A new proof of the characterization of the weighted Hardy inequality

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Abstract. Maz'ja and Sinnamon proved a characterization of the boundedness of the Hardy operator from $L^p(v)$ into $L^q(w)$ in the case $0 < q < p$, $1 < p < \infty$. We present here a new simple proof of the sufficiency part of that result.

Maz'ja [**2**] and Sinnamon [**3**] characterized the weights such that

$$\left(\int_0^\infty \left(\int_0^x f\right)^q w(x)\,dx\right)^{1/q} \le C\left(\int_0^\infty f^p(x)v(x)\,dx\right)^{1/p} \tag{1}$$

for all measurable $f \ge 0$ with a constant independent of f, where $0 < q < p$ and $1 < p < \infty$. The characterization reduces to the condition that the function

$$\Psi(x) = \left(\int_x^\infty w\right)^{1/p}\left(\int_0^x v^{1-p'}\right)^{1/p'}$$

belong to $L^r(w)$, where $1/r = 1/q - 1/p$. The proof of the necessity in [**2**] is not direct. However a direct and easy proof can be found in [**4**]. The proof of the sufficiency in [**2**] uses the Hölder inequality for three exponents and the characterization of Hardy's inequality for the case $p = q$. The proof of the sufficiency in [**4**] is simpler but uses the Hardy's inequality in the case $p = q$.

The aim of this paper is to present a new proof of the implication $\Psi \in L^r(w) \Rightarrow$ (1) that we believe is simple, elementary and standard because it is more similar to the proofs of the characterization in the easy case $p \le q$ and it does not use the characterization in the case $p = q$. The key point is to show that $\Psi \in L^r(w)$ implies that the function

$$\Phi(x) = \sup_{0<a<x} \Psi(a) = \sup_{0<a<x}\left(\int_a^\infty w\right)^{1/p}\left(\int_0^a v^{1-p'}\right)^{1/p'}$$

belongs also to $L^r(w)$. Once we have this fact, (1) follows by partitioning $(0,\infty)$ as it is usual and by using the Hölder inequality in a natural way. In this way we obtain a new characterization of (1), namely $\Phi \in L^r(w)$. We also think that the

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2000 *Mathematics Subject Classification.* Primary 47A15; Secondary 46A32, 47D20.

Key words and phrases. Hardy inequality, weights.

The first author was supported by CONICET and CAI+D-UNL.

The second and the third authors were supported by MCYT Grant BFM2001-1638 and Junta de Andalucía.

ideas in the proof could be of some help to overcome some difficulties appearing in the study of the Hardy's inequalities with $q < p$. In fact, the ideas of this paper are used in [**1**] to study the generalized Hardy-Steklov operator in weighted spaces.

Next we establish the theorem (due to Maz'ja and Sinnamon) and we give a complete simple proof of the theorem. We remark that the proof of the necessity, (a)⇒(b), is taken from [**4**] while the proofs of (b)⇒(c)⇒(a) are our genuine contribution. We include the proof of (a)⇒(b) to present a complete proof of the theorem.

THEOREM 1. *Let w and v be nonnegative measurable functions. Let q, p, p' and r be such that $0 < q < p < \infty$, $1 < p < \infty$, $p + p' = pp'$, $1/r = 1/q - 1/p$. The following statements are equivalent:*

(a) *There exists a positive constant C such that*

$$\left(\int_0^\infty \left(\int_0^x f\right)^q w(x)\,dx\right)^{1/q} \leq C\left(\int_0^\infty f^p(x)v(x)\,dx\right)^{1/p}$$

for all measurable $f \geq 0$.

(b) *The function*

$$\Psi(x) = \left(\int_x^\infty w\right)^{1/p}\left(\int_0^x v^{1-p'}\right)^{1/p'}$$

belongs to $L^r(w)$.

(c) *The function*

$$\Phi(x) = \sup_{0<a<x}\left(\int_a^\infty w\right)^{1/p}\left(\int_0^a v^{1-p'}\right)^{1/p'}$$

belongs to $L^r(w)$.

PROOF. (b)⇒(c). We may assume that w and $v^{1-p'}$ are integrable functions (the general case follows by approximating these weights by integrable functions). Observe that if

$$\Phi_1(x) = \sup_{0<a<x}\left(\int_a^x w\right)^{1/p}\left(\int_0^a v^{1-p'}\right)^{1/p'}$$

then

$$\Phi(x) \leq \Psi(x) + \Phi_1(x).$$

Therefore, it will suffice to prove that $\Phi_1 \in L^r(w)$. This fact follows from the inequality

$$w(\{x : \Phi_1(x) > \lambda\}) \leq 2w(\{x : \Psi(x) > \lambda\})$$

which we are going to prove now. Since

$$w(\{x : \Phi_1(x) > \lambda\}) \leq w(\{x : \Psi(x) > \lambda\}) + w(\{x : \Psi(x) \leq \lambda < \Phi_1(x)\}),$$

we only have to show that if $E = \{x : \Psi(x) \leq \lambda < \Phi_1(x)\}$ then

$$w(E) \leq w(\{x : \Psi(x) > \lambda\}).$$

To prove this last inequality we only have to establish that

$$\int_x^\infty w \leq w(\{y : \Psi(y) > \lambda\}) \tag{2}$$

for all $x \in E$. Let us fix a point $x \in E$. Then there exists a, $0 < a < x$, such that

$$\left(\int_x^\infty w\right)^{1/p} \left(\int_0^x v^{1-p'}\right)^{1/p'} \le \lambda < \left(\int_a^x w\right)^{1/p} \left(\int_0^a v^{1-p'}\right)^{1/p'}.$$

These inequalities imply that

$$\int_x^\infty w < \int_a^x w.$$

If $(a, x) \subset \{y : \Psi(y) > \lambda\}$ then (2) follows immediately. Assume that the set $F = \{y \in (a, x) : \Psi(y) \le \lambda\}$ is non-empty and let $y \in F$. Then

$$\left(\int_y^\infty w\right)^{1/p} \left(\int_0^y v^{1-p'}\right)^{1/p'} \le \lambda < \left(\int_a^x w\right)^{1/p} \left(\int_0^a v^{1-p'}\right)^{1/p'}.$$

Since $a < y$ it follows that

$$\int_y^\infty w \le \int_a^x w = \int_a^y w + \int_y^x w.$$

Thus

$$\int_x^\infty w \le \int_a^y w.$$

for all $y \in F$. If β is the infimum of F the last inequality implies that

$$\int_x^\infty w \le \int_a^\beta w.$$

This inequality implies that (2) holds since $(a, \beta) \subset \{y : \Psi(y) > \lambda\}$.

(c)⇒(a). It will suffice to prove it for integrable functions f. Let us choose a decreasing sequence x_k such that $x_0 = +\infty$ and $\int_0^{x_k} f = 2^{-k} \int_0^\infty f$. By the Hölder inequality with exponents p and p', we obtain

$$\begin{aligned}
\int_0^\infty \left(\int_0^x f\right)^q w(x)\,dx &\le \sum_k \int_{x_{k+1}}^{x_k} w(x)\,dx \left(\int_0^{x_k} f\right)^q \\
&\le 4^q \sum_k \int_{x_{k+1}}^{x_k} w(x)\,dx \left(\int_{x_{k+2}}^{x_{k+1}} f\right)^q \\
&\le 4^q \sum_k \left(\int_{x_{k+2}}^{x_{k+1}} f^p v\right)^{q/p} \int_{x_{k+1}}^{x_k} w(x)\,dx \left(\int_{x_{k+2}}^{x_{k+1}} v^{1-p'}\right)^{q/p'}.
\end{aligned}$$

Applying the Holder inequality with exponents p/q and r/q and the definition of Φ, we have that the last term is dominated by

$$4^q \left(\sum_k \int_{x_{k+2}}^{x_{k+1}} f^p v\right)^{q/p} \left(\sum_k \int_{x_{k+1}}^{x_k} w(x)\,dx \left(\int_{x_{k+1}}^{x_k} w\right)^{r/p} \left(\int_{x_{k+2}}^{x_{k+1}} v^{1-p'}\right)^{r/p'}\right)^{q/r}$$

$$\le 4^q \left(\int_0^\infty f^p v\right)^{q/p} \left(\sum_k \int_{x_{k+1}}^{x_k} \Phi^r w\right)^{q/r} \le 4^q \left(\int_0^\infty f^p v\right)^{q/p} \left(\int_0^\infty \Phi^r w\right)^{q/r}.$$

(a)⇒(b). Let w_0 and v_0 be nonnegative integrable functions such that $w_0 \le w$ and $v_0 \le v^{1-p'}$. Let

$$f(t) = \left(\int_t^\infty w_0\right)^{\frac{r}{pq}} \left(\int_0^t v_0\right)^{\frac{r}{p'q}-1} v_0(t).$$

Then

$$\begin{aligned}\int_0^x f(t)\,dt &\ge \left(\int_x^\infty w_0\right)^{\frac{r}{pq}} \int_0^x \left(\int_0^t v_0\right)^{\frac{r}{p'q}-1} v_0(t)\,dt \\ &= \frac{p'q}{r}\left(\int_x^\infty w_0\right)^{\frac{r}{pq}} \left(\int_0^x v_0\right)^{\frac{r}{p'q}}.\end{aligned}$$

Then by (a)

$$\begin{aligned}&\int_0^\infty \left(\frac{p'q}{r}\right)^q \left(\int_x^\infty w_0\right)^{r/p} \left(\int_0^x v_0\right)^{r/p'} w_0(x)\,dx \le \int_0^\infty \left(\int_0^x f\right)^q w(x)\,dx \\ &\le C^q \left(\int_0^\infty f^p v\right)^{\frac{q}{p}} = C^q \left(\int_0^\infty \left(\int_t^\infty w_0\right)^{r/q} \left(\int_0^t v_0\right)^{r/q'} v_0^p(t) v(t)\,dt\right)^{\frac{q}{p}} \\ &\le C^q \left(\int_0^\infty \left(\int_t^\infty w_0\right)^{r/q} \left(\int_0^t v_0\right)^{r/q'} v_0(t)\,dt\right)^{\frac{q}{p}} \\ &= C^q \left(\frac{p'}{q}\right)^{q/p} \left(\int_0^\infty \left(\int_x^\infty w_0\right)^{r/p} \left(\int_0^x v_0\right)^{r/p'} w_0(x)\,dx\right)^{\frac{q}{p}}.\end{aligned}$$

where the last equality is integration by parts. Since w_0 and v_0 are integrable functions, we get that

$$\int_0^\infty \left(\int_x^\infty w_0\right)^{r/p} \left(\int_0^x v_0\right)^{r/p'} w_0(x)\,dx \le C^r (r/p'q)^r (p'/q)^{r/p}.$$

Approximating w and $v^{1-p'}$ by increasing sequences of integrable functions we obtain (b). □

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