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Properties of the solutions to the Monge–Ampère equation

Liliana Forzani[a,b], Diego Maldonado[c,*]

[a] Unl-Imal-Conicet, Argentina
[b] School of Statistics, University of Minnesota, 313 Ford Hall, 224 Church Street, Minneapolis, MN-55455, USA
[c] Department of Mathematics, University of Kansas, 1460 Jayhawk Blvd., 405 Snow Hall, KS-66045, USA

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Abstract

We consider solutions to the equation $\det D^2\varphi = \mu$ when μ has a doubling property. We prove new geometric characterizations for this doubling property (by means of the so-called engulfing property) and deduce the quantitative behaviour of φ. Also, a constructive approach to the celebrated $C^{1,\beta}$-estimates proved by L. Caffarelli is presented, settling one of the open questions posed by Villani (Amer. Math. Soc. 58 (2003)).
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1. Introduction

Let $\varphi : \mathbb{R}^n \to \mathbb{R}$ be a convex function and let $\partial\varphi$ denote its normal mapping (or sub-differential). The Monge–Ampère measure μ_φ associated to φ is defined on any Borel set E by

$$\mu_\varphi(E) = |\partial\varphi(E)|,$$

where $|\cdot|$ stands for Lebesgue measure. For $x \in \mathbb{R}^n$, $p \in \partial\varphi(x)$ and $t > 0$, a section of φ centered in x at height t is the open convex set

$$S_\varphi(x, p, t) = \{y \in \mathbb{R}^n : \varphi(y) < \varphi(x) + p \cdot (y - x) + t\}.$$

* Corresponding author. Tel.: +1-785-864-5269; fax: +1-785-864-5255.
E-mail addresses: lforzani@math.unl.edu.ar (L. Forzani), maldonado@math.ukans.edu (D. Maldonado).

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Throughout this paper, we shall only consider functions φ whose sections are bounded sets. Geometrically, this means that the graph of φ does not contain half-lines. If φ is differentiable, then we identify $\partial\varphi(x)$ and $\nabla\varphi(x)$. In this case, we just write $S_\varphi(x,t)$ for the sections.

If we consider the archetypal convex function $\varphi_0(x)=\frac{1}{2}|x|^2$, then the Monge–Ampère measure associated to φ_0 is exactly Lebesgue measure, and for $x\in\mathbb{R}^n$ and $t>0$

$$S_{\varphi_0}(x,t)=B(x,\sqrt{2t}).$$

Hence, the family of sections of φ_0 consists of the usual balls in $\mathbb{R}^n$. Many conditions on a general φ have been proposed in order to preserve the harmony between measure theory and geometry enjoyed in the case of φ_0. The study of these properties began with the fundamental papers of Caffarelli [2,3], Caffarelli and Gutiérrez [4,5]; and was continued by Gutiérrez and Huang [9], and the Forzani and Maldonado [6,7]. Some of these conditions are imposed on the sections of φ. For instance, we say that the sections satisfy the *engulfing property* if there exists a $K>1$ such that for every section $S_\varphi(x,p,t)$ it holds:

$$y\in S_\varphi(x,p,t)\Rightarrow S_\varphi(x,p,t)\subset S_\varphi(y,q,Kt)$$

for all $q\in\partial\varphi(y)$. Also, some of the conditions are imposed on the measure μ_φ, for instance, we say that μ_φ satisfies the (DC)-doubling property if there exist constants $C>0$ and $0<\alpha<1$ such that for all sections $S_\varphi(x,p,t)$, we have

$$\mu_\varphi(S_\varphi(x,p,t))\leqslant C\mu_\varphi(\alpha S_\varphi(x,p,t)),$$

where $\alpha S_\varphi(x,p,t)$ denotes α-dilation with respect to the center of mass of $S_\varphi(x,p,t)$. This property of μ_φ plays a remarkable role in the regularity theory for solutions to the linearized Monge–Ampère equation, see [5,8]. In [9], Gutiérrez and Huang proved that the (DC)-doubling property for μ_φ implies the engulfing property for the sections of φ. On the other hand, in [6] the authors proved the converse of that result. The interplay between geometry and measure theory can be summarized in the following theorem (see [6,9] for these and other equivalent conditions).

Theorem 1. *Let $S_\varphi(x,p,t), x\in\mathbb{R}^n, p\in\partial\varphi(x), t>0$ be the bounded sections of a convex function φ. Then the following are equivalent*:

(i) *The sections of φ satisfy the engulfing property.*
(ii) *The measure μ_φ satisfies the* (DC)-*doubling property.*
(iii) *The Monge–Ampère measure μ_φ satisfies*

$$ct^n\leqslant|S_\varphi(x,p,t)|\mu_\varphi(S_\varphi(x,p,t))\leqslant t^n,$$

for all sections $S_\varphi(x,p,t)$ and some positive constants c,C.

Moreover, the (DC)-doubling property implies two important properties for φ when its sections are bounded sets: φ is strictly convex, and $\varphi\in C^{1,\beta}(Q)$, where $Q\subset\mathbb{R}^n$ is any compact set and β depends on Q. These results were first proved by Caffarelli [3]. See also Gutiérrez' book [8] for a comprehensive exposition of these and other results related to the Monge–Ampère equation.

On the other hand, if $\varphi : \mathbb{R}^n \to \mathbb{R}$ is strictly convex and differentiable, we set

$$\rho_\varphi(x, y) = \inf\{r : y \in S_\varphi(x, r), x \in S_\varphi(y, r)\}$$

and

$$d_\varphi(x, y) = (\nabla\varphi(x) - \nabla\varphi(y)) \cdot (x - y),$$

then it is immediate to check that

$$\rho_\varphi(x, y) \leqslant d_\varphi(x, y) \leqslant 2\rho_\varphi(x, y),$$

for every $x, y \in \mathbb{R}^n$. In [1] Aimar et al. proved that: if the sections of φ satisfy the engulfing property with constant K, then ρ_φ (as much as d_φ) is a quasi-distance on $\mathbb{R}^n$ whose balls are topologically equivalent to the sections of φ, that is, there exist positive constants $0 < \delta_1 < 1 < \delta_2$, depending only on K, such that

$$S_\varphi(x, \delta_1 t) \subset B_{\rho_\varphi}(x, t) \subset S_\varphi(x, \delta_2 t), \tag{1.1}$$

for every $x \in \mathbb{R}^n$ and $t > 0$. Moreover, the quasi-triangular constant of ρ_φ depends only on K. Conversely, if ρ is any quasi-distance on $\mathbb{R}^n$ whose balls are topologically equivalent to the sections of φ, then the sections of φ have the engulfing property; this is just due to the quasi-triangular inequality for ρ. Also, since the (DC)-doubling property of μ_φ implies another doubling condition of μ_φ on the sections, now with respect to the parameter t (see [8,9]), we have that the engulfing property turns $(\mathbb{R}^n, d_\varphi, \mu_\varphi)$ into a space of homogeneous type. Consequently, the real analysis (types of the Hardy–Littlewood maximal function, Calderón–Zygmund decomposition, BMO, Hardy spaces, singular integrals, Muckenhoupt's classes, etc.) with respect to μ_φ and the sections of φ follows in a standard way. This is another important application of convex functions satisfying the engulfing property.

To cite some other recent applications of these ideas, let us mention that in [7], the authors proved the following characterization for the engulfing property in dimension 1 which, in turn, is useful to characterize all doubling measures and quasi-symmetric mappings on $\mathbb{R}$.

Theorem 2. *Let $\varphi : \mathbb{R} \to \mathbb{R}$ be a strictly convex differentiable function. The following are equivalent*:

(i) (*Engulfing property of the sections of φ.*) *There exists a constant $K > 1$ such that if $x \in S_\varphi(y, t)$ then*

$$S_\varphi(y, t) \subset S_\varphi(x, Kt),$$

for every $x, y \in \mathbb{R}$ and $t > 0$.

(ii) *There exists a constant $K' > 1$ such that if $x, y \in \mathbb{R}$ and $t > 0$ verify $x \in S_\varphi(y, t)$, then $y \in S_\varphi(x, K't)$.*

(iii) *There exists a constant $K'' > 1$ such that for every $x, y \in \mathbb{R}$*

$$\begin{aligned} &\frac{K''+1}{K''}(\varphi(y) - \varphi(x) - \varphi'(x)(y - x)) \\ &\quad \leqslant (\varphi'(x) - \varphi'(y))(x - y) \\ &\quad \leqslant (K''+1)(\varphi(y) - \varphi(x) - \varphi'(x)(y - x)). \end{aligned}$$

Let us denote by $\mathrm{Eng}(n,K)$ the set of all convex functions $\varphi : \mathbb{R}^n \to \mathbb{R}$ whose bounded sections satisfy the engulfing property with constant K. Let us also define

$$\mathrm{Eng}(n) = \bigcup_{K>1} \mathrm{Eng}(n,K)$$

and

$$\mathrm{Eng}_0(n) = \bigcup_{K>1} \mathrm{Eng}_0(n,K),$$

where $\mathrm{Eng}_0(n,K) = \{\varphi \in \mathrm{Eng}(n,K) : \varphi(0) = 0, \nabla\varphi(0) = 0\}$.

The purpose of this paper is to exhibit new characterizations for the engulfing property and to describe the quantitative behaviour of functions in $\mathrm{Eng}(n)$. We do this by means of a multi-dimensional version of Theorem 2. Then, several properties of functions in $\mathrm{Eng}(n)$ are deduced. We also stress the importance of convex conjugate functions, in particular, we prove that $\mathrm{Eng}(n)$ is invariant under conjugation. The last part of the paper is devoted to the constructive estimates of Caffarelli's $C^{1,\beta}$-theory.

2. Examples of functions in Eng(*n*)

Let $\varphi : \mathbb{R}^n \to \mathbb{R}$ be a strictly convex differentiable function.

(i) If $\det D^2\varphi = p$, where p is a polynomial, then $\varphi \in \mathrm{Eng}(n,K)$ for some K depending only on the degree of p (in particular, K does not depend on the coefficients of p), see [8, p. 52].

(ii) If μ_φ verifies the μ_∞ property, i.e., given $\delta_1 \in (0,1)$, there exists $\delta_2 \in (0,1)$ such that for every section $S = S_\varphi(x,t)$ and every measurable set $E \subset S$,

$$\frac{|E|}{|S|} < \delta_2 \Rightarrow \frac{\mu_\varphi(E)}{\mu_\varphi(S)} < \delta_1$$

then $\varphi \in \mathrm{Eng}(n)$. To see how μ_∞ implies the (DC)-doubling condition, given $\delta_1 \in (0,1)$, pick $\alpha \in (0,1)$ such that

$$\frac{|S - \alpha S|}{|S|} = 1 - \alpha^n < \delta_2,$$

then

$$\frac{\mu_\varphi(S - \alpha S)}{\mu_\varphi(S)} < \delta_1$$

and the (DC)-doubling property follows with $C = 1/(1-\delta_1)$. By Theorem 1, we get $\varphi \in \mathrm{Eng}(n)$. This μ_∞ property plays an important role in the proof of Harnack's inequality for non-negative solutions to the linearized Monge–Ampère equation, see [5].

(iii) If $\varphi \in C^2(\mathbb{R}^n)$ and there exist constants $\lambda, \Lambda > 0$ such that

$$\lambda \leqslant \det D^2\varphi \leqslant \Lambda, \tag{2.2}$$

then $\varphi \in \mathrm{Eng}(n)$. This follows from the fact that in this case μ_φ clearly verifies the μ_∞ property. Actually, the same is true if we only ask (2.2) to hold in the Aleksandrov sense.

(iv) If $n = 1$ and $\varphi(x) = |x|^p$ with $p > 1$, then $\varphi \in \mathrm{Eng}(1)$. In general, if μ is a doubling measure on $\mathbb{R}$, then $\varphi_\mu(x) = \int_0^x \int_0^t \mathrm{d}\mu\,\mathrm{d}t$ belongs to $\mathrm{Eng}_0(1)$, see [7].

3. Some immediate properties of Eng(*n*)

Lemma 3. *Let* φ *be in* $\mathrm{Eng}(n, K)$.

(i) *If* $\lambda > 0$, *then* $\lambda\varphi \in \mathrm{Eng}(n, K)$.
(ii) *If* $\psi \in \mathrm{Eng}(n, K')$, *then* $\varphi + \psi \in \mathrm{Eng}(n, 2(K \vee K'))$.
(iii) *If for* $x, y \in \mathbb{R}^n$ *we set* $\varphi_{x,y}(s) = \varphi(sy + (1-s)x), s \in \mathbb{R}$, *then* $\varphi_{x,y} \in \mathrm{Eng}(1, K)$.
(iv) $\mathrm{Aff}(\mathbb{R}^n, \mathbb{R}^n)$ *acts on* $\mathrm{Eng}(n, K)$ *by composition.*
(v) $\mathrm{Aff}(\mathbb{R}^n, \mathbb{R})$ *acts on* $\mathrm{Eng}(n, K)$ *by addition.*

Proof. In order to prove (i), we observe that given $z \in \mathbb{R}^n$ and λ, $s > 0$, we have

$$S_{\lambda\varphi}(z, s) = S_\varphi(z, s/\lambda). \tag{3.3}$$

Now, if $y \in S_{\lambda\varphi}(x, t)$, then $y \in S_\varphi(x, t/\lambda)$. By the engulfing property of φ, $S_\varphi(x, t/\lambda) \subset S_\varphi(y, Kt/\lambda)$. And, according to (3.3), $S_{\lambda\varphi}(x, t) \subset S_{\lambda\varphi}(y, Kt)$. Hence, $\lambda\varphi \in \mathrm{Eng}(n, K)$.

To prove (ii), first note that for every $z \in \mathbb{R}^n$ and $s > 0$,

$$S_{\varphi+\psi}(z, s) \subset S_\varphi(z, s) \cap S_\psi(z, s) \subset S_{\varphi+\psi}(z, 2s). \tag{3.4}$$

In particular, the sections of $\varphi + \psi$ are bounded sets. Now, if $y \in S_{\varphi+\psi}(x, t)$ then $y \in S_\varphi(x, t) \cap S_\psi(x, t)$, which implies $S_\varphi(x, t) \subset S_\varphi(y, Kt)$ and $S_\psi(x, t) \subset S_\psi(y, K't)$. Therefore, setting $K'' = K \vee K'$ and using (3.4), we obtain

$$\begin{aligned} S_{\varphi+\psi}(x, t) &\subset S_\varphi(x, t) \cap S_\psi(x, t) \subset S_\varphi(y, Kt) \cap S_\psi(y, K't) \\ &\subset S_\varphi(y, K''t) \cap S_\psi(y, K''t) \subset S_{\varphi+\psi}(y, 2K''t). \end{aligned}$$

(iii) For $r, s \in \mathbb{R}$ and $t > 0$, we have

$$r \in S_{\varphi_{x,y}}(s, t) \Leftrightarrow ry + (1-r)x \in S_\varphi(sy + (1-s)x, t). \tag{3.5}$$

Thus, if $r \in S_{\varphi_{x,y}}(s, t)$ then, by (3.5) and the engulfing property of φ, we have $sy + (1-s)x \in S_\varphi(ry + (1-r)x, Kt)$. By (3.5) again, we have $s \in S_{\varphi_{x,y}}(r, Kt)$. The engulfing property for $\varphi_{x,y}$ now follows from Theorem 2, with constant K independent of x and y.

(iv) Given $T \in GL(n, \mathbb{R})$ and $b \in \mathbb{R}^n$, set $\varphi_A = \varphi \circ A$, where $Ax = Tx + b$. First note that for all $u, v \in \mathbb{R}^n$ and $s > 0$ we have

$$u \in S_{\varphi_A}(v, s) \Leftrightarrow Au \in S_\varphi(Av, s). \tag{3.6}$$

Now, $y \in S_{\varphi_A}(x, t) \Rightarrow Ay \in S_\varphi(Ax, t)$. By the engulfing property we have $S_\varphi(Ax, t) \subset S_\varphi(Ay, Kt)$. Now, this last inclusion and (3.6) imply that $S_{\varphi_A}(x, t) \subset S_{\varphi_A}(y, Kt)$, the engulfing property for φ_A. Finally, we use the condition $\det T \neq 0$ to assure the boundedness of the sections of φ_A. Thus, $\varphi_A \in \mathrm{Eng}(n, K)$.

(v) Fix $v \in \mathbb{R}^n, b \in \mathbb{R}$ and define $a(x) = v \cdot x + b$. Set $\psi(x) = \phi(x) + a(x)$. It is immediate that

$$S_\psi(x,t) = S_\varphi(x,t), \tag{3.7}$$

for every $x \in \mathbb{R}^n$ and $t > 0$. Thus, if $\phi \in \text{Eng}(n,K)$ then $\psi \in \text{Eng}(n,K)$. □

4. New characterizations for the engulfing property

The following result is the n-dimensional version of Theorem 2.

Theorem 4. *Let $\varphi : \mathbb{R}^n \to \mathbb{R}$ be a strictly convex differentiable function. The following are equivalent*:

(i) *There exists a constant $K > 1$ such that if $x \in S_\varphi(y,t)$ then*

$$S_\varphi(y,t) \subset S_\varphi(x,Kt),$$

for every $x, y \in \mathbb{R}^n$ and $t > 0$.(Engulfing property.)

(ii) *There exists a constant $K' > 1$ such that if $x, y \in \mathbb{R}^n$ and $t > 0$ verify $x \in S_\varphi(y,t)$, then $y \in S_\varphi(x,K't)$.*

(iii) *There exists a constant $K'' > 1$ such that for every $x, y \in \mathbb{R}^n$*

$$\begin{aligned} &\frac{K''+1}{K''}(\varphi(y) - \varphi(x) - \nabla\varphi(x)\cdot(y-x)) \\ &\quad \leqslant (\nabla\varphi(x) - \nabla\varphi(y))\cdot(x-y) \\ &\quad \leqslant (K''+1)(\varphi(y) - \varphi(x) - \nabla\varphi(x)\cdot(y-x)). \end{aligned}$$

Proof. The proof for (i) ⇒ (ii) is obvious since $y \in S_\varphi(y,t)$ for every $y \in \mathbb{R}^n$ and $t > 0$. Thus (ii) holds with $K' = K$.

Proof of (ii) ⇒ (iii): Given $x, y \in \mathbb{R}^n$ and $\varepsilon > 0$, we have

$$\begin{aligned} \varphi(x) < \varphi(x) + \varepsilon = \varphi(y) + \nabla\varphi(y)\cdot(x-y) + \varphi(x) - \varphi(y) \\ -\nabla\varphi(y)\cdot(x-y) + \varepsilon, \end{aligned}$$

(note that the convexity of φ implies $\varphi(x) - \varphi(y) - \nabla\varphi(y)\cdot(x-y) \geqslant 0$), this means that $x \in S_\varphi(y, \varphi(x) - \varphi(y) - \nabla\varphi(y)\cdot(x-y) + \varepsilon)$. By property (ii), we must have $y \in S_\varphi(x, K'(\varphi(x) - \varphi(y) - \nabla\varphi(y)\cdot(x-y) + \varepsilon))$, which means

$$\varphi(y) \leqslant \varphi(x) + \nabla\varphi(x)\cdot(y-x) + K'\varphi(x) - K'\varphi(y) - K'\nabla\varphi(y)\cdot(x-y) + K'\varepsilon.$$

Letting ε go to 0 and summing up we get

$$(K'+1)\varphi(y) \leqslant (K'+1)\varphi(x) + (\nabla\varphi(x) + K'\nabla\varphi(y))(y-x). \tag{4.8}$$

Now interchanging the roles of x and y, we obtain

$$(K'+1)\varphi(x) \leqslant (K'+1)\varphi(y) + (\nabla\varphi(y) + K'\nabla\varphi(x))(x-y). \tag{4.9}$$

From (4.8) and (4.9), we get

$$\frac{1}{K'+1}\nabla\varphi(x)\cdot(x-y)+\frac{K'}{K'+1}\nabla\varphi(y)(x-y)$$
$$\leqslant \varphi(x)-\varphi(y)$$
$$\leqslant \left(\frac{1}{K'+1}\nabla\varphi(y)+\frac{K'}{K'+1}\nabla\varphi(x)\right)(x-y). \tag{4.10}$$

By using the first inequality in (4.10) we get

$$\frac{1}{K'+1}(\nabla\varphi(x)-\nabla\varphi(y))\cdot(x-y)\leqslant \varphi(x)-\varphi(y)-\nabla\varphi(y)(x-y). \tag{4.11}$$

The second inequality in (4.10) yields

$$\varphi(x)-\varphi(y)-\nabla\varphi(x)\cdot(x-y)\leqslant \frac{1}{K'+1}(\nabla\varphi(y)-\nabla\varphi(x))(x-y), \tag{4.12}$$

which implies

$$\varphi(x)-\varphi(y)-\nabla\varphi(y)\cdot(x-y)\leqslant \frac{K'}{K'+1}(\nabla\varphi(x)-\nabla\varphi(y))(x-y). \tag{4.13}$$

Now (iii) follows from (4.13) and (4.11) with $K''=K'$.

Proof of (iii) $\Rightarrow$ (ii): Suppose $x\in S_\varphi(y,t)$, then

$$\varphi(x)-\varphi(y)-\varphi'(y)(x-y)<t,$$

now, by the second inequality in (iii), we get

$$(\nabla\varphi(x)-\nabla\varphi(y))(x-y)=(\nabla\varphi(y)-\nabla\varphi(x))\cdot(y-x)\leqslant (K''+1)t$$

and by using the first inequality in (iii),

$$\varphi(y)-\varphi(x)-\nabla\varphi(x)\cdot(y-x)\leqslant K''t.$$

That is, $y\in S_\varphi(x,K''t)$; and (ii) follows with $K'=K''$.

(ii) $\Rightarrow$ (i): Let $x\in S_\varphi(y,t)$. We want to prove the existence of a constant $K>1$ such that

$$S_\varphi(y,t)\subset S_\varphi(x,Kt).$$

Let us assume first that $\varphi(y)=0$ and $\nabla\varphi(y)=0$ (in particular, we get $\varphi\geqslant 0$). Consider the line going through x and z, $sx+(1-s)z$, $s\in\mathbb{R}$, and let $s_1,s_2\in\mathbb{R}$ such that

$$\varphi(s_1x+(1-s_1)z)=\varphi(s_2x+(1-s_2)z)=t.$$

By the strict convexity of φ, the segment $I=\{s\in\mathbb{R}: sx+(1-s)z\}\cap S_\varphi(y,t)$ equals a certain section $S_{\varphi_{x,z}}(h,l)$ of $\varphi_{x,z}$ (as defined in Lemma 3) for some $h\in I$ and $l>0$ such that

$$\varphi_{x,z}(s_1)-\varphi_{x,z}(h)-\varphi'_{x,z}(s_1-h)=\varphi_{x,z}(s_2)-\varphi_{x,z}(h)-\varphi'_{x,z}(s_2-h)=l, \tag{4.14}$$

which implies

$$\varphi_{x,z}(s_1)-\varphi_{x,z}(s_2)-\varphi'_{x,z}(h)(s_1-s_2)=0,$$

and, since $s_1 \neq s_2$ and $\varphi_{x,z}(s_1)=\varphi_{x,z}(s_2)=t$, we get $\varphi'_{x,z}(h)=0$; this equality, together with the non-negativity of φ and (4.14), implies $l \leqslant t$. Since $x \in S_\varphi(y,t)$, we have $\varphi_{x,z}(1)=\varphi(x)<t$. Hence,

$$1 \in S_{\varphi_{x,z}}(h,t). \tag{4.15}$$

On the other hand, since φ verifies (ii) with constant K' then it is straightforward that $\varphi_{x,z}$ verifies (ii) in Theorem 2 with the same constant K', and by Theorem 2 we get that $\varphi_{x,z} \in \mathrm{Eng}(1,K)$, where K depends only on K'. Actually, we can take $K=2K'(K'+1)$, see [7]. Thus, (4.15) implies

$$S_{\varphi_{x,z}}(h,t) \subset S_{\varphi_{x,z}}(1,Kt). \tag{4.16}$$

But, the fact that $z \in S_\varphi(y,t)$ can be written as $0 \in S_{\varphi_{x,z}}(h,t)$. And, by (4.16), we obtain $0 \in S_{\varphi_{x,z}}(1,Kt)$, which means $z \in S_\varphi(x,Kt)$, and we prove the Theorem when $\varphi(y)=0$ and $\nabla\varphi(y)=0$.

The general case for φ is treated as follows: given $y \in \mathbb{R}^n$, define the strictly convex auxiliary function φ_y as

$$\varphi_y(x)=\varphi(x)-\varphi(y)-\nabla\varphi(y)(x-y) \quad x \in \mathbb{R}^n,$$

then we have $\varphi_y(y)=0$ and $\nabla\varphi_y(y)=0$. Moreover, for every $x \in \mathbb{R}^n$ and $t>0$

$$S_{\varphi_y}(x,t)=S_\varphi(x,t),$$

and the theorem follows. □

Corollary 5. *$\varphi \in \mathrm{Eng}(n,K)$ if and only if for every $x,z \in \mathbb{R}^n$, $\varphi_{x,z} \in \mathrm{Eng}(1,K')$, K' independent of x and z.*

Proof. The proof is clear from Lemma 3 and the proof of Theorem 4. □

Given two objects $\mathscr{A}$ and $\mathscr{B}$ (numbers or functions), we shall write $\mathscr{A} \lesssim \mathscr{B}$ if there exists a constant c, depending only on K (the engulfing constant), such that $\mathscr{A} \leqslant c\mathscr{B}$. If $\mathscr{A} \lesssim \mathscr{B}$ and $\mathscr{B} \lesssim \mathscr{A}$, we shall write $\mathscr{A} \simeq \mathscr{B}$. Thus, the condition on Theorem 4 reads

$$\varphi(y)-\varphi(x)-\nabla\varphi(x)\cdot(y-x) \simeq (\nabla\varphi(x)-\nabla\varphi(y))\cdot(x-y), \tag{4.17}$$

for every $x, y \in \mathbb{R}^n$.

Corollary 6. *Set $B(x,y)=\varphi(y)-\varphi(x)-\nabla\varphi(x)\cdot(y-x)$. If $\varphi \in \mathrm{Eng}(n,K)$, then the function $\delta_\varphi(x,y)=\max\{B(x,y),B(y,x)\}$ is a quasi-distance in $\mathbb{R}^n$ and $\delta_\varphi \simeq d_\varphi$.*

Proof. The proof is immediate from Theorem 4. The function B is known as the *Bregman distance*. Even if the Bregman distance is not a distance, under the presence of the engulfing property it becomes essentially a quasi-distance. □

The following result relates the Euclidean balls and the d_φ-balls, providing the quantitative behaviour of φ.

Theorem 7. *Let $\varphi \in \mathrm{Eng}(n,K)$ and $r > 0$. For $y \in \mathbb{R}^n$ define $\varphi_y(x) = \varphi(x) - \varphi(y) - \nabla\varphi(y)(x-y)$. If $|x-y| \leqslant r$, then*

$$\left(\min_{z:|z-y|=r} \varphi_y(z)\right)\left(\frac{|x-y|}{r}\right)^{1+K} \leqslant \varphi(x) - \varphi(y) - \nabla\varphi(y)(x-y)$$
$$\leqslant \left(\max_{z:|z-y|=r} \varphi_y(z)\right)\left(\frac{|x-y|}{r}\right)^{1+1/K}. \tag{4.18}$$

If $|x-y| \geqslant r > 0$, then

$$\left(\min_{z:|z-y|=r} \varphi_y(z)\right)\left(\frac{|x-y|}{r}\right)^{1+1/K} \leqslant \varphi(x) - \varphi(y) - \nabla\varphi(y)(x-y)$$
$$\leqslant \left(\max_{z:|z-y|=r} \varphi_y(z)\right)\left(\frac{|x-y|}{r}\right)^{1+K}. \tag{4.19}$$

Proof. We shall first prove that if $\varphi \in \mathrm{Eng}_0(n,K)$ and $|x| \leqslant r$,

$$\left(\min_{z:|z|=r} \varphi(z)\right)\left(\frac{|x|}{r}\right)^{1+K} \leqslant \varphi(x) \leqslant \left(\max_{z:|z|=r} \varphi(z)\right)\left(\frac{|x|}{r}\right)^{1+1/K}, \tag{4.20}$$

and, if $|x| \geqslant r > 0$, then

$$\left(\min_{z:|z|=r} \varphi(z)\right)\left(\frac{|x|}{r}\right)^{1+1/K} \leqslant \varphi(x) \leqslant \left(\max_{z:|z|=r} \varphi(z)\right)\left(\frac{|x|}{r}\right)^{1+K}. \tag{4.21}$$

Consider first a function $\phi \in \mathrm{Eng}_0(1,K)$. By Theorem 4 we know that

$$\frac{1}{K}\phi(t) \leqslant \phi'(t)t - \phi(t) \leqslant K\phi(t), \tag{4.22}$$

for every $t \in \mathbb{R}$. Let us work out the second inequality in the first place. For $t > 0$, we get

$$\frac{\phi'(t)}{\phi(t)} \leqslant (1+K)\frac{1}{t}$$

recognizing the derivatives of the corresponding logarithms, we get that the function $\phi(t)/t^{1+K}$ is decreasing in $(0,\infty)$. Now, given $x \in \mathbb{R}^n$, write $x = tx_0$, where $|x_0| = 1$, and define $\phi(t) = \varphi(tx_0)$. By Lemma 3, $\phi \in \mathrm{Eng}_0(1,K)$. If $|x| \leqslant r$, then $t \leqslant r$ and we use the mentioned monotonicity to get

$$\phi(r)/r^{1+K} \leqslant \phi(t)/t^{1+K}$$

which is

$$\varphi(rx_0)\frac{1}{r^{1+K}} \leqslant \varphi(tx_0)\frac{1}{t^{1+K}} = \varphi(x)\frac{1}{|x|^{1+K}}$$

and the first inequality in (4.18) follows. The other inequalities are proven in similar fashion, by remarking that the function $\phi(t)/t^{1+1/K}$ is increasing in $(0,\infty)$.

In order to finish the proof we need to consider the general case $\varphi \in \mathrm{Eng}(n,K)$. In this case, given $y \in \mathbb{R}^n$, define $\psi_y(x) = \varphi(x+y) - \varphi(y) - \nabla\varphi(y)\cdot x$. Thus, by

Lemma 3, $\psi_y \in \mathrm{Eng}_0(n,K)$ and we complete the proof by applying (4.21) and (4.20) to the function ψ_y. □

We have the following immediate consequence of Theorem 7

Corollary 8. *Let $\varphi \in \mathrm{Eng}(n,K)$. For $y \in \mathbb{R}^n$, φ_y defined as in Theorem 7, and $r > 0$,*

$$S_\varphi(y, m(\varphi, y, r)) \subset B(y,r) \subset S_\varphi(y, M(\varphi, y, r)), \tag{4.23}$$

where $m(\varphi, y, r) = \min_{z:|z-y|=r} \varphi_y(z)$ and $M(\varphi, y, r) = \max_{z:|z-y|=r} \varphi_y(z)$.

5. More properties of Eng(n). The convex conjugate

As we saw, given $\varphi \in \mathrm{Eng}_0(n,K)$, the inequalities (4.22) imply that

$$\varphi(x) \simeq \nabla\varphi(x) \cdot x$$

(in particular, if $n=1$ the functions in $\mathrm{Eng}_0(1,K)$ verify the Δ_2-condition), now we could ask if similar inequalities hold up to the second derivative, that is, is it true that $xD^2\varphi(x)x \simeq \varphi(x)$ (provided that φ is twice differentiable)? As we will see, the answer is no.

Notice that $\mathrm{Eng}_0(n,K)$ is not contained in $C^2(\mathbb{R}^n)$ (take, for instance, $\varphi(x) = |x|^p$ with $2 > p > 1$). To prove that the estimate $xD^2\varphi(x)x \simeq \varphi(x)$ does not hold in general, consider $n=1$ and pick a continuous doubling weight w on $\mathbb{R}$ which vanishes at certain point $x_0 \neq 0$. Set $\varphi(x) \doteq \int_0^x \int_0^s w(t)\,\mathrm{d}t\,\mathrm{d}s \in \mathrm{Eng}_0(1,K)$ (see [7]), now we cannot have $w(x)x^2 \simeq \varphi(x)$, since φ is strictly positive when $x \neq 0$ and $w(x_0) = 0$. However, an integral version of the inequalities $xD^2\varphi(x)x \simeq \varphi(x)$ does hold. More precisely, we have

Theorem 9. *Let $\varphi \in \mathrm{Eng}(n,K) \cap C^2(\mathbb{R}^n)$. Then*

$$\varphi(x) - \varphi(y) - \nabla\varphi(y)(x-y) \simeq \int_0^1 t(x-y)D^2\varphi(tx + (1-t)y)t(x-y)\,\mathrm{d}t. \tag{5.24}$$

Proof. Consider first $\varphi \in \mathrm{Eng}_0(n,K)$, fix $x \in \mathbb{R}^n$ and define $f(t) = \varphi(tx) \in \mathrm{Eng}_0(1,K)$. As proved in [7], there exist positive constants c_K, C_K depending only on K such that

$$c_K f(1) \leqslant \int_0^1 t^2 f''(t)\,\mathrm{d}t \leqslant C_K f(1)$$

which yields

$$c_K \varphi(x) \leqslant \int_0^1 txD^2\varphi(tx)tx\,\mathrm{d}t \leqslant C_K \varphi(x). \tag{5.25}$$

To complete the proof, given any $\varphi \in \mathrm{Eng}(n,K)$, fix $y \in \mathbb{R}^n$ and define $\psi_y(x) = \varphi(x+y) - \varphi(y) - \nabla\varphi(y)x \in \mathrm{Eng}_0(n,K)$ and apply (5.25) to ψ_y. □

We immediately have

Corollary 10. *Let* $\varphi \in \mathrm{Eng}(n,K) \cap C^2(\mathbb{R}^n)$. *Then*

$$d_\varphi(x,y) \simeq \int_0^1 t(x-y)D^2\varphi(tx+(1-t)y)t(x-y)\,\mathrm{d}t.$$

Lemma 11. *If* $\varphi \in \mathrm{Eng}(n,K)$, *then* $\nabla\varphi : \mathbb{R}^n \to \mathbb{R}^n$ *is a continuous bijection.*

Proof. The continuity of $\nabla\varphi$ follows from Caffarelli's results mentioned in the Introduction. Injectivity of $\nabla\varphi$ follows from the strict convexity of φ. We could also use that $\varphi \in \mathrm{Eng}(n,K)$ to turn ρ_φ into a quasi-distance, consequently

$$\nabla\varphi(x) = \nabla\varphi(y) \Rightarrow \rho_\varphi(x,y) = 0 \Rightarrow x = y.$$

To prove that $\nabla\varphi$ is onto, note that it is enough to suppose $\varphi \in \mathrm{Eng}_0(n,K)$ (subtract a hyperplane from φ). Thus, (4.21), with $r=1$, gives

$$\lim_{|x|\to+\infty} \frac{\varphi(x)}{|x|} = +\infty. \tag{5.26}$$

Now, given $a \in \mathbb{R}^n$ we can minimize $h(x) \doteq \varphi(x) - ax$ to get that $a \in \nabla\varphi(\mathbb{R}^n)$. $\square$

Theorem 12. *Let* φ *be in* $\mathrm{Eng}(n,K)$. *If* φ^* *denotes the conjugate of* φ, *then* $\varphi^* \in \mathrm{Eng}(n,K^*)$ *with* K^* *depending only on* K. *Moreover, the sections of* φ *and* φ^* *are related as follows: for every* $x \in \mathbb{R}^n, t > 0$

$$\nabla\varphi(S_\varphi(x,t/K)) \subset S_{\varphi^*}(\nabla\varphi(x),t) \subset \nabla\varphi(S_\varphi(x,Kt)). \tag{5.27}$$

Proof. Recall that

$$\varphi^*(x) = \sup_{z\in\mathbb{R}^n}(xz - \varphi(z)).$$

Since φ has the engulfing property, we know that φ is a strictly convex differentiable function. By Theorem 26.5 in [10], we get that φ^* is also a strictly convex differentiable function whose domain is $\nabla\varphi(\mathbb{R}^n)$ which, by Lemma 11, equals $\mathbb{R}^n$. We also have

$$\nabla\varphi(\nabla\varphi^*(x)) = \nabla\varphi^*(\nabla\varphi(x)) = x \quad \forall x \in \mathbb{R}^n \tag{5.28}$$

and

$$\varphi^*(\nabla\varphi(x)) = \nabla\varphi(x)x - \varphi(x) \quad \forall x \in \mathbb{R}^n, \tag{5.29}$$

(remark that (5.29) and (4.22) imply $\varphi^*(\nabla\varphi(x)) \simeq \varphi(x)$). Moreover, $(\varphi^*)^* = \varphi$. We first note that for every $x, y \in \mathbb{R}^n$

$$y \in S_\varphi(x,t) \Leftrightarrow \nabla\varphi(x) \in S_{\varphi^*}(\nabla\varphi(y),t). \tag{5.30}$$

To prove (5.30), we do as follows: $y \in S_\varphi(x,t)$ if and only if

$$\varphi(y) < \varphi(x) + \nabla\varphi(x)(y-x) + t = \varphi(x) - \nabla\varphi(x)x + \nabla\varphi(x)y + t$$

Now, we use (5.29) to get the equivalent condition

$$\varphi(y)y - \varphi^*(\nabla\varphi(y)) < -\varphi^*(\nabla\varphi(x)) + \nabla\varphi(x)y + t$$

which is the same as

$$\varphi^*(\nabla\varphi(x)) < \varphi^*(\nabla\varphi(y)) + y\cdot(\nabla\varphi(x) - \nabla\varphi(y)) + t$$

and, by (5.28), this means $\nabla\varphi(x)\in S_{\varphi^*}(\nabla\varphi(y),t)$. Thus, (5.30) and (ii) in Theorem 4 imply that $\varphi^*\in \mathrm{Eng}(n,K^*)$, for some K^* depending only on K. Following up the constants we can take $K^* = 2K(K+1)$.

The next step is to prove the following inclusions for every $x\in\mathbb{R}^n, t>0$:

$$\nabla\varphi(S_\varphi(x,t)) \subset S_{\varphi^*}(\nabla\varphi(x),Kt) \subset \nabla\varphi(S_\varphi(x,K^2t)). \tag{5.31}$$

To prove the first inclusion, let us take $z\in\nabla\varphi(S_\varphi(x,t))$. Then, $z=\nabla\varphi(y)$ for some $y\in S_\varphi(x,t)$; and, by the engulfing property for φ, $x\in S_\varphi(y,Kt)$. Now, by (5.30), $z=\nabla\varphi(y)\in S_{\varphi^*}(\nabla\varphi(x),Kt)$.

To prove the second inclusion, take $z\in S_{\varphi^*}(\nabla\varphi(x),Kt)$. By (5.28), $z=\nabla\varphi(y)$ for some $y\in\mathbb{R}^n$. Then $\nabla\varphi(y)\in S_{\varphi^*}(\nabla\varphi(x),Kt)$, and by using (5.30) we get $x\in S_\varphi(y,Kt)$. Again by the engulfing property, $y\in S_\varphi(x,K^2t)$, which implies $z\in\nabla\varphi(S_\varphi(x,K^2,t))$. Applying $\nabla\varphi^*$ in (5.31), we obtain

$$S_\varphi(x,t)\subset\nabla\varphi^*(S_{\varphi^*}(\nabla\varphi(x),Kt))\subset S_\varphi(x,K^2t). \qquad \square \tag{5.32}$$

Corollary 13. *If $\varphi\in\mathrm{Eng}(n)$, then $\nabla\varphi:\mathbb{R}^n\to\mathbb{R}^n$ is a homeomorphism.*

Proof. Immediate from Lemma 11 and Theorem 12, since the continuous inverse of $\nabla\varphi$ is $\nabla\varphi^*$. $\square$

Corollary 14. *If $\varphi,\psi\in\mathrm{Eng}(n)$, then the infimal convolution $\varphi\odot\psi\in\mathrm{Eng}(n)$.*

Proof. Recall that the infimal convolution of two convex functions φ and ψ is the convex function defined by

$$\varphi\odot\psi(x) = \inf_{y\in\mathbb{R}^n}\{\varphi(y)-\psi(x-y)\},$$

and we always have $(\varphi\odot\psi)^* = \varphi^*+\psi^*$. Thus, we get the result applying Lemma 3 and Theorem 12. $\square$

6. A constructive approach to Caffarelli's $C^{1,\beta}$ regularity result

As mentioned in the Introduction, Caffarelli proved the $C^{1,\beta}$ regularity of any convex function $\varphi\in\mathrm{Eng}(n,K)$. His proof is based on a compactness argument that does not provide an estimate for β or the $C^{1,\beta}$ norm of φ on compact sets. The task of finding the explicit size of these constants was posed as an open problem in Villani's recent book (see [11, p. 141]).

In this section we will get such estimates, in terms of K, through Theorem 7. To illustrate the main idea, let us take a look at the case $n=1$. Consider $\varphi \in \mathrm{Eng}_0(1,K)$, $|x| \leqslant 1$, and denote by $M(\varphi,1)$ the maximum between $\varphi(1)$ and $\varphi(-1)$. Then, by (4.18), we get $\varphi(x) \leqslant M(\varphi,1)|x|^{1+1/K}$. On the other hand, by (4.22), we have $0 \leqslant \varphi'(x)x \leqslant (K+1)\varphi(x)$. Consequently, for every x with $|x| \leqslant 1$, we get $|\varphi'(x)| \leqslant (K+1)M(\varphi,1)|x|^{1/K}$. Which is the $C^{1/K}$ regularity of φ' about 0. Before stating the general result some notation is in order. Given a convex function $\phi \in \mathrm{Eng}(n,K)$, $y \in \mathbb{R}^n$, and $r > 0$, set

$$M(\phi, y, r) = \max_{z:|z-y|=r} \{\phi(z) - \phi(y) - \nabla\phi(y)\cdot(z-y)\}$$

and

$$m(\phi, y, r) = \min_{z:|z-y|=r} \{\phi(z) - \phi(y) - \nabla\phi(y)\cdot(z-y)\}$$

Theorem 15. *Let $\varphi \in \mathrm{Eng}(n,K), \varphi^* \in \mathrm{Eng}(n,K^*)$, and $y \in \mathbb{R}^n$. For every $z \in \mathbb{R}^n$ with $|z-y| \leqslant r$, we have*

$$\frac{|\nabla\varphi(z) - \nabla\varphi(y)|}{|z-y|^{1/1+K^*}} \leqslant C(r, K, m(\psi_y^*, 0, 1), M(\varphi, y, r)),$$

where ψ_y^* is the convex conjugate to

$$\psi_y(x) = \varphi(x+y) - \varphi(y) - \nabla\varphi(y)\cdot x.$$

Proof. As usual, let us begin considering the case $\varphi \in \mathrm{Eng}_0(n,K)$ and $y=0$. Take x with $|x| \leqslant r$, by (4.18) we get

$$\varphi(x) \leqslant M(\varphi,0,r)\left(\frac{|x|}{r}\right)^{1+1/K} \leqslant M(\varphi,0,r).$$

Next, observe that if $|\nabla\varphi(x)| \geqslant 1$, then, by (4.19) applied to φ^*,

$$m(\varphi^*,0,1)|\nabla\varphi(x)|^{1+1/K^*} \leqslant \varphi^*(\nabla\varphi(x)) \leqslant K\varphi(x) \leqslant KM(\varphi,0,r),$$

where we used (5.29) and Theorem 4 to write $\varphi^*(\nabla\varphi(x)) = \nabla\varphi(x)x - \varphi(x) \leqslant K\varphi(x)$. All this gives,

$$|\nabla\varphi(x)| \leqslant \max\left\{1, \left(\frac{KM(\varphi,0,r)}{m(\varphi^*,0,1)}\right)^{K^*/K^*+1}\right\} \doteq C_1 = C_1(\varphi,r,K).$$

Now we can apply (4.18) to φ^* with C_1 and at $\nabla\varphi(x)$ to get

$$m(\varphi^*,0,C_1)\left(\frac{|\nabla\varphi(x)|}{C_1}\right)^{1+K^*} \leqslant \varphi^*(\nabla\varphi(x)) \leqslant K\varphi(x)$$

that is,

$$|\nabla\varphi(x)| \leqslant C_1\left(\frac{K}{m(\varphi^*,0,C_1)}\right)^{1/1+K^*} \varphi(x)^{1/1+K^*}$$

and dividing by $|x|^{1/1+K^*}$,

$$\frac{|\nabla\varphi(x)|}{|x|^{1/1+K^*}} \leqslant C_1 \left(\frac{K}{m(\varphi^*,0,C_1)}\right)^{1/1+K^*} \left(\frac{\varphi(x)}{|x|}\right)^{1/1+K^*}.$$

Note that $\varphi(x)/|x| = \varphi(|x|x/|x|)/|x|$ and for any $z \in \mathbb{R}^n$ the function $t \to \varphi(tz)/t$ is increasing. Therefore, since $|x| \leqslant r$, we get

$$\frac{|\nabla\varphi(x)|}{|x|^{1/1+K^*}} \leqslant C_1 \left(\frac{K}{m(\varphi^*,0,C_1)}\right)^{1/1+K^*} \left(\frac{\varphi(rx/|x|)}{r}\right)^{1/1+K^*}$$

$$\leqslant C_1 \left(\frac{K}{m(\varphi^*,0,C_1)}\right)^{1/1+K^*} \left(\frac{M(\varphi,0,r)}{r}\right)^{1/1+K^*}. \tag{6.33}$$

To complete the proof, given $\varphi \in \mathrm{Eng}(n,K)$ and $y \in \mathbb{R}^n$, set $\psi_y(x) = \varphi(x+y) - \varphi(y) - \nabla\varphi(y)x \in \mathrm{Eng}_0(n,K)$ and $z = x + y$ to get, for $|z - y| \leqslant r$,

$$\frac{|\nabla\varphi(z) - \nabla\varphi(y)|}{|z-y|^{1/1+K^*}} \leqslant C_y \left(\frac{K}{m(\psi_y^*,0,C_y)}\right)^{1/1+K^*} \left(\frac{M(\varphi,y,r)}{r}\right)^{1/1+K^*},$$

where

$$C_y \doteq \max\left\{1, \left(\frac{KM(\varphi,y,r)}{m(\psi_y^*,0,1)}\right)^{K^*/1+K^*}\right\}.$$

Thus, $\nabla\varphi$ is in C^β with $\beta = 1/1 + K^*$ and $K^* = 2K(K+1)$. □

7. Further remarks

If the Monge–Ampère measure μ_φ satisfies the (DC)-doubling condition with constants C and α, then $\varphi \in \mathrm{Eng}(n,K)$ with

$$K = \frac{2^{n+2} w_n w_{n-1}}{\alpha_n^{n+1}} \frac{C}{(1-\alpha)^n} + 1,$$

where w_k is the volume of the k-dimensional unit ball and $\alpha_n = n^{-3/2}$. In the case $\lambda \leqslant \det D^2\varphi \leqslant \Lambda$, if we set $\alpha = 1/2$ we get $C = 2^n \Lambda/\lambda$. These constants can be easily followed up from [8].

Although we consider solutions to $\det D^2\varphi = \mu$ in $\mathbb{R}^n$, the main results in this paper can be proved (after slight modifications) for solutions to the Monge–Ampère equation in a bounded convex domain $\Omega \subset \mathbb{R}^n$.

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