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Discrete Applied Mathematics 134 (2004) 193–212
DISCRETE APPLIED MATHEMATICS
www.elsevier.com/locate/dam

Lift and project relaxations for the matching and related polytopes☆

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Received 24 July 2001; received in revised form 3 March 2003; accepted 17 March 2003

Abstract

We compare lift and project methods given by Lovász and Schrijver (the N_+ and N procedures) and by Balas, Ceria and Cornuéjols (the disjunctive procedure) when working on the matching, perfect matching and covering polytopes. When the underlying graph is the complete graph of $n=2s+1$ nodes we obtain that the disjunctive index for all problems is s^2, the N_+-index for the matching and perfect matching problems is s (extending a result by Stephen and Tunçel), the N-index for the perfect matching problem is s, and the N_+ and N indices for the covering problem and the N-index for the matching problem are strictly greater than s.

Keywords: Matching; Covering; Polyhedral combinatorics; Sequential tightening procedures

1. Introduction

Lift and project procedures have been widely used in polyhedral combinatorics. Starting from a given polyhedron $\mathscr{K}$ in $[0,1]^m$, these methods attempt to give a description of $\text{conv}(\mathscr{K}^0)$, where $\mathscr{K}^0 = \{x \in \{0,1\}^m : x \in \mathscr{K}\}$, through a finite number of "lift and project" steps. In each step the current polyhedron—initially $\mathscr{K}$—is "lifted" to a higher-dimensional space, where it is tightened, and then it is "projected" back.

☆ Supported by grants of Consejo Nacional de Investigaciones Científicas y Técnicas, Universidad Nacional del Litoral, and Universidad Nacional de Rosario (Argentina).
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0166-218X/03/$ - see front matter
doi:10.1016/S0166-218X(03)00337-8

Lovász and Schrijver [5] introduced two such procedures—N and N_+— by lifting the original polyhedron $\mathscr{K}$ to a higher dimensional space requiring about as many as the square of the original variables. Both procedures obtain $\text{conv}(\mathscr{K}^0)$ in at most m steps, but one of them—N_+—combines linear restrictions with non-linear restrictions from the cone of positive semidefinite (psd) matrices.

Balas et al. [1] presented another lift and project procedure, which we call disjunctive or BCC, requiring just about twice as many as the number of original variables in the lifting step. Although this procedure generally obtains at each step a weaker relaxation than those of Lovász and Schrijver's, it also gets $\text{conv}(\mathscr{K}^0)$ in at most m steps.

Although differences among these methods are clearly established from their definitions, it is not clear at all the role these differences play in obtaining the convex hull. Recently, Goemans and Tunçel [4] gave a detailed study of the N_+ and N procedures from a geometric perspective, but the conditions they found are somewhat hard to verify for particular problems.

We have chosen to compare the performance of these procedures on three specific problems having strong similarities, namely, the matching, perfect matching and covering (of nodes by edges) problems. In these problems the complete description by linear inequalities of the convex hulls of integer solutions is known and all of them involve "odd set constraints". Thus, these constraints become the key for comparing the performances of BCC, N_+ and N procedures.

On the other hand, although the N_+ procedure on the matching polytope has been analyzed [6], the perfect matching and covering polytopes have never been studied in this way, as far as we know.

The three procedures on the three problems present interesting differences and this is the main motivation of this paper.

In Table 1 we show our results on the minimum number of steps required by each procedure for obtaining the convex hull of feasible solutions on each problem, when the underlying graph is the complete graph of $2s+1$ nodes.

Our paper is organized as follows: in the next Section we briefly introduce the three problems we work on. In Section 3 we consider the Balas, Ceria and Cornuéjols procedure and observe the consistent behavior over the three problems shown in Table 1. The N and N_+ operators are introduced in Section 4.

In Section 5 we analyze the N_+ operator on the matching, perfect matching and covering problems. In Section 6 we do the same for the N operator. Finally in Section 7, we extend the results obtained to ranks of valid inequalities for $\text{conv}(\mathscr{K}^0)$.

Table 1
The index for the different procedures and polytopes

	BCC	N	N_+
Matching	s^2	$>s$	s
Perfect matching	s^2	s	s
Covering	s^2	$>s$	$>s$

2. The matching, perfect matching and covering polytopes

In this section we introduce further notation that will be used throughout the paper and establish basic results.

Let $G=(V,E)$ be an undirected simple graph (no self-loops and no multiple edges) with node set V and edge set E. We will think of an edge $e\in E$ as a set of two vertices and write $e=[u,v]$, where $u,v\in e$ are the endpoints of e. Mostly we use $n=|V|$, $m=|E|$ and, in order to simplify the notation, we assume that $V=\{1,2,\dots,n\}$. Again abusing the notation, we identify E with the set $\{1,2,\dots,m\}$ and think of vectors $x\in\mathbb{R}^m$ either as $x=(x_1,\dots,x_m)$ or as $x=(x_e)_{e\in E}$. For $x=(x_e)_{e\in E}\in\mathbb{R}^m$ and $F\subset E$, $x(F)$ denotes the sum $x(F)=\sum_{e\in F}x_e$. Usually, we will make no distinction between an actual subset of edges and its characteristic vector in $\{0,1\}^m$.

For a given subset U of V, $E(U)$ denotes the set of edges whose endpoints are in U, and $\delta(U)$ denotes the set of edges incident on exactly one element of U. When U reduces to a point, we set $\delta(u)=\delta(\{u\})$. If U and U' are disjoint subsets of nodes, the set of edges joining U and U' is denoted by $(U:U')$.

A *matching* on G is a subset $\mathscr{M}$ of edges such that no two of them are incident on a common node. Thus, the characteristic function on $\mathscr{M}$, defined by a 0–1 vector $x=(x_e)_{e\in E}\in\mathbb{R}^m$, satisfies the $m+n$ inequalities

$$x(\delta(u))\leqslant 1 \quad \text{for all } u\in V,$$

$$x_e\geqslant 0 \quad \text{for all } e\in E. \tag{1}$$

The inequalities in (1) describe the initial linear relaxation of the matching polytope, which we will denote by $\mathscr{K}_{\leqslant}$. The set of 0–1 vectors of $\mathscr{K}_{\leqslant}$ will be denoted by $\mathscr{K}^0_{\leqslant}$.

Edmonds [3] showed that $\mathrm{conv}(\mathscr{K}^0_{\leqslant})$ is described by inequalities (1) and the odd set inequalities

$$x(E(U))\leqslant\frac{|U|-1}{2} \tag{2}$$

for all odd subsets U of V with $|U|\geqslant 3$.

A subset $\mathscr{C}$ of E is a *covering* (of nodes by edges) if for every node $v\in V$ there exists $e\in\mathscr{C}$ incident on v. In this case, the initial relaxation $\mathscr{K}_{\geqslant}$ is described by the inequalities

$$x(\delta(u))\geqslant 1 \quad \text{for all } u\in V,$$

$$0\leqslant x_e\leqslant 1 \quad \text{for all } e\in E \tag{3}$$

and we will denote by $\mathscr{K}^0_{\geqslant}$ the set of 0–1 vectors of $\mathscr{K}_{\geqslant}$.

As shown by Balinski [2], the inequalities describing $\mathrm{conv}(\mathscr{K}^0_{\geqslant})$ are those of (3) together with the odd set inequalities

$$x(E(U)\cup\delta(U))\geqslant\frac{|U|+1}{2} \tag{4}$$

for all odd subsets U of V with $|U|\geqslant 3$.

Finally, a subset $\mathscr{P}$ of edges is a *perfect matching* if it is both a matching and a covering. The corresponding initial relaxation $\mathscr{K}_=$ is described by the inequalities

$$x(\delta(u)) = 1 \quad \text{for all } u \in V,$$

$$x_e \geqslant 0 \quad \text{for all } e \in E \tag{5}$$

and we will denote by $\mathscr{K}^0_=$ the set of 0–1 vectors of $\mathscr{K}_=$. Again, $\text{conv}(\mathscr{K}^0_=)$ can be described by inequalities (5) and either the set of odd inequalities (2) or (4), which become equivalent in this case.

Clearly, $\mathscr{K}^0_=$ is empty if n is odd. However, since our goal is to study the "performance" of each procedure, we analyze this case as well.

When there is no need to distinguish between problems, we will use $\mathscr{K}$ and $\mathscr{K}^0$ to denote the original relaxation and the set of 0–1 solutions of them.

In all three problems—matching, perfect matching and covering—the inequalities describing the convex hull of 0–1 solutions are associated to odd subsets of V. Conversely, for each problem we may associate to any odd subset of nodes $U \subset V$ a valid inequality for $\mathscr{K}^0$. We will refer to these inequalities as U-inequalities.

Another similarity between these three problems is the fact that $\mathscr{K} = \text{conv}(\mathscr{K}^0)$ if and only if the graph G is bipartite.

3. The BCC or disjunctive operator

Balas et al. [1] described a lift and project procedure for polytopes $\mathscr{K}$ of the form

$$\mathscr{K} = \{x \in \mathbb{R}^m\colon Ax \leqslant b \text{ and } 0 \leqslant x_i \leqslant 1 \text{ for all } i = 1, \ldots, m\}$$
$$= \{x \in \mathbb{R}^m\colon \tilde{A}x \leqslant \tilde{b}\}$$

as follows:

For fixed j, $1 \leqslant j \leqslant m$, the inequalities $\tilde{A}x \leqslant \tilde{b}$ are multiplied by x_j and $1 - x_j$, obtaining a system of non-linear inequalities. Then x_j^2 is replaced by x_j and products of the form $x_j x_i$ are replaced by new variables y_i for $i \neq j$, obtaining a system of linear inequalities in the variables x and y. This polytope is projected back onto the x-space, by eliminating the y variables (using non-negative linear combinations). The resulting polytope is denoted by $P_j(\mathscr{K})$.

Recalling that $\mathscr{K}^0 = \mathscr{K} \cap \{0,1\}^m$, the following result gives an alternative definition of $P_j(\mathscr{K})$:

Theorem 1 (Balas, Ceria, Cornuéjols).

$$P_j(\mathscr{K}) = \text{conv}(\{x \in \mathscr{K}\colon x_j \in \{0,1\}\}).$$

In particular, $\text{conv}(\mathscr{K}^0) \subset P_j(\mathscr{K}) \subset \mathscr{K}$.

In order to study the successive iterations, for $F = \{i_1, \ldots, i_k\} \subset \{1, \ldots, m\}$ we let

$$P_F(\mathscr{K}) = \text{conv}(\{x \in \mathscr{K}\colon x_i \in \{0,1\} \text{ for } i \in F\}).$$

In [1] it is proved that

$$P_F(\mathscr{K}) = P_{i_1}(P_{i_2}(\ldots(P_{i_k}(\mathscr{K}))\ldots))$$

and, as a consequence,

$$P_{1,\ldots,m}(\mathscr{K}) = \mathrm{conv}(\mathscr{K}^0),$$

which allows the definition of the BCC or disjunctive index of a relaxation $\mathscr{K}$ as

$$\min\{|F|\colon F \subset \{1,\ldots,m\} \text{ and } P_F(\mathscr{K}) = \mathrm{conv}(\mathscr{K}^0)\}.$$

Given a valid inequality for $\mathrm{conv}(\mathscr{K}^0)$ of the form $\alpha x \leqslant \beta$, we can also define its BCC or disjunctive rank as

$$\min\{|F|\colon F \subset \{1,\ldots,m\} \text{ and } \alpha x \leqslant \beta \text{ for all } x \in P_F(\mathscr{K})\}.$$

Trivially, the disjunctive index of $\mathscr{K}$ is the maximum among disjunctive ranks of valid inequalities for $\mathrm{conv}(\mathscr{K}^0)$.

3.1. The disjunctive index on the three problems

Throughout this section $\mathscr{K}$ and $\mathscr{K}^0$ will denote, respectively, the original relaxation and the set of 0–1 solutions of any of the three problems.

Given a general graph $G = (V,E)$, with $|V| = n$, we denote by $\gamma(G)$ the minimum number of edges that must be taken off from E in order to obtain a bipartite graph. Moreover, given $U \subset V$ we denote by G_U the subgraph of G induced by the node set U.

Recalling that if $|U| = 2r+1$, the U-inequality valid for $\mathrm{conv}(\mathscr{K}^0)$ in the matching and the perfect matching problem is

$$x(E(U)) \leqslant r$$

and the one corresponding to the covering problem is

$$x(E(U) \cup \delta(U)) \geqslant r+1,$$

we have the following upper bound for the disjunctive rank of U-inequalities:

Lemma 2. *Given $G = (V,E)$ and $U \subset V$ with $|U| = 2r+1 \leqslant n$, the disjunctive rank of the U-inequality is at most $\gamma(G_U)$.*

Proof. The proof is by induction on $|U|$. Clearly, it holds for $|U| = 3$, so let us assume it holds whenever $|U| = 2r-1$, and consider $U \subset V$ with $|U| = 2r+1$ and $F \subset E(U)$ such that $G_F = (V, E \setminus F)$ is bipartite.

Given $x \in P_F(\mathscr{K})$, if $x_a = 0$ for all $a \in F$, let (U_1, U_2) be the corresponding partition of U in G_F. It is easy to see that if $x \in \mathscr{K}_=$ or $x \in \mathscr{K}_\leqslant$ then

$$x(E(U)) \leqslant \min(|U_1|, |U_2|) \leqslant r$$

and if $x \in \mathcal{K}_{\geqslant}$ then

$$x(E(U) \cup \delta(U)) \geqslant \max(|U_1|, |U_2|) \geqslant r + 1.$$

If $x_a = 1$ for some $a \in F$, it is enough to observe that if $x \in \mathcal{K}_{\leqslant}$ or $x \in \mathcal{K}_{=}$,

$$x(E(U)) = x_a + x(E(\tilde{U})),$$

and if $x \in \mathcal{K}_{\geqslant}$ then

$$x(E(U) \cup \delta(U)) \geqslant x_a + x(E(\tilde{U}) \cup \delta(\tilde{U})),$$

where $\tilde{U} = U \setminus \{a\}$. By inductive hypothesis, the $\tilde{U}$-inequality is valid and this completes the proof. □

The main result of this section is:

Theorem 3. *If $G = K_n$ and $n = 2s + 1$, the disjunctive index of $\mathcal{K}$ is s^2.*

Proof. Since $\gamma(G) = s^2$, the previous lemma gives an upper bound.

On the other hand, given $F \subset E$ such that $|F| < s^2$, the graph $G_F = (V, E \setminus F)$ cannot be bipartite.

Let C be an odd circuit in G_F of length $2r + 1$, $r \geqslant 1$, and let $\mathcal{M}$ be a perfect matching between the nodes of V which are not in C ($\mathcal{M}$ can be void). We define $\tilde{x}$ on E by

$$\tilde{x}_e = \begin{cases} 1/2 & \text{if } e \in C, \\ 1 & \text{if } e \in \mathcal{M}, \\ 0 & \text{otherwise.} \end{cases}$$

Clearly, $\tilde{x} \in P_F(\mathcal{K})$ and

$$\tilde{x}(E) = \frac{2r+1}{2} + s - r = s + \frac{1}{2}.$$

so, $\tilde{x}$ violates the V-inequality. □

4. The N and N_+ operators, description and properties

In this section we will briefly overview the operators N and N_+ introduced by Lovász and Schrijver [5]. There, the authors work with convex cones in $\mathbb{R}^{m+1}$, homogenizing the inequalities by introducing a variable x_0.

Thus, the vectors x are now of the form $(x_0, x_1, \ldots, x_m)$, an inequality of the form

$$\sum_{i=1}^{m} a_i x_i \geqslant b$$

is translated as

$$\sum_{i=1}^{m} a_i x_i \geqslant b x_0$$

and we will usually work with vectors x satisfying

$$0 \leqslant x_i \leqslant x_0 \quad \text{for all } i = 1, \ldots, m. \tag{6}$$

Let $\mathbf{e}_i$ denote the ith unit vector of $\mathbb{R}^{m+1}$ for $i = 0, 1, \ldots, m$, and let $\mathbf{f}_i = \mathbf{e}_0 - \mathbf{e}_i$ for $i = 1, \ldots, m$. Given a convex cone $\mathscr{K}$, whose points satisfy the inequalities (6), $M(\mathscr{K})$ is defined as the cone of matrices Y in $\mathbb{R}^{(m+1)\times(m+1)}$ satisfying

(1) Y is symmetric,
(2) $\text{diag}(Y) = Y\mathbf{e}_0 \in \mathscr{K}$,
(3) $Y\mathbf{e}_i \in \mathscr{K}$ and $Y\mathbf{f}_i \in \mathscr{K}$ for $i = 1, \ldots, m$

and the cone $N(\mathscr{K})$ is defined by

$$N(\mathscr{K}) = \{Y\mathbf{e}_0 \colon\ Y \in M(\mathscr{K})\}.$$

$N_+(\mathscr{K})$ is defined in a similar way: the cone $M_+(\mathscr{K})$ is the set of those matrices in $M(\mathscr{K})$ that are also positive semidefinite, and

$$N_+(\mathscr{K}) = \{Y\mathbf{e}_0 \colon\ Y \in M_+(\mathscr{K})\}.$$

We notice that $N(\mathscr{K})$ is always a polyhedral cone, whereas, in general, $N_+(\mathscr{K})$ is not. Clearly $N_+(\mathscr{K}) \subset N(\mathscr{K})$. We also observe the monotonicity of N and N_+, in the sense that if $\mathscr{K} \subset \mathscr{K}'$ then $N(\mathscr{K}) \subset N(\mathscr{K}')$ and $N_+(\mathscr{K}) \subset N_+(\mathscr{K}')$.

If the rth iteration of N and N_+ is denoted, respectively, by $N^r(\mathscr{K})$ and $N_+^r(\mathscr{K})$, we have:

Theorem 4 (Lóvasz–Schrijver).

$$N_+^m(\mathscr{K}) = N^m(\mathscr{K}) = \text{conv}(\mathscr{K}^0),$$

where $\mathscr{K}^0$ *is the set of* 0–1 *solutions in* $\mathscr{K}$.

The N-index is defined as the minimum s such that $N^s(\mathscr{K}) = \text{conv}(\mathscr{K}^0)$, and, similarly, the N-rank of a valid inequality for $\text{conv}(\mathscr{K}^0)$ is r if it is valid for $N^r(\mathscr{K})$ and it is not valid for $N^{r-1}(\mathscr{K})$. Analogous definitions hold for the N_+-index and N_+-rank.

In order to compare the operators N and N_+ with the disjunctive operator, we will assume an embedding of

$$\{x \in \mathbb{R}^m \colon\ 0 \leqslant x_i \leqslant 1 \text{ for all } i = 1, \ldots, m\}$$

in the set

$$\{x \in \mathbb{R}^{m+1} \colon\ 0 \leqslant x_i \leqslant x_0 \text{ for all } i = 1, \ldots, m\}.$$

Defining for $i = 1, \ldots, m$,

$$\mathscr{H}_i = \{x \in \mathbb{R}^{m+1} \colon\ x_i = 0\} \quad \text{and} \quad \mathscr{G}_i = \{x \in \mathbb{R}^{m+1} \colon\ x_i = x_0\}, \tag{7}$$

the following result by Lovász and Schrijver [5] provides a comparison with the disjunctive operator:

Lemma 5. *For* $i = 1,\dots,m$,

$$\mathrm{conv}(\mathscr{K}^0) \subset N_+(\mathscr{K}) \subset N(\mathscr{K}) \subset \mathrm{conv}(\mathscr{K} \cap (\mathscr{H}_i \cup \mathscr{G}_i)) = P_i(\mathscr{K}).$$

It follows from these inclusions that:

Corollary 6. *For a given* $\mathscr{K}$, *its indices according to the operators are ordered, i.e.*

$$N_+\text{-}index \leqslant N\text{-}index \leqslant disjunctive\ index.$$

Trivially, the same relation holds for the ranks of valid inequalities.

We now state and prove some results which will be needed later.

4.1. The"glue" lemma

In the following sections we often apply the N_+ and N operators to initial relaxations where the underlying graph $G=(V,E)$ can be viewed as a "glue" between two disjoint graphs, and we will make frequent use of the following:

Lemma 7 ("Glue" Lemma). *Suppose* $G^1=(V^1,E^1)$ *and* $G^2=(V^2,E^2)$ *are two disjoint complete graphs, and let* $G=(V,E)$ *be the graph where* $V=V^1\cup V^2$ *and the edges in* E *are either in* $E^1\cup E^2$ *or have an endpoint in* V^1 *and the other in* V^2. *Let* $\mathscr{K}^1$, $\mathscr{K}^2$ *and* $\mathscr{K}$ *be the corresponding initial relaxations for any of the problems (matching, perfect matching or covering), and let* $x^k \in N(\mathscr{K}^k)$ *for* $k=1,2$, *with* $x_0^1 = x_0^2 = 1$.

If z *is defined by*

$$z_0 = 1,$$

$$z_e = \begin{cases} x_e^1 & if\ e \in E^1, \\ x_e^2 & if\ e\in E^2, \\ 0 & otherwise, \end{cases}$$

then $z \in N^r(\mathscr{K})$ *(resp.* $z\in N_+^r(\mathscr{K})$*) if and only if* $x^k \in N^r(\mathscr{K}^k)$ *(resp.* $x^k \in N_+^r(\mathscr{K}^k)$*), for* $k=1,2$.

Proof. If $z\in N(\mathscr{K})$, there exists a matrix $Y \in M(\mathscr{K})$ with $\mathrm{diag}(Y)=z$ and having the form

$$Y = \begin{bmatrix} 1 & x^{1\mathrm{T}} & x^{2\mathrm{T}} & 0 \\ x^1 & Y^1 & W & 0 \\ x^2 & W^{\mathrm{T}} & Y^2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

It is easy to prove that for $k = 1,2$ the matrix

$$Y_*^k = \begin{bmatrix} 1 & x^{k\mathrm{T}} \\ x^k & Y^k \end{bmatrix}$$

is in $M(\mathscr{K}^k)$ and so $x^k \in N(\mathscr{K}^k)$.

Conversely for $k = 1,2$, let Y_*^k be a matrix in $\mathscr{M}(\mathscr{K}^k)$ such that $\mathrm{diag}(Y_*^k) = x^k$ and

$$Y_*^k = \begin{bmatrix} 1 & x^{k\mathrm{T}} \\ x^k & Y^k \end{bmatrix}.$$

If Y is defined by

$$Y = \begin{bmatrix} 1 & x^{1\mathrm{T}} & x^{2\mathrm{T}} & 0 \\ x^1 & Y^1 & x^1 x^{2\mathrm{T}} & 0 \\ x^2 & x^2 x^{1\mathrm{T}} & Y^2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

then

(1) Y is symmetric
(2) $\mathrm{diag}(Y) = z$ and $Y\mathbf{e}_0 = z$;
(3) $Y\mathbf{e}_e \in \mathscr{K}$ and $Y\mathbf{f}_e \in \mathscr{K}$ for all $e \in V$.

Thus $Y \in \mathscr{M}(\mathscr{K})$ and therefore $z \in N(\mathscr{K})$.

When including psd restrictions, we may use the Schur complement to obtain that Y_*^k is positive semidefinite if and only if $Z^j = Y^k - x^k x^{k\mathrm{T}}$ is positive semidefinite. Also, the block diagonal matrix

$$\begin{bmatrix} Z^1 & \mathbf{0} \\ \mathbf{0} & Z^2 \end{bmatrix}$$

is positive semidefinite if and only if the matrices Z^k are positive semidefinite. This fact completes the proof when $r = 1$ and the lemma follows by induction on r. □

Due to the symmetry of the problems we are working with, points having some kind of symmetry arise naturally. We will find it particularly useful to study for $\alpha \in \mathbb{R}_+$ points which we will call *α-symmetric*: points $x \in \mathbb{R}^{m+1}$ whose coordinates are of the form

$$x_0 = 1 \quad \text{and} \quad x_i = \alpha \quad \text{for } i = 1,\ldots,m.$$

If x is α-symmetric and $x \in N^r(K)$, it is easy to prove that there is always a matrix $Y \in M^r(\mathscr{K})$, of the form

$$Y_{00} = 1$$

$$Y_{0e} = Y_{e0} = a \quad \text{for } e \in E,$$

$$Y_{ee'} = \begin{cases} a & \text{if } |e \cap e'| = 2, \\ b & \text{if } |e \cap e'| = 1, \\ c & \text{if } |e \cap e'| = 0 \end{cases} \tag{8}$$

with $a = \alpha$.

In the following section, we study in detail some properties of this matrix.

4.2. Properties of the matrix Y for given a, b and c

We proceed to describe the eigenvalues and eigenspaces of the matrix Y introduced previously (Eq. (8)) when a, b and c are given and $G = K_n$ with $n \geqslant 3$.

If $\mathbf{j} \in \mathbb{R}^m$ is the vector consisting of all 1's, and writing

$$Y = \begin{bmatrix} 1 & a\mathbf{j}^{\mathrm{T}} \\ a\mathbf{j} & M \end{bmatrix},$$

we first study the eigenvalues of the matrix $M \in \mathbb{R}^{m\times m}$.

M could be written more conveniently as

$$M = (a - c)I + (b - c)A + c\mathbf{J},$$

where I is the identity matrix, A is the adjacency matrix in the line graph $L(K_n)$, and $\mathbf{J}$ is the matrix with all 1's.

It is well known that the matrix A has eigenvalues

$$\lambda_A = 2n - 4, \quad \mu_A = n - 4 \quad \text{and} \quad v_A = -2$$

with multiplicities 1, $n - 1$ and $m - n = n(n - 3)/2$, respectively. Furthermore, the eigenspace corresponding to λ_A is spanned by $\mathbf{j}$.

Thus, since $\mathbf{Jj} = m\mathbf{j}$, $\mathbf{j}$ is an eigenvector of M with eigenvalue

$$\lambda_M = (a - c) + \lambda_A(b - c) + mc = a + 2(n - 2)b + \frac{(n - 2)(n - 3)}{2}\, c.$$

The other eigenspaces of A are orthogonal to $\mathbf{j}$, and therefore they are contained in the null space of $\mathbf{J}$ and are eigenspaces of M as well. Hence, the other eigenvalues of M are

$$\mu_M = (a - c) + \mu_A(b - c)$$

and

$$v_M = (a - c) + v_A(b - c).$$

Turning back to the matrix $Y \in \mathbb{R}^{(m+1)\times(m+1)}$, we observe that an eigenvector $v = (v_1, \ldots, v_m)$ of M orthogonal to $\mathbf{j}$ gives rise to an eigenvector of the form $(0, v)=$

$(0, v_1, \dots, v_m)$ of Y with the same eigenvalue, i.e.

$$\mu = \mu_M = a + (n-4)b - (n-3)c$$

and

$$v = v_M = a - 2b + c$$

are eigenvalues of Y with multiplicities $n-1$ and $n(n-3)/2$, respectively.

Looking now for eigenvectors of Y of the form $(1, t\mathbf{j}^{\mathrm{T}}) = (1, t, \dots, t)$ with eigenvalue λ, we are led to the equations

$$1 + amt = 1 + (a\mathbf{j}^{\mathrm{T}})(t\mathbf{j}) = \lambda \quad \text{and} \quad (a + \lambda_M t)\mathbf{j} = a\mathbf{j} + M(t\mathbf{j}) = \lambda t\mathbf{j},$$

which for $a \neq 0$ have solutions

$$\lambda_{\pm} = \frac{\lambda_M + 1 \pm \sqrt{\Delta}}{2} \quad \text{and} \quad t_{\pm} = \frac{\lambda_M - 1 \pm \sqrt{\Delta}}{n(n-1)a},$$

where $\Delta = (\lambda_M - 1)^2 + 2n(n-1)a^2$.

Summing up, we have:

Theorem 8. *Using the previous notation, if $G = K_n$, $n \geqslant 3$, and $a \neq 0$, the eigenvalues of the matrix Y defined in (8) are*

(1) λ_+ *with multiplicity* 1,
(2) λ_- *with multiplicity* 1,
(3) μ *with multiplicity* $n-1$, *and*
(4) v *with multiplicity* $n(n-3)/2$ (*for* $n \geqslant 4$).

If $a = 0$, the eigenvalues in (1) and (2) should be replaced by 1 and $2(n-2)b + (n-2)(n-3)c/2$. [1]

As a consequence, we have:

Corollary 9. *If $G = \mathscr{K}_n$, $n \geqslant 4$, then Y is positive semidefinite if and only if the following three conditions hold simultaneously*:

(1) $a + c \geqslant 2b$,
(2) $a + (n-4)b \geqslant (n-3)c$, *and*
(3) $2a + 4(n-2)b + (n-2)(n-3)c \geqslant n(n-1)a^2$.

5. The N_+-index on the three problems

In this section $G = K_n$, with $n = 2s + 1$.

Since the disjunctive index for any of the problems is 1 for $n = 3$, by Corollary 6 the N- and N_+-indices are also 1. So, in what follows we consider $n \geqslant 5$.

[1] Of course, it is possible that some of the eigenvalues coincide for particular choices of a, b, c.

Working on the matching problem, Stephen and Tunçel proved [6] that the N_+-index of $\mathscr{K}_{\leqslant}$ is s, i.e.

$$N_+^s(\mathscr{K}_{\leqslant}) = \operatorname{conv}(\mathscr{K}^0_{\leqslant}) \subsetneq N_+^{s-1}(\mathscr{K}_{\leqslant}).$$

More specifically, they proved that the $\frac{1}{2s}$-symmetric point is in $N_+^{s-1}(\mathscr{K}_{\leqslant})$.

For the perfect matching problem we observe that the $\frac{1}{2s}$-symmetric point is in $N_+^{s-1}(\mathscr{K}_{=})$ and so the N_+-index of $\mathscr{K}_{=}$ is at least s. By the symmetry of $\mathscr{K}_{=}$, if any x were in $N_+^s(\mathscr{K}_{=})$, the $\frac{1}{2s}$-symmetric point would also be in $N_+^s(\mathscr{K}_{=})$, but since this point violates the V-inequality we have

$$N_+^s(\mathscr{K}_{=}) = \operatorname{conv}(\mathscr{K}^0_{=}) = \emptyset \subsetneq N_+^{s-1}(\mathscr{K}_{=}).$$

Formally,

Lemma 10. *The N_+-index on the perfect matching problem is s.*

Since $\mathscr{K}_{=}$ is a subset of $\mathscr{K}_{\geqslant}$, by monotonicity of the N_+ operator, the N_+-index on the covering problem is at least s, but we will show that we have a little more:

Theorem 11. *If $G = K_n$ and $n = 2s+1 \geqslant 5$, then the N_+-index on the covering problem is strictly greater than s.*

Before the proof of this theorem we observe that if there exists $x' \in N_+^r(\mathscr{K})$, with $x'_0 > 0$ and satisfying $x'(E) < \frac{n+1}{2} x'_0$, we may find a symmetric point satisfying the same conditions.

Let us set $n' = n+2$, $m' = n'(n'-1)/2$, $K_{n'} = (V', E')$, and let us denote by $\mathscr{K}$ and $\mathscr{K}'$ the corresponding initial relaxations of the form $\mathscr{K}_{\geqslant}$.

Lemma 12. *Suppose that for some $n = 2s+1$ and r, there exists $x \in N_+^r(\mathscr{K})$ α-symmetric such that*

$$x(E) < \frac{n+1}{2}. \tag{9}$$

Then there exists $\xi \in N_+^{r+1}(\mathscr{K}')$ α'-symmetric satisfying

$$\xi(E') < \frac{n'+1}{2}.$$

Proof. Suppose $V = \{1, 2, \ldots, n\}$ and $V' = V \cup \{n+1, n+2\}$. Fixing $e' = [n+1, n+2]$, by the "Glue" Lemma (Lemma 7) and taking averages, we see that the vector $\zeta \in \mathbb{R}^{m'+1}$ with coordinates

$$\zeta_0 = 1,$$

$$\zeta_e = \begin{cases} 0 & \text{if } e = e', \\ \beta & \text{if } |e \cap e'| = 1, \\ \gamma & \text{if } e \cap e' = \emptyset, \end{cases}$$

where

$$\beta = \frac{1+(n-1)\alpha}{2n} \quad \text{and} \quad \gamma = \frac{n-2}{n}\,\alpha$$

is in $N_+^r(\mathscr{K}')$. Using symmetry, we may take any $e' \in E'$.

We now obtain a matrix Y' of the form given in Corollary 9, with n replaced by $n' = n+2$, and coefficients a, b and c which are functions of α, in such a way that the matrix Y' is in $M_+^{r+1}(\mathscr{K}')$.

To this end, we ensure $Y'\mathbf{e}_e \in \operatorname{conv}(\mathscr{K}'^0)$ for $e \in E'$ by requiring

$$a + 2nb + \frac{n(n-1)}{2}\,c = \frac{n+3}{2}\,a \tag{10}$$

and we ensure $Y'\mathbf{f}_e \in M_+^r(\mathscr{K}_{\geqslant})$, by requiring

$$a - b = (1-a)\beta \quad \text{and} \quad a - c = (1-a)\gamma. \tag{11}$$

Eqs. (10) and (11) have solutions

$$a = \frac{1}{d}\,(2 + n(n-1)\alpha),$$

$$b = \frac{1}{2nd}\,(1 + 2n - n^2 + (n-1)^3\alpha),$$

$$c = \frac{1}{nd}\,(2n - (n^2 - 5n + 2)\alpha),$$

where $d = (n+1)^2 + (n-1)n\alpha$.

Since $x \in \mathscr{K}_{\geqslant}$, we have $\alpha \geqslant 1/(n-1)$. Since x also satisfies inequality (9), we have $\alpha < (n+1)/(n(n-1))$. Now it is simple to check that a, b and c are non-negative and, by Corollary 9, Y' is positive semidefinite. Hence taking $\alpha' = a$ the lemma is proved. □

The proof of Theorem 11 will be completed once we show there is a point $x \in N_+^2(\mathscr{K}_{\geqslant})$ such that $x(E) < s+1$.

Lemma 13. *For $n = 5$ the 13/48-symmetric point is in $N_+^2(\mathscr{K}_{\geqslant})$.*

Proof. The proof consists of several simple verifications and we shall skip many details.

Let x be the 13/48-symmetric point for $n = 5$. We first observe that $x = Y\mathbf{e}_0$, where Y has the form of Corollary 9 with

$$a = 13/48, \quad b = 1/36 \quad \text{and} \quad c = 1/8.$$

For $e \in E$, $Y\mathbf{e}_e$ is a multiple of the vector ξ with coordinates

$$\xi_0 = 1,$$

$$\xi_{e'} = \begin{cases} 1 & \text{if } e = e', \\ 4/39 & \text{if } |e \cap e'| = 1, \\ 6/13 & \text{otherwise,} \end{cases}$$

so that it is in conv($\mathscr{K}^0_{\geqslant}$). Moreover, $Y\mathbf{f}_e$ is a multiple of the vector ζ with coordinates

$$\zeta_0 = 1,$$

$$\zeta_{e'} = \begin{cases} 0 & \text{if } e = e', \\ 1/3 & \text{if } |e \cap e'| = 1, \\ 1/5 & \text{otherwise.} \end{cases}$$

In turn, ζ is the projection of the matrix Y' defined by

$$Y'\mathbf{e}_0 = (Y')^{\mathrm{T}}\,\mathbf{e}_0 = \zeta,$$

and for $e', e'' \in E$,

$$Y'_{e'e''} = Y'_{e''e'} = \begin{cases} 1/3 & \text{if } e' = e'' \text{ and } |e \cap e'| = 1, \\ 1/5 & \text{if } e' = e'' \text{ and } e \cap e' = \emptyset, \\ 1/6 & \text{if } |e \cap e'| = |e \cap e''| = 1 \text{ and } e' \cap e'' = \emptyset, \\ 2/15 & \text{if } |e \cap e'| = 1 \text{ and } e' \cap e'' = e \cap e'' = \emptyset, \\ 1/30 & \text{if } |e \cap e'| = |e' \cap e''| = 1 \text{ and } e \cap e'' = \emptyset, \\ 0 & \text{otherwise,} \end{cases}$$

so that $\zeta \in N_+(\mathscr{K}_{\geqslant})$ and $x \in N_+^2(\mathscr{K}_{\geqslant})$. □

The behavior of the N_+ operator on the covering problem is worse than on the matching or perfect matching problems. However, the exact N_+ index on the covering problem is still an open problem.

It is worth noting that Goemans and Tunçel [4] found upper bounds for the N_+-ranks of valid inequalities:

Theorem 14 (Goemans–Tunçel). *Let $a = (a_0, a_1, \ldots, a_m)$ be a vector with $a_0 \geqslant 0$ and $a_i \leqslant 0$ for all $i = 1, \ldots, m$, and set $S = \{i : a_i < 0\}$. For $I \subset \{1, \ldots, m\}$ let us set*

$$\mathscr{K}(I) = \{x \in \mathscr{K} : x_i = x_0 \text{ for all } i \in I\}.$$

If $a^{\mathrm{T}}x \geqslant 0$ is valid for $\mathscr{K}(J)$ for all $J \subset S$ such that

(1) $|J| = r$, *or*
(2) $|J| < r$ *and* $a_0 + \sum_{j \in J} a_j \leqslant 0$,

then $a^{\mathrm{T}}x \geqslant 0$ is valid for $N_+^r(\mathscr{K})$.

The bounds given by this Theorem are exact for the U-inequalities on the matching and perfect matching problems, but for the covering problem the bound is worse than the disjunctive rank.

We now turn to the study of the N-index.

6. The N-index on the three problems

We assume again that $G = K_n$, $n = 2s + 1$.

On the covering problem, as a natural consequence of Theorem 11 and Corollary 6, we have $\text{conv}(\mathscr{K}^0_{\geqslant}) \subsetneq N^s(\mathscr{K}_{\geqslant})$. In terms of indices we can only say that the N-index on the covering problem is also greater than s.

We will see that when dealing with the perfect matching problem, the N_+ and N indices coincide, whereas they are different for the matching problem.

Let us work firstly on the perfect matching problem. Setting $N^0(\mathscr{K}) = \mathscr{K}$ for any $\mathscr{K}$, we have:

Lemma 15. *For every $s \geqslant 1$, $N^{s-1}(\mathscr{K}_=) = \{x^s\}$ where x^s is the $\frac{1}{2s}$-symmetric point.*

Proof. As we have already pointed out, $x^s \in N_+^{s-1}(\mathscr{K}_=)$ for $s > 1$, and trivially $x^s \in N^{s-1}(\mathscr{K}_=)$. To show there is no other point in $N^{s-1}(\mathscr{K}_=)$, we use induction on s. If $s = 1$, it is easy to see that the equations defining $\mathscr{K}_=$ determine a unique point.

For larger values of s, suppose $x \in N^{s-1}(\mathscr{K}_=)$, so that there exists a matrix $Y \in M^{s-1}(\mathscr{K}_=)$ such that $Y\mathbf{e}_0 = x$ and $Y\mathbf{e}_e \in M^{s-2}(\mathscr{K}_=) \cap \mathscr{G}_e$[2] for any $e \in E$. Thus $(Y\mathbf{e}_e)_{e'} = x_e/(n-3)$ for any $e' \in E$ with $e \cap e' = \emptyset$. Since $(Y\mathbf{e}_e)_{e'} = (Y\mathbf{e}_{e'})_e$, we must have $x_e = x_{e'}$ for all $e, e' \in E$ with $e \cap e' = \emptyset$. By transitivity (since $n \geqslant 5$), the value of x_e is constant. $\square$

From the previous lemma we have

$$N^s(\mathscr{K}_=) = \text{conv}(\mathscr{K}^0_=) = \emptyset \subsetneq N^{s-1}(\mathscr{K}_=),$$

and therefore

Corollary 16. *If $n = 2s + 1$, the N-index of the perfect matching problem is s.*

We now turn to the matching problem. We will prove that the N-index is at least $s + 1$ which is the best-known lower bound for this index.

We first observe that the bounds obtained by Lovász and Schrijver [5] could be carried over to the N-index for the matching problem, since a matching in a graph can be viewed as a stable set in the corresponding line graph. However, since $\mathscr{K}_{\leqslant}$ is contained in the edge relaxation, only the upper bounds could be used.

Lemma 17. *Suppose $2s + 1 = n$ and $\gamma \leqslant (2s-1)/(4s^2-2)$. If $\bar{x} \in \mathbb{R}_+^{m+1}$ is γ-symmetric then $\bar{x} \in N^s(\mathscr{K}_{\leqslant})$.*

Proof. Since $(1, 0, 0, \ldots, 0) \in \mathscr{K}_{\leqslant}$, by convexity it is enough to take $\gamma = (2s-1)/(4s^2-2)$, and we may write

$$\gamma = \frac{1}{2s+1} + \frac{\delta}{2s+1}$$

with $\delta = 1/(4s^2-2)$.

[2] The definition of $\mathscr{G}_e$ is given in Eq. (7) of Section 4.

We observe that since $\gamma \leqslant 1/(2s)$, $\bar{x} \in \mathscr{K}_{\leqslant}$, and in order to show that $\bar{x} \in N^s(\mathscr{K}_{\leqslant})$, we will show that the matrix Y defined by

$$Y\mathbf{e}_0 = Y^{\mathrm{T}}\mathbf{e}_0 = \bar{x},$$

and for $e, e' \in E$,

$$Y_{ee'} = \begin{cases} \gamma & \text{if } e = e', \\ \delta & \text{if } e \cap e' = \emptyset, \\ 0 & \text{otherwise,} \end{cases}$$

satisfies $Y \in M^s(\mathscr{K}_{\leqslant})$.

For fixed $e \in E$ we have

$$\sum_{e' \in E: e \cap e' = \emptyset} Y_{ee'} = (2s-1)(s-1)\delta = \gamma(s-1),$$

so that the "Glue" Lemma (Lemma 7) implies $Y\mathbf{e}_e \in \mathrm{conv}(\mathscr{K}^0_{\leqslant})$.

On the other hand, if $z = Y\mathbf{f}_e$, we have

$$z_0 = 1 - \gamma,$$

$$z_{e'} = \begin{cases} 0 & \text{if } e = e', \\ \gamma - \delta & \text{if } e \cap e' = \emptyset, \\ \gamma & \text{otherwise} \end{cases}$$

and we may write $z = (1-\gamma)(\lambda x^1 + (1-\lambda)x^2)$, where

$$\lambda = \frac{2s(\gamma - \delta)}{1 - \gamma} < 1,$$

$$x^1_0 = x^2_0 = 1,$$

$$x^1_{e'} = \begin{cases} 0 & \text{if } e = e', \\ 1/(2s) & \text{otherwise,} \end{cases}$$

$$x^2_{e'} = \begin{cases} 0 & \text{if } e = e' \text{ or } e \cap e' = \emptyset, \\ 1/(2s-1) & \text{otherwise.} \end{cases}$$

A simple check shows that $x^1 \in N^{s-1}(\mathscr{K}_{\leqslant})$ and $x^2 \in \mathrm{conv}(\mathscr{K}^0_{\leqslant})$ (if $n = 2s+1 \geqslant 5$). Therefore $z \in N^{s-1}(\mathscr{K}_{\leqslant})$ and $Y \in M^s(\mathscr{K}_{\leqslant})$, proving the lemma. □

As a consequence, if γ is such that

$$\frac{1}{2s+1} < \gamma \leqslant \frac{2s-1}{4s^2-2},$$

then $\bar{x} \notin \mathrm{conv}(\mathscr{K}^0_{\leqslant})$. It follows that:

Corollary 18. *If $|V| = 2s+1$, the N-index of the matching problem is greater than s.*

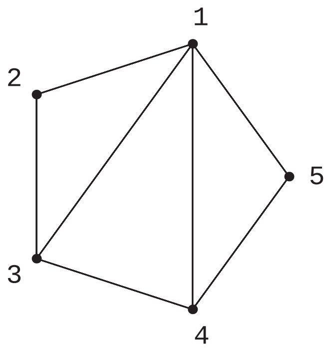

Fig. 1. Graph where the N and disjunctive indices for the matching problem do not coincide.

The N-index is still unknown for a general K_n. Nevertheless for $G=K_5$, we found that the $\frac{6}{29}$-symmetric point is in $N^3(\mathscr{K}_{\leqslant})$. So in this case the N-index is 4 and coincides with the disjunctive index. However, these indices need not coincide, as can be verified for the graph of Fig. 1, where the disjunctive index is 2 whereas the N-index is 1.

7. The rank of the U-inequalities

In this section we will study the N_+, the N and the disjunctive rank of the U-inequalities defining the 0–1 solutions on the three problems, when $G=K_n$ with n even or odd.

To this end, for $U \subset V$ we will denote by G_U the subgraph induced by the nodes in U and by $\mathscr{K}_U$ the corresponding initial relaxation on G_U.

Let us first make some remarks on constraints (2) and (4) of Section 2 when dealing with the perfect matching and covering problems when n is even.

In the perfect matching problem, the U-inequality and the $V \setminus U$-inequality are equivalent. Thus, for the perfect matching problem we will only consider U's with $3 \leqslant |U| \leqslant n/2$.

In the covering problem, if $|U| = n-1$ the U-inequality can be obtained from the initial relaxation. Thus, for the covering problem we restrict our attention to U's with $3 \leqslant |U| < n-1$.

The next lemma is valid for the three operators, and so we denote by T any of them:

Lemma 19. *Let $G=K_n$ and $U \subset V$ with $|U| = 2s+1 < n$. Then the T-rank of the U-inequality is at least the T-index of $\mathscr{K}_U$.*

Proof. By Corollary 3.3 in [4] it is known that if F is any face of $[0,1]^m$ then

$$T^r(\mathscr{K} \cap F) = T^r(\mathscr{K}) \cap F$$

and so it is enough to prove that given a point x in $\mathscr{K}_U$ which violates the U-inequality, we can find a point y in $\mathscr{K}$ which also violates the U-inequality. The point y may be defined as follows:

(1) For the matching problem we set $y_e = x_e$ for $e \in E(U)$ and $y_e = 0$, otherwise.
(2) For the perfect matching problem:
   (a) If n is odd, $|V \setminus U|$ is even and we may find a perfect matching $\mathscr{M}'$ on $E(V \setminus U)$. In this case we set

$$y_e = \begin{cases} x_e & \text{if } e \in E(U), \\ 1 & \text{if } e \in \mathscr{M}', \\ 0 & \text{otherwise.} \end{cases}$$

   (b) If n is even, since we are assuming $|U| \leqslant n/2$, we take $U' \subset V \setminus U$ such that $|U| = |U'|$. We may find a copy of x in the graph $G_{U'}$ and a perfect matching $\mathscr{M}'$ between the nodes in $V \setminus (U \cup U')$, and define y by

$$y_e = \begin{cases} x_e & \text{if } e \in E(U') \cup E(U), \\ 1 & \text{if } e \in \mathscr{M}', \\ 0 & \text{otherwise.} \end{cases}$$

(3) For the covering problem, we set $y_e = x_e$ for $e \in E(U)$ and $y_e = 1$ for $e \in E(V \setminus U)$ (recall that if n is even we are assuming $|U| < n - 1$). □

From the lemma above and Lemma 2, it follows that

Lemma 20. *On the three problems, if $|U| = 2s+1$ the disjunctive rank of the U-inequality is s^2.*

When considering the N and N_+ procedures, we cannot improve the previous result for the covering problem.

However, for the perfect matching problem it is also true that the T-rank of the U-inequality and the T-index of $\mathscr{K}_U$ always coincide. This follows from Lemma 19 and the N_+-rank of the matching problem if $T = N_+$, but we will need a special treatment for the N operator.

Theorem 21. *Let $G = K_n$ and $\mathscr{K} = \mathscr{K}_=$. If $U \subset V$ with $|U| = 2s + 1$, then the N-rank of the U-inequality is s.*

Proof. By Lemma 19 we know the N-rank is at least s, so we need an upper bound.
Let us consider first some special cases:

- If $|U| = n$ the theorem has already been proved (Corollary 16).
- If $n = |U| + 2 = 2(s+1) + 1$, we know by Lemma 15 that $N^s(\mathscr{K}_=) = \{x^{s+1}\}$ where $x_e^{s+1} = 1/2(s+1)$, and that this point satisfies the U-inequality, so the N-rank is at most s.

Now, for the remaining cases, i.e n odd and $|U| < n-2$ or n even and $|U| \leqslant n/2$, we will show any $x \in N^s(\mathscr{K}_=)$ satisfies the U-inequalities.

This is certainly true if $s=1$ (and $n \geqslant 5$) since the disjunctive rank is 1 for $s=1$ (Lemma 20). Using induction, let us consider the case $s>1$ assuming the result is true for $s-1$.

If U is fixed and $|U| = 2s+1$, we set $U' = V \setminus U$. We may assume that $|U'| \geqslant 4$, since if n is odd we have studied the case when $|U'| = 2$ and if n is even we may assume $|U'| = |U| \geqslant 5$.

Suppose now that $x \in N^s(\mathscr{K}_=)$ and $Y \in M^s(\mathscr{K}_=)$ is such that $Y\mathbf{e}_0 = x$. Using the symmetry of the problem, by taking averages we may assume:

(1) There exist constants $\alpha > 0$, $\beta \geqslant 0$ and $\gamma \geqslant 0$, so that x has coordinates

$$x_0 = 1,$$

$$x_e = \begin{cases} \alpha & \text{if } e \in E(U), \\ \beta & \text{if } e \in E(U'), \\ \gamma & \text{if } e \in (U : U'). \end{cases}$$

(2) There exist non-negative constants $y_{\alpha\alpha}$, $y_{\alpha\beta}$, $y_{\alpha\gamma}$, $y_{\beta\beta}$, $y_{\beta\gamma}$ and $y_{\gamma\gamma}$, so that the coordinates $Y_{ee'}$ for $e, e' \in E$ are described by
   (a) If $e \in E(U)$:

$$Y_{ee'} = \begin{cases} \alpha & \text{if } e = e', \\ y_{\alpha\alpha} & \text{if } e' \in E(U \setminus \{e\}), \\ y_{\alpha\gamma} & \text{if } e' \in (U \setminus \{e\} : U'), \\ y_{\alpha\beta} & \text{if } e' \in E(U'). \end{cases}$$

   (b) If $e \in E(U')$:

$$Y_{ee'} = \begin{cases} \beta & \text{if } e = e', \\ y_{\alpha\beta} & \text{if } e' \in E(U), \\ y_{\beta\beta} & \text{if } e' \in (E(U') \setminus \{e\}), \\ y_{\beta\gamma} & \text{if } e' \in (U : U' \setminus \{e\}). \end{cases}$$

   (c) If $e = [u, v]$, with $u \in U$ and $v \in U'$:

$$Y_{ee'} = \begin{cases} \gamma & \text{if } e = e', \\ y_{\alpha\gamma} & \text{if } e' \in E(U \setminus \{u\}), \\ y_{\beta\gamma} & \text{if } e' \in E(U' \setminus \{v\}), \\ y_{\gamma\gamma} & \text{if } e' \in (U \setminus \{u\} : U' \setminus \{v\}). \end{cases}$$

From assumption (1) and since $x = Y\mathbf{e}_0 \in N^s(\mathscr{K}_=)$, we see that proving that x satisfies the inequality for U is equivalent to proving

$$\alpha \leqslant \frac{1}{2s+1}.$$

Also from assumption (1) we have

$$2s\alpha + |U'|\gamma = 1. \tag{12}$$

On the other hand, from assumption (2a) and $Y\mathbf{e}_e \in N^{s-1}(\mathscr{K}_=)$, we must have (by the induction hypothesis)

$$y_{\alpha\alpha} \leqslant \frac{\alpha}{2s-1}, \tag{13}$$

and, from assumptions (2a) and (2c),

$$(2s-2)\, y_{\alpha\alpha} + |U'|\, y_{\alpha\gamma} = \alpha,$$

$$(2s-1)\, y_{\alpha\gamma} + (|U'|-1) y_{\gamma\gamma} = \gamma. \tag{14}$$

Solving Eqs. (12) and (14) for γ, $y_{\alpha\gamma}$ and $y_{\gamma\gamma}$ in terms of α and $y_{\alpha\gamma}$, we obtain

$$y_{\gamma\gamma} = \frac{1-(4s-1)\alpha + 2(s-1)(2s-1)y_{\alpha\alpha}}{|U'|(|U'|-1)}. \tag{15}$$

Since $y_{\gamma\gamma} \geqslant 0$, using bound (13) we obtain from Eq. (15)

$$1-(2s+1)\alpha \geqslant 0,$$

proving the theorem. □

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