RESTRICTED WEAK TYPE INEQUALITIES FOR CONVOLUTION MAXIMAL OPERATORS IN WEIGHTED L^p SPACES

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Abstract

Let $\varphi : \mathbb{R} \to [0, \infty)$ be an integrable function such that $\varphi\chi_{(-\infty,0)} = 0$ and φ is decreasing in $(0, \infty)$. Let $\tau_h f(x) = f(x - h)$, with $h \in \mathbb{R} \setminus \{0\}$ and $f_R(x) = (1/R) f(x/R)$, with $R > 0$. In this paper we study the pair of weights (u, v) such that the operators $M_{\tau_h\varphi} f(x) = \sup_{R>0} |f| * [\tau_h\varphi]_R(x)$ are of restricted weak type (p, p) with respect to (u, v), $1 \leqslant p < \infty$. As particular cases, these operators include some maximal operators related to Cesàro convergence. We also characterize those functions φ for which $M_{\tau_h\varphi}$ is of (restricted) weak type (p, p) with respect to the Lebesgue measure. Unlike the case of the Cesàro maximal operators, it follows from the characterization that the interval of those p such that $M_{\tau_h\varphi}$ is of weak type (p, p) can be left-closed, $[p_0, \infty]$, or left-open, $(p_0, \infty]$, without having restricted weak type (p_0, p_0).

1. Introduction

Let φ be a non-negative integrable function on the real line and let us denote by $\varphi_R(x) = (1/R)\varphi(x/R)$, $R > 0$. It is well known that for all $f \in L^p(\mathbb{R})$, $1 \leqslant p < \infty$, the convolutions $f * \varphi_R$ converge in $L^p(\mathbb{R})$ to $(\int \varphi) f$ as R goes to zero. The study of the almost everywhere (a.e.) convergence of $f * \varphi_R$ is harder and we need to add certain assumptions on φ. For instance, if φ has support in $[0, \infty)$ and it is decreasing in $(0, \infty)$ then $f * \varphi_R$ converges to $(\int \varphi) f$ a.e. as $R \to 0^+$, $f \in L^p(\mathbb{R})$, $1 \leqslant p < \infty$. This result follows from the fact that the maximal operator

$$M_\varphi f(x) = \sup_{R>0} |f| * \varphi_R$$

is of weak type (p, p), $1 \leqslant p < \infty$.

Let us consider now the maximal operator associated with the translation $\tau_h\varphi(x) = \varphi(x - h)$, that is,

$$M_{\tau_h\varphi} f(x) = \sup_{R>0} |f| * [\tau_h\varphi]_R(x), \qquad h \in \mathbb{R}\setminus\{0\}.$$

We note the following facts.

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(a) The support of $\tau_h\varphi$ is not necessarily contained in $[0,\infty)$; if it is, then $\tau_h\varphi$ is not necessarily bounded for a decreasing function in $(0,\infty)$.

(b) Examples of such operators are

$$M_\alpha^- f(x) = \sup_{R>0} \frac{1}{R} \int_{x-2R}^{x-R} |f(y)| \left(\frac{x-R-y}{R}\right)^\alpha dy, \quad -1<\alpha<0,$$

and

$$\widetilde{M}_\alpha^+ f(x) = \sup_{R>0} \frac{1}{R} \int_{x}^{x+R} |f(y)| \left(\frac{x+R-y}{R}\right)^\alpha dy, \quad -1<\alpha<0.$$

These operators coincide with $M_{\tau_h\varphi}$ with $h=1$ and $h=-1$ respectively where $\varphi(t) = t^\alpha \chi_{(0,1]}(t)$. These operators are related to the Cesàro convergence of singular integrals and Cesàro continuity [**1**,**5**]. It is known that they are of restricted weak type $(1/(1+\alpha), 1/(1+\alpha))$, of strong type (p,p) for $p>1/(1+\alpha)$ and they are not of weak type $(1/(1+\alpha), 1/(1+\alpha))$; see, for instance, [**5**].

(c) Weighted weak type inequalities for M_α^- and $\widetilde{M}_\alpha^+$ have been studied in [**3**, **8**].

It follows from (a) that one cannot apply the classical theory to study the boundedness of $M_{\tau_h\varphi}$ nor, consequently, the a.e. convergence of $f*[\tau_h\varphi]_R$ (however, the convolutions $f*[\tau_h\varphi]_R$ converge in $L^p(\mathbb{R})$, $1\leqslant p<\infty$, since $\tau_h\varphi$ is integrable). On the other hand, (b) and (c) lead to us to study the following questions.

(1) Is the behaviour of the maximal operator $M_{\tau_h\varphi}$ with respect to the Lebesgue measure analogous to that of M_α^- and $\widetilde{M}_\alpha^+$? More precisely, is it always true that for all φ there exists $p_0\geqslant 1$ such that $M_{\tau_h\varphi}$ is of weak type (p,p) if and only if $p>p_0$ and $M_{\tau_h\varphi}$ is of restricted weak type (p_0,p_0)?

(2) Weighted weak type inequalities for $M_{\tau_h\varphi}$ in L^p-spaces.

(3) Restricted weak type inequalities for $M_{\tau_h\varphi}$ in weighted L^p-spaces.

As for the first question, we shall see in this paper that the behaviour of $M_{\tau_h\varphi}$ is not always analogous to that of M_α^- and $\widetilde{M}_\alpha^+$. We shall demonstrate with examples of φ that the following situations are possible for $p_0\geqslant 1$:

(i) $M_{\tau_h\varphi}$ is of weak type (p,p) if and only if $p>p_0$ and $M_{\tau_h\varphi}$ is not of restricted weak type (p_0,p_0);

(ii) $M_{\tau_h\varphi}$ is of weak type (p,p) if and only if $p>p_0$ and $M_{\tau_h\varphi}$ is of restricted weak type (p,p), and this is the case if and only if $p\geqslant p_0$ (the case of M_α^- and $\widetilde{M}_\alpha^+$);

(iii) $M_{\tau_h\varphi}$ is of weak type (p,p) if and only if $p\geqslant p_0$, and $M_{\tau_h\varphi}$ is not of restricted weak type (p,p) if $p<p_0$.

These examples will be given in Corollary 2.13, the proof of which uses answers to (2) and (3).

Since question (2) was studied in [**2**], we shall investigate only (3) in this paper. We present our results in the next section. For the sake of completeness, we start with the results from [**2**].

Throughout the paper, p' stands for the conjugate exponent of p, $1 < p < \infty$, and the letter C means a positive constant that may change from one line to another. If E is a Lebesgue measurable set then $|E|$ stands for the Lebesgue measure of E. Given a positive measurable function u, the maximal operator M_u^- is defined by

$$M_u^- f(x) = \sup_{a<x} \frac{\int_a^x |f|u}{\int_a^x u}.$$

We shall use that this operator is of weak type (1,1) with respect to the measure $u(x)dx$ [**13**].

2. Statement of results

The problem of a characterization of two weighted weak type inequalities for $M_{\tau_h\varphi}$ was solved in [**2**] for functions φ belonging to a subset of

$$\mathcal{F}^+ = \left\{\varphi : \mathbb{R} \to [0,\infty) : \ \varphi\chi_{(-\infty,0)} = 0, \ \varphi \text{ decreasing in } (0,\infty), 0 < \int \varphi = A < \infty\right\}.$$

The characterization depends on the behaviour of φ near zero and on the sign of h. In particular the following theorem was proved [**2**, Theorems 1.6, 1.7 and 1.8].

THEOREM 2.1 *Let* $1 < p < \infty$, $h \in \mathbb{R}\backslash\{0\}$, $0 < \gamma \leqslant |h|$, $\delta \in (0,1)$ *and* $\varphi \in \mathcal{E}^+_{\gamma,\delta} = \{\varphi \in \mathcal{F}^+ : \ \varphi(\gamma) > 0$ and $t^\delta\varphi(t)$ is increasing in $(0,\gamma]\}$. *Let u and v be positive measurable functions* (*weights*).

(i) *If $h > 0$, then $M_{\tau_h\varphi}$ is of weak type (p,p) with respect to the pairs of weights (u,v), that is, there exists $C > 0$ such that*

$$\int_{\{M_{\tau_h\varphi} f>\lambda\}} u \leqslant C\lambda^{-p} \int_{\mathbb{R}} |f|^p v \tag{1}$$

for all $\lambda > 0$ and for all $f \in L^p(v)$ if and only if $(u,v) \in A^-_{p,\varphi,\gamma}$, that is, there exists $C > 0$ such that for all $a < b < c$

$$\left(\int_b^c u\right)^{1/p} \left(\int_a^b v^{1-p'}(y)\varphi^{p'}\left(\frac{b-y}{c-a}\gamma\right) dy\right)^{1/p'} \leqslant C\ \frac{c-a}{\gamma}, \qquad \frac{1}{p}+\frac{1}{p'}=1.$$

(ii) *If $h < 0$ and* $\operatorname{supp}(\varphi) \subset (0,|h|]$, *then* (1) *holds if and only if $(u,v) \in \widetilde{A}^+_{p,\varphi,\gamma}$, that is, there exists $C > 0$ such that for all $a < b < c$*

$$\left(\int_a^b u\right)^{1/p} \left(\int_b^c v^{1-p'}(y)\varphi^{p'}\left(\frac{c-y}{c-a}\gamma\right) dy\right)^{1/p'} \leqslant C\ \frac{c-a}{\gamma}.$$

(iii) *If $h < 0$ and* $\operatorname{supp}(\varphi) \cap (|h|,\infty) \neq \emptyset$, *then* (1) *holds if and only if $(u,v) \in \widetilde{A}^+_{p,\varphi,\gamma} \cap A_p$, where A_p is the Muckenhoupt's class of weights* [**9**], *that is, $(u,v) \in A_p$ if there exists C such that for all $a < b$*

$$\left(\int_a^b u\right)^{1/p} \left(\int_a^b v^{1-p'}\right)^{1/p'} \leqslant C\ (b-a).$$

Notice that $A^-_{p,\varphi,\gamma}$ and $\widetilde{A}^+_{p,\varphi,\gamma}$ are related to the Sawyer's classes A^-_p and A^+_p [**7**, **12**] which are the classes of the good weights for the one-sided Hardy–Littlewood maximal operators

$$M^- f(x) = \sup_{a<x}(x-a)^{-1}\int_a^x |f|$$

and

$$M^+ f(x) = \sup_{b>x}(b-x)^{-1}\int_x^b |f|.$$

In fact, if $1 < p < \infty$, $\varphi = \chi_{[0,1]}$ and $\gamma = 1$ then $A^-_p = A^-_{p,\varphi,\gamma}$ and $A^+_p = \widetilde{A}^+_{p,\varphi,\gamma}$. For future use, we recall that $(u, v) \in A^-_1$ (A^+_1) if and only if $M^+u \leqslant Cv$ $(M^-u \leqslant Cv)$ a.e. The Muckenhoupt A_1 class [**9**] is defined in the same way with M^+ replaced by the Hardy–Littlewood maximal operator

$$Mf(x) = \sup_{a<x<b}(b-a)^{-1}\int_a^b |f|.$$

When $\varphi(0+) = \lim_{t\to 0^+}\varphi(t) < +\infty$, the characterization given in Theorem 2.1 is simpler as the following theorem shows; see, [**2**, Theorem 1.5].

THEOREM 2.2 *Let* $1 \leqslant p < \infty$, $\varphi \in \mathcal{F}^+$ *and* $\varphi(0+) < +\infty$.

(i) *If* $h > 0$, *then* (1) *holds if and only if* (u, v) *belongs to* A^-_p.

(ii) *If* $h < 0$ *and* $\operatorname{supp}(\varphi) \subset (0, |h|]$, *then* (1) *holds if and only if* (u, v) *belongs to* A^+_p.

(iii) *If* $h < 0$ *and* $\operatorname{supp}(\varphi) \cap (|h|, \infty) \neq \emptyset$, *then* (1) *holds if and only if* $(u, v) \in A_p$.

Observe that if p_0 is such that $\varphi \notin L^{p_0'}(0,\gamma)$, where $1/p_0 + 1/p_0' = 1$, and $\operatorname{ess\,inf}_{x\in(a,b)} v^{1-p_0'}(x) > 0$ for some interval (a, b) then the conditions $A^-_{p_0,\varphi,\gamma}$ and $\widetilde{A}^+_{p_0,\varphi,\gamma}$ do not hold and therefore the two weighted weak type (p, p) inequalities for $M_{\tau_h\varphi}$ are not true for $1 < p \leqslant p_0$. However it is still possible to have restricted weak type (p_0, p_0). This happens for M^-_α and $\widetilde{M}^+_\alpha$ with $p_0 = 1/(1+\alpha)$ (see [**1**, **3**, **8**]). This is our motivation for studying a characterization of the restricted weak type inequalities in weighted L^p-spaces for the general operator $M_{\tau_h\varphi}$. In order to simplify the statements of the results we start with a definition.

DEFINITION 2.3 It is said that an operator T is of restricted weak type (p, p) with respect to the pair of weights (u, v) if there exists $C > 0$ such that

$$\int_{\{|T\chi_E|>\lambda\}} u \leqslant C\lambda^{-p}\int_{\mathbb{R}} \chi_E v \tag{2}$$

for all $\lambda > 0$ and for all measurable sets E.

In the proofs of our results, we need the characterization of the weighted restricted weak type inequalities for the one-sided Hardy–Littlewood maximal operators M^+ and M^-, and for the two-sided Hardy–Littlewood maximal operator M (see [**4**, Theorem 3 and Lemma 2.8; **6**, Proposition 1] for M and [**10**, **11**] for the corresponding results for M^+ and M^-). These characterizations are collected in the next theorem.

THEOREM 2.4 *Let* $1 \leqslant p < \infty$.

(i) *The one-sided Hardy–Littlewood maximal operator* M^- *is of restricted weak type* (p, p) *with respect to the pair of weights* (u, v) *if and only if* $(u, v) \in RA_p^-$, *that is, there exists* $C > 0$ *such that for all* $a < b < c$ *and all measurable sets* E,

$$\left(\int_b^c u\right) |E \cap (a, b)|^p \leqslant C\ (c - a)^p \int_a^b \chi_E v.$$

(ii) *The one-sided Hardy–Littlewood maximal operator* M^- *is of restricted weak type* (p, p) *with respect to the pair of weights* (u, v) *if and only if* $(u, v) \in RA_p^+$, *that is, there exists* $C > 0$ *such that for all* $a < b < c$ *and all measurable sets* E,

$$\left(\int_a^b u\right) |E \cap (b, c)|^p \leqslant C\ (c - a)^p \int_b^c \chi_E v.$$

(iii) *The Hardy–Littlewood maximal operator* M *is of restricted weak type* (p, p) *with respect to the pair of weights* (u, v) *if and only if* $(u, v) \in RA_p$, *that is, if there exists* C *such that for all* $a < b$ *and all measurable sets* E,

$$\left(\int_a^b u\right) |E \cap (a, b)|^p \leqslant C\ (b - a)^p \int_a^b \chi_E v.$$

Now we are ready to state our first result which characterizes the weighted restricted weak type inequalities when $\varphi(0+) < \infty$.

THEOREM 2.5 *Let* $1 \leqslant p < \infty$, $\varphi \in \mathcal{F}^+$ *and* $\varphi(0+) < +\infty$.

(i) *If* $h > 0$, *then* $M_{\tau_h\varphi}$ *is of restricted weak type* (p, p) *with respect to the pairs of weights* (u, v) *if and only if* $(u, v) \in RA_p^-$.

(ii) *If* $h < 0$ *and* $\operatorname{supp}(\varphi) \subset (0, |h|]$, *then* $M_{\tau_h\varphi}$ *is of restricted weak type* (p, p) *with respect to the pairs of weights* (u, v) *if and only if* $(u, v) \in RA_p^+$.

(iii) *If* $h < 0$ *and* $\operatorname{supp}(\varphi) \cap (|h|, \infty) \neq \emptyset$, *then* $M_{\tau_h\varphi}$ *is of restricted weak type* (p, p) *with respect to the pairs of weights* (u, v) *if and only if* $(u, v) \in RA_p$.

Now we state our main result, that is, without assuming that $\varphi(0^+) < \infty$.

THEOREM 2.6 *Let* $1 \leqslant p < \infty$, $h \in \mathbb{R}\backslash\{0\}$, $0 < \gamma \leqslant |h|$, $\delta \in (0, 1)$ *and* $\varphi \in \mathcal{E}^+_{\gamma,\delta}$.

(i) *If* $h > 0$, *then* $M_{\tau_h\varphi}$ *is of restricted weak type* (p, p) *with respect to the pairs of weights* (u, v) *if and only if* $(u, v) \in RA^-_{p,\varphi,\gamma}$, *that is, there exists* $C > 0$ *such that for all* $a < b < c$ *and all measurable sets* E,

$$\left(\int_b^c u\right) \left(\int_a^b \chi_E(y)\varphi\left(\frac{b - y}{c - a}\gamma\right) dy\right)^p \leqslant C\ \left(\frac{c - a}{\gamma}\right)^p \int_a^b \chi_E v.$$

(ii) *If* $h < 0$ *and* $\operatorname{supp}(\varphi) \subset (0, |h|]$, *then* $M_{\tau_h\varphi}$ *is of restricted weak type* (p, p) *with respect to the pairs of weights* (u, v) *if and only if* $(u, v) \in R\widetilde{A}^+_{p,\varphi,\gamma}$, *that is, there exists* $C > 0$ *such that, for all* $a < b < c$ *and all measurable sets* E,

$$\left(\int_a^b u\right) \left(\int_b^c \chi_E(y)\varphi\left(\frac{c - y}{c - a}\gamma\right) dy\right)^p \leqslant C\ \left(\frac{c - a}{\gamma}\right)^p \int_b^c \chi_E v.$$

(iii) *If* $h < 0$ *and* $\operatorname{supp}(\varphi) \cap (|h|, \infty) \neq \emptyset$, *then* $M_{\tau_h\varphi}$ *is of restricted weak type* (p, p) *with respect to the pairs of weights* (u, v) *if and only if* $(u, v) \in R\widetilde{A}^{+}_{p,\varphi,\gamma} \cap RA_p$.

REMARK 2.7 We observe that RA^-_p and RA^+_p are equal to $RA^-_{p,\varphi,\gamma}$ and $R\widetilde{A}^{+}_{p,\varphi,\gamma}$ with $\varphi = \chi_{[0,1]}$ and $\gamma = 1$, respectively.

The proof of Theorems 2.5 and 2.6 will be given in section 3. The last section is dedicated to the proof of the relations between the classes of weights for $M_{\tau_h\varphi}$ and those for M^+ and M^-. The results read as follows (we distinguish the cases when $p > 1$ and $p = 1$).

PROPOSITION 2.8 *Let* $\gamma > 0$, $\delta \in (0, 1)$ *and* $p(1 - \delta) \geqslant 1$. *Assume that* $\varphi \in \mathcal{E}^+_{\gamma,\delta}$. *Then*

(i) $RA^-_{p(1-\delta)} \subset RA^-_{p,\varphi,\gamma} \subset RA^-_p$,

(ii) $RA^+_{p(1-\delta)} \subset R\widetilde{A}^{+}_{p,\varphi,\gamma} \subset RA^+_p$.

PROPOSITION 2.9 *Let* $\gamma > 0$ *and* $\varphi \in \mathcal{F}^+$. *Then*

(i) $(u, v) \in RA^-_{1,\varphi,\gamma}$ *if and only if* $(u, v) \in A^-_1$ *and* $\varphi(0+) < \infty$.

(ii) $(u, v) \in R\widetilde{A}^{+}_{1,\varphi,\gamma}$ *if and only if* $(u, v) \in A^+_1$ *and* $\varphi(0+) < \infty$.

It is clear (see [**14**]) that, for the operator $M_{\tau_h\varphi}$, the weak type $(1, 1)$ inequality is equivalent to the restricted one. Therefore, Proposition 2.9 together with Theorem 2.6 characterize the weighted weak type $(1, 1)$ inequality. In particular, $\varphi(0+) < \infty$ is necessary.

It is worth noticing that it is possible to state and prove the corresponding theorems for the class $\mathcal{F}^- = \{\varphi : \varphi(-x) \in \mathcal{F}^+\}$. Then the results for $\varphi \in \mathcal{F} = \{\varphi(x) = \psi(x) + \psi(-x) : \psi \in \mathcal{F}^+\}$ can be obtained. It follows that if $\varphi \in \mathcal{F}$ and the support of φ is equal to $\mathbb{R}$, then the class of weights characterizing the (restricted) weak type (p, p) inequality is contained in (RA_p) A_p. Therefore, it is interesting to characterize the weights w in the Muckenhoupt class (RA_p) A_p such that (w, w) is a good pair for the (restricted) weak type (p, p) inequality for $M_{\tau_h\varphi}$.

PROPOSITION 2.10 *Let* $\gamma > 0$ *and* $\varphi \in \mathcal{F}^+$. *Assume that* w *belongs to* A_p, $1 < p < \infty$, *that is, the pair* (w, w) *belong to* A_p. *The following statements are equivalent:*

(i) $w \in A^-_{p,\varphi,\gamma}$;

(ii) $w \in \widetilde{A}^{+}_{p,\varphi,\gamma}$;

(iii) *there exists* $C > 0$ *such that*

$$\int_a^b w^{1-p'}(y)\varphi^{p'}\left(\frac{b-y}{b-a}\gamma\right) dy \leqslant C \int_a^b w^{1-p'}(y)\, dy$$

for all $a < b$.

PROPOSITION 2.11 *Let* $\gamma > 0$ *and* $\varphi \in \mathcal{F}^+$. *Assume that* w *belongs to* RA_p, $1 < p < \infty$. *The following statements are equivalent:*

(i) $w \in RA^-_{p,\varphi,\gamma}$;

(ii) $w \in R\widetilde{A}^{+}_{p,\varphi,\gamma}$;

(iii) *there exists* $C > 0$ *such that*

$$\int_a^b \chi_E(y)\varphi\left(\frac{b-y}{b-a}\gamma\right)\,dy \leqslant C\|w^{-1}\chi_{(a,b)}\|_{p',\infty;w}\left(\int_a^b \chi_E w\right)^{1/p}$$

for all $a < b$ *and any measurable subset* E*, where*

$$\|f\|_{p',\infty;w} = \sup_{t>0} t\left(\int_{\{x:|f(x)|>t\}} w\right)^{1/p'}.$$

Propositions 2.10 and 2.11 together with Theorems 2.1 and 2.6 allow to describe the class of functions φ for which $M_{\tau_h\varphi}$ is of weak or restricted weak type (p, p) with respect to the Lebesgue measure.

THEOREM 2.12 *Let* $h \in \mathbb{R}\backslash\{0\}$, $0 < \gamma \leqslant |h|$, $\delta \in (0, 1)$ *and* $\varphi \in \mathcal{E}^+_{\gamma,\delta}$.

(i) $M_{\tau_h\varphi}$ *is of weak type* (p, p), $1 < p < \infty$, *with respect to the Lebesgue measure if and only if* $\varphi^{p'}$ *is integrable on* $(0, \gamma)$ (*or on any bounded interval* $(0, a)$).

(ii) $M_{\tau_h\varphi}$ *is of restricted weak type* (p, p), $1 \leqslant p < \infty$, *with respect to the Lebesgue measure if and only if there exists* $C > 0$ *such that* $\int_0^t \varphi(s)\,ds \leqslant C\, t^{1/p}$ *for all* $t \in (0, \gamma)$ (*or for all* $t \in (0, a)$, $a < \infty$).

We first note that the conditions on φ in Theorem 2.12 describe only the behaviour of φ near to zero. Furthermore, this theorem answers question (1) from the introduction. The following corollary provides examples of functions φ with a different behaviour near the left endpoint of the interval of numbers p where $M_{\tau_h\varphi}$ is of weak type (p, p).

COROLLARY 2.13 *Let* $p \geqslant 1$. *Let* $-1 < \alpha \leqslant 0$ *and* $\beta \in \mathbb{R}$ *with* $\beta \geqslant 0$ *if* $\alpha = 0$. *Let* η, γ *and* δ *be such that* $\varphi(t) = t^\alpha(\log 1/t)^\beta\chi_{(0,\eta)}(t) \in \mathcal{E}^+_{\gamma,\delta}$.

(i) *If* $-1 < \alpha \leqslant 0$ *and* $\beta > 0$, *then* $M_{\tau_h\varphi}$ *is of weak type* (p, p) *with respect to the Lebesgue measure if and only if* $p > 1/(1+\alpha)$ *and it is not of restricted weak type* (p, p) *for* $p \leqslant 1/(1+\alpha)$.

(ii) *If* $-1 < \alpha \leqslant 0$ *and* $\beta = 0$, *then* $M_{\tau_h\varphi}$ *is of weak type* (p, p) *with respect to the Lebesgue measure if and only if* $p > 1/(1+\alpha)$; *it is further of restricted weak type* $(1/(1+\alpha), 1/(1+\alpha))$*and it is not of restricted weak type* (p, p) *for* $p < 1/(1+\alpha)$.

(iii) *If* $-1 < \alpha < 0$ *and* $\beta < \alpha$, *then* $M_{\tau_h\varphi}$ *is of weak type* (p, p) *with respect to the Lebesgue measure if and only if* $p \geqslant 1/(1+\alpha)$ *and it is not of restricted weak type* (p, p) *for* $p \leqslant 1/(1+\alpha)$.

(iv) *If* $-1 < \alpha < 0$ *and* $\alpha \leqslant \beta < 0$, *then* $M_{\tau_h\varphi}$ *is of weak type* (p, p) *with respect to the Lebesgue measure if and only if* $p > 1/(1+\alpha)$; *it is further of restricted weak type* $(1/(1+\alpha), 1/(1+\alpha))$ *and it is not of restricted weak type* (p, p) *for* $p < 1/(1+\alpha)$.

To prove this corollary, we just have to check when the conditions in Theorem 2.12 are satisfied.

3. Proof of Theorems 2.5 and 2.6

Proof of Theorem 2.5. The proof of Theorem 2.5 is an immediate consequence of [**2**, Lemma 2.1] and Theorem 2.4 stated in section 2. We reproduce here [**2**, Lemma 2.1].

LEMMA 3.1 *Let $\varphi \in \mathcal{F}^+$ and $\varphi(0+) < +\infty$. Let $\ell > 0$ be such that $\varphi(\ell) > 0$. There exist positive constants C_1 and C_2 such that the following hold.*

(i) *If $h > 0$,*

$$C_1\varphi(\ell)hM^- f(x) \leqslant M_{\tau_h\varphi} f(x) \leqslant \left(\varphi(0)h + \int_0^\infty \varphi\right) M^- f(x).$$

(ii) *If $h < 0$ and $\operatorname{supp}(\varphi) \subset (0, |h|]$,*

$$C_2\varphi(\ell)|h|M^+ f(x) \leqslant M_{\tau_h\varphi} f(x) \leqslant \varphi(0)|h|M^+ f(x).$$

(iii) *If $h < 0$, $\operatorname{supp}(\varphi) \cap (|h|, \infty) \neq \emptyset$ and $\ell > |h|$*

$$2\varphi(\ell)\min\{|h|, \ell + h\}Mf(x) \leqslant M_{\tau_h\varphi} f(x) \leqslant 2\left(\varphi(0)|h| + \int_{|h|}^\infty \varphi\right) Mf(x).$$

Proof of Theorem 2.6. The proof of Theorem 2.6 follows the lines of that of Theorem 2.1 (see [**2**, proofs of Theorems 1.6, 1.7 and 1.8]). We shall give only the proof of (i) because the proofs of (ii) and (iii) can be obtained in the same way (following [**2**, proofs of Theorems 1.6, 1.7 and 1.8]).

As in [**2**], we write $\varphi = \varphi\chi_{(0,\gamma]} + \varphi\chi_{(\gamma,\infty)}$. Then if we define $M_{\varphi,h,\gamma} := M_{\tau_h(\varphi\chi_{(0,\gamma]})}$ and $M_{\varphi,h,\infty} := M_{\tau_h(\varphi\chi_{(\gamma,\infty)})}$ we get the following inequalities:

$$\max\left\{M_{\varphi,h,\gamma}, M_{\varphi,h,\infty}\right\} \leqslant M_{\tau_h\varphi} \leqslant M_{\varphi,h,\gamma} + M_{\varphi,h,\infty}. \tag{3}$$

Therefore, $M_{\tau_h\varphi}$ satisfies (2) if and only if (2) holds for $M_{\varphi,h,\gamma}$ and $M_{\varphi,h,\infty}$. The study of $M_{\varphi,h,\infty}$ is completely analogous to that of $M_{\tau_h\varphi}$ with $\varphi(0+) < \infty$. The difficult part is concentrated in the local operator $M_{\varphi,h,\gamma}$.

To prove (i) in Theorem 2.6, we start studying the local part $M_{\varphi,h,\gamma}$. More precisely, we shall prove the following theorem.

THEOREM 3.2 *Let $1 \leqslant p < \infty$, $h > 0$, $0 < \gamma \leqslant h$, $\delta \in (0, 1)$ and $\varphi \in \mathcal{E}^+_{\gamma,\delta}$. The following statements are equivalent:*

(a) *$M_{\varphi,h,\gamma}$ is of restricted weak type (p, p) with respect to the pairs of weights (u, v);*

(b) $(u, v) \in RA^-_{p,\varphi,\gamma}$.

Proof. Notice that if $\beta = (h + \gamma)/h > 1$, then $M_{\varphi,h,\gamma}$ can be written as

$$M_{\varphi,h,\gamma} f(x) = \sup_{R>0} \frac{1}{R} \int_{x-\beta hR}^{x-hR} |f(y)|\varphi\left(\frac{x - hR - y}{R}\right) dy.$$

As in [**2**], we define the following non-centred version of $M_{\varphi,h,\gamma}$:

$$N_{\varphi,h,\gamma} f(x) = \sup_{(a,b)\in\mathcal{A}_x} \frac{\gamma}{b - a} \int_a^b |f(y)|\varphi\left(\frac{b - y}{b - a}\gamma\right) dy,$$

where $\mathcal{A}_x = \{(a, b) : b < x \text{ and } b - a \geqslant \gamma(x - b)/h\}$. In [**2**, Proposition 3.2] it was proved that for $\varphi \in \mathcal{E}^+_{\gamma,\delta}$ there exists $C > 0$ such that

$$M_{\varphi,h,\gamma} f(x) \leqslant N_{\varphi,h,\gamma} f(x) \leqslant CM_{\varphi,h,\gamma} f(x). \tag{4}$$

Therefore, (a) is equivalent to the same inequality involving $N_{\varphi,h,\gamma}$.

(a) ⇒ (b). Let $a < b < c$ and $f = \chi_{E\cap(a,b)}$. First, assume that $b - a \geqslant \gamma(c-b)/h$. Since $t\varphi(t)$ is increasing, we have, for all $x \in (b, c)$,

$$N_{\varphi,h,\gamma} f(x) \geqslant \frac{\gamma}{b-a}\int_a^b \chi_E(y)\varphi\left(\frac{b-y}{b-a}\gamma\right) dy$$
$$\geqslant \frac{\gamma}{c-a}\int_a^b \chi_E(y)\varphi\left(\frac{b-y}{c-a}\gamma\right) dy = \lambda.$$

Assume now that $b - a < \gamma(c-b)/h$ and let $\overline{a} < a$ be such that $b - \overline{a} = \gamma(c-a)/h$. For all $x \in (b, c)$, we obtain

$$N_{\varphi,h,\gamma} f(x) \geqslant \frac{\gamma}{b-\overline{a}}\int_a^b \chi_E(y)\varphi\left(\frac{b-y}{b-\overline{a}}\gamma\right) dy = \frac{h}{c-a}\int_a^b \chi_E(y)\varphi\left(\frac{b-y}{c-a}h\right) dy$$
$$\geqslant \frac{\gamma}{c-a}\int_a^b \chi_E(y)\varphi\left(\frac{b-y}{c-a}\gamma\right) = \lambda.$$

Applying (a) with $N_{\varphi,h,\gamma}$ we have, in both cases,

$$\left(\int_b^c u\right)\left(\int_a^b \chi_E(y)\varphi\left(\frac{b-y}{c-a}\gamma\right) dy\right)^p \leqslant C\left(\frac{c-a}{\gamma}\right)^p \int_a^b \chi_E v.$$

(b) ⇒ (a). In order to prove this implication we need the following proposition.

PROPOSITION 3.3 *Let* $1 \leqslant p < \infty$, $h > 0$, $0 < \gamma \leqslant h$, $\delta \in (0,1)$ *and* $\varphi \in \mathcal{E}^+_{\gamma,\delta}$. *Assume that* $(u, v) \in RA^-_{p,\varphi,\gamma}$. *Then, there exists* $C > 0$ *such that for every measurable set* E

$$N_{\varphi,h,\gamma}\chi_E(x) \leqslant C\left[M_u^-\left(\chi_E v u^{-1}\right)(x)\right]^{1/p}.$$

Before proving the proposition, we give the proof of $(b) \Rightarrow (a)$. By inequality (4) and the proposition we have

$$\int_{\{M_{\varphi,h,\gamma}\chi_E>\lambda\}} u \leqslant \int_{\{N_{\varphi,h,\gamma}\chi_E>\lambda\}} u \leqslant \int_{\{M_u^-(\chi_E v u^{-1})>\left(\frac{\lambda}{C}\right)^p\}} u.$$

Now (a) follows from these inequalities and the fact that M_u^- is of weak type (1,1) with respect to the measure $u(x)dx$.

Proof of Proposition 3.3. Let $x \in \mathbb{R}$ and $(a, b) \in \mathcal{A}_x$. First, let us assume that $4\int_b^x u > \int_a^x u$. Since $(u, v) \in RA^-_{p,\varphi,\gamma}$, we have

$$\int_a^b \chi_E(y)\varphi\left(\frac{b-y}{b-a}\gamma\right) dy \leqslant C\frac{x-a}{\gamma}\left(\int_a^b \chi_E v\right)^{1/p}\left(\int_b^x u\right)^{-1/p}$$
$$\leqslant C\frac{x-a}{\gamma}\left(\int_a^x \chi_E v\right)^{1/p}\left(\int_a^x u\right)^{-1/p}.$$

Now, $(a,b) \in \mathcal{A}_x$ implies that $x-a = x-b+b-a \leqslant \frac{h}{\gamma}(b-a)+(b-a) = \frac{h+\gamma}{\gamma}(b-a)$. Therefore

$$\int_a^b \chi_E(y)\varphi\left(\frac{b-y}{b-a}\gamma\right)dy \leqslant C\,\frac{b-a}{\gamma}\left(\frac{\gamma+h}{\gamma}\left[M_u^-\left(\chi_E v u^{-1}\right)(x)\right]^{1/p}\right).$$

Now, assume that $4\int_b^x u \leqslant \int_a^x u$. Let $\{x_i\}$ be the increasing sequence in $[a,x]$ defined by $x_0 = a$ and

$$\int_{x_{i+1}}^x u = \int_{x_i}^{x_{i+1}} u = \frac{1}{2}\int_{x_i}^x u.$$

Let N be such that $x_N \leqslant b < x_{N+1}$ (observe that $N \geqslant 2$). Then we have

$$\int_a^b \chi_E(y)\varphi\left(\frac{b-y}{b-a}\gamma\right)dy = \sum_{i=0}^{N-2}\int_{x_i}^{x_{i+1}} \cdots\, dy + \int_{x_{N-1}}^b \cdots\, dy = I + II.$$

We first estimate II. By $RA^-_{p,\varphi,\gamma}$, the monotonicity of φ and $t\varphi(t)$ on $(0,\gamma]$, and the inequality $\int_{x_{N-1}}^x u \leqslant 4\int_b^x u$, we get

$$\begin{aligned}
II &\leqslant \int_{x_{N-1}}^b \chi_E(y)\varphi\left(\frac{b-y}{x-a}\gamma\right)dy \\
&\leqslant \frac{x-a}{x-x_{N-1}}\int_{x_{N-1}}^b \chi_E(y)\varphi\left(\frac{b-y}{x-x_{N-1}}\gamma\right)dy \\
&\leqslant C\frac{x-a}{\gamma}\left(\int_{x_{N-1}}^b \chi_E v\right)^{1/p}\left(\int_b^x u\right)^{-1/p} \\
&\leqslant C\frac{x-a}{\gamma}\left(\int_{x_{N-1}}^x \chi_E v\right)^{1/p}\left(\int_{x_{N-1}}^x u\right)^{-1/p} \\
&\leqslant C\left(\frac{b-a}{\gamma}\right)\left(\frac{\gamma+h}{\gamma}\right)\left[M_u^-\left(\chi_E v u^{-1}\right)(x)\right]^{1/p}.
\end{aligned}$$

Now we shall estimate I. Notice that for each i, $0 \leqslant i \leqslant N-2$, there exists $q_i = (Ux_{i+1}-b)/(U-1)$ where $U = (b-a)/(x_{i+1}-x_i) > 1$ such that $q_i \in [x_i, x_{i+1}]$ and $(b-y)/(b-a) \geqslant (x_{i+1}-y)/(x_{i+1}-x_i)$ if and only if $y \geqslant q_i$. We can thus write

$$\int_{x_i}^{x_{i+1}} \chi_E(y)\varphi\left(\frac{b-y}{b-a}\gamma\right)dy = \int_{x_i}^{q_i} \cdots\, dy + \int_{q_i}^{x_{i+1}} \cdots\, dy = III + IV.$$

Since φ is decreasing, the hypothesis $(u, v) \in RA^-_{p,\varphi,\gamma}$ and the definition of the sequence $\{x_i\}$ give

$$\begin{aligned} IV &\leqslant \int_{q_i}^{x_{i+1}} \chi_E(y)\varphi\left(\frac{x_{i+1}-y}{x_{i+1}-x_i}\gamma\right) dy \leqslant \int_{x_i}^{x_{i+1}} \cdots dy \\ &\leqslant C\frac{x_{i+2}-x_i}{\gamma}\left(\int_{x_i}^{x_{i+1}} \chi_E v\right)^{1/p}\left(\int_{x_{i+1}}^{x_{i+2}} u\right)^{-1/p} \\ &\leqslant C\frac{x_{i+2}-x_i}{\gamma}\left(\int_{x_i}^{x} \chi_E v\right)^{1/p}\left(\int_{x_i}^{x} u\right)^{-1/p} \\ &\leqslant C\,\frac{x_{i+2}-x_i}{\gamma}\left[M_u^-\left(\chi_E v u^{-1}\right)(x)\right]^{1/p}. \end{aligned}$$

To estimate III, we shall use that $(b-y)/(b-a) < (x_{i+1}-y)/(x_{i+1}-x_i)$ if and only if $y < q_i$ and the fact that $t^\delta\varphi(t)$ is increasing in $(0, \gamma]$. Then

$$III = \int_{x_i}^{q_i} \chi_E(y)\varphi\left(\frac{b-y}{b-a}\gamma\right) dy \leqslant \int_{x_i}^{q_i} \chi_E(y)\varphi\left(\frac{x_{i+1}-y}{x_{i+1}-x_i}\gamma\right) g(y)\, dy,$$

where $g(y) = \left(\frac{b-y}{b-a}\right)^{-\delta}\left(\frac{x_{i+1}-y}{x_{i+1}-x_i}\right)^{\delta}$. Since g is decreasing in (x_i, q_i), we have

$$III \leqslant \left(\frac{b-x_i}{b-a}\right)^{-\delta}\int_{x_i}^{x_{i+1}} \chi_E(y)\varphi\left(\frac{x_{i+1}-y}{x_{i+1}-x_i}\gamma\right) dy.$$

Using the same argument as in the boundedness of IV and the increasingness of $(b-y)^{-\delta}$ we get that

Is this referring to an integral?

$$\begin{aligned} III &\leqslant C\left(\frac{b-x_i}{b-a}\right)^{-\delta}\frac{x_{i+2}-x_i}{\gamma}\left[M_u^-\left(\chi_E v u^{-1}\right)(x)\right]^{1/p} \\ &\leqslant \frac{C}{\gamma}\left(\int_{x_i}^{x_{i+2}}\left(\frac{b-y}{b-a}\right)^{-\delta} dy\right)\left[M_u^-\left(\chi_E v u^{-1}\right)(x)\right]^{1/p}. \end{aligned}$$

Now, summing over i, we get

$$I \leqslant 2C\,\frac{b-a}{\gamma}\frac{2-\delta}{1-\delta}\left[M_u^-\left(\chi_E v u^{-1}\right)(x)\right]^{1/p}.$$

Finally, putting together the estimates of I and II, we are done.

As a consequence of Theorem 2.5 we get the following characterization of the restricted weak type inequalities for $M_{\varphi,h,\infty}$.

THEOREM 3.4 *Let $\varphi \in \mathcal{F}^+$, $h > 0$ and $0 < \gamma \leqslant h$. Then $M_{\varphi,h,\infty}$ is of restricted weak type (p, p) with respect to the pairs of weights (u, v) if and only if $(u, v) \in RA^-_p$.*

Proof. Let $\psi = \tau_{-\gamma}(\varphi\chi_{(\gamma,\infty)})$. It is clear that $\psi \in \mathcal{F}^+$ and $\tau_h(\varphi\chi_{(\gamma,\infty)}) = \tau_{h+\gamma}(\psi)$. Then $M_{\varphi,h,\infty}$ is equal to the operator $M_{\tau_{h+\gamma}\psi}$. Therefore, since $h+\gamma > 0$ and $\psi(0+) = \varphi(\gamma) < +\infty$, applying Theorem 2.5 (i) we are done.

Proof of Theorem 2.6(i). If (2) holds, then the same estimate is true for $M_{\varphi,h,\gamma}$ and, by Theorem 3.2, $(u, v) \in RA^-_{p,\varphi,\gamma}$.

Conversely, by (3), we only have to show that $M_{\varphi,h,\gamma}$ and $M_{\varphi,h,\infty}$ satisfy (2). First, by Theorem 3.2, $(u, v) \in RA^-_{p,\varphi,\gamma}$ implies that $M_{\varphi,h,\gamma}$ satisfies (2). On the other hand, since φ is decreasing, we have

$$\left(\varphi(\gamma)\int_a^b \chi_E\right)^p \leqslant \left(\int_a^b \chi_E(y)\varphi\left(\frac{b-y}{c-a}\gamma\right)\,dy\right)^p,$$

whence $RA^-_{p,\varphi,\gamma} \subset RA^-_p$. Now, Theorem 3.4 gives that $M_{\varphi,h,\infty}$ is of restricted weak type (p, p) with respect to the pair (u, v).

4. Proofs of Propositions 2.8, 2.9, 2.10 and 2.11, Theorem 2.12 and Corollary 2.13

Proof of Proposition 2.8(i). We only prove the first inclusion in (i) as the second was already established in the proof of Theorem 2.6. Let $a < b < c$, let E be a measurable set and $E' = E \cap (a, b)$. The integral

$$\int_a^b \chi_E(y)\varphi\left(\frac{b-y}{c-a}\gamma\right)\,dy = \int_{E'} \varphi\left(\frac{b-y}{c-a}\gamma\right)\,dy$$

is not greater than the integral

$$\int_{b-|E'|}^b \varphi\left(\frac{b-y}{c-a}\gamma\right)\,dy = \frac{c-a}{\gamma}\int_0^{(|E'|/(c-a))\gamma} \varphi(s)\,ds,$$

since the measure of E' is equal to the measure of the interval $(b - |E'|, b)$ and the function $y \to \varphi\left(\frac{b-y}{c-a}\gamma\right)$ is increasing in (a, b). This remark and the fact that $\varphi \in \mathcal{E}^+_{\gamma,\delta}$ gives

$$\begin{aligned}\int_a^b \chi_E(y)\varphi\left(\frac{b-y}{c-a}\gamma\right)\,dy &\leqslant \int_{b-|E'|}^b \varphi\left(\frac{b-y}{c-a}\gamma\right)\,dy\\ &= \frac{c-a}{\gamma}\int_0^{(|E'|/(c-a))\gamma} \varphi(s)\,ds\\ &\leqslant \frac{c-a}{\gamma}\varphi(\gamma)\gamma^\delta\int_0^{(|E'|/(c-a))\gamma} s^{-\delta}\,ds\\ &= \frac{\varphi(\gamma)\gamma^\delta}{1-\delta}\left(\frac{c-a}{\gamma}\right)^\delta |E'|^{1-\delta}.\end{aligned}$$

Raising the last inequality to the pth power and multiplying by $\int_b^c u$ we get that

$$\left(\int_b^c u\right)\left(\int_a^b \chi_E(y)\varphi\left(\frac{b-y}{c-a}\gamma\right)\,dy\right)^p \leqslant \left(\int_b^c u\right)\left(\frac{\varphi(\gamma)\gamma^\delta}{1-\delta}\right)^p\left(\frac{c-a}{\gamma}\right)^{p\delta}|E'|^{p(1-\delta)}.$$

If the pair (u, v) belong to $RA^-_{p(1-\delta)}$ the last term is dominated by

$$C\left(\frac{\varphi(\gamma)\gamma^\delta}{1-\delta}\right)^p\left(\frac{c-a}{\gamma}\right)^{p\delta}\left(\frac{c-a}{\gamma}\right)^{p(1-\delta)}\int_a^b v\chi_E,$$

and consequently $(u, v) \in RA^-_{p,\varphi,\gamma}$.

Proof of Proposition 2.8(ii). We shall begin by proving the second inclusion. Assume that $(u,v)\in R\widetilde{A}^+_{p,\varphi,\gamma}$. Let $a<b<c$ and let E be a measurable set. Since the function $y\to\varphi\left(\frac{c-y}{c-a}\gamma\right)$ is increasing in $[a,c)$ it follows that $\varphi(\gamma)\leqslant\varphi\left(\frac{c-y}{c-a}\gamma\right)$ for all $y\in(b,c)$. Using that $(u,v)\in R\widetilde{A}^+_{p,\varphi,\gamma}$ we have

$$\left(\int_a^b u\right)\left(\int_b^c \chi_E(y)\,dy\right)^p(\varphi(\gamma))^p\leqslant\left(\int_a^b u\right)\left(\int_b^c \chi_E(y)\varphi\left(\frac{c-y}{c-a}\gamma\right)dy\right)^p$$
$$\leqslant C\left(\frac{c-a}{\gamma}\right)^p\int_b^c\chi_E v.$$

Therefore $(u,v)\in RA^+_p$.

We prove now the first inclusion in 2.8 (ii). Assume that $(u,v)\in RA^+_{p(1-\delta)}$. Let $a<b<c$, let E be a measurable set and $E'=E\cap(b,c)$. Since the function $y\to\varphi\left(\frac{c-y}{c-a}\gamma\right)$ is increasing in (b,c), the measure of E' is equal to the measure of the interval $(c-|E'|,c)$ and we obtain

$$\int_b^c\chi_E(y)\varphi\left(\frac{c-y}{c-a}\gamma\right)dy=\int_{E'}\varphi\left(\frac{c-y}{c-a}\gamma\right)dy$$
$$\leqslant\int_{c-|E'|}^c\varphi\left(\frac{c-y}{c-a}\gamma\right)dy=\frac{c-a}{\gamma}\int_0^{(|E'|/(c-a))\gamma}\varphi(s)\,ds.$$

This inequality and the fact that $\varphi\in\mathcal{E}^+_{\gamma,\delta}$ gives

$$\int_b^c\chi_E(y)\varphi\left(\frac{c-y}{c-a}\gamma\right)dy\leqslant\frac{c-a}{\gamma}\varphi(\gamma)\gamma^\delta\int_0^{(|E'|/(c-a))\gamma}s^{-\delta}\,ds$$
$$=\frac{\varphi(\gamma)\gamma^\delta}{1-\delta}\left(\frac{c-a}{\gamma}\right)^\delta|E'|^{1-\delta}.$$

Raising the last inequality to the pth power, multiplying by $\int_a^b u$ and using that $(u,v)\in RA^+_{p(1-\delta)}$ we get that

$$\left(\int_a^b u\right)\left(\int_b^c\chi_E(y)\varphi\left(\frac{c-y}{c-a}\gamma\right)dy\right)^p\leqslant\left(\int_a^b u\right)\left(\frac{\varphi(\gamma)\gamma^\delta}{1-\delta}\right)^p\left(\frac{c-a}{\gamma}\right)^{p\delta}|E'|^{p(1-\delta)}$$
$$\leqslant C\left(\frac{\varphi(\gamma)\gamma^\delta}{1-\delta}\right)^p\left(\frac{c-a}{\gamma}\right)^{p\delta}\left(\frac{c-a}{\gamma}\right)^{p(1-\delta)}\int_b^c v\chi_E=C\left(\frac{c-a}{\gamma}\right)^p\int_b^c v\chi_E.$$

Therefore $(u,v)\in R\widetilde{A}^+_{p,\varphi,\gamma}$.

Proof of Proposition 2.9. We only prove (i). Let $(u,v)\in RA^-_{1,\varphi,\gamma}$. Given $n\in\mathbb{N}$, let $E_n=\{v\leqslant n\}$ and $v_n=v\chi_{E_n}$. Let $a<b<c$, where b is a Lebesgue point of χ_{E_n} and v_n for all $n\in\mathbb{N}$. Now, applying the condition $RA^-_{1,\varphi,\gamma}$ with $E=E_n$ we get

$$\left(\frac{1}{c-a}\int_b^c u\right)\left(\int_a^b\chi_{E_n}(y)\varphi\left(\frac{b-y}{c-a}\gamma\right)dy\right)\leqslant C\int_a^b v_n.$$

Since φ is decreasing,

$$\left(\frac{1}{c-a}\int_b^c u\right)\varphi\left(\frac{b-a}{c-a}\gamma\right)\frac{1}{b-a}\left(\int_a^b \chi_{E_n}\right)\leqslant C\,\frac{1}{b-a}\int_a^b v_n.$$

On letting $a\uparrow b$, we have

$$\left(\frac{1}{c-b}\int_b^c u\right)\varphi(0+)\chi_{E_n}(b)\leqslant Cv_n(b).$$

Finally, on letting $n\to\infty$, we obtain

$$\left(\frac{1}{c-b}\int_b^c u\right)\varphi(0+)\leqslant Cv(b)$$

for almost every $b\in\mathbb{R}$, and we are done.

Conversely, if $(u,v)\in A_1^-$ and $\varphi(0+)<\infty$ then, by Theorem 2.2, $M_{\tau_h\varphi}$ with $h>0$ is of weak type $(1,1)$ with respect to (u,v) and therefore is of restricted weak type $(1,1)$ with respect to (u,v), or equivalently $(u,v)\in RA^-_{1,\varphi,\gamma}$.

Proof of Proposition 2.10. (i) $\Rightarrow$ (iii). Let $a<b$ and choose c such that $b=\frac{1}{2}(a+c)$. As φ is decreasing, w is in $A^-_{p,\varphi,\gamma}$ and by the Hölder inequality we have

$$\begin{aligned}\left(\int_b^c w\right)^{1/p}&\left(\int_a^b w^{1-p'}(y)\varphi^{p'}\left(\frac{b-y}{b-a}\gamma\right)dy\right)^{1/p'}\\&\leqslant\left(\int_b^c w\right)^{1/p}\left(\int_a^b w^{1-p'}(y)\varphi^{p'}\left(\frac{b-y}{c-a}\gamma\right)dy\right)^{1/p'}\\&\leqslant C\frac{c-a}{\gamma}=C\frac{c-b}{\gamma}\leqslant\frac{C}{\gamma}\left(\int_b^c w\right)^{1/p}\left(\int_b^c w^{1-p'}\right)^{1/p'}.\end{aligned}$$

Therefore

$$\int_a^b w^{1-p'}(y)\varphi^{p'}\left(\frac{b-y}{b-a}\gamma\right)dy\leqslant\frac{C}{\gamma}\int_b^c w^{1-p'}.$$

As w is in A_p we have that $w^{1-p'}\in A_{p'}$ and then $w^{1-p'}$ is a doubling weight. Consequently,

$$\int_b^c w^{1-p'}\leqslant C\int_a^b w^{1-p'}.$$

Putting together the last two inequalities we obtain (iii).

(iii) $\Rightarrow$ (i). Let $a<b<c$. We have to show that

$$\left(\int_b^c w\right)^{1/p}\left(\int_a^b w^{1-p'}(y)\varphi^{p'}\left(\frac{b-y}{c-a}\gamma\right)dy\right)^{1/p'}\leqslant C\frac{c-a}{\gamma}.$$

Let us take $\bar a \leqslant a$ and $\bar c \geqslant c$ such that $b = (\bar a + \bar c)/2$ and $\bar c - \bar a \leqslant 2(c-a)$. Since φ is decreasing, $t^\delta \varphi(t)$ is increasing in $(0, \gamma]$, and the fact that (iii) holds together with $w \in A_p$ gives

$$\begin{aligned}
&\left(\int_b^c w\right)^{1/p} \left(\int_a^b w^{1-p'}(y)\varphi^{p'}\left(\frac{b-y}{c-a}\gamma\right) dy\right)^{1/p'} \\
&\qquad \leqslant \left(\int_b^{\bar c} w\right)^{1/p} \left(\int_{\bar a}^b w^{1-p'}(y)\varphi^{p'}\left(\frac{b-y}{\bar c-\bar a}\gamma\right) dy\right)^{1/p'} \\
&\qquad \leqslant 2^\delta \left(\int_b^{\bar c} w\right)^{1/p} \left(\int_{\bar a}^b w^{1-p'}(y)\varphi^{p'}\left(\frac{b-y}{b-\bar a}\gamma\right) dy\right)^{1/p'} \\
&\qquad \leqslant C \left(\int_b^{\bar c} w\right)^{1/p} \left(\int_{\bar a}^b w^{1-p'}(y) dy\right)^{1/p'} \leqslant C(\bar c - \bar a) \leqslant \frac{C}{\gamma}(c-a).
\end{aligned}$$

(ii) $\Rightarrow$ (iii). Let $a < b$ and let $\bar a < a$ be such that $a = \frac{1}{2}(\bar a + b)$. As φ is decreasing and w is in $\widetilde{A}^+_{p,\varphi,\gamma}$,

$$\begin{aligned}
&\left(\int_{\bar a}^a w\right)^{1/p} \left(\int_a^b w^{1-p'}(y)\varphi^{p'}\left(\frac{b-y}{b-a}\gamma\right) dy\right)^{1/p'} \\
&\qquad \leqslant \left(\int_{\bar a}^a w\right)^{1/p} \left(\int_a^b w^{1-p'}(y)\varphi^{p'}\left(\frac{b-y}{b-\bar a}\gamma\right) dy\right)^{1/p'} \\
&\qquad \leqslant C\frac{b-\bar a}{\gamma} \leqslant \frac{C}{\gamma}(\bar a - a) \\
&\qquad \leqslant \frac{C}{\gamma} \left(\int_{\bar a}^a w\right)^{1/p} \left(\int_{\bar a}^a w^{1-p'}(y)\, dy\right)^{1/p'}
\end{aligned}$$

Therefore

$$\int_a^b w^{1-p'}(y)\varphi^{p'}\left(\frac{b-y}{b-a}\gamma\right) dy \leqslant C \int_{\bar a}^a w^{1-p'}(y)\, dy.$$

Now (iii) follows from the fact that $w^{1-p'}$ is a doubling weight because $w^{1-p'} \in A_{p'}$.

(iii) $\Rightarrow$ (ii). Let $a < b < c$. We have to show that

$$\left(\int_a^b w\right)^{1/p} \left(\int_b^c w^{1-p'}(y)\varphi^{p'}\left(\frac{c-y}{c-a}\gamma\right) dy\right)^{1/p'} \leqslant C\frac{c-a}{\gamma}.$$

Let us take $\bar a \leqslant a$ and $\bar b \leqslant b$ such that $\bar b = (\bar a + c)/2$ and $c - \bar a \leqslant 2(c-a)$. As w is a doubling weight (because $w \in A_p$) we have that

$$\int_a^b w \leqslant C \int_{\bar a}^{\bar b} w.$$

Therefore

$$\left(\int_a^b w\right)^{1/p}\left(\int_b^c w^{1-p'}(y)\varphi^{p'}\left(\frac{c-y}{c-a}\gamma\right)dy\right)^{1/p'} \leqslant C\left(\int_{\bar a}^{\bar b} w\right)^{1/p}\left(\int_{\bar b}^c w^{1-p'}(y)\varphi^{p'}\left(\frac{c-y}{c-a}\gamma\right)dy\right)^{1/p'}.$$

Since φ is decreasing, $t^\delta\varphi(t)$ is increasing in $(0,\gamma)$, the fact that (iii) holds together with $w\in A_p$ gives that the last term is dominated by

$$\begin{aligned}C\left(\int_{\bar a}^{\bar b} w\right)^{1/p}&\left(\int_{\bar b}^c w^{1-p'}(y)\varphi^{p'}\left(\frac{c-y}{c-\bar a}\gamma\right)dy\right)^{1/p'}\\ &\leqslant 2^\delta C\left(\int_{\bar a}^{\bar b} w\right)^{1/p}\left(\int_{\bar b}^c w^{1-p'}(y)\varphi^{p'}\left(\frac{c-y}{c-\bar b}\gamma\right)dy\right)^{1/p'}\\ &\leqslant C\left(\int_{\bar a}^{\bar b} w\right)^{1/p}\left(\int_{\bar b}^c w^{1-p'}(y)dy\right)^{1/p'}\leqslant C(c-\bar a)\leqslant\frac{C}{\gamma}(c-a).\end{aligned}$$

Putting together all the inequalities, we obtain (ii).

Proof of Proposition 2.11. (i) $\Rightarrow$ (iii). Let $a<b$ and choose c such that $b=\frac{1}{2}(a+c)$. As φ is decreasing, w is in $RA^-_{p,\varphi,\gamma}$ and by the Hölder inequality in $L(p,q)$ spaces we have

$$\begin{aligned}\left(\int_b^c w\right)\left(\int_a^b \chi_E(y)\varphi\left(\frac{b-y}{b-a}\gamma\right)dy\right)^p&\leqslant\left(\int_b^c w\right)\left(\int_a^b \chi_E(y)\varphi\left(\frac{b-y}{c-a}\gamma\right)dy\right)^p\\ &\leqslant C\left(\frac{c-a}{\gamma}\right)^p\int_a^b\chi_E w\leqslant C\left(\frac{b-a}{\gamma}\right)^p\int_a^b\chi_E w\\ &\leqslant C\left(\int_a^b ww^{-1}\right)^p\int_a^b\chi_E w\leqslant C\left(\int_a^b w\right)\|w^{-1}\chi_{(a,b)}\|^p_{p',\infty;w}\int_a^b\chi_E w.\end{aligned}$$

As w is in RA_p we have that w is a doubling weight. Therefore

$$\int_a^b w\leqslant C\int_b^c w.$$

Putting together the inequalities we obtain

$$\left(\int_b^c w\right)\left(\int_a^b \chi_E(y)\varphi\left(\frac{b-y}{b-a}\gamma\right)dy\right)^p\leqslant C\left(\int_b^c w\right)\|w^{-1}\chi_{(a,b)}\|^p_{p',\infty;w}\int_a^b\chi_E w.$$

Now it is clear that (iii) follows from the last inequality.

(iii) $\Rightarrow$ (i). The proof is similar to that of (iii) $\Rightarrow$ (i) in Proposition 2.10, but (see [**6**]) it uses the equivalence of RA_p to the existence of a C such that

$$\left(\int_b^c w\right)^{1/p}\|w^{-1}\chi_{(b,c)}\|_{p',\infty;w}\leqslant C(c-b)$$

for the whole interval (b,c).

Let $a < b < c$. We have to show that

$$\left(\int_b^c w\right)\left(\int_a^b \chi_E(y)\varphi\left(\frac{b-y}{c-a}\gamma\right)\,dy\right)^p \leqslant C\,\left(\frac{c-a}{\gamma}\right)^p \int_a^b \chi_E w.$$

Let us take $\bar{a} \leqslant a$ and $\bar{c} \geqslant c$ such that $b = (\bar{a}+\bar{c})/2$ and $\bar{c}-\bar{a} \leqslant 2(c-a)$. Since φ is decreasing, $t^\delta\varphi(t)$ is increasing in $(0,\gamma)$, the fact that (iii) holds together with $w \in RA_p$ gives

$$\begin{aligned}
&\left(\int_b^c w\right)\left(\int_a^b \chi_E(y)\varphi\left(\frac{b-y}{c-a}\gamma\right)\,dy\right)^p \\
&\qquad\leqslant \left(\int_b^{\bar{c}} w\right)\left(\int_{\bar{a}}^b \chi_E(y)\varphi\left(\frac{b-y}{\bar{c}-\bar{a}}\gamma\right)\,dy\right)^p \\
&\qquad\leqslant 2^\delta\left(\int_b^{\bar{c}} w\right)\left(\int_{\bar{a}}^b \chi_E(y)\varphi\left(\frac{b-y}{b-\bar{a}}\gamma\right)\,dy\right)^p \\
&\qquad\leqslant C\left(\int_b^{\bar{c}} w\right)\|w^{-1}\chi_{(\bar{a},b)}\|^p_{p',\infty;w}\int_{\bar{a}}^b \chi_E w \\
&\qquad\leqslant C(\bar{c}-\bar{a})^p\int_{\bar{a}}^b \chi_E w \leqslant C\left(\frac{c-a}{\gamma}\right)^p\int_{\bar{a}}^b \chi_E w.
\end{aligned}$$

(ii) $\Rightarrow$ (iii). Let $a < b$ and let $\bar{a}$ be such that $a = \frac{1}{2}(\bar{a}+b)$. As φ is decreasing, w is in $R\widetilde{A}^+_{p,\varphi,\gamma}$ and by the Hölder inequality in $L(p,q)$ spaces we have

$$\begin{aligned}
&\left(\int_{\bar{a}}^a w\right)\left(\int_a^b \chi_E(y)\varphi\left(\frac{b-y}{b-a}\gamma\right)\,dy\right)^p \\
&\qquad\leqslant \left(\int_{\bar{a}}^a w\right)\left(\int_a^b \chi_E(y)\varphi\left(\frac{b-y}{b-\bar{a}}\gamma\right)\,dy\right)^p \\
&\qquad\leqslant C\left(\frac{b-\bar{a}}{\gamma}\right)^p\int_a^b \chi_E w \leqslant C\left(\frac{b-a}{\gamma}\right)^p\int_a^b \chi_E w \\
&\qquad\leqslant \frac{C}{\gamma^p}\left(\int_a^b w\right)\|w^{-1}\chi_{(a,b)}\|^p_{p',\infty;w}\int_a^b \chi_E w.
\end{aligned}$$

As w is in RA_p we have that w is a doubling weight. Therefore

$$\int_a^b w \leqslant C\int_{\bar{a}}^a w.$$

Putting together the last inequalities we obtain

$$\left(\int_{\bar{a}}^a w\right)\left(\int_a^b \chi_E(y)\varphi\left(\frac{b-y}{b-a}\gamma\right)\,dy\right)^p \leqslant \frac{C}{\gamma^p}\left(\int_{\bar{a}}^a w\right)\|w^{-1}\chi_{(a,b)}\|^p_{p',\infty;w}\int_a^b w\chi_E.$$

Now (iii) follows from this inequality.

(iii) $\Rightarrow$ (ii). Let $a < b < c$. We have to show that

$$\left(\int_a^b w\right)\left(\int_b^c \chi_E(y)\varphi\left(\frac{c-y}{c-a}\gamma\right)\,dy\right)^p \leqslant C\left(\frac{c-a}{\gamma}\right)^p\int_b^c w\chi_E$$

for any measurable set E. We may assume that $E \subset (b,c)$.

Let us take $\bar{a} \leqslant a$ and $\bar{b} \leqslant b$ such that $\bar{b} = (\bar{a} + c)/2$ and $c - \bar{a} \leqslant 2(c - a)$. As w is a doubling weight (because $w \in RA_p$) we have that

$$\int_a^b w \leqslant C \int_{\bar{a}}^{\bar{b}} w.$$

Therefore

$$\left(\int_a^b w\right)\left(\int_b^c \chi_E(y)\varphi\left(\frac{c-y}{c-a}\gamma\right)\,dy\right)^p \leqslant C\left(\int_{\bar{a}}^{\bar{b}} w\right)\left(\int_{\bar{b}}^c \chi_E(y)\varphi\left(\frac{c-y}{c-a}\gamma\right)\,dy\right)^p.$$

Since φ is decreasing, $t^\delta\varphi(t)$ is increasing in $(0,\gamma)$, the fact that (iii) holds together with $w \in RA_p$ gives that the last term is dominated by

$$\begin{aligned}
C&\left(\int_{\bar{a}}^{\bar{b}} w\right)\left(\int_{\bar{b}}^c \chi_E(y)\varphi\left(\frac{c-y}{c-\bar{a}}\gamma\right)\,dy\right)^p \\
&\leqslant 2^\delta C\left(\int_{\bar{a}}^{\bar{b}} w\right)\left(\int_{\bar{b}}^c \chi_E(y)\varphi\left(\frac{c-y}{c-\bar{b}}\gamma\right)\,dy\right)^p \\
&\leqslant C\left(\int_{\bar{a}}^{\bar{b}} w\right)\|w^{-1}\chi_{(\bar{b},c)}\|^p_{p',\infty;w}\left(\int_{\bar{b}}^c \chi_E w\right) \\
&\leqslant C(c-\bar{a})^p\left(\int_{\bar{b}}^c \chi_E w\right) \\
&\leqslant C\left(\frac{c-a}{\gamma}\right)^p\int_{\bar{b}}^c \chi_E w = C\left(\frac{c-a}{\gamma}\right)^p\int_b^c \chi_E w,
\end{aligned}$$

where in the last inequality we have used that $E \subset (b,c)$. Putting together all the inequalities, we obtain (ii).

Proof of Theorem 2.12. By Theorem 2.1 and Proposition 2.10, $M_{\tau_h\varphi}$ is of weak type (p,p) with respect to the Lebesgue measure if and only if there exists C such that

$$\int_a^b \varphi^{p'}\left(\frac{b-y}{b-a}\gamma\right)\,dy \leqslant C(b-a)$$

for all $a < b$. By a change of variables, this is equivalent to the fact that $\varphi^{p'}$ is integrable on $(0,\gamma)$.

In order to prove (ii), we use Proposition 2.11 and Theorem 2.6 to obtain that $M_{\tau_h\varphi}$ is of restricted weak type (p,p), with respect to the Lebesgue measure if and only if there exists C such that

$$\int_E \varphi\left(\frac{b-y}{b-a}\gamma\right)\,dy \leqslant C(b-a)^{1/p'}|E|^{1/p}$$

for all $a < b$ and any subset $E \subset (a,b)$. Since φ is decreasing, the above inequality holds for all E if and only if it holds for any interval $(b-s,b) \subset (a,b)$, that is, if and only if there exists C such that

$$\frac{b-a}{\gamma}\int_0^{(s/(b-a))\gamma}\varphi = \int_{b-s}^b \varphi\left(\frac{b-y}{b-a}\gamma\right)\,dy \leqslant C(b-a)^{1/p'}s^{1/p}$$

for all $s \in (0,b-a)$. Setting $t = \dfrac{s}{b-a}\gamma$, we are done.

Proof of Corollary 2.13. The existence of η, γ and δ is easily verified by differentiating φ.

We only prove (i). We observe that the limit

$$\lim_{t\to 0^+} \frac{\int_0^t \varphi}{t^{1/p}} = \lim_{t\to 0^+} p\frac{\varphi(t)}{t^{(1/p)-1}} = \ell \in [0,\infty]$$

is finite if and only if $p > 1/(1+\alpha)$. It follows from Theorem 2.12 (ii) that $M_{\tau_h\varphi}$ is of restricted weak type (p, p) if and only if $p > 1/(1+\alpha)$.

On the other hand, $M_{\tau_h\varphi}$ is of weak type (p, p) if $p > 1/(1+\alpha)$. In fact, if we choose $\varepsilon > 0$ such that $\alpha - \varepsilon\beta > -1/p'$, then $\varphi^{p'}$ is integrable on $(0,\gamma)$ since $\varphi(s) \leqslant Cs^{\alpha-\varepsilon\beta}$ for small s. By Theorem 2.12 (i), $M_{\tau_h\varphi}$ is of weak type (p, p).

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