $f(V(G)) \leq j$?

TOTAL $\{k\}$-DOMINATION PROBLEM ($\{k\}$-DOM-T)
Inst.: $G = (V(G), E(G))$, $j \in \mathbb{N}$
Quest.: Does G have a total $\{k\}$-dominating function f with $f(V(G)) \leq j$?

It is clear that, for $k = 1$, the above problems become the well-known Total Domination Problem DOM-T. It is known that k-DOM-T and $\{k\}$-DOM-T are NP-complete for each value of k, even for bipartite graphs (see [8,15]). In this work we study these problems in some subclasses of bipartite graphs.

In Section 2, we consider bipartite planar graphs and provide NP-completeness results not only for k-DOM-T and $\{k\}$-DOM-T, but also for DOM-T. For the latest, we obtain an inapproximability result for bipartite subcubic graphs.

In Section 3, we introduce a more general variation of total domination (total (r, m)-domination) that allows us to design a specific linear time algorithm for bipartite distance-hereditary graphs which in particular, returns a minimum total $\{k\}$-dominating function for a given bipartite distance-hereditary graph. The motivation of considering this subclass of bipartite graphs is given by the following reasoning (for the definition of clique-width and q expression, see the Appendix):

Theorem 1 *([5,14]). Let $q \in \mathbb{Z}_+$. Every LinEMSOL(τ_1) problem $\mathcal{P}$ on the family of graphs with clique-width at most q can be solved in polynomial time. Moreover, if the q-expression can be found in linear time, the problem $\mathcal{P}$ can be solved in linear time.*

We can prove that given $k, q \in \mathbb{Z}_+$, k-DOM-T and $\{k\}$-DOM-T can be solved in polynomial time for the family of graphs with clique-width at most q (see Theorem 16 in the Appendix) and, in particular, in linear time for distance-hereditary graphs since it is known that they have clique-width bounded by 3 and moreover, a 3-expression can be found in linear time for them [6]. The main contribution of Section 3 is a specific linear time algorithm to find a minimum total $\{k\}$-dominating function for bipartite distance-hereditary graphs.

1.1. First results

Let us remark that it is not hard to see that $\gamma_{\{k\},t}(G) \leq k \cdot \gamma_t(G)$, for every graph G and positive integer k. An open problem concerning these type of bounds is to characterize graphs that verify this inequality by an equality. The next result—that will be used at the end of Section 3—provides a tool in that direction.

Lemma 2. *Let G be a graph, k a positive integer. Then, $\gamma_{\{k\},t}(G) = k \cdot \gamma_t(G)$ if and only if there exists a minimum weight total $\{k\}$-dominating function f of G such that $f(v) \in \{0, k\}$ for all $v \in V(G)$.*

Proof. First, let f be a minimum weight total $\{k\}$-dominating function of G such that $f(v) \in \{0, k\}$ for all $v \in V(G)$. Note that $f(N(v))$ is a multiple of k for every vertex v of G, thus the function $g = \frac{f}{k}$ is a total dominating function and $k \cdot g(V(G)) = f(V(G)) = \gamma_{\{k\},t}(G)$. Hence, $\gamma_{\{k\},t}(G) \geq k \cdot \gamma_t(G)$. From the observation above, it holds $\gamma_{\{k\},t}(G) = k \cdot \gamma_t(G)$.

Conversely, if g is a minimum weight total dominating function of G, then $f = k \cdot g$ is a total $\{k\}$-dominating function of G with $f(V(G)) = k \cdot g(V(G)) = k \cdot \gamma_t(G) = \gamma_{\{k\},t}(G)$ and the lemma holds. $\square$

Next, we provide an equality that relates the total $\{k\}$-domination and the k-tuple total domination numbers through a graph product. Given two graphs G and H, the *lexicographic product* $G \circ H$ is defined on the vertex set $V(G) \times V(H)$ where two vertices (u_1, v_1) and (u_2, v_2) are adjacent if and only if either u_1 is adjacent to u_2 in G, or $u_1 = u_2$ and v_1 is adjacent to v_2 in H.

In particular, if G is a graph with $V(G) = \{v_1, \ldots, v_n\}$ and S_k is the edgeless graph with $V(S_k) = \{1, \ldots, k\}$, we denote a vertex (v_r, j), $r \in \{1, \ldots, n\}$ and $j \in \{1, \ldots, k\}$ of $G \circ S_k$ by v_r^j.

Theorem 3. *For any graph G and $k \in \mathbb{Z}_+$, $\gamma_{\{k\},t}(G) = \gamma_{\times k,t}(G \circ S_k)$.*

Proof. Let f be a total $\{k\}$-dominating function with minimum weight of G and $V' = \bigcup_{r=1}^{n} \{v_r^j : j = 1, \ldots, f(v_r)\} \subseteq V(G \circ S_k)$.

It is clear that $|V'| = \gamma_{\{k\},t}(G)$. In addition, as $f(N(v_r)) \geq k$, it holds $|N(v_r^j) \cap V'| \geq k$ for all $r \in \{1, \ldots, n\}$ and $j \in \{1, \ldots, k\}$. Thus, the function that assigns 1 to the vertices in V' and zero otherwise is a k-tuple total dominating function of $G \circ S_k$ implying $\gamma_{\{k\},t}(G) \geq \gamma_{\times k,t}(G \circ S_k)$.

Conversely, let f be a k-tuple total dominating function of $G \circ S_k$ and $V' \subseteq V(G \circ S_k)$ such that $v \in V'$ if and only if $f(v) = 1$. It is immediate to check that the function $f : V(G) \mapsto \{0, 1, \ldots, k\}$ defined by $f(v_r) = |V' \cap \{v_r^j : j = 1, \ldots, k\}|$ is a total $\{k\}$-dominating function of G and then $\gamma_{\{k\},t}(G) \leq \gamma_{\times k,t}(G \circ S_k)$. $\square$

2. NP-completeness and inapproximability results

A *vertex cover* of a graph is a subset of vertices intersecting all the edges. The minimum cardinality of a vertex cover in a graph G is called *vertex cover number* of G and denoted by $\tau(G)$. The related decision problem is the well-known Vertex Cover Problem (VCP), which is NP-complete for planar graphs [7]. By reducing VCP for planar graphs to DOM-T for bipartite planar graphs, we have the following result.

Theorem 4. *DOM-T is NP-complete for bipartite planar graphs.*

Proof. We transform a planar graph $G = (V, E)$ into a bipartite planar graph G' as follows: subdivide each edge of G and add a pendant vertex to each vertex arising from

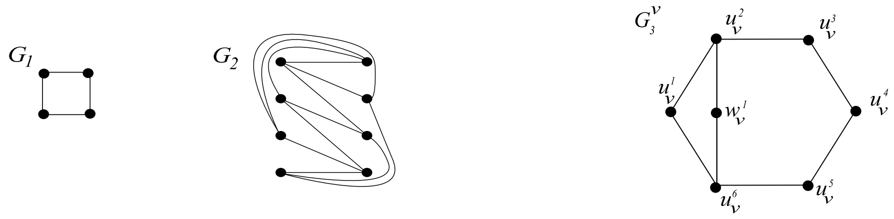

Fig. 1. Graphs G_1 and G_2 of Lemma 6 and G^v_3 of Lemma 8.

the subdivision. Clearly, G' is a bipartite planar graph and it can be obtained in polynomial time.

We will prove that $\tau(G) + |E(G)| = \gamma_t(G')$ by proving that G has a vertex cover S with $|S| \geq j$ if and only if G' has a total dominating function f with $f(V(G')) \geq j + |E(G)|$.

Let S be a vertex cover of G of size at least j and let $f : V(G') \mapsto \{0, 1\}$ such that $\{v \in V(G) : \ f(v) = 1\} = S \cup N$, where N is the subset of $V(G')$ of vertices arising from the subdivision. Note that $|N| = |E(G)|$. It is clear that f is a total dominating function of G' with weight at least $j + |E(G)|$. Conversely, let f be a total dominating function of G' with weight at least $j + |E(G)|$. Notice that $N \subseteq \{v \in V(G') : \ f(v) = 1\}$. W.l.o.g. we can assume that the set $\{v \in V(G') : \ f(v) = 1\}$ does not contain any of the added pendant vertices. Then, it is clear that $\{v \in V(G') : \ f(v) = 1\} - N$ is a vertex cover of G and $|\{v \in V(G') : \ f(v) = 1\} - N| \geq j$.

Then, $\tau(G) + |E(G)| = \gamma_t(G')$ and the theorem holds. □

A similar approach as the one used to prove Theorem 4 can be used to show the following inapproximability result. Recall that APX is the class of problems approximable in polynomial time to within some constant, and that a problem Π is APX-hard if every problem in APX reduces to Π via an AP-reduction. APX-hard problems do not admit a polynomial-time approximation scheme (PTAS), unless P=NP. To show that a problem is APX-hard, it suffices to show that an APX-complete problem is L-reducible to it [2].

Recall that, given two NP optimization problems $\prod$ and $\prod'$, we say that $\prod$ is L-reducible to $\prod'$ if there exists a polynomial-time transformation from instances of $\prod$ to instances of $\prod'$ and positive constants α and β such that for every instance X of $\prod$, we have: $opt_{\prod'}(f(X)) \leq \alpha \cdot opt_{\prod}(X)$, and for every feasible solution y' of $f(X)$ with objective value c_2 we can compute in polynomial time a solution y of X with objective value c_1 such that $|opt_{\prod}(X) - c_1| \leq \beta \cdot |opt_{\prod'}(f(X)) - c_2|$.

In what follows, we consider VCP and DOM-T as optimization problems. We have:

Theorem 5. *DOM-T is APX-hard for bipartite subcubic graphs.*

Proof. Since VCP is APX-complete for cubic graphs [1], it suffices to show that VCP in cubic graphs is L-reducible to DOM-T in bipartite subcubic graphs. Consider the polynomial-time transformation described in Theorem 4, that starts from an instance of VCP given by a cubic graph G (not necessarily planar) and computes an instance G' of DOM-T. By Theorem 4, we have $\gamma_t(G') = \tau(G) + |E(G)|$. Moreover, since G is cubic, every vertex in a vertex cover of G covers exactly 3 edges, hence $\tau(G) \geq \frac{|E(G)|}{3}$. This implies that $\gamma_t(G') = \tau(G) + |E(G)| \leq 4\tau(G)$, hence the first condition in the definition of L-reducibility is satisfied with $\alpha = 4$. The second condition in the definition of L-reducibility states that for every total dominating set D, we can compute in polynomial time a vertex cover S of G such that $|S| - \tau(G) \leq \beta \cdot (|D| - \gamma_t(G'))$ for some $\beta > 0$. We claim that this can be achieved with $\beta = 1$. Indeed, the proof of Theorem 4 shows how one can transform in polynomial time any total dominating set D in G' to a vertex cover S of G such that $|S| \leq |D| - |E(G)|$. Therefore, $|S| - \tau(G) \leq |D| - |E(G)| - \tau(G) = |D| - \gamma_t(G')$. This shows that VCP in cubic graphs is L-reducible to DOM-T in bipartite subcubic graphs, and completes the proof. □

Notice that for a bipartite planar graph and an integer $k \geq 4$, there is no k-tuple total dominating function. For the remaining values of k and a given graph G, we construct a graph $W(G)$ by adding to each $v \in V(G)$, a graph G_v with 2^k vertices and isomorphic to G_{k-1}, with $k = 2, 3$ (see Fig. 1), and the edge $v1_v$, where 1_v is any vertex in the outer face of G_v.

Lemma 6. *For $k = 2, 3$ and any graph G, $\gamma_{\times(k-1),t}(G) = \gamma_{\times k,t}(W(G)) - 2^k|V(G)|$.*

Proof. Let f be a $(k-1)$-tuple total dominating function of G and define $\tilde{f} : V(W(G)) \to \{0, 1\}$ such that $\tilde{f}(v) = f(v)$ for $v \in V(G)$ and $\tilde{f}(u) = 1$ for $u \in \bigcup_{v \in V(G)} V(G_v)$. It turns out that $\tilde{f}$ is a k-tuple total dominating function of $W(G)$. Then $\gamma_{\times k,t}(W(G)) \leq \gamma_{\times(k-1),t}(G) + 2^k|V(G)|$.

Conversely, let $\tilde{f}$ be a k-tuple total dominating function of $W(G)$. Notice that $\tilde{f}(u) = 1$ for $u \in \bigcup_{v \in V(G)} V(G_v)$. Define $f : V(G) \to \{0, 1\}$ such that $f(v) = \tilde{f}(v)$ for $v \in V(G)$. It is not difficult to see that f is a $(k-1)$-tuple total dominating function of G and $f(V(G)) = \tilde{f}(V(G)) - 2^k|V(G)|$. Thus $\gamma_{\times(k-1),t}(G) \leq \gamma_{\times k,t}(W(G)) - 2^k|V(G)|$.

Hence we have proved that $\gamma_{\times k,t}(W(G)) = \gamma_{\times(k-1),t}(G) + 2^k|V(G)|$ and the result follows. □

When G is bipartite planar, it is clear that $W(G)$ is also bipartite planar. Thus, as a consequence of the lemma above we have:

Theorem 7. *k-DOM-T is NP-complete for bipartite planar graphs, for $k \in \{2, 3\}$.*

Proof. Clearly, k-DOM-T on bipartite planar graphs is NP. As a consequence of Lemma 6, we can prove that this problem is NP-complete. $\square$

Lemma 8. *For any k and graph G, $\gamma_{\{k\},t}(H(G)) = \gamma_{\{\lfloor \frac{k}{2} \rfloor\},t}(G) + |V(G)|\gamma_{\{k\},t}(C_6)$.*

Proof. Given a graph G, define a graph $H(G)$ by adding to each vertex $v \in V(G)$, a graph G_3^v and an edge vu_v^1 (see Fig. 1).

Clearly, when G is a bipartite planar graph, $H(G)$ also is. Besides, it is clear that $H(G)$ can be built in polynomial time.

Let $g : V(G) \to \{0, \dots, k\}$ be a minimum total $\{\lfloor \frac{k}{2} \rfloor\}$-dominating function of G. We define $\hat{g} : V(H(G)) \to \{0, \dots, k\}$ as follows: for each $v \in V(G)$, $\hat{g}(v) = g(v)$, $\hat{g}(w_v^1) = 0$, $\hat{g}(u_v^1) = \hat{g}(u_v^2) = \hat{g}(u_v^5) = \hat{g}(u_v^6) = \left\lceil \frac{k}{2} \right\rceil$, and $\hat{g}(u_v^3) = \hat{g}(u_v^4) = \left\lfloor \frac{k}{2} \right\rfloor$. It is not hard to see that $\hat{g}$ is a total $\{k\}$-dominating function of $H(G)$. Therefore,

$$\gamma_{\{k\},t}(H(G)) \leq \hat{g}(V(H(G))) = \gamma_{\{\lfloor \frac{k}{2} \rfloor\},t}(G) + |V(G)|\gamma_{\{k\},t}(C_6).$$

To see the converse inequality, let $\hat{h} : V(H(G)) \to \{0, \dots, k\}$ be a total $\{k\}$-dominating function of $H(G)$. Since $N(w_v^1) \subseteq N(u_v^1)$ for every $v \in V(G)$, it is not difficult to prove that we can assume $\hat{h}(w_v^1) = 0$ for all $v \in V(G)$. We will construct a total $\{k\}$-dominating function $\hat{f}$ of $H(G)$ such that $\hat{f}(V(H(G))) \leq \hat{h}(V(H(G)))$, according to the following procedure: for each $v \in V(G)$:

Case 1: $\hat{h}(u_v^1) \geq \left\lceil \frac{k}{2} \right\rceil$. First, observe that $N(v) \cap V(G) \neq \emptyset$ since G has no isolated vertices. Besides, note that $\hat{h}(\{u_v^2, u_v^4, u_v^6\}) \geq \frac{3k}{2}$ and $\hat{h}(\{u_v^4\}) = \hat{h}(u_v^3) + \hat{h}(u_v^5) \geq k$. Then $\hat{h}(\{u_v^2, u_v^3, u_v^4, u_v^5, u_v^6\}) \geq \left\lceil \frac{5k}{2} \right\rceil$, which implies $\hat{h}(V(G_3^v)) \geq \left\lceil \frac{5k}{2} \right\rceil + \left\lceil \frac{k}{2} \right\rceil + (\hat{h}(u_v^1) - \left\lceil \frac{k}{2} \right\rceil)$. We define $\hat{f}(u_v^1) = \hat{f}(u_v^2) = \hat{f}(u_v^5) = \hat{f}(u_v^6) = \left\lceil \frac{k}{2} \right\rceil$, $\hat{f}(u_v^3) = \hat{f}(u_v^4) = \left\lfloor \frac{k}{2} \right\rfloor$, $\hat{f}(x_v) = \min\{\hat{h}(x_v) + \hat{h}(u_v^1) - \left\lceil \frac{k}{2} \right\rceil, k\}$ for some $x_v \in N(v) \cap V(G)$ and $\hat{f}(z) = \hat{h}(z)$ for all the remaining vertices.

Case 2: $0 \leq \hat{h}(u_v^1) \leq \left\lceil \frac{k}{2} \right\rceil - 1$. First, observe that $\hat{h}(N(w_v^1)) = \hat{h}(u_v^2) + \hat{h}(u_v^6) \geq k$, $\hat{h}(N(u_v^2)) = \hat{h}(u_v^1) + \hat{h}(w_v^1) + \hat{h}(u_v^3) = \hat{h}(u_v^1) + \hat{h}(u_v^3) \geq k$, $\hat{h}(N(u_v^3)) = \hat{h}(u_v^2) + \hat{h}(u_v^4) \geq k$, $\hat{h}(N(u_v^4)) = \hat{h}(u_v^3) + \hat{h}(u_v^5) \geq k$, $\hat{h}(N(u_v^5)) = \hat{h}(u_v^4) + \hat{h}(w_v^6) \geq k$, and $\hat{h}(N(u_v^6)) = \hat{h}(u_v^1) + \hat{h}(w_v^1) + \hat{h}(u_v^5) = \hat{h}(u_v^1) + \hat{h}(u_v^5) \geq k$.
Therefore, $\hat{h}(V(G_3^v)) \geq \gamma_{\{k\},t}(C_6)$.
Then, we define $\hat{f}(u_v^1) = \hat{f}(u_v^2) = \hat{f}(u_v^5) = \hat{f}(u_v^6) = \left\lceil \frac{k}{2} \right\rceil$, $\hat{f}(u_v^3) = \hat{f}(u_v^4) = \left\lfloor \frac{k}{2} \right\rfloor$ and $\hat{f}(z) = \hat{h}(z)$ for all the remaining vertices.

From its construction, in both cases $\hat{f}$ is a $\{k\}$-dominating function of $H(G)$ such that $\hat{f}(H(G))) \leq \hat{h}(V(H(G)))$, as desired. Besides, $\hat{f}(u_v^1) = \left\lceil \frac{k}{2} \right\rceil$ for all $v \in V(G)$ which implies that the restriction of $\hat{f}$ to G is a total $\{\lfloor \frac{k}{2} \rfloor\}$-dominating function of G. As $\hat{f}(V(G_3^v)) = \gamma_{\{k\},t}(C_6)$ for all $v \in V(G)$, we have $\hat{f}(V(H(G))) \geq \gamma_{\{\lfloor \frac{k}{2} \rfloor\},t}(G) + |V(G)|\gamma_{\{k\},t}(C_6)$, hence

$$\gamma_{\{k\},t}(H(G)) \geq \gamma_{\{\lfloor \frac{k}{2} \rfloor\},t}(G) + |V(G)|\gamma_{\{k\},t}(C_6). \quad \square$$

As a consequence of the lemma above, we obtain:

Theorem 9. *For every fixed $k \in \mathbb{Z}_+$, $\{k\}$-DOM-T is NP-complete for bipartite planar graphs.*

Proof. Clearly, $\{k\}$-DOM-T on bipartite planar graphs is in NP and, from Theorem 4, DOM-T is NP-complete on bipartite planar graphs. Besides, it is not difficult to prove that $\gamma_{\{k\},t}(C_6) = 3k + 1$.

Now, Lemma 8 implies that, given a positive integer m, $\gamma_{\{\lfloor \frac{k}{2} \rfloor\},t}(G) \leq m$ if and only if $\gamma_{\{k\},t}(H(G)) - |V(G)|\gamma_{\{k\},t}(C_6) \leq m$. $\square$

3. Bipartite distance-hereditary graphs

A graph G is *distance-hereditary* if for each induced connected subgraph G' of G and all $x, y \in V(G')$, the distances in G and in G' between x and y coincide. A graph is *bipartite distance-hereditary* (BDH, for short) if it is distance-hereditary and bipartite. It is known that a graph G is distance-hereditary if and only if it can be constructed from K_1 (a single vertex) by a sequence of three operations: adding a pendant vertex, creating a true twin vertex and creating a false twin vertex [3].

A *pruning sequence* of a graph G is a total ordering $\sigma = [x_1, \dots, x_{|V(G)|}]$ of $V(G)$ and a sequence Q of words $q_i = (x_i, Z, y_i)$ for $i = 1, \dots, |V(G)| - 1$, where $Z \in \{P, F, T\}$ and such that, for $i \in \{1, \dots, |V(G)| - 1\}$, if $G_i = G \setminus \{x_1, \dots, x_{i-1}\}$ then, $Z = P$ if x_i is a pendant vertex and $y_i = s(x_i)$ its neighbour in G_i, $Z = F$ if x_i and y_i are false twins in G_i, and $Z = T$ if x_i and y_i are true twins in G_i.

Distance-hereditary graphs are characterized as those graphs that admit a pruning sequence [12] that can be obtained in $O(|V(G)| + |E(G)|)$-time [11]. On the other hand, BDH graphs are characterized as the graphs that can be constructed from K_1 by a sequence of additions of false twins and pendant vertices. Then, a pruning sequence of a connected BDH graph has no words (x, T, y), except possibly $(x_{|V(G)|-1}, T, y_{|V(G)|-1})$.

As mentioned in Section 1, we know that k-DOM-T and $\{k\}$-DOM-T can be solved in linear time for BDH graphs. However, there is not a specific algorithm for this graph class that solves these problems. In this section, we present a simple and easy to implement linear time algorithm that, in particular, returns a minimum total $\{k\}$-dominating function for a given BDH graph.

To this end, let us introduce a more general variation of total domination.

Definition 10. Let G be a graph, k a positive integer and $r(v), m(v) \in \{0, \dots, k\}$ for each $v \in V(G)$. A *total* (r, m)*-dominating function* of G is a function $f : V \mapsto \{0, \dots, k\}$ such that $f(N(v)) \geq r(v)$ and $f(v) \geq m(v)$ for all $v \in V(G)$. The minimum weight of a total (r, m)-dominating function of G is called the *total* (r, m)*-domination number* of G and denoted by $\gamma_{(r,m),t}(G)$.

Algorithm 1 (r, m)-TotalDomBDH(G, k, r, m, f).

Require: A connected BDH graph G with $|V(G)| \geq 2$, $k \in \mathbb{Z}^+$, $r, m : V(G) \mapsto \{0, \dots, k\}$.
Ensure: A minimum total (r, m)-dominating function f of G.

Obtain a pruning sequence with $Q = [q_1, \dots, q_{|V(G)|-1}]$ of G
if $|V(G)| \geq 3$ **then**
  for $i = 1$ **to** $|V(G)| - 2$ **do**
    if $q_i = (x_i, P, y_i)$ **then**
      $r(y_i) = \max\{r(y_i) - m(x_i), 0\}$ and $m(y_i) = \max\{m(y_i), r(x_i)\}$
    else
      for $v \in N(x_i)$ **do**
        $r(v) = \max\{r(v) - m(x_i), 0\}$ and $r(y_i) = \max\{r(y_i), r(x_i)\}$
      end for
    end if
    $G = G - x_i$ and $f(x_i) = m(x_i)$
  end for
else
  $f(x_1) = \max\{r(x_2), m(x_1)\}$ and $f(x_2) = \max\{r(x_1), m(x_2)\}$
end if

Algorithm 1 is based on the following results:

Remark 11. Let $V(K_2) = \{v_1, v_2\}$, k be a positive integer and $r(v_i), m(v_i) \in \{0, \dots, k\}$ for $i = 1, 2$. Then, a minimum total (r, m)-dominating function f of K_2 is defined by $f(v_i) = \max\{r(v_j), m(v_i)\}$ with $i, j = 1, 2$ and $i \neq j$.

Lemma 12. *Let G be a connected graph with $|V(G)| \geq 3$, k a positive integer and $r(x), m(x) \in \{0, \dots, k\}$ for every $x \in V(G)$. Let $v, v' \in V$ such that $N(v) \subseteq N(v')$. Then, there exists a minimum total (r, m)-dominating function f of G such that $f(v) = m(v)$.*

Proof. Let f' be a minimum total (r, m)-dominating function of G such that $f'(v) > m(v)$. Consider $f : V \mapsto \{0, \dots, k\}$ such that $f(v') = \min\{f'(v') + f'(v) - m(v), k\}$, $f(v) = m(v)$ and $f(x) = f'(x)$ o.w. It is not difficult to prove that f is a total (r, m)-dominating function of G and $f(V(G)) \leq f'(V(G))$. $\square$

Proposition 13. *Let G be a connected graph with $|V(G)| \geq 3$, k a positive integer and $r(x), m(x) \in \{0, \dots, k\}$ for every $x \in V(G)$. We have:*

- *When w is a pendant vertex of G and u its neighbour, $\gamma_{(r,m),t}(G) = \gamma_{(r',m'),t}(G - w) + m(w)$ where $r'(u) = \max\{r(u) - m(w), 0\}$, $m'(u) = \max\{m(u), r(w)\}$ and $r'(x) = r(x)$ and $m'(x) = m(x)$ if $x \in V(G) - \{w, u\}$.*
- *When v and v' are false twins in G, $\gamma_{(r,m),t}(G) = \gamma_{(r',m'),t} \times (G - v') + m(v')$ where $r'(v) = \max\{r(v), r(v')\}$, $r'(u) = \max\{r(u) - m(v'), 0\}$ if $u \in N(v')$, $r'(x) = r(x)$ if $x \in V - (\{v, v'\} \cup N(v'))$ and $m'(x) = m(x)$ for every $x \in V(G) - v'$.*

Proof. Let f be a minimum total (r, m)-dominating function of G and w a pendant vertex of G. W.l.o.g from Lemma 12 we suppose that $f(w) = m(w)$. Consider f', the restriction of f to $V - w$. Note that $f'(u) = f(u) \geq \max\{m(u), r(w)\} = m'(u)$ and $f'(N(u)) = f(N(u) - w) = f(N(u)) - f(w) = f(N(u)) - m(w) \geq \max\{r(u) - m(w), 0\} = r'(u)$. Thus, f' is a total (r', m')-dominating function of $G - v'$ and $f'(V - w) = f(V) - m(w)$. Thus $\gamma_{(r,m),t}(G) \geq \gamma_{(r',m'),t}(G - w) + m(w)$.

To prove the converse inequality it is enough to see that if f' is a total (r', m)-dominating function of $G - w$, then the function $f : V \mapsto \{0, \dots, k\}$ such that $f(w) = m(w)$ and $f(x) = f'(x)$ o.w. is a total (r, m)-dominating function of G.

Let f be a minimum total (r, m)-dominating function of G and v and v' false twins in G. W.l.o.g from Lemma 12 we suppose that $f(v) = m(v)$. Consider f', the restriction of f to $V - v$. Note that $f'(N(v)) = f(N(v')) \geq \max\{r(v), r(v')\} = r'(v)$ and $f'(N(u)) = f(N(u)) - f(v) = f(N(u)) - m(v) \geq \max\{r(u) - m(v), 0\} = r'(u)$. Thus f' is total (r', m')-dominating function of $G - v'$ and $f'(V - v) = f(V) - m(v)$. Thus $\gamma_{(r,m),t}(G) \geq \gamma_{(r',m'),t}(G - v) + m(v)$.

To prove the converse inequality it is enough to see that if f' is a total (r', m)-dominating function of $G - v$, then the function $f : V \mapsto \{0, \dots, k\}$ such that $f(v) = m(v)$ and $f(x) = f'(x)$ o.w. is a total (r, m)-dominating function of G. $\square$

Finally, we have:

Theorem 14. *Algorithm 1 returns a minimum weight total (r, m)-dominating function for a connected BDH graph G in $O(|V(G)| + |E(G)|)$-time.*

From Proposition 13, the correctness of Algorithm 1 holds.

As a total $(k, 0)$-dominating function is a total $\{k\}$-dominating function, we obtain as a corollary of the above theorem, that Algorithm 1 returns a minimum total $\{k\}$-dominating function of any given connected BDH graph in linear time. Notice that in this case, the total $(k, 0)$-dominating function f returned by Algorithm 1 verifies $f(v) \in \{0, k\}$. Then, from Lemma 2 we know how to calculate its weight:

Proposition 15. *Let G be a BDH graph and k a positive integer. Then, $\gamma_{\{k\},t}(G) = k \cdot \gamma_t(G)$.*

Appendix

The vocabulary $\{E\}$ consisting of one binary relation symbol E is denoted by τ_1. For a graph G, $G(\tau_1)$ denotes the presentation of G as a τ_1-structure $< V, E >$, where V is the domain of the logical structure $(V(G))$ and E is the binary relation corresponding to the adjacency matrix of G.

Regarding graph properties, if a formula can be defined using vertices and sets of vertices of a graph, the logical operators OR, AND, NOT (denoted by $\vee$, $\wedge$, $\neg$), the logical quantifiers $\forall$ and $\exists$ over vertices and sets of vertices, the membership relation $\in$ to check whether an element belongs to a set, the equality operator $=$ for vertices and the binary adjacency relation *adj*, where $adj(u,v)$ holds if and only if vertices u and v are adjacent, then the formula is expressible in τ_1-monadic second-order logic, MSOL(τ_1) for short.

An optimization problem P is a LinEMSOL(τ_1) optimization problem over graphs, if it can be defined in the following form: Given a graph G presented as a τ_1-structure and functions $f_1,\ldots,f_m$ associating integer values to the vertices of G, find an assignment z to the free variables in θ such that

$$\sum_{\substack{1\le i\le l\\ 1\le j\le m}} a_{ij}|z(X_i)|_j = opt\{\sum_{\substack{1\le i\le l\\ 1\le j\le m}} a_{ij}|z'(X_i)|_j : \theta(X_1,\ldots,X_l) \text{ is true for } G \text{ and } z'\},$$

where θ is an MSOL(τ_1) formula having free set variables $X_1,\ldots,X_l$, $a_{ij}: i\in\{1,\ldots,l\}$, $j\in\{1,\ldots,m\}$ are integer numbers and $|z(X_i)|_j := \sum_{a\in z(X_i)} f_j(a)$. More details can be found for example in [5] and in [14].

It has been shown that MSOL(τ_1) is particularly useful when combined with the concept of the graph parameter clique-width.

With every p-graph G, an algebraic expression built using the following operations can be associated: creation of a vertex with label i, disjoint union, renaming label i to label j and connecting all vertices with label i to all vertices with label j, for $i\neq j$.

If all the labels in the expression of G are in $\{1,\ldots,q\}$ for positive integer q, the expression is called a *q-expression* of G. It is clear to see that there is a $|V(G)|$-expression which defines G, for every graph G. For a positive integer q, $\mathcal{C}(q)$ denotes the class of p-graphs which can be defined by q-expressions. The *clique-width* of a p-graph G, denoted by $cwd(G)$, is defined by $cwd(G)=\min\{q : G\in\mathcal{C}(q)\}$.

We can prove:

Theorem 16. *Let $k,q\in\mathbb{Z}_+$. Then, k-DOM-T and $\{k\}$-DOM-T can be solved in polynomial time for the family of graphs with clique-width at most q.*

Proof. Based on Theorem 1, we first prove that k-DOM-T is a LinEMSOL(τ_1) optimization problem.

Given a graph G presented as a τ_1-structure $G(\tau_1)$ and one evaluation function (the constant function that associates 1's to the vertices of G) and denoting by $X(u)$ the atomic formula indicating that $u\in X$, finding the k-tuple total domination number of G, $\gamma_{\times k,t}(G)$, is equivalent to finding an assignment z to the free set variable X in θ such that $|z(X)|_1=\min\{|z'(X)|_1 : \theta(X)$ is true for G and $z'\}$, where

$$\theta(X)=\forall v\left(\bigwedge_{1\le r\le k} A_r(X,v,u_1,\ldots,u_r)\right),$$

with $A_1(X,v,u_1):=\exists u_1\,[X(u_1)\wedge adj(v,u_1)]$, and for each $r>1$

$$A_r(X,v,u_1,\ldots,u_r) := \exists u_r\left[X(u_r)\wedge adj(v,u_r)\wedge \bigwedge_{1\le i\le r-1}\neg(u_r=u_i)\right].$$

Hence for fixed q, k-DOM-T can be solved in polynomial time on the family of graphs with clique-width bounded by q.

Finally, let us consider the following graph operation: for disjoint graphs G and H and $v\in V(G)$, $G[H/v]$ denotes the graph obtained by the *substitution* in G of v by H, i.e. $V(G[H/v])=V(G)\cup V(H)-\{v\}$ and

$$E(G[H/v]) = E(H)\cup\{e : e\in E(G) \text{ and } e \text{ is not incident with } v\}\cup \{uw : u\in V(H),\ w\in V(G) \text{ and } w \text{ is adjacent to } v \text{ in } G\}.$$

In [5] it is also proved that $cwd(G[H/v]) = \max\{cwd(G),cwd(H)\}$ for every pair of disjoint graphs G and H and $v\in V(G)$. This, together with the fact that $cwd(S_k)=1$ for every k, imply that, if G is a graph having clique-width bounded by q, then $G\circ S_k$ also is for every k, concluding that also $\{k\}$-DOM-T can be solved in polynomial time on the family of graphs with clique-width bounded by q, for fixed q. □

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