1.

Weighted inequalities for a maximal function in the real line

A. L. Bernardis* F. J. Martín-Reyes†

Abstract

We consider the maximal operator defined in the real line by $M_\alpha f(x) = \sup_{R>0} \frac{1}{(2R)^{1+\alpha}} \int_{R<|x-y|<2R} |f(y)|(|x-y|-R)^\alpha \, dy, \quad -1 < \alpha \leq 0$, which is related to the Cesàro convergence of the singular integrals. We characterize the weights w for which M_α is of weak type, strong type and restricted weak type (p,p) with respect to the measure $w(x)\,dx$.

1 Introduction

In this paper we are interested in the study of the boundedness in weighted L^p-spaces of the maximal operator M_α acting on measurable functions on $\mathbb{R}$ and defined by

$$M_\alpha f(x) = \sup_{R>0} \frac{1}{(2R)^{1+\alpha}} \int_{R<|x-y|<2R} |f(y)|(|x-y|-R)^\alpha \, dy, \quad -1 < \alpha \leq 0.$$

This operator is interesting by itself and it is useful in the study of the Cesàro-α convergence of singular integrals associated to Calderón-Zygmund kernels (see [1]). Furthermore, M_α is, up to constants, a particular case of the maximal function of positive convolution operators associated with approximations of the identity given by

$$M_\varphi f(x) = \sup_{R>0} \frac{1}{R} \int_{\mathbb{R}} \varphi\left(\frac{x-y}{R}\right) f(y)\, dy.$$

The operator M_φ was studied in [4] providing access to the study of the Cesàro continuity of order less than one.

*Supported by CONICET, Prog. CAI+D - UNL and PICT 98 (Código: 03-04186)

†Partially supported by Proy. FOMEC, D.G.I.C.Y.T. grant (PB97-1097) and Junta de Andalucía.

On one hand, it follows from [4, Theorem 1] that if $\alpha > -1$ then M_α is of restricted weak type $(\frac{1}{1+\alpha}, \frac{1}{1+\alpha})$ and, consequently, it is of strong type (p,p) for $p > \frac{1}{1+\alpha}$. On the other hand, it was proved in [1] that if w is in the Muckenhoupt $A_{p(1+\alpha)}$ class and $p > \frac{1}{1+\alpha}$ then M_α is of strong type (p,p) with respect to $w(x)dx$, while if $w \in A_1$ then M_α is of restricted weak type $(\frac{1}{1+\alpha}, \frac{1}{1+\alpha})$ with respect to $w(x)dx$. The aim of this paper is to characterize the weighted inequalities of restricted weak type, weak type and strong type for M_α. Our results refer only to the case of equal weights.

The study of the boundedness of M_α in weighted L^p-spaces has two main difficulties. The first one is the kernel $(|x-y|-R)^\alpha$. The second one is to find a noncentered maximal operator pointwise equivalent to M_α as in the case of the Hardy-Littlewood maximal operator, i.e., as in the case $\alpha = 0$.

The paper is organized as follows: we introduce in §2 a noncentered version of M_α and we prove that it is pointwise equivalent to M_α; Sections 3 and 4 are devoted to characterize the weighted weak and strong type (p,p) inequalities, while the restricted weak type inequalities with weights are studied in §5. The main results in the paper are in Theorems 3.1 and 4.3, where we prove the equivalence for $p > 1$ of the weighted weak type (p,p) inequality, the weighted strong type (p,p) inequality for M_α and the fact that w satisfies the following condition: there exists $C > 0$ such that for any interval I

$$\left(\int_I w(s)\,ds\right)^{1/p} \left(\int_I w^{1-p'}(s)|s-x|^{\alpha p'}\,ds\right)^{1/p'} \le C|I|^{1+\alpha},$$

where x is the center of I, $|I|$ is the length of I and $1/p + 1/p' = 1$. In the final section we observe some relations between the good weights for M_α and the Muckenhoupt A_p-weights.

Throughout the paper, we shall use the following notations: If x and R are real numbers with $R > 0$, the interval $(x-R, x+R)$ is denoted by $I(x,R)$. If $I = I(x,R)$ and λ is a positive number then $\lambda I = I(x, \lambda R)$ while ∂I is the border of I, i.e., the set $\{x-R, x+R\}$. If $s, t \in \mathbb{R}$ and $A \subset \mathbb{R}$, $d(s,t)$ and $d(s,A)$ are the euclidean distances from s to t and to A, respectively. By $|A|$ and $w(A)$ we denote the measure of A and the integral $\int_A w(s)\,ds$, respectively. If $1 < p < \infty$ then p' denotes its conjugate exponent. Finally, the letter C means a positive constant nonnecesarily the same at each occurrence.

2 The noncentered maximal function

Observe first that with the notations introduced in §1 we have that

$$M_\alpha f(x) = \sup_{R>0} \frac{1}{|I(x,R)|^{1+\alpha}} \int_{2I(x,R)\setminus I(x,R)} |f(s)| d(s, I(x,R))^\alpha \, ds.$$

Notice also that $M_0 f \leq M_\alpha f$ (since $\alpha \leq 0$) and that $M_0 f$ is pointwise equivalent to the Hardy-Littlewood maximal function

$$Mf(x) = \sup_{R>0} \frac{1}{|I(x,R)|} \int_{I(x,R)} |f(s)| \, ds.$$

We define the noncentered maximal operator N_α associated with M_α as

$$N_\alpha f(x) = \sup_{I: x \in \frac{1}{2} I} \frac{1}{|I|^{1+\alpha}} \int_{2I \setminus I} |f(s)| d(s, I)^\alpha \, ds,$$

where the supremum is taken over all the bounded intervals such that $x \in \frac{1}{2} I$. The next proposition shows that M_α and N_α are pointwise equivalent.

(2.1) **Proposition:** *Let $-1 < \alpha \leq 0$. There exists a constant C depending only on α such that $M_\alpha f \leq N_\alpha f \leq C M_\alpha f$, for all measurable function f.*

Proof: The first inequality is obvious. Let $I = I(z, R)$ be an interval such that $x \in \frac{1}{2} I$. Without loss of generality we may assume that $x \in (z - \frac{R}{2}, z]$. Then

$$\int_{2I \setminus I} |f(s)| d(s, I)^\alpha ds = \int_{z-2R}^{z-R} |f(s)| (z - R - s)^\alpha ds + \int_{z+R}^{z+2R} |f(s)| (s - z - R)^\alpha ds = I + II.$$

On one hand, if $L = x - z + R$ we have $R/2 < L \leq R$ and $x - 2L \geq z - 2R$. Thus

$$\begin{aligned} I &\leq \int_{z-2R}^{x-2L} |f(s)| (z - R - s)^\alpha ds + \int_{x-2L}^{x-L} |f(s)| (x - L - s)^\alpha \, ds \\ &\leq (R/2)^\alpha \int_{x-L-R}^{x+L+R} |f(s)| ds + \int_{2I(x,L) \setminus I(x,L)} |f(s)| d(s, I(x,L))^\alpha ds. \end{aligned}$$

On the other hand, if $T = z + R - x$ then $R \leq T < 3/2R$ and $x + 2T \geq z + 2R$. Therefore

$$II \leq \int_{x+T}^{x+2T} |f(s)| (s - x - T)^\alpha ds \leq \int_{2I(x,T) \setminus I(x,T)} |f(s)| d(s, I(x,T))^\alpha ds.$$

Putting together the inequalities we get

$$\begin{aligned}\frac{1}{|I|^{1+\alpha}}\int_{2I\setminus I}|f(s)|d(s,I)^\alpha ds \le C\frac{1}{|I|}&\int_{I(x,L+R)}|f(s)|ds\\ &+\frac{1}{|I|^{1+\alpha}}\int_{2I(x,L)\setminus I(x,L)}|f(s)|d(s,I(x,L))^\alpha ds\\ &+\frac{1}{|I|^{1+\alpha}}\int_{2I(x,T)\setminus I(x,T)}|f(s)|d(s,I(x,T))^\alpha ds.\end{aligned}$$

Since the lengths of the intervals I, $I(x,L+R)$, $I(x,L)$ and $I(x,T)$ are essentially the same, the right hand-side is dominated by $C[Mf(x)+M_\alpha f(x)]\le CM_\alpha f(x)$ and we are done. □

3 Weighted weak type inequalities

The first main result of the paper characterizes the weighted weak type inequalities for the maximal operator M_α by means of a Muckenhoupt type condition.

(3.1) **Theorem:** *Let w be a nonnegative measurable function on $\mathbb{R}$ and let $-1<\alpha\le 0$. If $1<p<\infty$ then the following are equivalent:*

(i) M_α is of weak type (p,p) with respect to $w(x)\,dx$, i.e., there exists C such that $w(\{M_\alpha f>\lambda\})\le C\lambda^{-p}\int|f|^p w$, for all $\lambda>0$ and all $f\in L^p(w)$.

(ii) w satisfies $A_{p,\alpha}$, i.e., there exists C such that for any interval I

$$\left(\int_{\frac{1}{2}I}w\right)^{1/p}\left(\int_{2I\setminus I}w^{1-p'}(s)d(s,I)^{\alpha p'}ds\right)^{1/p'}\le C|I|^{1+\alpha}.$$

(3.2) **Remark:** Observe that for $\alpha<0$ the weighted weak type (p,p) inequality is not possible for $1<p\le\frac{1}{1+\alpha}$ unless $w=0$ a.e., since if $1<p\le\frac{1}{1+\alpha}$ then (ii) does not hold. As a consequence, we have that the weighted weak type $(1,1)$ inequality for M_α with $\alpha<0$ never holds. In other words, the weak type $(1,1)$ inequality makes sense only for $\alpha=0$. In this case (M_0 is pointwise equivalent to the Hardy-Littlewood maximal operator) the weighted weak type inequalities are characterized by the well known Muckenhoupt A_p-conditions. This is the reason why we do not include the case $p=1$ in the statement of the theorem.

Proof of Theorem 3.1: By 2.1, statement (i) is equivalent to the weighted weak type (p,p) inequality for N_α. Then (ii) follows from (i) by standard arguments, i.e., roughly speaking, applying (i) (with N_α) to the functions $w^{1-p'}(s)d(s,I)^{\alpha(p'-1)}\chi_{2I\setminus I}(s)$. In order to prove (ii) $\Rightarrow$ (i) we need to know that w is a doubling weight, i.e., $w(2I) \le Cw(I)$ for all interval I.

(3.3) **Lemma:** *If $1 < p < \infty$, $-1 < \alpha \le 0$ and w satisfies $A_{p,\alpha}$ then w is a doubling weight.*

We postpone the proof of Lemma 3.3 and continue with the proof of the theorem. Assume that (ii) holds. Let $x \in \mathbb{R}$ and let I be any interval with center x. By the Hölder inequality and the $A_{p,\alpha}$ condition we obtain

$$\int_{2I\setminus I} |f(s)|d(s,I)^\alpha ds \le \left(\int_{2I\setminus I} |f|^p w\right)^{1/p} \left(\int_{2I\setminus I} w^{1-p'}(s)d(s,I)^{\alpha p'}ds\right)^{1/p'}$$
$$\le C\left(\int_{2I\setminus I} |f|^p w\right)^{1/p} \left(\int_{\frac{1}{2}I} w\right)^{-1/p} |I|^{1+\alpha}.$$

Since w is a doubling weight (Lemma 3.3), we get

$$\frac{1}{|I|^{1+\alpha}} \int_{2I\setminus I} |f(s)|d(s,I)^\alpha ds \le C\left(\frac{\int_{2I}|f|^p w}{\int_{2I} w}\right)^{1/p}.$$

Therefore

$$M_\alpha f(x) \le C\left[\mathcal{M}_w(|f|^p)\right]^{1/p}(x),$$

where $\mathcal{M}_w g(x) = \sup_{R>0}\left[\frac{1}{w(I(x,R))}\int_{I(x,R)}|g|w\right]$. Now (i) follows from the above inequality and the well known fact that $\mathcal{M}_w$ is of weak type $(1,1)$ with respect to $w(x)dx$. □

Proof of Lemma 3.3: If $I = I(x,R)$ we obtain, by $A_{p,\alpha}$ and the Hölder inequality, that

$$\left(\int_{\frac{1}{2}I} w\right)^{1/p} \left(\int_{2I\setminus I} w^{1-p'}(s)d(s,I)^{\alpha p'}ds\right)^{1/p'} \le C\int_{x+R}^{x+2R} d(s,I)^\alpha ds$$
$$\le C\left(\int_{x+R}^{x+2R} w\right)^{1/p} \left(\int_{x+R}^{x+2R} w^{1-p'}(s)d(s,I)^{\alpha p'}ds\right)^{1/p'}.$$

Since $(x+R, x+2R)$ is contained in $2I \setminus I$ we have

$$w((x-R/2, x+R/2)) \le Cw((x+R, x+2R)),$$

for every $x \in \mathbb{R}$ and all positive R. Applying this property to the intervals $(x-2R, x-R)$ and $(x-R, x-R/2)$ instead of $(x-R/2, x+R/2)$ we obtain that

$$w((x-2R, x-R/2)) \le C[w(\frac{1}{2}I) + w(\frac{1}{4}I)] \le Cw(\frac{1}{2}I).$$

Analogously, we have

$$w((x+R/2, x+2R)) \leq Cw(\frac{1}{2}I).$$

Summing the inequalities we get that $w(2I \setminus \frac{1}{2}I) \leq Cw(\frac{1}{2}I)$. Thus $w(2I) \leq Cw(\frac{1}{2}I) \leq Cw(I)$. $\square$

4 Weighted strong type inequalities

We start establishing different characterizations of the $A_{p,\alpha}$ condition which are a key step for the study of the strong type inequalities. In order to state the result we recall that if μ is a Borel measure then it is said that a nonnegative measurable function w satisfies $A_p(\mu)$, $1 < p < \infty$, if there exists a positive constant C such that

$$\left(\int_I w\, d\mu\right)^{1/p} \left(\int_I w^{1-p'} d\mu\right)^{1/p'} \leq C\mu(I),$$

for all bounded intervals I. See for instance [5]. (If μ is the Lebesgue measure then $A_p(\mu)$ is the Muckenhoupt A_p condition).

(4.1) **Proposition:** *Let $-1 < \alpha \leq 0$ and $p > 1$. Let w be a nonnegative measurable function. The following statements are equivalent:*

(a) w satisfies $A_{p,\alpha}$.

(b) There exists C such that for any interval I

$$\left(\int_I w\right)^{1/p} \left(\int_I w^{1-p'}(s) d(s,x)^{\alpha p'} ds\right)^{1/p'} \leq C|I|^{1+\alpha},$$

where x is the center of I.

(c) The functions $s \to w(s)d(s,z)^{-\alpha}$ satisfy $A_p(\mu_z)$ with a constant independent of $z \in \mathbb{R}$, where $d\mu_z = d(s,z)^\alpha ds$, i.e., there exists C such that for any interval I and all $z \in \mathbb{R}$

$$\left(\int_I w\right)^{1/p} \left(\int_I w^{1-p'}(s) d(s,z)^{\alpha p'} ds\right)^{1/p'} \leq C \int_I d(s,z)^\alpha ds.$$

Proof: It is clear that (c) $\Rightarrow$ (b). Therefore we only have to prove (a) $\Rightarrow$ (b) $\Rightarrow$ (c) and (b) $\Rightarrow$ (a).

(a) $\Rightarrow$ (b). Let $I=(a,b)$, $I^-=(a,x)$ and $I^+=(x,b)$, where x is the center of I. It suffices to establish that

$$\left(\int_I w\right)^{1/p}\left(\int_{I^*} w^{1-p'}(s)d(s,x)^{\alpha p'}ds\right)^{1/p'} \leq C|I|^{1+\alpha}$$

for $I^*=I^-$ and $I^*=I^+$. We shall only prove the inequality for I^- since the another one is proved in a similar way. Let J be the interval with left end point equals x and the same length as I. It is clear that $I^- \subset 2J\setminus J$ and $d(s,x)=d(s,J)$ for all $s\in I^-$. These properties together with the fact that w satisfies $A_{p,\alpha}$ (and, therefore is a doubling weight) give

$$\begin{aligned}\left(\int_I w\right)^{1/p}&\left(\int_{I^-} w^{1-p'}(s)d(s,x)^{\alpha p'}ds\right)^{1/p'}\\ &\leq C\left(\int_{\frac{1}{2}J} w\right)^{1/p}\left(\int_{2J\setminus J} w^{1-p'}(s)d(s,J)^{\alpha p'}ds\right)^{1/p'} \leq C|I|^{1+\alpha}.\end{aligned}$$

(b) $\Rightarrow$ (c). Let $I=(a,b)$. We shall consider the following two cases: (1) $z\in[a,b]$ and (2) $z\notin[a,b]$.

Case (1): Let $J\supset I$ an interval centered at z such that $|I|\leq|J|\leq 2|I|$. Enlarging the interval I to J and applying (b) we obtain

$$\left(\int_I w\right)^{1/p}\left(\int_I w^{1-p'}(s)d(s,z)^{\alpha p'}ds\right)^{1/p'} \leq C|J|^{1+\alpha}=C|I|^{1+\alpha}.$$

If $z=a$ or $z=b$ we are done. If $z\in(a,b)$ we have

$$\begin{aligned}|I|^{1+\alpha}&\leq C\left[(b-z)^{1+\alpha}+(z-a)^{1+\alpha}\right]=C\left[(b-z)^{\alpha}\int_z^b ds+(z-a)^{\alpha}\int_a^z ds\right]\\ &\leq C\left[\int_z^b (s-z)^{\alpha}ds+\int_a^z (z-s)^{\alpha}ds\right]=C\int_a^b d(s,z)^{\alpha}\,ds.\end{aligned}$$

Putting together the last inequalities we obtain (c) for $z\in(a,b)$.

Case (2): We shall prove it only for $z>b$. First observe that the function $g(s)=\left(\frac{d(s,z)}{d(s,b)}\right)^{\alpha p'}$ is decreasing in the interval (a,b). Therefore,

$$\left(\int_I w^{1-p'}(s)d(s,z)^{\alpha p'}ds\right)^{1/p'} \leq \left(\frac{z-a}{b-a}\right)^{\alpha}\left(\int_I w^{1-p'}(s)d(s,b)^{\alpha p'}ds\right)^{1/p'}.$$

Using *Case* (1) ($z = b$) and the fact that $\alpha \le 0$ we obtain

$$\left(\int_I w\right)^{1/p}\left(\int_I w^{1-p'}(s)d(s,z)^{\alpha p'}ds\right)^{1/p'} \le C(z-a)^\alpha|I| \le C\int_I d(z,s)^\alpha ds.$$

(b) $\Rightarrow$ (a). First we observe that (b) implies that w is doubling. Now, let $I = I(x,R)$ any interval. Applying (b) we have

$$\left(\int_{x-2R}^{x} w\right)^{1/p}\left(\int_{x-2R}^{x} w^{1-p'}(s)d(s,x-R)^{\alpha p'}ds\right)^{1/p'} \le CR^{1+\alpha}.$$

Restricting the interval $(x-2R,x)$ in the second integral and applying that w is doubling we obtain

$$\left(\int_{x-R/2}^{x+R/2} w\right)^{1/p}\left(\int_{x-2R}^{x-R} w^{1-p'}(s)d(s,I)^{\alpha p'}ds\right)^{1/p'} \le CR^{1+\alpha}.$$

Analogously we get the same inequality changing the interval $(x-2R,x-R)$ by $(x+R,x+2R)$. Finally, (a) follows adding both inequalities. $\square$

As a consequence of the characterizations obtained in 4.1 we have the following proposition.

(4.2) **Proposition:** *Let $-1 < \alpha \le 0$, $1 < p < \infty$ and let w be a nonnegative measurable function on the real line. If w satisfies $A_{p,\alpha}$ then there exists $\epsilon > 0$, $0 < \epsilon < p-1$, such that w satisfies $A_{p-\epsilon,\alpha}$.*

Proof: We know by Proposition 4.1 that $w(s)d(s,z)^{-\alpha}$ satisfies $A_p(\mu_z)$ with an $A_p(\mu_z)$-constant independent of z. Then (see [5] p.5) there exists $\epsilon > 0$ depending only on the $A_p(\mu_z)$-constant such that $w(s)d(s,z)^{-\alpha}$ satisfies $A_{p-\epsilon}(\mu_z)$ where the $A_{p-\epsilon}(\mu_z)$-constant depends only on the $A_p(\mu_z)$-constant and ϵ. Applying again Proposition 4.1 we obtain that w satisfies $A_{p-\epsilon,\alpha}$. $\square$.

It is clear that Marcinkiewicz's Interpolation Theorem, 3.1, 4.1 and 4.2 give immediately the characterization of the weighted strong type inequality.

(4.3) **Theorem:** *Let $-1 < \alpha \le 0$ and $p > 1$. Let w be a nonnegative measurable function on $\mathbb{R}$. The following statements are equivalent:*

(a) M_α is of strong type (p,p) with respect to $w(x)\,dx$, i.e., there exists C such that $\int |M_\alpha f|^p w \le C \int |f|^p w$, for all $f \in L^p(w)$.

(b) w satisfies $A_{p,\alpha}$ or, equivalently, there exists C such that for any interval I

$$\left(\int_I w\right)^{1/p}\left(\int_I w^{1-p'}(s)d(s,x)^{\alpha p'}ds\right)^{1/p'} \le C|I|^{1+\alpha},$$

where x is the center of I.

5 Restricted weak type inequalities

As we said above, the operator M_α is not of weak type $(\frac{1}{1+\alpha}, \frac{1}{1+\alpha})$ with respect to the Lebesgue measure if $\alpha < 0$ but it is of restricted weak type $(\frac{1}{1+\alpha}, \frac{1}{1+\alpha})$; in other words, M_α satisfies the weak type inequality for characteristic functions or, equivalently, M_α applies the Lorentz space $L_{\frac{1}{1+\alpha},1}(dx)$ into the Lorentz space $L_{\frac{1}{1+\alpha},\infty}(dx)$. Therefore, it is interesting to study the weights w such that $w(\{x : \ M_\alpha\chi_E(x) > \lambda\}) \le C\lambda^{-p}w(E)$ for all $\lambda > 0$ and all measurable sets $E \subset \mathbb{R}$.

(5.1) **Theorem:** *Let w be a nonnegative measurable function on $\mathbb{R}$ and let $-1 < \alpha \le 0$. If $1 \le p < \infty$ then the following are equivalent:*

(a) M_α is of restricted weak type (p,p) with respect to $w(x)\,dx$, i.e., there exists C such that $w(\{x : \ M_\alpha\chi_E(x) > \lambda\}) \le C\lambda^{-p}w(E)$ for all $\lambda > 0$ and all measurable $E \subset \mathbb{R}$.

(b) w satisfies $RA_{p,\alpha}$, i.e., there exists C such that for every interval I and all measurable $E \subset \mathbb{R}$

$$\left(\int_{\frac{1}{2}I} w\right)\left(\int_{E\cap(2I\setminus I)} d(s,I)^\alpha ds\right)^p \le C|I|^{(1+\alpha)p}\int_{E\cap(2I\setminus I)} w.$$

Proof: By using 2.1 we see that (b) follows from (a) since $N_\alpha\chi_{E\cap(2I\setminus I)}(x) \ge \frac{1}{|I|^{1+\alpha}}\int_{E\cap(2I\setminus I)} d(s,I)^\alpha\, ds$, for all $x \in \frac{1}{2}I$. Assume now that (b) holds. We shall need the following lemma.

(5.2) **Lemma:** *If $-1 < \alpha \le 0$ and w satisfies $RA_{p,\alpha}$ then w is a doubling weight.*

We postpone the proof of the lemma and continue with the proof of the theorem. By (b) and the fact that w is a doubling weight we have

$$\frac{\int_{E\cap(2I\setminus I)} d(s,I)^\alpha\, ds}{|I|^{1+\alpha}} \le C\left(\frac{w(E\cap(2I\setminus I))}{w(2I)}\right)^{1/p}.$$

Therefore $M_\alpha\chi_E \le C\left(\mathcal{M}_w\chi_E\right)^{1/p}$. As in the proof of 3.1 we obtain (a) applying that $\mathcal{M}_w$ is of weak type (1,1) with respect to $w(x)dx$. □

Proof of Lemma 5.2: Let $I = I(x,R)$ be any interval. Applying that w satisfies $RA_{p,\alpha}$ with $E = (x+R, x+2R)$ we obtain

$$\left(\int_{x-R/2}^{x+R/2} w\right)\left(\int_{x+R}^{x+2R} d(s,I)^\alpha\, ds\right)^p \le C|I|^{(1+\alpha)p}\int_{x+R}^{x+2R} w,$$

and therefore $w((x-R/2,x+R/2))\leq Cw((x+R,x+2R))$. Now, we continue as in the proof of Lemma 3.3. $\square$

We can give equivalent formulations of the $RA_{p,\alpha}$ condition in the same way as we did with the $A_{p,\alpha}$ condition in 4.1. We collect them in the next proposition. We omit the proof since it is similar to the proof of 4.1.

(5.3) **Proposition:** *Let $-1<\alpha\leq 0$ and $p\geq 1$. Let w be a nonnegative measurable function. The following statements are equivalent:*

(a) w satisfies $RA_{p,\alpha}$.

(b) There exists C such that for any interval I and all measurable $E\subset I$

$$\left(\int_I w\right)\left(\int_E d(s,x)^\alpha ds\right)^p\leq C|I|^{(1+\alpha)p}\int_E w,$$

where x is the center of I.

(c) There exists C such that for any interval I, all measurable $E\subset I$ and all $z\in\mathbb{R}$

$$\left(\int_I w\right)\left(\int_E d(s,z)^\alpha ds\right)^p\leq C\left(\int_I d(s,z)^\alpha\right)^p\int_E w.$$

6 Further results

This section is devoted to establish several relations among the classes of weights considered in the previous sections. Some of them are proven easily; for instance:

(a) If $1<p<\infty$ and $\alpha\leq\beta$ then $A_{p,\alpha}\subset A_{p,\beta}$.

(b) If $1<p\leq q<\infty$ then $A_{p,\alpha}\subset A_{q,\alpha}$.

Others relations appear in the next proposition.

(6.1) **Proposition:** *If $-1<\alpha\leq 0$ and $p(1+\alpha)>1$ then $A_{p(1+\alpha)}\subset A_{p,\alpha}\subset A_p$ and $A_{p,\alpha}\neq A_r$ for all $r>p(1+\alpha)$.*

Proof: Taking $\beta=0$ in (a) we obtain $A_{p,\alpha}\subset A_p$ and applying the Hölder inequality we get that $A_r\subset A_{p,\alpha}$ for all r with $1<r<p(1+\alpha)$. Keeping in mind that $w\in A_{p(1+\alpha)}$ implies $w\in A_r$ for some $r<p(1+\alpha)$ (see [3] or [5]) we have that $A_{p(1+\alpha)}\subset A_{p,\alpha}$. In order to see that $A_{p,\alpha}\neq A_r$ for all $r>p(1+\alpha)$ we consider the functions $w(x)=|x|^\gamma$. It is well known (see [3]) that $w\in A_r$ if, and only if, $-1<\gamma<r-1$. On the other hand, if $w\in A_{p,\alpha}$ then (Proposition 4.1 (b))

$$\int_{-a}^{a} w^{1-p'}(s)d(s,0)^{\alpha p'}\,ds=\int_{-a}^{a}|s|^{\gamma(1-p')+\alpha p'}\,ds<\infty.$$

This implies that $\gamma < p(1+\alpha) - 1$. Therefore, if $p(1+\alpha) - 1 < \gamma < r - 1$ then $w \in A_r$ and $w \notin A_{p,\alpha}$. $\square$

(6.2) **Remark:** The same argument in the proof of the above proposition shows that if $p(1+\alpha) > 1$ then $w(x) = |x|^\gamma \in A_{p,\alpha}$ if, and only if, $-1 < \gamma < p(1+\alpha) - 1$. However we do not know if $A_{p(1+\alpha)}$ is equal to $A_{p,\alpha}$ for $\alpha < 0$.

Now we check the same kind of relations among the classes $RA_{p,\alpha}$. Clearly, properties (a) and (b) also hold for the classes $RA_{p,\alpha}$. If we denote the classes $RA_{p,0}$ by RA_p we can prove the following proposition.

(6.3) **Proposition:** *If $-1 < \alpha \leq 0$ and $p(1+\alpha) \geq 1$ then $RA_{p(1+\alpha)} \subset RA_{p,\alpha} \subset RA_p$ and $RA_{p,\alpha} \neq RA_r$ for all $r > p(1+\alpha)$.*

Proof: The relation $RA_{p,\alpha} \subset RA_p$ is obvious. Now, let $w \in RA_{p(1+\alpha)}$. Then (Proposition 5.3 (b) for $\alpha = 0$)

$$\left(\int_I w\right) |E|^{p(1+\alpha)} \leq C|I|^{p(1+\alpha)} \int_E w$$

for all interval I and all measurable $E \subset I = I(x,R)$. Since $\int_E d(s,x)^\alpha\, ds \leq |E|^{1+\alpha}$ we obtain that $w \in RA_{p,\alpha}$, by Proposition 5.3. In order to prove that $RA_r \neq RA_{p,\alpha}$ for all $r > p(1+\alpha)$, let us consider $w(x) = |x|^{r-1}$. It was noticed in [2] that $w \in RA_r$. However w does not belong to $RA_{p,\alpha}$ because if $w \in RA_{p,\alpha}$ then (Proposition 5.3 (b))

$$\left(\int_{-a}^{a} w\right) \left(\int_0^\epsilon d(s,0)^\alpha\, ds\right)^p \leq C(2a)^{p(1+\alpha)} \int_0^\epsilon w,$$

for all a and ϵ with $0 < \epsilon < a$ or, equivalently, $(a/\epsilon)^r \leq C\,(a/\epsilon)^{p(1+\alpha)}$, $0 < \epsilon < a$, which is a contradiction since $r > p(1+\alpha)$. $\square$

(6.4) **Remark:** With the same arguments as above we can easily see that $w(x) = |x|^\gamma \in RA_{p,\alpha}$ if and only if $-1 < \gamma \leq p(1+\alpha) - 1$. On the other hand, as in the case of the $A_{p,\alpha}$ classes, we do not know if $RA_{p(1+\alpha)}$ is equal to $RA_{p,\alpha}$ when $\alpha < 0$ but the equality holds when p is the endpoint, i.e., if $p = \frac{1}{1+\alpha}$.

(6.5) **Proposition:** $RA_{\frac{1}{1+\alpha},\alpha} = RA_1 = A_1$.

Proof: First, notice that by 6.3 ($p = \frac{1}{1+\alpha}$), we have that $RA_1 \subset RA_{\frac{1}{1+\alpha},\alpha}$. Second, RA_1 is clearly equivalent to A_1 since the restricted weak type $(1,1)$ is equivalent to the weak type $(1,1)$. It only remains to show that $RA_{\frac{1}{1+\alpha},\alpha} \subset A_1$. Let $w \in RA_{\frac{1}{1+\alpha},\alpha}$ and let I be any bounded interval. Applying Proposition 5.3 (c) to $E = (z - \epsilon, z + \epsilon) \subset I$ we get

$$w(I) \left(\int_{z-\epsilon}^{z+\epsilon} d(s,z)^{\alpha} \, ds \right)^{\frac{1}{1+\alpha}} \leq C|I| \int_{z-\epsilon}^{z+\epsilon} w.$$

Thus $\frac{1}{|I|} \int_I w \leq C \frac{1}{2\epsilon} \int_{z-\epsilon}^{z+\epsilon} w$. If we let ϵ go to 0 we obtain $\frac{1}{|I|} \int_I w \leq Cw(z)$ for almost every $z \in I$, i.e., w is in the class A_1 of Muckenhoupt. □.

References

[1] A. L. Bernardis and F. J. Martín-Reyes, *Singular integrals in the Cesáro sense*, Preprint.

[2] H.-M. Chung, R. A. Hunt and D. S. Kurtz, *The Hardy-Littlewood maximal function on $L(p, q)$ spaces with weights*, Indiana University Mathematics Journal, **31**, 1982, 109–120.

[3] J. García-Cuerva and J. L. Rubio de Francia, *Weighted norm inequalities and related topics*, North-Holland, Amsterdam, New York, Oxford, 1985.

[4] W. Jourkat and J. Troutman, *Maximal inequalities related to generalized a.e.*, Trans. Amer. Math. Soc., **252**, 1979, 49–64.

[5] J.-O. Stromberg y A. Torchinsky, *Weighted Hardy inequalities*, Lecture Notes in Mathematics, **1381**, 1989, Springer-Verlag Berlin Heidelberg.

PEMA - INTEC, Güemes 3450, (3000) Santa Fe, Argentina.
E-mail address: `bernard@alpha.arcride.edu.ar`

Departamento de Análisis Matemático, Facultad de Ciencias, Universidad de Málaga, 29071 Málaga, Spain.
E-mail address: `martin@anamat.cie.uma.es`