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The disjunctive procedure and blocker duality ☆

Néstor E. Aguilera[a], Mariana S. Escalante[b], Graciela L. Nasini[c, *]

[a] CONICET and UNL, IMAL/INTEC, Güemes 3450, 3000 Santa Fe, Argentina

[b] CONICET and UNR, Depto. de Matemática Facultad de Ciencias Exactas, Ingeniería y Agrimensura, Av. Pellegrini 250, 2000 Rosario, Argentina

[c] UNR, Depto. de Matemática Facultad de Ciencias Exactas, Ingeniería y Agrimensura, Av. Pellegrini 250, 2000 Rosario, Argentina

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Abstract

In this paper we relate two rather different branches of polyhedral theory in linear optimization problems: the blocking type polyhedra and the disjunctive procedure of Balas et al. For this purpose, we define a disjunctive procedure over blocking type polyhedra with vertices in $[0,1]^n$, study its properties, and analyze its behavior under blocker duality. We compare the indices of the procedure over a pair of blocking clutter polyhedra, obtaining that they coincide.

Keywords: Disjunctive procedure; Blocker duality; Clutter; Blocking type polyhedra

1. Introduction

Polyhedral theory is a well established set of tools for studying linear integer optimization problems. In this paper we find connections between two rather different branches of it: the blocking type polyhedra, as defined in [3], and the disjunctive procedure of Balas et al. described in [1].

Let us recall some definitions and results that we will need.

For $U \subset \mathbb{R}^n$, we denote by $\mathrm{conv}(U)$ the convex hull of the elements of U, and by $\mathrm{cone}(U)$ the (nonnegative) cone generated by U.

Given a polyhedron $\mathscr{K} \subset \mathbb{R}^n$ we indicate by $V(\mathscr{K})$ the set of its vertices, and by $\mathscr{K}^*$ the polyhedron

$$\mathscr{K}^* = \mathrm{conv}(\mathscr{K} \cap \mathbf{Z}^n).$$

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* Corresponding author.

E-mail addresses: mariana@fceia.unr.edu.ar (M.S. Escalante), nasini@fceia.unr.edu.ar (G.L. Nasini).

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A polyhedron $\mathcal{K}$ in $\mathbb{R}^n$ is of *blocking type* if $\mathcal{K} \subset \mathbb{R}^n_+$ and if $y \geqslant x \in \mathcal{K}$ implies $y \in \mathcal{K}$. We furthermore notice that if $\mathcal{K}$ is of blocking type, then $\mathcal{K}^*$ is also of blocking type.

In the following lemma we summarize some alternative definitions for blocking type polyhedra.

Lemma 1.1. *Given a polyhedron $\mathcal{K} \neq \emptyset$, $\mathcal{K} \subset \mathbb{R}^n_+$, the following are equivalent*:

(i) *$\mathcal{K}$ is a blocking type polyhedron*;
(ii) *$\{e_1, \ldots, e_n\}$, the canonical basis of $\mathbb{R}^n$, is the normalized extreme ray set of $\mathcal{K}$*;
(iii) *$\mathcal{K} = \{x \in \mathbb{R}^n_+ : Ax \geqslant 1\}$ where A is a nonnegative matrix, with no zero rows.*

If $\mathcal{K}$ is a blocking type polyhedron, then the *blocker* of $\mathcal{K}$, $\mathcal{K}^B$, is defined by

$$\mathcal{K}^B = \{\pi \in \mathbb{R}^n_+ : \pi \cdot x \geqslant 1 \text{ for all } x \in \mathcal{K}\}.$$

It is known [3] that

Theorem 1.2. *Let $\mathcal{K} = \{x \in \mathbb{R}^n_+ : Ax \geqslant 1\}$ with A a nonnegative matrix with no zero rows and let B be a $|V(\mathcal{K})| \times n$ matrix whose rows are the vertices of $\mathcal{K}$. Then*

(i) $\mathcal{K}^B = \{\pi \in \mathbb{R}^n_+ : B\pi \geqslant 1\}$,
(ii) $(\mathcal{K}^B)^B = \mathcal{K}$.

In the following, we will refer to $\mathcal{K}$ and $\mathcal{K}^B$ as a blocking pair of polyhedra.

A particularly interesting blocker relationship arises when $\mathcal{K} = \{x \in \mathbb{R}^n_+ : Ax \geqslant 1\}$ and the rows of A are the characteristic vectors of a clutter. Recall that, for $N = \{1, \ldots, n\}$, a *clutter* is a set of noncomparable subsets of N and, if $\mathcal{F}$ is a clutter of N, the *blocking clutter* of $\mathcal{F}$, denoted by $B(\mathcal{F})$, is the clutter of the minimal subsets H of N satisfying

$$H \cap F \neq \emptyset \quad \text{for all } F \in \mathcal{F}.$$

Since for any clutter $\mathcal{F}$, $B(B(\mathcal{F})) = \mathcal{F}$ [2] when either $\mathcal{H} = B(\mathcal{F})$ or $\mathcal{F} = B(\mathcal{H})$, we may simply refer to $\mathcal{F}$ and $\mathcal{H}$ as a pair of *blocking clutters*. In this case, denoting by A and B (respectively) the matrices whose rows are the characteristic vectors of the elements of $\mathcal{F}$ and $\mathcal{H}$, it is proved in [3] that if

$$Q = \{x \in \mathbb{R}^n_+ : Ax \geqslant 1\} \quad \text{and} \quad Q_B = \{x \in \mathbb{R}^n_+ : Bx \geqslant 1\},$$

then Q^* and Q_B (similarly Q and Q_B^*) are a blocking pair of polyhedra. These relationships can be summarized in the following diagram:

$$\begin{array}{ccc} & \text{blocker} & \\ Q & \leftrightarrow & Q^B = (Q_B)^* \\ \text{convex hull} \downarrow & & \uparrow \text{convex hull} \\ Q^* & \leftrightarrow & (Q^*)^B = Q_B \\ & \text{blocker} & \end{array}$$

Let us now introduce the disjunctive procedure of Balas et al. developed in [1] defined on polytopes of the form

$$\mathscr{K} = \{x \in \mathbb{R}^n_+ \colon Ax \geqslant b,\ x_i \leqslant 1 \quad \text{for } i = 1, \ldots, n\} = \{x \in \mathbb{R}^n \colon \tilde{A}x \geqslant \tilde{b}\}.$$

This lift and project procedure can be described as follows:

For fixed j, $1 \leqslant j \leqslant n$, the inequalities $\tilde{A}x \geqslant \tilde{b}$ are multiplied by x_j and $1 - x_j$, obtaining a system of, in general, nonlinear inequalities. Then, x_j^2 is replaced by x_j and products of the form $x_i x_j$ are replaced by new variables y_i for $i \neq j$, obtaining a system of linear inequalities in the variables x and y. The polytope $M_j(\mathscr{K})$, defined by this system of linear inequalities, is projected back onto the x-space, by eliminating the y variables. The resulting polytope is denoted by $P_j(\mathscr{K})$.

The following result, proved in [1], gives an alternative definition for P_j, much more geometrical in nature, and is central to our discussion:

Theorem 1.3. *For any* $j \in \{1, \ldots, n\}$,

$$P_j(\mathscr{K}) = \operatorname{conv}(\mathscr{K} \cap \{x \in \mathbb{R}^n_+ \colon x_j \in \{0, 1\}\}).$$

In particular, $\mathscr{K} \cap \{0,1\}^n \subset P_j(\mathscr{K}) \subset \mathscr{K}$.

Defining for $F = \{i_1, \ldots, i_k\} \subset \{1, \ldots, n\}$

$$P_F(\mathscr{K}) = \operatorname{conv}(\{x \in \mathscr{K} \colon x_i \in \{0,1\} \text{ for } i \in F\}),$$

in [1] it was proved that

$$P_F(\mathscr{K}) = P_{i_1}(P_{i_2}(\ldots(P_{i_k}(\mathscr{K})),$$

and in particular,

$$P_{\{1,\ldots,n\}}(\mathscr{K}) = \mathscr{K}^*.$$

Our aim is to relate blocker/blocking duality and the disjunctive procedure of Balas et al. Since this procedure is defined for polytopes in $[0,1]^n$, we need to define an extension for particular blocking polyhedra.

We will say that $\mathscr{K}$ is a $[0,1]$ *blocking type polyhedron* if $\mathscr{K}$ is a blocking type polyhedron whose vertices belong to $[0,1]^n$.

Given a $[0,1]$ blocking type polyhedron $\mathscr{K}$, we define the disjunctive procedure over $\mathscr{K}$ as

$$\bar{P}_j(\mathscr{K}) = P_j(\mathscr{K}_0) + \text{cone}(\{e_1,\ldots,e_n\}),$$

where $\mathscr{K}_0 = \mathscr{K} \cap [0,1]^n$ and $j \in \{1,\ldots,n\}$.

In Section 2 we will work on this new operator, proving that it preserves the properties of the one defined for bounded polyhedra by Balas et al. In particular, we will find that starting from a $[0,1]$ blocking type polyhedron $\mathscr{K}$, after the successive applications of $\bar{P}_j$, for $j = 1,\ldots,n$, we obtain $\mathscr{K}^*$. Therefore, we can define the *disjunctive index* of $\mathscr{K}$ as the minimum number of iterations of $\bar{P}_j$ needed so as to find the convex hull of the integer points in $\mathscr{K}$.

It is well known that if $\mathscr{K}$ is defined by a clutter, then all the vertices of $\mathscr{K}$ are 0–1 if and only if all the vertices of $\mathscr{K}_B$ are 0–1, i.e. the disjunctive index of $\mathscr{K}$ is zero if and only if the disjunctive index of $\mathscr{K}_B$ is zero.

Our main goal in this paper is to compare the disjunctive index of blocking pair of polyhedra.

For this purpose, we will need to analyze $\bar{P}_j([\bar{P}_j(\mathscr{K})]^B)$, thus we will need to work on a family of polyhedra where it can be computed.

Therefore, we define the family $\mathscr{S}$ of polyhedra $\mathscr{K}$ such that $\mathscr{K}$ and $\mathscr{K}^B$ are $[0,1]$ blocking type polyhedra.

One of the main results of the paper, proved in Section 3, is

Theorem 1.4. *If $\mathscr{K} \in \mathscr{S}$ and $F \subset \{1,\ldots,n\}$ then*

$$\bar{P}_F([\bar{P}_F(\mathscr{K})]^B) \subset \mathscr{K}^B.$$

We also give examples where the inclusion is strict and show that equality holds when $\mathscr{K}$ is defined by a clutter.

Finally, we are able to present the main result of the paper whose proof can be found at the end of Section 3.

Theorem 1.5. *Given a pair of blocking clutters their corresponding polyhedra have the same disjunctive index.*

2. Extending P_j to blocking polyhedra

The purpose of this section is to define an extension of the procedure of Balas et al. [1] for blocking type polyhedra.

If $\mathscr{K} \subset \mathbb{R}^n_+$ is a blocking type polyhedron then, by (1.1),

$$\mathscr{K} = \tilde{\mathscr{K}} + \text{cone}(\{e_1,\ldots,e_n\}),$$

where $\tilde{\mathscr{K}}$ is a polytope such that $V(\mathscr{K}) \subset \tilde{\mathscr{K}} \subset \mathscr{K}$.

If $\tilde{\mathscr{K}} \subset [0,1]^n$, a natural extension, $\tilde{P}_j$, of P_j is

$$\tilde{P}_j(\mathscr{K}) = P_j(\tilde{\mathscr{K}}) + \text{cone}(\{e_1,\ldots,e_n\}).$$

However, since we would like to keep the elements of $\mathscr{K}^*$ through the successive applications of the procedure, we will have to restrict our choice of $\tilde{\mathscr{K}}$. Consider for example

$$\mathscr{K} = \{x \in \mathbb{R}^2_+ \colon 2x_1 + x_2 \geqslant 1\}.$$

In this case we could take

$$\mathscr{K} = \tilde{\mathscr{K}} + \text{cone}(\{e_1, e_2\}),$$

where

$$\tilde{\mathscr{K}} = \{x \in \mathbb{R}^2_+ \colon 2x_1 + x_2 = 1\}$$

and then notice that applying $\tilde{P}_1$, the vertex $(1,0)$ is lost, so it is not useful as a relaxation of $\mathscr{K}^*$.

In the following, for any $\mathscr{K}$, we define $\mathscr{K}_0 = \mathscr{K} \cap [0,1]^n$.

As already mentioned in Section 1, we have

Definition 2.1. Given a $[0,1]$ blocking type polyhedron $\mathscr{K}$, we define the disjunctive procedure over $\mathscr{K}$ as

$$\bar{P}_j(\mathscr{K}) = P_j(\mathscr{K}_0) + \text{cone}(\{e_1,\ldots,e_n\}).$$

It follows directly from the definition of $\bar{P}_j$ that if $\mathscr{K}$ is a $[0,1]$ blocking type polyhedron then $\bar{P}_j(\mathscr{K})$ is a blocking type polyhedron.

The following lemma is the key for proving that the procedure $\bar{P}_j$ preserves the basic properties of P_j.

Lemma 2.2. *If $\mathscr{K}$ is a $[0,1]$ blocking type polyhedron, then for any j*

$$(\bar{P}_j(\mathscr{K}))_0 = P_j(\mathscr{K}_0).$$

Proof. Since $P_j(\mathscr{K}_0) \subset \bar{P}_j(\mathscr{K})$ and $P_j(\mathscr{K}_0) \subset [0,1]^n$, we must have $P_j(\mathscr{K}_0) \subset (\bar{P}_j(\mathscr{K}))_0$. We need to show that $(\bar{P}_j(\mathscr{K}))_0 \subset P_j(\mathscr{K}_0)$.

To this end, consider $x' \in (\bar{P}_j(\mathscr{K}))_0$. Then $x' \in [0,1]^n$ and, by definition of $\bar{P}_j(\mathscr{K})$,

$$x' = x + \sum_{k=1}^{n} \tau_k e_k$$

for some $x \in P_j(\mathscr{K}_0)$ and $\tau \in \mathbb{R}^n_+$. In order to show that $x' \in P_j(\mathscr{K}_0)$ we study the case $x' = x + \lambda e_i$ for $i \neq j$ and $\lambda > 0$ (the case $i = j$ is easier.)

Since $x \in P_j(\mathscr{K}_0)$, by Theorem 1.3 we can write

$$x' = \alpha x^0 + (1-\alpha)x^1 + \lambda e_i,$$

with $x^0 \in \mathscr{K}_0$, $x^0_j = 0$, $x^1 \in \mathscr{K}_0$, $x^1_j = 1$ and consider only $0 < \alpha < 1$.

Since $x'_i \leqslant 1$, it is possible to take s and t such that

$$\max\,(0, \lambda - (1-\alpha)(1-x'_i)) \leqslant \alpha t \leqslant \min\,(\lambda, (1-x^0_i)\alpha),$$
$$s = \frac{\lambda - \alpha t}{1-\alpha}.$$

Thus

$$x' = \alpha(x^0 + te_i) + (1-\alpha)(x^1 + se_i)$$

is a convex combination of elements in $P_j(\mathscr{K}_0)$. □

Defining for $F = \{i_1, \ldots, i_k\} \subset \{1, \ldots, n\}$

$$\bar{P}_F(\mathscr{K}) = P_F(\mathscr{K}) + \mathrm{cone}(\{e_1, \ldots, e_n\}),$$

we have the following

Corollary 2.3. *If* $F = \{i_1, \ldots, i_k\} \subset \{1, \ldots, n\}$ *then*

(i) $\bar{P}_F(\mathscr{K}) = \bar{P}_{i_1}(\bar{P}_{i_2}(\ldots(\bar{P}_{i_k}(\mathscr{K})),$
(ii) $\mathscr{K}^* \subset \bar{P}_F(\mathscr{K}) \subset \mathscr{K}$,
(iii) $\bar{P}_{\{1,\ldots,n\}}(\mathscr{K}) = \mathscr{K}^*$.

Proof. Follows directly from Lemma 2.2 and the properties of the disjunctive procedure given in Section 1. □

Let us give a description of $\bar{P}_j(\mathscr{K})$ in terms of $\mathscr{K} \cap \{x\colon x_j = 0\}$ and $\mathscr{K} \cap \{x\colon x_j = 1\}$.

Lemma 2.4. *If* $\mathscr{K}^0_j = \mathscr{K} \cap \{x\colon x_j = 0\}$, $\mathscr{K}^1_j = \mathscr{K} \cap \{x\colon x_j = 1\}$ *and* $\bar{\mathscr{K}}^1_j = \mathscr{K} \cap \{x\colon x_j \geqslant 1\}$, *then*

$$\bar{P}_j(\mathscr{K}) = \mathrm{conv}(\mathscr{K}^0_j \cup \bar{\mathscr{K}}^1_j) \tag{2.1}$$

$$= \mathrm{conv}(\mathscr{K}^0_j \cup \mathscr{K}^1_j) + \mathrm{cone}(\{e_j\}). \tag{2.2}$$

Proof. Let $x \in \bar{P}_j(\mathscr{K})$. By Theorem 1.3, we can write

$$x = \alpha x^1 + (1-\alpha)x^0 + \sum_{i=1}^{n} \lambda_i e_i$$

with $x^0 \in \{x \in \mathscr{K}_0\colon x_j = 0\}$, $x^1 \in \{x \in \mathscr{K}_0\colon x_j = 1\}$ and $0 \leqslant \alpha \leqslant 1$.

Obviously, $x \in \mathrm{conv}(\mathscr{K}^0_j \cup \bar{\mathscr{K}}^1_j)$ if $\alpha = 0$. On the other hand, if $\alpha \neq 0$ then

$$x = \alpha x^1 + (1-\alpha)x^0 + \sum_{i=1}^{n} \lambda_i e_i = \alpha\left(x^1 + \sum_{i=1}^{n} \frac{\lambda_i}{\alpha} e_i\right) + (1-\alpha)x^0$$

and since $x^1 + \sum_{i=1}^{n}(\lambda_i/\alpha)e_i \in \bar{\mathscr{K}}^1_j$ and $x^0 \in \mathscr{K}^0_j$, it follows that $x \in \mathrm{conv}(\mathscr{K}^0_j \cup \bar{\mathscr{K}}^1_j)$.

Now let

$$x = \alpha z + (1-\alpha)w$$

with $z \in \bar{\mathscr{K}}^1_j$, $w \in \mathscr{K}^0_j$ and $0 \leqslant \alpha \leqslant 1$. Since, in particular, $z, w \in \mathscr{K}$,

$$z = z^0 + \sum_{i=1}^{n} \beta_i e_i \quad \text{and} \quad w = w^0 + \sum_{i=1}^{n} \gamma_i e_i,$$

where $\gamma, \beta \in \mathbb{R}^n_+$, and $z^0, w^0 \in \mathscr{K}_0$ are such that

$$z_j = z^0_j + \beta_j = 1 + \delta \geqslant 1 \quad \text{and} \quad w_j = w^0_j + \gamma_j = 0$$

with $\delta \geqslant 0$. Note that in this case $\beta_j - \delta = 1 - z^0_j \geqslant 0$ and we can write

$$\begin{aligned} x &= \alpha \left(z^0 + \sum_{i=1}^{n} \beta_i e_i \right) + (1-\alpha) \left(w^0 + \sum_{i=1}^{n} \gamma_i e_i \right) \\ &= \alpha (z^0 + (\beta_j - \delta) e_j) + (1-\alpha) w^0 + \alpha \delta e_j + \sum_{i=1,\, i \neq j}^{n} (\alpha \beta_i + (1-\alpha)\gamma_i) e_i. \end{aligned}$$

Since

$$(z^0 + (\beta_j - \delta) e_j) \in \{ y \in \mathscr{K}_0 \colon y_j = 1 \} \quad \text{and} \quad w^0 \in \{ y \in \mathscr{K}_0 \colon y_j = 0 \},$$

we must have

$$x \in P_j(\mathscr{K}_0) + \text{cone}(\{e_1, \ldots, e_n\}).$$

We omit the proof of (2.2) since it uses the same techniques used for proving (2.1) □

The following lemma gives a characterization of the valid inequalities for $\mathscr{K}$ and it will be useful in the following section.

Lemma 2.5. *Let $\mathscr{K}$ be a $[0,1]$ blocking type polyhedron such that*

$$\mathscr{K} = \{ x \in \mathbb{R}^n_+ \colon a^k \cdot x \geqslant 1 \ for\ k = 1, \ldots, m \}$$

with $a^k \in \mathbb{R}^n_+$, and let $\mathscr{K}^0_j = \mathscr{K} \cap \{x \colon x_j = 0\}$ and $\bar{\mathscr{K}}^1_j = \mathscr{K} \cap \{x \colon x_j \geqslant 1\}$.

If $a \cdot x \geqslant 1$ is a valid inequality for $\mathscr{K}^0_j$ with nonnegative coefficients, there exist $\mu \in \mathbb{R}^m_+$ and $\lambda_i \geqslant 0$, $i \neq j$, such that

$$a_i = \sum_{k=1}^{m} \mu_k a^k_i + \lambda_i \quad for\ i \neq j$$

with

$$\sum_{k=1}^{m} \mu_k = 1 \quad and \quad \sum_{k=1}^{m} \mu_k a^k_j + \lambda_j \geqslant 0.$$

Similarly, if $a \cdot x \geqslant 1$ is a valid inequality for $\bar{\mathscr{K}}_j^1$, with nonnegative coefficients, there exist $\eta \in \mathbb{R}_+^m$ and $\gamma \in \mathbb{R}_+^n$ such that

$$a_i = \sum_{k=1}^{m} \eta_k a_i^k + \gamma_i \quad \textit{for all } i$$

with

$$\sum_{k=1}^{m} \eta_k + \gamma_j = 1.$$

Proof. The proof follows easily from a general theorem on valid inequalities for polyhedra described by inequalities (see e.g. [4, pp. 208, Part 2, Chap. 1]). □

We close this section with two illustrative remarks on this new operator.

Remark 2.6. In order to compare the descriptions by linear inequalities of $P_j(\mathscr{K}_0)$ and $\bar{P}_j(\mathscr{K})$, suppose that

$$a^i \cdot x \geqslant c^i \quad \text{for } i = 1, \ldots, k,$$

are the inequalities we need to add to a description of $\mathscr{K}_0$ in order to obtain $P_j(\mathscr{K}_0)$, i.e.

$$P_j(\mathscr{K}_0) = \mathscr{K}_0 \cap \{x\colon a^i \cdot x \geqslant c^i \text{ for } i = 1, \ldots, k\}.$$

Since $\mathscr{K}_0 = \mathscr{K} \cap [0,1]^n$ it would be natural to relate $R = \mathscr{K} \cap \{x\colon a^i \cdot x \geqslant c^i \text{ for } i = 1, \ldots, k\}$ and $\bar{P}_j(\mathscr{K})$. Nevertheless, it can be shown by simple examples that it may not be true that $R = \bar{P}_j(\mathscr{K})$.

Remark 2.7. Following the description of the P_j procedure in Section 1, the $\bar{P}_j$ procedure may be described as the projection of a higher dimensional polyhedron defined as

$$\bar{M}_j(\mathscr{K}) = \{(x, y, z) : (y, z) \in M_j(\mathscr{K}_0), x \geqslant y\}.$$

On the other hand, since $\bar{P}_j(\mathscr{K})$ may be also characterized as

$$\bar{P}_j(\mathscr{K}) = \text{proy}_x\{(x, y)\colon y \in P_j(\mathscr{K}_0),\ y \leqslant x\},$$

the Fourier–Motzkin procedure allows us to obtain a description by linear inequalities of $\bar{P}_j(\mathscr{K})$ in terms of a given one of $P_j(\mathscr{K}_0)$.

Thus, we have the following result, whose proof we omit:

Lemma 2.8. *Let $\mathscr{K}$ be a $[0,1]$ blocking type polyhedra such that $P_j(\mathscr{K}_0)$ is defined by the inequalities*

$$a^k \cdot y \geqslant 1 \qquad \textit{for } k = 1, \ldots, m,$$

$$0 \leqslant y_i \leqslant 1 \qquad \textit{for } i = 1, \ldots, n$$

with $a^k \in \mathbb{R}^n_+$. Then, $\bar{P}_j(\mathscr{K})$ is defined by inequalities of the form

$$\sum_{i\in N_p'} a_i^k x_i \geqslant 1 - \sum_{i\in N_p} a_i^k$$

for all $N_p \subset \{1,\ldots,n\}$ such that $|N_p| = p$, for $p = 0,\ldots,n-1$ and $N_p' = \{1,\ldots,n\}\backslash N_p$, and the nonnegativity constraints.

3. On special [0, 1] blocking type polyhedra

The purpose of the remaining part of the paper is to study the relationship between the disjunctive procedure over particular blocking type polyhedra and their blockers.

If $\mathscr{K}$ is a $[0,1]$ blocking type polyhedron, we know that $\bar{P}_j(\mathscr{K}) \subset \mathscr{K}$ and therefore

$$[\bar{P}_j(\mathscr{K})]^B \supset \mathscr{K}^B,$$

so it is natural to ask whether there are any inclusions between $\mathscr{K}^B$ and

$$\bar{P}_j([\bar{P}_j(\mathscr{K})]^B)$$

when it exists.

We can easily check with an example that even though $\bar{P}_j(\mathscr{K})$ is a $[0,1]$ blocking type polyhedron, its blocker is not necessarily of this type.

Applying Theorem 1.2 to

$$\bar{P}_j(\mathscr{K}) = \{x \in \mathbb{R}^n_+ \colon Ax \geqslant 1\},$$

we see that A is the matrix whose rows are the vertices of $(\bar{P}_j(\mathscr{K}))^B$. Then, if we want $(\bar{P}_j(\mathscr{K}))^B$ to be a $[0,1]$ blocking type we should have the coefficients of the matrix A in $[0,1]$, arriving to the following

Definition 3.1. A polyhedron $\mathscr{K}$ is in the class $\mathscr{S}$ if $\mathscr{K}$ and $\mathscr{K}^B$ are $[0,1]$ blocking type polyhedron.

For this family of polyhedra we have the following result which is the key for stating Theorem 3.3.

Theorem 3.2. *If $\mathscr{K} \in \mathscr{S}$ then $\bar{P}_j(\mathscr{K}) \in \mathscr{S}$.*

Proof. In order to obtain that $\bar{P}_j(\mathscr{K}) \in \mathscr{S}$ we have to show that any valid inequality for $\bar{P}_j(\mathscr{K})$ is dominated by a valid inequality whose coefficients belong to $[0,1]$.

Now suppose that $\mathscr{K} = \{x \in \mathbb{R}^n_+ \colon a^k \cdot x \geqslant 1,\ k = 1,\ldots,m\}$ where $a^k \in [0,1]^n$ for all $k = 1,\ldots,m$. If $a^* \cdot x \geqslant 1$ is a valid inequality for $\bar{P}_j(\mathscr{K})$ with $a^* \in \mathbb{R}^n$, then, by Lemma 2.4, it is a valid inequality for $\mathscr{K}_j^0 = \mathscr{K} \cap \{x\colon x_j = 0\}$ and $\bar{\mathscr{K}}_j^1 = \mathscr{K} \cap \{x\colon x_j \geqslant 1\}$.

If $\mathscr{K}_j^0 \neq \emptyset$, since a^* has nonnegative components, by Lemma 2.5 we can rewrite $a^* \cdot x \geqslant 1$ as

$$\sum_{i=1}^{n}\sum_{k=1}^{m}(\mu_k a_i^k + \lambda_i)x_i \geqslant 1$$

with $\mu_k \geqslant 0$, $\lambda_i \geqslant 0$ for $i \neq j$, $\lambda_j \in \mathbb{R}$, $\sum_{k=1}^{n} \mu_k a_j^k + \lambda_j \geqslant 0$ and $\sum_{k=1}^{n} \mu_k = 1$.

Hence, the inequality $a^* \cdot x \geqslant 1$ is dominated by

$$\sum_{i=1,\, i\neq j}^{n} \sum_{k=1}^{m} \mu_k a_i^k x_i \geqslant 1,$$

a valid inequality for $\bar{P}_j(\mathscr{K})$ with coefficients in $[0,1]$.

On the other hand, if $\mathscr{K}_j^0 = \emptyset$, we can easily show that $\mathscr{K} = \bar{\mathscr{K}}_j^1$. In fact, it is obvious that $\bar{\mathscr{K}}_j^1 \subset \mathscr{K}$. Since the extreme rays of $\mathscr{K}$ are $\{e_1,\ldots,e_n\}$ and $\mathscr{K} \subset \mathbb{R}_+^n$, in a description of $\mathscr{K}$ by inequalities without redundancies there will be exactly one of them of the form

$$bx_j \geqslant c$$

with $c=0$ or $c=1$ and $0 < b \leqslant 1$. Since there are no vertices with $x_j = 0$ ($\mathscr{K}_j^0 = \emptyset$), we must have $c=1$. Finally, the point $(1,\ldots,1) \in \mathscr{K}$, and therefore $b=1$. □

Now we can prove one of the main results of the paper.

Theorem 3.3. *If* $\mathscr{K} \in \mathscr{S}$ *then*

$$\bar{P}_F([\bar{P}_F(\mathscr{K})]^B) \subset \mathscr{K}^B$$

for any set $F \subset \{1,\ldots,n\}$.

Proof. We begin by proving the result for $F = \{j\} \subset \{1,\ldots,n\}$, that is,

$$\bar{P}_j([\bar{P}_j(\mathscr{K})]^B) \subset \mathscr{K}^B. \tag{3.1}$$

By Lemma 2.4,

$$\bar{P}_j([\bar{P}_j(\mathscr{K})]^B) = \mathrm{conv}(\{a \in [\bar{P}_j(\mathscr{K})]^B\colon a_j \in \{0,1\}\}) + \mathrm{cone}(\{e_j\}),$$

so it will be enough to prove that if $a \cdot x \geqslant 1$ is a valid inequality for $\bar{P}_j(\mathscr{K})$ such that $a_j = 0$ or $a_j = 1$, then it is a valid inequality for $\mathscr{K}$.

If $a \in [\bar{P}_j(\mathscr{K})]^B$ and $a_j = 0$ then, for any $x \in \mathscr{K}$,

$$a \cdot x = a \cdot (x_1,\ldots,x_n) = a \cdot (x_1, x_2, \ldots, x_{j-1}, 1, x_{j+1}, \ldots, x_n) \geqslant 1,$$

and therefore $a \cdot x \geqslant 1$ is a valid inequality for $\mathscr{K}$.

Suppose now $a \in [\bar{P}_j(\mathscr{K})]^B$ with $a_j = 1$, and let $\mathscr{K}_j^0$ and $\bar{\mathscr{K}}_j^1$ be as in Lemma 2.4.

If $\mathscr{K}_j^0 \neq \emptyset$, it is possible to write

$$1 = a_j = \sum_{k=1}^{m} \mu_k a_j^k + \lambda \quad \text{and} \quad 1 = \sum_{k=1}^{m} \mu_k, \tag{3.2}$$

with $\mu_k \in \mathbb{R}^m_+$ and $\lambda \in \mathbb{R}$. Since $\mathscr{K} \in \mathscr{S}$, we have $a^k \in [0,1]^n$ for all k, and we conclude, from (3.2), that $\lambda \geqslant 0$. Therefore, the inequality $a \cdot x \geqslant 1$ is dominated by a nonnegative combination of valid inequalities for $\mathscr{K}$.

If $\mathscr{K}^0_j = \emptyset$, we saw in the proof of the Theorem 3.2 that $\mathscr{K} = \bar{\mathscr{K}}^1_j$, and therefore

$$[\bar{P}_j(\mathscr{K})]^B = [\bar{\mathscr{K}}^1_j]^B = \mathscr{K}^B \quad \text{and} \quad \bar{P}_j([\bar{P}_j(\mathscr{K})]^B) \subset \mathscr{K}^B.$$

Now, for $p \geqslant 2$, let us consider $F = \{i_1, i_2, \ldots, i_p\} \subset \{1, \ldots, n\}$.

Since $\mathscr{K} \in \mathscr{S}$, by Theorem 3.2 we must have $\bar{P}_T(\mathscr{K}) \in \mathscr{S}$ for any $T \subset \{1, \ldots, n\}$.

Applying $\bar{P}_{i_p}$ to $[\bar{P}_F(\mathscr{K})]^B$ and using (3.1), we have

$$\bar{P}_{i_p}([\bar{P}_F(\mathscr{K})]^B) = \bar{P}_{i_p}([\bar{P}_{i_p}(\bar{P}_{F\setminus\{i_p\}}(\mathscr{K}))]^B) \subset [\bar{P}_{F\setminus\{i_p\}}(\mathscr{K})]^B.$$

By monotonicity of the procedure and using recursion, we obtain

$$\bar{P}_F([\bar{P}_F(\mathscr{K})]^B) \subset \mathscr{K}^B. \qquad \square$$

In Theorem 3.3, in order to compute $\bar{P}_j([\bar{P}_j(\mathscr{K})]^B)$ we could have relaxed the condition $\mathscr{K} \in \mathscr{S}$ by $\bar{P}_j(\mathscr{K}) \in \mathscr{S}$. However, in this case, the relationship stated in that theorem is not always true as the following example shows.

Example 3.4. Let us consider

$$\mathscr{K} = \{x \in \mathbb{R}^2_+ : 2x_1 + x_2 \geqslant 1, x_1 + 2x_2 \geqslant 1\},$$

then

$$\bar{P}_1(\mathscr{K}) = \{x \in \mathbb{R}^2_+ : x_1 + x_2 \geqslant 1\} = \bar{P}_2(\mathscr{K}).$$

Since $V(\bar{P}_1(\mathscr{K})) = \{(0,1),(1,0)\}$,

$$[\bar{P}_1(\mathscr{K})]^B = \{\pi \in \mathbb{R}^2_+ : \pi_1 \geqslant 1, \pi_2 \geqslant 1\}$$

and therefore

$$\bar{P}_1([\bar{P}_1(\mathscr{K})]^B) = [\bar{P}_1(\mathscr{K})]^B.$$

On the other hand, the blocker of $\mathscr{K}$ is

$$\mathscr{K}^B = \{\pi \in \mathbb{R}^2_+ : \tfrac{1}{3}\pi_1 + \tfrac{1}{3}\pi_2 \geqslant 1, \pi_1 \geqslant 1, \pi_2 \geqslant 1\},$$

so that

$$\bar{P}_1([\bar{P}_1(\mathscr{K})]^B) \not\subset \mathscr{K}^B.$$

In the following example, we show that the inclusion in Theorem 3.3 may be strict.

Example 3.5. Let A be the matrix

$$A=\begin{pmatrix} 1/2 & 1/2 & 1/2 & 0 \\ 1/2 & 1/2 & 0 & 1/2 \\ 1/2 & 0 & 1/2 & 1/2 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

and consider the polyhedron $\mathscr{K}=\{x\in\mathbb{R}^4_+ \colon Ax\geqslant 1\}$.

It is easy to check that

$$\bar{P}_2([\bar{P}_2(\mathscr{K})]^B)\subsetneq \mathscr{K}^B.$$

Note that if Q is defined by a clutter (see Section 1), then $Q\in\mathscr{S}$. Moreover, in the following lemma we prove that for this family of polyhedra the inclusion in Theorem 3.3 is actually a set equality.

Lemma 3.6. *If Q is defined by a clutter,*

$$\bar{P}_F([\bar{P}_F(Q)]^B)=Q^B,$$

for any $F\subset\{1,\dots,n\}$.

Proof. By Theorem 3.3 we only need to show that $Q^B\subset\bar{P}_F([\bar{P}_F(Q)]^B)$. Since $\bar{P}_F(Q)\subset Q$,

$$Q^B\subset[\bar{P}_F(Q)]^B.$$

Now $Q^B=(Q_B)^*$, and therefore, $\bar{P}_F(Q^B)=Q^B$. Finally, applying $\bar{P}_F$ to (3), the monotonicity of $\bar{P}_j$ implies

$$Q^B=\bar{P}_F(Q^B)\subset\bar{P}_F([\bar{P}_F(Q)]^B). \qquad \square$$

This lemma leads us naturally to what in our view is the most interesting result of the paper.

Theorem 3.7. *Given a pair of blocking clutters their corresponding polyhedra have the same disjunctive index.*

Proof. Let Q and Q_B the polyhedra defined by a pair of blocking clutters. Since we can interchange the roles of Q and Q_B, we only need to prove that if $\bar{P}_R(Q)=Q^*$ for some $R=\{i_1,\dots,i_p\}\subset N$, then

$$\bar{P}_R(Q_B)=Q_B^*.$$

But, since $Q_B = (Q^*)^B$,

$$\bar{P}_R(Q_B) = \bar{P}_R((Q^*)^B),$$

so that, by hypothesis,

$$\bar{P}_R(Q_B) = \bar{P}_R([\bar{P}_R(Q)]^B).$$

The result now follows by using Lemma 3.6 and that $Q^B = Q_B^*$. □

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