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ORLICZ BOUNDEDNESS FOR CERTAIN CLASSICAL OPERATORS

BY

E. HARBOURE, O. SALINAS AND B. VIVIANI (Santa Fe)

Abstract. Let ϕ and ψ be functions defined on $[0,\infty)$ taking the value zero at zero and with non-negative continuous derivative. Under very mild extra assumptions we find necessary and sufficient conditions for the fractional maximal operator M_Ω^α, associated to an open bounded set Ω, to be bounded from the Orlicz space $L^\psi(\Omega)$ into $L^\phi(\Omega)$, $0 \le \alpha < n$. For functions ϕ of finite upper type these results can be extended to the Hilbert transform $\widetilde{f}$ on the one-dimensional torus and to the fractional integral operator I_Ω^α, $0<\alpha<n$. Since these operators are linear and self-adjoint we get, by duality, boundedness results near infinity, deriving in this way some generalized Trudinger type inequalities.

1. Introduction and preliminaries. Let Ω be an open bounded set in $\mathbb{R}^n$. For $0 \le \alpha < n$ we consider the following centered maximal operators associated to Ω:

$$\mathcal{M}_\Omega^\alpha f(x) = \sup_{B(x,r)\subset\Omega} \frac{1}{|B(x,r)|^{1-\alpha/n}} \int_{B(x,r)} |f(y)|\,dy, \qquad x\in\Omega, \tag{1.1}$$

$$M_\Omega^\alpha f(x) = \sup_{r>0} \frac{1}{|\Omega\cap B(x,r)|^{1-\alpha/n}} \int_{B(x,r)} |f(y)|\,dy, \qquad x\in\Omega, \tag{1.2}$$

for $f\in L^1(\Omega)$, where $B(x,r)$ denotes the euclidean ball centered at $x\in\Omega$ with radius $r>0$ and, as usual, $|E|$ is the Lebesgue measure of the set E. When $\alpha=0$ we will drop the index α, writing only $\mathcal{M}_\Omega$ or M_Ω.

Regarding these operators it may be useful to make some comments.

First, the main reason to consider these two maximal operators is that, at least for good domains, they enclose in between other maximal functions. In fact, if M_α denotes the classical centered fractional maximal function in $\mathbb{R}^n$ (i.e. any of the above when $\Omega=\mathbb{R}^n$) and if we use a bar to indicate the corresponding non-centered maximal operator, the following chain of

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inequalities holds for $x \in \Omega$ and f supported in Ω and measurable:

$$\mathcal{M}_\Omega^\alpha f(x) \le \overline{\mathcal{M}}_\Omega^\alpha f(x) \le \overline{M}_\alpha f(x) \simeq M_\alpha f(x) \le M_\Omega^\alpha f(x) \le \overline{M}_\Omega^\alpha f(x).$$

Moreover if Ω has the property that there exists a constant C such that for any $z \in \Omega$ and $r > 0$,

$$|\Omega \cap B(z, 2r)| \le C|\Omega \cap B(z, r)|,$$

i.e. Ω with the euclidean metric and Lebesgue measure is a space of homogeneous type, the last two maximal functions above are equivalent and actually they agree with the fractional maximal function usually defined on this kind of spaces. Examples of such domains are cubes, balls or more generally, any open convex set in $\mathbb{R}^n$.

Secondly, no matter what the shape of Ω is, the centered maximal function M_Ω^α is of weak type $(1, n/(n-\alpha))$, $0 \le \alpha < n$, as a consequence of the Besicovitch covering lemma (see for example [G]). For the non-centered version, this lemma cannot be applied. Instead, for $n > 1$, a Vitali covering lemma is used, involving some kind of "doubling condition" on the measure of balls.

Finally we make the obvious remark that all the operators above are of strong type $(n/\alpha, \infty)$ for $0 < \alpha < n$, and (∞, ∞) when $\alpha = 0$.

Related to the maximal operators above, we will work with the Hilbert transform on the one-dimensional torus $\mathcal{T}$ and the fractional integral on a bounded measurable subset Ω of $\mathbb{R}^n$, that is,

$$\widetilde{f}(\theta) = \frac{1}{\pi} \lim_{\varepsilon \to 0} \int\limits_{\pi > |\theta - t| > \varepsilon} f(t) \cot\left(\frac{\theta - t}{2}\right) dt, \qquad \theta \in [-\pi, \pi], \tag{1.3}$$

where f is an integrable function defined on $[-\pi, \pi]$ and extended periodically to $\mathbb{R}$, and for $0 < \alpha < n$,

$$I_\Omega^\alpha f(x) = \int\limits_\Omega \frac{f(y)}{|x-y|^{n-\alpha}}\, dy, \qquad x \in \Omega. \tag{1.4}$$

Now, we turn our attention to the relevant function spaces. We recall that a *growth function*, that is, a non-negative increasing function ϕ defined on $[0, \infty)$ with $\lim_{t \to 0^+} \phi(t) = 0$, is said to be of *lower type p* if there exists a constant C such that

$$\phi(st) \le C s^p \phi(t) \qquad \text{for } s \in [0,1] \text{ and } t \ge 0. \tag{1.5}$$

Similarly, ϕ is said to be of *upper type q* if there is a constant C such that

$$\phi(st) \le C s^q \phi(t) \qquad \text{for } s \ge 1 \text{ and } t > 0. \tag{1.6}$$

Whenever there is a $p > 0$ satisfying (1.5) we shall say the ϕ is of *positive lower type*, and in the case that there is a finite q for which (1.6) holds, ϕ will be said of *finite upper type*. The latter condition is known to be equivalent to the so-called Δ_2 *condition*, that is, $\phi(2t) \le C\phi(t)$ for some constant C and any $t > 0$.

Frequently, only the behavior of ϕ away from the origin will matter; we then say that ϕ is of finite upper type at infinity and so on.

For a non-negative increasing function ϕ defined on $[0, \infty)$ with $\lim_{t\to 0^+} \phi(t) = 0$ we denote by $L^\phi(\Omega)$ the class of all measurable functions on Ω for which $\int_\Omega \phi(C|f|) < \infty$ for some positive constant C. It is clear that for Ω of finite measure the space $L^\phi(\Omega)$ will remain the same if we change the values of ϕ in a neighborhood of the origin since for any $\lambda > 0$,

$$\int\limits_{\Omega\cap\{x:|f|\le\lambda\}} \phi(C|f|) \le \phi(C\lambda)|\Omega| < \infty.$$

The *Luxemburg norm* is introduced as the quantity

$$\|f\|_{L^\phi(\Omega)} = \inf\Big\{\lambda > 0 : \int\limits_\Omega \phi(|f|/\lambda) \le 1\Big\},$$

That this quantity is finite for $f \in L^\phi(\Omega)$ is a consequence of the Lebesgue dominated convergence theorem. When ϕ is a convex growth function it gives a norm on $L^\phi(\Omega)$ which makes $L^\phi(\Omega)$ a Banach space. If we just know that ϕ is of positive lower type, the quantity $\|\ \|_{L^\phi}$ defines a translation invariant quasi-metric, turning $L^\phi(\Omega)$ into a metrizable topological vector space. Moreover the metric can be chosen to be translation invariant. We notice that when $\phi(t) = t^p$, $p \ge 1$, we obtain the Lebesgue spaces. We will keep the classical notation $L^p(\Omega)$ when we need to refer to these specific cases.

Finally, notice that if T is an operator defined on integrable functions and satisfying the sublinearity conditions

$$T(\lambda f) = \lambda T(f), \qquad \lambda > 0, \tag{1.7}$$

$$T(f+g) \le T(f) + T(g), \tag{1.8}$$

in the almost everywhere sense, an inequality of the type $\|T(f)\|_{L^\phi} \le C\|f\|_{L^\psi}$ implies L^ψ-L^ϕ continuity of T and conversely. Moreover if the operator further satisfies a monotonicity condition of the type

$$|f| \le g \ \Rightarrow\ T(f) \le T(g), \tag{1.9}$$

then L^ψ-L^ϕ continuity and boundedness are each equivalent to the following mapping property of the operator: $f \in L^\psi(\Omega) \Rightarrow T(f) \in L^\phi(\Omega)$. We write these remarks in the following proposition and outline part of the proof.

(1.10) Proposition. *Let T be an operator defined on integrable functions and satisfying* (1.7)–(1.9). *Let ϕ, ψ be two growth functions such that*

ψ is convex and ϕ has a positive lower type. Then the following statements are equivalent:

(1.11) *T is continuous from $L^{\psi}(\Omega)$ into $L^{\phi}(\Omega)$.*

(1.12) $\|T(f)\|_{L^{\phi}(\Omega)} \le C\|f\|_{L^{\psi}(\Omega)}$.

(1.13) $f \in L^{\psi}(\Omega) \Rightarrow Tf \in L^{\phi}(\Omega)$.

Proof. To show (1.11)$\Rightarrow$(1.12), assume that (1.12) does not hold. Then for any $n \in \mathbb{N}$ we can find functions $f_n \in L^{\psi}$ such that

$$\|T(f_n)\|_{L^{\phi}(\Omega)} \ge n^3\|f_n\|_{L^{\psi}(\Omega)}. \tag{1.14}$$

Because of (1.9) we have $T(f_n) \le T(|f_n|)$ and, making use of (1.7) and (1.8), we also get $-T(f_n) \le T(-f_n)$ to conclude that $|T(f_n)| \le T(|f_n|)$. So the sequence $\{f_n\}$ can be assumed to be non-negative.

Also, by (1.14), taking $g_n = f_n/\|f_n\|_{L^{\psi}(\Omega)}$ we get a sequence of non-negative functions with L^{ψ} norm 1 such that

$$\|T(g_n/n^2)\|_{L^{\phi}(\Omega)} \ge n.$$

Now the series $g = \sum g_n/n^2$ defines an $L^{\psi}(\Omega)$-function since it is an absolutely convergent series in a Banach space. Moreover the series also converges almost everywhere to a non-negative function. Clearly $g_n/n^2 \le g$ and hence (1.9) together with the last inequality gives

$$\|T(g)\|_{L^{\phi}(\Omega)} \ge n \quad \forall n \in \mathbb{N},$$

so $T(g)$ does not belong to $L^{\phi}(\Omega)$.

The proof that (1.13) implies (1.12) is quite similar to the above. The statement (1.11) follows easily from (1.12) and the inequality $|Tf - Tg| \le T(|f-g|)$, which is inmediate from (1.8) and (1.9). Finally, it is obvious that (1.13) follows from (1.12). ■

Finally, we remark that all the operators under consideration, except for the Hilbert transform, satisfy the three conditions stated in the proposition.

2. Main theorems. Let a and b be positive continuous functions defined on $[0,\infty)$. We also suppose that b is non-decreasing and $b(s) \to \infty$ as $s \to \infty$. We define

$$\phi(t) = \int_0^t a(s)\,ds \quad \text{and} \quad \psi(t) = \int_0^t b(s)\,ds \tag{2.1}$$

for $t \ge 0$. Assume, in addition, that ϕ is of positive lower type.

(2.2) Theorem. *Under the above assumptions the following conditions are equivalent*:

(2.3) *There exists a constant C such that*

$$\int_1^t \frac{a(s)}{s}\,ds \le Cb(Ct) \quad \textit{for every } t \ge 1.$$

(2.4) *There exists a constant C such that*

$$\int_\Omega \phi(M_\Omega f(x))\,dx \le C + C\int_\Omega \psi(C|f(x)|)\,dx \quad \textit{for every } f \textit{ in } L^1(\Omega).$$

(2.5) *There exists a constant C such that*

$$\|M_\Omega f\|_{L^\phi(\Omega)} \le C\|f\|_{L^\psi(\Omega)} \quad \textit{for every } f \textit{ in } L^1(\Omega).$$

(2.6) *There exists a constant C such that*

$$\int \phi(\mathcal{M}_\Omega f(x))\,dx \le C + C\int_\Omega \psi(C|f(x)|)\,dx \quad \textit{for every } f \textit{ in } L^1(\Omega).$$

(2.7) *There exists a constant C such that*

$$\|\mathcal{M}_\Omega f\|_{L^\phi(\Omega)} \le C\|f\|_{L^\psi(\Omega)} \quad \textit{for every } f \textit{ in } L^1(\Omega).$$

(2.8) Remark. The proof that (2.3) implies (2.4) follows the lines of the similar result contained in Theorem 2.1 of [K] (moreover, this reasoning can be applied to any operator of weak type $(1,1)$ and (∞,∞)). However, the converse is proved by a direct and simpler argument.

The above theorem can be extended to the case of the fractional maximal operators defined in (1.1) and (1.2) as follows:

(2.9) Theorem. *Let ϕ and ψ be as in* (2.1) *and $0<\alpha<n$. If $s^{n/\alpha-1}/b(s)$ is increasing and tends to infinity as $s\to\infty$, then the following statements are equivalent*:

(2.10) *There exist constants C_1 and C_2 such that*

$$\int_1^{C_1 t^{1-\alpha/n} b(t)^{-\alpha/n}} \frac{a(s)}{s^{n/(n-\alpha)}}\,ds \le C_2 b(C_2 t)^{n/(n-\alpha)} \quad \textit{for every } t \ge 1.$$

(2.11) *There exists a constant C such that*

$$\|M_\Omega^\alpha f\|_{L^\phi(\Omega)} \le C\|f\|_{L^\psi(\Omega)} \quad \textit{for every } f \textit{ in } L^1(\Omega).$$

(2.12) *There exists a constant C such that*

$$\|\mathcal{M}_\Omega^\alpha f\|_{L^\phi(\Omega)} \le C\|f\|_{L^\psi(\Omega)} \quad \textit{for every } f \textit{ in } L^1(\Omega).$$

(2.13) Remark. It is easy to see that (2.4) with M_Ω replaced by a general sublinear operator implies the corresponding version of (2.5). However, the converse is not always true. Indeed, the above theorem proves that M_Ω^α

is bounded from $L^1(\log^+ L)^{(n-\alpha)/n}$ into $L^{n/(n-\alpha)}$, $0<\alpha<n$, but inequality (2.4) for M_Ω^α is clearly false, as can be seen by taking functions λg for some fixed g in $L^1(\log^+ L)^{(n-\alpha)/n}$ and $\lambda>1$, in place of f.

(2.14) Remark. Theorems (2.2) and (2.9) are even true under the weaker assumptions that $b(s)$ and $s^{n/\alpha-1}/b(s)$ are quasi-increasing. We say that a non-negative function ϱ is *quasi-increasing* if there exist constants $c_1>0$ and $c_2>0$ such that

$$\varrho(t_1)\le c_1\varrho(c_1t_2) \quad \text{for all } t_2\ge t_1\ge c_2.$$

From Theorem (2.9) together with Remark (2.14) we can obtain the following corollary.

(2.15) Corollary. *Let $0<\alpha<n$ and ϕ be as in* (2.1). *If $\phi(s)/s^{n/(n-\alpha)}$ is non-decreasing and tends to infinity as $s\to\infty$, then the followings statements are equivalent*:

(2.16) *There exists a constant C such that*

$$\int_1^t \frac{a(s)}{s^{n/(n-\alpha)}}\,ds\le C\,\frac{\phi(Ct)}{t^{n/(n-\alpha)}} \quad \textit{for every } t\ge 1.$$

(2.17) *There exists a constant C such that*

$$\|M_\Omega^\alpha f\|_{L^\phi(\Omega)}\le C\|f\|_{L^\psi(\Omega)} \quad \textit{for every } f\in L^1(\Omega),$$

where $\psi^{-1}(t)=t^{\alpha/n}\phi^{-1}(t)$.

The good-λ type inequalities relating the distribution functions of the Hilbert transform to the Hardy–Littlewood maximal function, and the fractional integral to the fractional maximal operator, together with Theorems (2.2) and (2.9), allow us to obtain the following results for the Hilbert transform and the fractional integral operator I_Ω^α defined in (1.3) and (1.4) respectively.

(2.18) Theorem. *Let ϕ and ψ be as in* (2.1). *Assume further that ϕ is of finite upper type at infinity. Then the following statements are equivalent*:

(2.19) *There exists a constant C such that*

$$\int_1^t \frac{a(s)}{s}\,ds\le Cb(Ct) \quad \textit{for every } t\ge 1.$$

(2.20) *There exists a constant C such that*

$$\int_{\mathcal T}\phi(|\widetilde f(\theta)|)\,d\theta\le C+C\int_{\mathcal T}\psi(|f(\theta)|)\,d\theta \quad \textit{for every } f\in L^1(\mathcal T).$$

(2.21) *There exists a constant C such that*

$$\|\widetilde f\|_{L^\phi(\mathcal T)}\le C\|f\|_{L^\psi(\mathcal T)} \quad \textit{for every } f\in L^1(\mathcal T).$$

(2.22) THEOREM. *Let* ϕ *and* ψ *be as in* (2.9) *and* $0 < \alpha < n$. *Assume further that* ϕ *is of finite upper type at infinity. Then the following statements are equivalent*:

(2.23) *There exist constants* C_1 *and* C_2 *such that*

$$\int_1^{C_1 t^{1-\alpha/n} b(t)^{-\alpha/n}} \frac{a(s)}{s^{n/(n-\alpha)}}\, ds \le C_2 b(C_2 t)^{n/(n-\alpha)} \quad \textit{for every } t \ge 1.$$

(2.24) *There exists a constant* C *such that*

$$\|I_\Omega^\alpha f\|_{L^\phi(\Omega)} \le C\|f\|_{L^\psi(\Omega)} \quad \textit{for every } f \textit{ in } L^1(\Omega).$$

Recall that if a linear operator T is bounded from a Banach space X into a Banach space Y, then its adjoint T^* is bounded from the dual space Y^* into X^*. On the other hand, it is well known (see, for instance, [RR]) that, for every Young function ϕ of finite upper type, the Orlicz space $L^{\widetilde{\phi}}$ coincides with the dual space of L^ϕ ($\widetilde{\phi}$ denotes the complementary function of ϕ, defined by $\widetilde{\phi}(y) = \sup\{x|y| - \phi(x) : x \ge 0\}$). These facts allow us to obtain the following corollaries of Theorems (2.18) and (2.22).

(2.25) COROLLARY OF THEOREM (2.18). *Let* ϕ *and* ψ *be as in* (2.1). *Assume that both are Young functions of finite upper type at infinity. If* a *and* b *satisfy* (2.19), *then there exists a constant* C *such that*

$$\|\widetilde{f}\|_{L^{\tilde{\psi}}(\mathcal{T})} \le C\|f\|_{L^{\tilde{\phi}}(\mathcal{T})} \quad \textit{for every } f \in L^1(\mathcal{T}). \tag{2.26}$$

(2.27) COROLLARY OF THEOREM (2.22). *Let* ϕ *and* ψ *be as in Theorem* (2.9). *Assume further that both are Young functions of finite upper type at infinity. If* a *and* b *satisfy* (2.23), *then there exists a constant* C *such that*

$$\|I_\Omega^\alpha f\|_{L^{\tilde{\psi}}(\Omega)} \le C\|f\|_{L^{\tilde{\phi}}(\Omega)} \quad \textit{for every } f \in L^1(\Omega). \tag{2.28}$$

(2.29) REMARK. Note that when ϕ and ψ are as in (2.1) (respectively as in Theorem (2.9)), and $\widetilde{\phi}$ and $\widetilde{\psi}$ are Young functions with ψ, ϕ and $\widetilde{\phi}$ of finite upper type, the duality argument can be applied to prove that (2.26) (resp. (2.28)) is equivalent to (2.19) (respectively (2.23)).

Some examples. We present several pairs of functions ψ, ϕ to which our theorems can be applied, and, in some cases, we relate them to some previously known results appearing in the literature. In what follows we will use the symbol "$\sim$" to indicate that the functions involved behave in the same way at infinity.

1) The following pairs of functions satisfy the condition (2.3) (they were taken from [K]):

$$\psi(t) = \phi(t) = \frac{1}{p}t^p \quad \text{for } 1 < p < \infty, \tag{2.30}$$

$$\text{(2.31)} \qquad \begin{cases} \psi(t) \sim t(\log t)^{\delta}, \\ \phi(t) \sim \dfrac{t}{(\log t)^{1-\delta}}, \end{cases} \qquad \text{for } 0 < \delta \leq 1,$$

$$\text{(2.32)} \qquad \begin{cases} \psi(t) \sim tL_n(t), \\ \phi(t) \sim \dfrac{t}{L_1(t)\ldots L_n(t)}, \end{cases} \qquad \text{for } n \geq 2,$$

where $L_1(t) = \log t$ and $L_n(t) = \log L_{n-1}(t)$.

Thus, each of these pairs can be used to obtain (2.4) or, equivalently, (2.5) for the Hardy–Littlewood maximal operator. Since ϕ is of finite upper type in each case, the same pairs allow us to get (2.20) or (2.21) for the conjugate function $\widetilde{f}$. In particular, the case $\delta = 1$ of (2.31) gives the well known result about boundedness between $L \log L$ and L^1 of the Hardy–Littlewood maximal operator acting on functions with compact support. Corollary (2.25), applied to the same pair, yields another well known result, namely

$$\int_{\mathcal{T}} e^{C|\widetilde{f}(\theta)|/\|f\|_\infty}\, d\theta \leq 1,$$

where $\mathcal{T}$ denotes the one-dimensional torus.

2) The pairs of functions given by

$$\text{(2.33)} \qquad \begin{cases} \psi(t) \sim t(\log t)^{(\delta+1)(n-\alpha)/n}(\log\log t)^{\xi(n-\alpha)/n}, \\ \phi(t) \sim t^{n/(n-\alpha)}(\log t)^{\delta}(\log\log t)^{\xi}, \end{cases}$$

and

$$\text{(2.34)} \qquad \begin{cases} \psi(t) \sim t(\log\log t)^{(\delta+1)(n-\alpha)/n}, \\ \phi(t) \sim t^{n/(n-\alpha)}\dfrac{(\log\log t)^{\delta}}{\log t}, \end{cases}$$

with $0 \leq \alpha < n$, $\delta > -1$ and $\xi \in \mathbb{R}$, satisfy (2.3) for $\alpha = 0$ and (2.10) for $0 < \alpha < n$. Then, for $\alpha = 0$, Theorems (2.2) and (2.18) give us boundedness results for M and $\widetilde{f}$ respectively. Similarly, for the case $0 < \alpha < n$, Theorem (2.9) gives us inequalities (2.11) for M_Ω^α and (2.12) for $\mathcal{M}_\Omega^\alpha$, while Theorem (2.22) insures that the inequality (2.24) holds for the fractional integral operator.

3) The function

$$\text{(2.35)} \qquad \phi(t) \sim \exp(t^{1/\beta}), \qquad \beta > 0,$$

satisfies (2.16) since it is convex (see [S], p. 515). Then, according to Corollary (2.15), we get (2.17) for M_Ω^α with $\psi(t) \sim t^{n/\alpha}(\log t)^{-\beta n/\alpha}$.

4) It is not too difficult to show that the complementary functions for the pairs given in (2.33) and (2.34) with $0<\alpha<n$ satisfy

$$\text{(2.36)}\qquad \begin{cases} \widetilde{\psi}(t) \sim \exp(t^{n/((\delta+1)(n-\alpha))}(\log t)^{-\xi(\delta+1)}), \\ \widetilde{\phi}(t) \sim t^{n/\alpha}(\log t)^{-\delta(n-\alpha)/\alpha}(\log\log t)^{-\xi(n-\alpha)/\alpha}, \end{cases}$$

and

$$\text{(2.37)}\qquad \begin{cases} \widetilde{\psi}(t) \sim \exp(\exp(t^{n/((n-\alpha)(\delta+1))})), \\ \widetilde{\phi}(t) \sim t^{n/\alpha}(\log t)^{(n-\alpha)/\alpha}(\log\log t)^{-\delta(n-\alpha)/\alpha}, \end{cases}$$

respectively. Furthermore all the functions in (2.33) and (2.34) are of finite upper type. Thus, we can apply Corollary (2.27) to these pairs, recovering results obtained by different methods in [MS], [EK], [EGO1] and [EGO2]. Note, however, a difference in presentation: those authors do not state boundedness results in a direct way. In fact, they are interested in integrability results and prove that, for the pairs (ψ,ϕ) mentioned above, there exists a constant C such that

$$\int_{\Omega} \phi(C|I_\alpha f(x)|)\,dx<\infty$$

for every f in L^ψ. However, according to Proposition (1.10), this statement is equivalent to I_Ω^α being bounded from L^ψ into L^ϕ. In particular all of these results can be considered as extensions of the Trudinger inequality for I_Ω^α (see [GT]):

$$\int_{\Omega} \exp\left(\frac{I_\Omega^\alpha f}{C_1\|f\|_{n/(n-\alpha)}}\right)^{n/(n-\alpha)} \le C_2|\Omega|.$$

In fact, this inequality is a direct consequence of (2.28) applied to the pair (2.36) when $\delta=\xi=0$.

5) For $\alpha=0$, the functions in (2.33) with $\delta>0$ and $\xi\in\mathbb{R}$ give us the boundedness of $\widetilde{f}$ for the pair

$$\text{(2.38)}\qquad \begin{cases} \widetilde{\psi}(t) \sim \exp(t^{1/(\delta+1)}/(\log t)^{\xi/(\delta+1)}), \\ \widetilde{\phi}(t) \sim \exp(t^{1/\delta}/(\log t)^{\xi/\delta}), \end{cases}$$

and when $\delta=0$ and $\xi>0$ for the pair

$$\text{(2.39)}\qquad \begin{cases} \widetilde{\psi}(t) \sim \exp(t/(\log t)^{\xi}), \\ \widetilde{\phi}(t) \sim \exp(\exp t^{1/\xi}). \end{cases}$$

3. Proofs. For simplicity, we will use the same letter, C, to indicate constants, perhaps different, when there is no possibility of confusion.

In the proofs we are going to use the following property of the function ψ defined in (2.1):

$$\frac{1}{2}b\left(\frac{t}{2}\right) \le \frac{\psi(t)}{t} \le b(t). \tag{3.1}$$

It follows easily from the definition and the fact that b is non-decreasing.

Proof of Theorem (2.2). First we prove that (2.3) implies (2.4). Given $\lambda > 0$, we can write

$$f = h_\lambda + g_\lambda,$$

where $h_\lambda = f\chi_{\{|f|\le\lambda\}}$. Then, since

$$M_\Omega f \le M_\Omega(h_\lambda) + M_\Omega(g_\lambda),$$

the (∞,∞) boundedness of M_Ω allows us to obtain

$$|\{x\in\Omega : M_\Omega f(x) > 2\lambda\}| \le |\{x\in\Omega : M_\Omega g_\lambda(x) > \lambda\}|. \tag{3.2}$$

With this estimate, the weak type $(1,1)$ of M_Ω, (2.3) and (3.1), we have

$$\begin{aligned}
\int_\Omega \phi\big(\tfrac{1}{2}M_\Omega f(x)\big)\,dx &\le \int_0^\infty a(\lambda)|\{x\in\Omega : M_\Omega g_\lambda(x) > \lambda\}|\,d\lambda \\
&\le C + C\int_1^\infty \frac{a(\lambda)}{\lambda}\Big(\int_\Omega |g_\lambda(x)|\,dx\Big)\,d\lambda \\
&\le C + C\int_\Omega |f(x)|\bigg(\int_1^{|f(x)|}\frac{a(\lambda)}{\lambda}\,d\lambda\bigg)\,dx \\
&\le C + C\int_\Omega |f(x)|b(C|f(x)|)\,dx \le C + C\int_\Omega \psi(C|f(x)|)\,dx,
\end{aligned}$$

proving (2.4). If we take $f/(C\|f\|_{L^\psi(\Omega)})$ instead of f in (2.4) then, using the lower type of ϕ, we obtain

$$\int_\Omega \phi\bigg(\frac{M_\Omega f(x)}{C\|f\|_{L^\psi(\Omega)}}\bigg)\,dx \le 1$$

for an appropriate constant C, and that is (2.5). Moreover, from the fact that

$$\mathcal{M}_\Omega f \le M_\Omega f,$$

it is clear that (2.4) implies (2.6). It is easy to see that (2.5) and (2.6) each imply (2.7).

Finally, we show that (2.3) follows from (2.7). Without loss of generality, we can assume that Ω contains the unit ball $B_0 = \{x\in\mathbb{R}^n : |x|\le 1\}$. For $\delta > 0$, take $f_\delta(x) = w_n^{-1}\delta^{-n}\chi_{B_0}(x/\delta)$, where $w_n = |B_0|$. Now, given $\delta < 1/8$

and $\lambda \in (2^n w_n^{-1}, w_n^{-1}\delta^{-n}2^{-n-1})$, we have

$$
\begin{aligned}
|\{x : \mathcal{M}_\Omega f_\delta(x) > \lambda\}| &\geq \Big|\Big\{x : \delta < |x| < 1/4 \\
&\qquad \text{and } |B(x, 2|x|)|^{-1} \int\limits_{B(x,2|x|)} f_\delta(y)\, dy > \lambda\Big\}\Big| \\
&= |\{x : \delta < |x| < 1/4 \text{ and } 2^{-n}|x|^{-n}/w_n > \lambda\}| \\
&= |\{x : \delta < |x| < 1/4 \text{ and } |x| < 1/(2w_n^{1/n}\lambda^{1/n})\}| \\
&= C\bigg(\frac{1}{2^n\lambda w_n} - \delta^n\bigg) \geq \frac{C}{\lambda}.
\end{aligned}
$$

From this inequality and (2.7), using the notation $\|\ \|_\psi$ instead of $\|\ \|_{L^\psi(\Omega)}$, we get

$$
\begin{aligned}
1 &\geq \int\limits_\Omega \phi\bigg(\frac{\mathcal{M}_\Omega f_\delta(x)}{C\|f_\delta\|_\psi}\bigg)\, dx \\
&= \int\limits_0^\infty a(\lambda)|\{x \in \Omega : \mathcal{M}_\Omega f_\delta(x) > C\|f_\delta\|_\psi \lambda\}|\, d\lambda \\
&= \int\limits_0^\infty a\bigg(\frac{\lambda}{C\|f_\delta\|_\psi}\bigg)|\{x \in \Omega : \mathcal{M}_\Omega f_\delta(x) > \lambda\}|\, \frac{d\lambda}{C\|f_\delta\|_\psi} \\
&\geq C \int\limits_{w_n^{-1}2^n}^{w_n^{-1}\delta^{-n}2^{-n-1}} a\bigg(\frac{\lambda}{C\|f_\delta\|_\psi}\bigg)\, \frac{d\lambda}{\lambda\|f_\delta\|_\psi} \\
&= C \int\limits_{C_1/\|f_\delta\|_\psi}^{C_2\delta^{-n}/\|f_\delta\|_\psi} a(\lambda)\, \frac{d\lambda}{\lambda\|f_\delta\|_\psi}.
\end{aligned}
$$

Therefore, since $\|f_\delta\|_\psi = w_n^{-1}\delta^{-n}/\psi^{-1}(\delta^{-n}w_n^{-1})$, it follows that

$$
\int\limits_{C_1'\delta^n\psi^{-1}(\delta^{-n}w_n^{-1})}^{C_2'\psi^{-1}(\delta^{-n}w_n^{-1})} \frac{a(\lambda)}{\lambda}\, d\lambda \leq C\frac{\delta^{-n}}{\psi^{-1}(\delta^{-n}w_n^{-1})}.
$$

Taking $t = C_2'\psi^{-1}(\delta^{-n}w_n^{-1})$, we obtain

$$
\int\limits_{C_1't/(C_2'\psi(t/C_2')w_n)}^{t} \frac{a(\lambda)}{\lambda}\, d\lambda \leq Cb(Ct).
$$

Since $t/\psi(t) \leq 2/b(t/2) \to 0$ as $t \to \infty$ by hypothesis, there exists $t_0 > 0$ such that $C_1't/(C_2'\psi(t/C_2')w_n) \leq 1$ for every $t \geq t_0$, so the above inequality allows us to write

$$\int_1^t \frac{a(\lambda)}{\lambda} d\lambda \leq Cb(Ct) \quad \text{for } t \geq t_0.$$

From the fact that $a(\lambda)$ is a continuous function and b is non-decreasing we conclude that the last inequality holds for $t \geq 1$ (perhaps with a different constant), which is (2.3). ■

Proof of Theorem (2.9). To show that (2.10) implies (2.11), we shall apply a similar reasoning to that used in the proof of (2.3)⇒(2.4) of Theorem (2.2).

First, suppose that $\|f\|_{L^\psi(\Omega)} = 1$. Taking $\widetilde{f} = f/(2C_2)$, where C_2 is the constant appearing in (2.10), for $\lambda > 0$ given, we wish to estimate the measure of the set

$$\{x \in \Omega : M_\Omega^\alpha \widetilde{f}(x) > 2\lambda\}.$$

For this purpose we decompose $\widetilde{f}$ as

$$\widetilde{f} = h_s + g_s,$$

where $h_s = \widetilde{f}\chi_{\{|\widetilde{f}| \leq s\}}$ and s is to be fixed later as a function of λ. Clearly,

$$M_\Omega^\alpha \widetilde{f} \leq M_\Omega^\alpha(h_s) + M_\Omega^\alpha(g_s).$$

Since M_Ω^α maps $L^{n/\alpha}(\Omega)$ into $L^\infty(\Omega)$ continuously and $s^{n/\alpha-1}/b(s)$ is increasing, we can write

$$\begin{aligned}
(3.3)\qquad (M_\Omega^\alpha(h_s)(x))^{n/\alpha} &\leq \|h_s\|_{n/\alpha}^{n/\alpha} \leq \int_0^\infty \lambda^{n/\alpha-1} |\{x \in \Omega : |h_s(x)| > \lambda\}| \, d\lambda \\
&\leq \int_0^s \frac{\lambda^{n/\alpha-1}}{b(\lambda)} b(\lambda) |\{x \in \Omega : |\widetilde{f}(x)| > \lambda\}| \, d\lambda \\
&\leq \frac{s^{n/\alpha-1}}{b(s)} \int_\Omega \psi(|\widetilde{f}(x)|) \, dx \leq \frac{s^{n/\alpha-1}}{b(s)} \int_\Omega \psi(|f(x)|) \, dx \\
&= \frac{s^{n/\alpha-1}}{b(s)},
\end{aligned}$$

where the last inequality follows upon assuming $C_2 > 1$ (which is always possible) since ψ is increasing. Now, we want to choose s such that

$$M_\Omega^\alpha h_s(x) \leq \frac{\lambda}{C_1} \quad \forall x \in \Omega,$$

where C_1 is the constant appearing in (2.10). From (3.3), it is enough to choose s satisfying

$$(3.4)\qquad C_1 s^{1-\alpha/n} b(s)^{-\alpha/n} = \lambda.$$

Notice that this choice is possible for λ greater than a constant $\lambda_0 > 1$, since the function $s^{n/\alpha-1}/b(s)$ is continuous for $s > 0$ and tends to infinity as $s \to \infty$ by hypothesis. Moreover since it is also increasing there is only one such s. Therefore

$$(3.5) \qquad |\{x \in \Omega : M_\Omega^\alpha f(x) > 2\lambda/C_1\}| \le |\{x \in \Omega : M_\Omega^\alpha g_s(x) > \lambda/C_1\}|$$

for every $\lambda > \lambda_0$ and $s = s(\lambda)$ defined by (3.4). From this estimate and the weak type $(1, n/(n-\alpha))$ of M_Ω^α, we get

$$\begin{aligned}\int_\Omega \phi\Big(\frac{C_1}{2} M_\Omega^\alpha \widetilde{f}(x)\Big)\, dx &= \Big(\int_0^{\lambda_0} + \int_{\lambda_0}^\infty\Big) a(\lambda)|\{x \in \Omega : M_\Omega^\alpha \widetilde{f}(x) > 2\lambda/C_1\}|\, d\lambda \\ &\le C + \int_1^\infty a(\lambda)|\{x \in \Omega : M_\Omega^\alpha g_s(x) > \lambda/C_1\}|\, d\lambda \\ &\le C + C\int_1^\infty \frac{a(\lambda)}{\lambda^{n/(n-\alpha)}}\Big(\int_\Omega |g_s(x)|\, dx\Big)^{n/(n-\alpha)} d\lambda, \\ &= C + C\int_1^\infty \Big(\int_\Omega \frac{|g_s(x)|a(\lambda)^{(n-\alpha)/n}}{\lambda}\, dx\Big)^{n/(n-\alpha)} d\lambda\end{aligned}$$

where C_1 is the constant appearing in (2.10). Applying Minkowski's integral inequality and (2.10), we have

$$\begin{aligned}&\int_\Omega \phi\Big(\frac{C_1}{2} M_\Omega^\alpha \widetilde{f}(x)\Big)\, dx \\ &\le C + C\bigg(\int_\Omega \Big(\int_1^\infty |g_s(x)|^{n/(n-\alpha)} \frac{a(\lambda)}{\lambda^{n/(n-\alpha)}}\, d\lambda\Big)^{(n-\alpha)/n} dx\bigg)^{n/(n-\alpha)} \\ &\le C + C\bigg(\int_\Omega |\widetilde{f}(x)|\Big(\int_1^{s^{-1}(|\widetilde{f}(x)|)} \frac{a(\lambda)}{\lambda^{n/(n-\alpha)}}\, d\lambda\Big)^{(n-\alpha)/n} dx\bigg)^{n/(n-\alpha)} \\ &= C + C\bigg(\int_\Omega |\widetilde{f}(x)|\Big(\int_1^{C_1|\widetilde{f}(x)|^{1-\alpha/n} b^{-\alpha/n}(|\widetilde{f}(x)|)} \frac{a(\lambda)}{\lambda^{n/(n-\alpha)}}\, d\lambda\Big)^{(n-\alpha)/n} dx\bigg)^{n/(n-\alpha)} \\ &\le C + CC_2 \int_\Omega |\widetilde{f}(x)| b(C_2|\widetilde{f}(x)|)\, dx \\ &\le C + C\int_\Omega \psi(2C_2|\widetilde{f}(x)|)\, dx = C + C\int_\Omega \psi(|f(x)|)\, dx.\end{aligned}$$

From the above inequality and the fact that ϕ is of positive lower type, it is easy to see that there exists a constant C such that

$$\int_{\Omega} \phi(CM_{\Omega}^{\alpha} f(x))\,dx \le 1$$

for every f with $\|f\|_{L^{\psi}(\Omega)} = 1$. Finally, it is clear that the last assumption on f can be removed by taking $f/\|f\|_{L^{\psi}(\Omega)}$ instead of f. This completes the proof of (2.11).

Since

$$\mathcal{M}_{\Omega}^{\alpha} f(x) \le M_{\Omega}^{\alpha} f(x) \quad \text{for a.e. } x \in \Omega,$$

it follows that (2.11) implies (2.12).

In order to prove that (2.12) implies (2.10), we assume, without loss of generality, that Ω contains the unit ball B_0. For $\delta > 0$, we define

$$f_{\delta}(x) = w_n \delta^{-n} b(\delta^{-n})^{-\alpha/n} \chi_{B_0}(x/\delta),$$

where $w_n = |B_0|$. Given δ small enough, and λ belonging to the interval

$$J_{\delta} = (d_0 b(\delta^{-n})^{-\alpha/n}, d_1 b(\delta^{-n})^{-\alpha/n} \delta^{-n+\alpha})$$

where $d_0 = 2^{n-\alpha} w^{\alpha/n-1}$ and $d_1 = 2^{-(1+1/n)(n-\alpha)} w_n^{\alpha/n-1}$, we have

$$\begin{aligned}
|\{x : \mathcal{M}_{\Omega}^{\alpha} f_{\delta}(x) > \lambda\}| &\ge \Big|\Big\{x : \delta < |x| < 1/4 \\
&\qquad \text{and } |B(x, 2|x|)|^{-1+\alpha/n} \int_{B(x,2|x|)} f_{\delta}(y)\,dy > \lambda\Big\}\Big| \\
&= |\{x : \delta < |x| < 1/4 \text{ and } (2|x|)^{\alpha-n} w_n^{\alpha/n-1} b(\delta^{-n})^{-\alpha/n} > \lambda\}| \\
&= \left|\left\{x : \delta < |x| < \frac{1}{2} w_n^{-1/n} \left(\frac{b(\delta^{-n})^{-\alpha/n}}{\lambda}\right)^{1/(n-\alpha)}\right\}\right| \\
&= C\left(\frac{1}{2^n} w_n^{-1} \left(\frac{b(\delta^{-n})^{-\alpha/n}}{\lambda}\right)^{n/(n-\alpha)} - \delta^n\right) \\
&\ge C\left(\frac{b(\delta^{-n})^{-\alpha/n}}{\lambda}\right)^{n/(n-\alpha)}.
\end{aligned}$$

From (2.12) and this estimate, we obtain

$$\begin{aligned}
1 &\ge \int_{\Omega} \phi\left(\frac{\mathcal{M}_{\Omega}^{\alpha} f_{\delta}(x)}{C\|f_{\delta}\|_{L^{\psi}(\Omega)}}\right) dx \\
&= \int_0^{\infty} a(\lambda) |\{x \in \Omega : \mathcal{M}_{\Omega}^{\alpha} f_{\delta}(x) > \lambda C \|f_{\delta}\|_{L^{\psi}(\Omega)}\}|\,d\lambda
\end{aligned}$$

$$= \int_0^\infty a\left(\frac{s}{C\|f_\delta\|_{L^\psi(\Omega)}}\right)|\{x \in \Omega : \mathcal{M}^\alpha_\Omega f_\delta(x) > s\}| \frac{ds}{C\|f_\delta\|_{L^\psi(\Omega)}}$$

$$\geq \frac{Cb(\delta^{-n})^{-\alpha/(n-\alpha)}}{\|f_\delta\|_{L^\psi(\Omega)}} \int_{d_0 b(\delta^{-n})^{-\alpha/n}}^{d_1 b(\delta^{-n})^{-\alpha/n}\delta^{\alpha-n}} a\left(\frac{s}{C\|f_\delta\|_{L^\psi(\Omega)}}\right) \frac{ds}{s^{n/(n-\alpha)}}$$

$$= \frac{Cb(\delta^{-n})^{-\alpha/(n-\alpha)}}{\|f_\delta\|^{n/(n-\alpha)}_{L^\psi(\Omega)}} \int_{\widetilde{d}_0 b(\delta^{-n})^{-\alpha/n}/\|f_\delta\|_{L^\psi(\Omega)}}^{\widetilde{d}_1 b(\delta^{-n})^{-\alpha/n}\delta^{\alpha-n}/\|f_\delta\|_{L^\psi(\Omega)}} \frac{a(s)}{s^{n/(n-\alpha)}}\, ds.$$

Then, since $\|f_\delta\|_{L^\psi(\Omega)} = b(\delta^{-n})^{-\alpha/n}\delta^{-n}w_n^{-1}/\psi^{-1}(\delta^{-n}w_n^{-1})$, we get

$$\int_{\widetilde{d}_0 w_n \delta^n \psi^{-1}(\delta^{-n}w_n^{-1})}^{\widetilde{d}_1 w_n \delta^\alpha \psi^{-1}(\delta^{-n}w_n^{-1})} \frac{a(s)}{s^{n/(n-\alpha)}}\, ds \leq C\left(\frac{\delta^{-n}}{\psi^{-1}(\delta^{-n}w_n^{-1})}\right)^{n/(n-\alpha)}.$$

Taking $t = \psi^{-1}(\delta^{-n}w_n^{-1})$ and applying (3.1), we have

$$\int_{\widetilde{d}_0 2/b(t/2)}^{\widetilde{d}_1 w_n^{1-\alpha/n} b(t)^{-\alpha/n} t^{1-\alpha/n}} \frac{a(s)}{s^{n/(n-\alpha)}}\, ds \leq \int_{\widetilde{d}_0 t/\psi(t)}^{\widetilde{d}_1 w_n^{1-\alpha/n}\psi(t)^{-\alpha/n} t} \frac{a(s)}{s^{n/(n-\alpha)}}\, ds$$

$$\leq C\left(\frac{\psi(t)}{t}\right)^{n/(n-\alpha)} \leq Cb(t)^{n/(n-\alpha)}.$$

But, since $b(t) \to \infty$ as $t \to \infty$, there exists $t_0 > 0$ such that

$$\int_1^{C_1 b(t)^{-\alpha/n} t^{1-\alpha/n}} \frac{a(s)}{s^{n/(n-\alpha)}} ds \leq C_2 b(C_2 t)^{n/(n-\alpha)} \quad \text{for every } t \geq t_0.$$

Finally, using the fact that the functions $s^{n/\alpha-1}/b(s)$ and $b(s)$ are non-decreasing, we clearly deduce the above inequality for $t \geq 1$, which completes the proof of the theorem. ∎

Proof of Corollary (2.15). Assume (2.16). First we observe that from the definition of ϕ in (2.17), we have

$$\psi(t) = \phi(t\psi(t)^{-\alpha/n}). \tag{3.6}$$

Then, taking $b(t) = \psi'(t)$ and using (3.1), for β a constant to be determined later, we have

$$\int_1^{\beta b(t)^{-\alpha/n} t^{1-\alpha/n}} \frac{a(s)}{s^{n/(n-\alpha)}}\, ds \leq C\frac{\phi(C\beta b(t)^{-\alpha/n} t^{1-\alpha/n})}{(\beta b(t)^{-\alpha/n} t^{1-\alpha/n})^{n/(n-\alpha)}}$$

$$\leq C\frac{\phi(C\beta t\psi(t)^{-\alpha/n})}{(\beta b(t)^{-\alpha/n} t^{1-\alpha/n})^{n/(n-\alpha)}}$$

for t large enough. Now, choosing $\beta = 1/C$, from (3.1) and (3.6) we obtain

$$\int_1^{(1/C)b(t)^{-\alpha/n}t^{1-\alpha/n}} \frac{a(s)}{s^{n/(n-\alpha)}}\, ds \le C^{1+n/(n-\alpha)} \frac{\psi(t)}{tb(t)^{-\alpha/(n-\alpha)}} \le C^{1+n/(n-\alpha)} b(t)^{n/(n-\alpha)}. \tag{3.7}$$

If we can prove that $b(s)$ and $s^{n/\alpha-1}/b(s)$ are quasi-increasing and tend to infinity as $s \to \infty$, then from Remark (2.14) and Theorem (2.9) we shall obtain (2.17). In order to see this, first notice that since $\phi(t)/t^{n/(n-\alpha)} \to \infty$ as $t \to \infty$, it follows that $\phi(t) \to \infty$ as $t \to \infty$. Consequently, we have $s/(\phi^{-1}(s))^{n/(n-\alpha)} \to \infty$ as $s \to \infty$. So, from the definition of ψ we have $\psi(s)/s \to \infty$ as $s \to \infty$, and moreover, $\psi(s)/s$ is non-decreasing. These facts together with (3.1) allow us to conclude that $b(s)$ is quasi-increasing and tends to infinity as $s \to \infty$. On the other hand, using again (3.1) and suitable changes of variables, it is easy to see that $s^{n/\alpha-1}/b(s)$ is equivalent to $\psi^{-1}(t)^{n/\alpha}/t$, which is $(\phi^{-1}(t))^{n/\alpha}$. Thus $s^{n/\alpha-1}/b(s)$ is quasi-increasing and tends to infinity as $s \to \infty$.

Let us now prove that (2.17) implies (2.16). As before, we can prove that $b(s)$ and $s^{n/\alpha-1}/b(s)$ are quasi-increasing and tend to infinity as $s \to \infty$; then, from Remark (2.14) and Theorem (2.9), we have

$$\int_1^{C_1 t^{1-\alpha/n} b(t)^{-\alpha/n}} \frac{a(s)}{s^{n/(n-\alpha)}} ds \le C_2 b(C_2 t)^{n/(n-\alpha)} \quad \text{for every } t \ge 1.$$

By a reverse reasoning to that used to prove (3.7), we obtain (2.16). ■

To prove Theorems (2.18) and (2.22) we shall need the following result.

(3.8) Lemma. *Let ϕ be as in* (2.1) *and assume that ϕ is of finite upper type at infinity. Let h and g be two non-negative measurable functions defined on (A,μ), a measure space with $\mu(A) < \infty$, and such that*:

(3.9) (Good-λ inequality) *There exist positive constants C and δ such that*

$$\mu(\{x \in A : h(x) > 2\lambda \text{ and } g(x) \le \beta\lambda\}) \le C\beta^\delta \mu(\{x \in A : h(x) > \lambda\})$$

for all $\lambda > 0$ and $0 < \beta < 1$.

Then there exists a constant C such that

$$\int_A \phi(h(x))\, dx \le C\mu(A) + C \int_A \phi(Cg(x))\, dx.$$

Proof. First notice that we may assume that inequality (1.6) is satisfied for some finite q with $s = 2$ and any $t \ge 1$, i.e., $\phi(2t) \le C\phi(t)$ for all $t \ge 1$.

Using now (3.9), for $M > 0$ we have

$$\begin{aligned}
&\int_0^M a(\lambda)\mu(\{x\in A : h(x)>\lambda\})\,d\lambda \\
&\qquad = 2\int_0^{M/2} a(2\lambda)\mu(\{x\in A : h(x)>2\lambda\})\,d\lambda \\
&\qquad = 2\int_0^{M/2} a(2\lambda)\mu(\{x\in A : h(x)>2\lambda \text{ and } g(x)\le\beta\lambda\})\,d\lambda \\
&\qquad\quad + 2\int_0^{M/2} a(2\lambda)\mu(\{x\in A : h(x)>2\lambda \text{ and } g(x)>\beta\lambda\})\,d\lambda \\
&\qquad \le 2C\beta^\delta\int_0^{M} a(2\lambda)\mu(\{x\in A : h(x)>\lambda\})\,d\lambda \\
&\qquad\quad + \frac{2}{\beta}\int_0^{\infty} a\Big(\frac{2}{\beta}\lambda\Big)\mu(\{x\in A : g(x)>\lambda\})\,d\lambda.
\end{aligned}$$

Take $h_M = h\chi_{\{h\le M\}} + M\chi_{\{h>M\}}$; then from the above inequality and the fact that ϕ is of finite upper type at infinity, we get

$$\begin{aligned}
&\int_0^M a(\lambda)|\{x\in A : h(x)>\lambda\}| \\
&\quad \le 2C\beta^\delta\int_0^{\infty} a(2\lambda)|\{x\in A : h_M(x)>\lambda\}|\,d\lambda + \int_A \phi\Big(\frac{2}{\beta}g(x)\Big)\,d\mu(x) \\
&\quad \le C\beta^\delta\mu(A) + C\beta^\delta\int_{\{x\in A: h_M(x)>1\}} \phi(2h_M(x))\,d\mu(x) + \int_A \phi\Big(\frac{2}{\beta}g(x)\Big)\,d\mu(x) \\
&\quad \le C\mu(A) + C\beta^\delta\int_A \phi(h_M(x))\,d\mu(x) + \int_A \phi\Big(\frac{2}{\beta}g(x)\Big)\,d\mu(x) \\
&\quad \le C\mu(A) + C\beta^\delta\int_0^M a(\lambda)\mu(\{x\in A : h(x)>\lambda\})\,d\lambda + \int_A \phi\Big(\frac{2}{\beta}g(x)\Big)\,d\mu(x).
\end{aligned}$$

Taking β small enough and M tending to infinity, we clearly arrive at the desired conclusion. ∎

Proof of Theorem (2.18). First we assume (2.19). It is well known that the one-dimensional torus is a space of homogeneous type. In this context, it was proved in [A] that a good-λ inequality of the type (3.4) holds between Mf, the Hardy–Littlewood maximal function, and T^*f, the maximal operator associated to a singular integral. Since $\widetilde{f}$ is in fact a singular integral for this particular space, we may apply Lemma (3.8) to get

$$\int_{\mathcal{T}} \phi(|\widetilde{f}^*(\theta)|)\,d\theta \le C + C\int_{\mathcal{T}} \phi(CMf(\theta))\,d\theta.$$

Now, on the one hand we have $|\widetilde{f}(\theta)| \le |\widetilde{f}^*(\theta)|$ a.e. and on the other hand it is easy to see that the Hardy–Littlewood maximal function associated to that space is equivalent to the maximal function $M_{(-\pi,\pi)}$ appearing in Theorem (2.2). Therefore, using that theorem we conclude that (2.20) holds. Clearly, from the fact that ϕ is of positive lower type, (2.20) implies (2.21).

Let us see that (2.19) follows from (2.21). Taking $f_\delta(\theta) = \delta^{-1}\chi_{[-\delta,\delta]}(\theta)$, it is easy to check that

$$\widetilde{f_\delta}(\theta) = \frac{1}{\delta}\log\left|\frac{\sin\frac{\theta-\delta}{2}}{\sin\frac{\theta+\delta}{2}}\right|.$$

Now, for each θ in $(0,\pi/2)$, we take $F_\theta(\delta) = \widetilde{f_\delta}(\theta)$. Then, using linear approximation, we get

$$F_\theta(\delta) = -\frac{\sin\theta}{2\sin^2\dfrac{\theta}{2}} - \frac{1}{2}\frac{\sin\theta\sin\delta_0}{(\cos\delta_0-\cos\theta)^2}\delta \tag{3.10}$$

for $0<\delta_0<\delta$. Since $2\theta/\pi \le \sin\theta \le \theta$ for $\theta\in(0,\pi/2)$, we have

$$\frac{4}{\pi\theta} \le \frac{\sin\theta}{2\sin^2(\frac{\theta}{2})} \le \frac{\pi^2}{2\theta}.$$

Then, from (3.10), we get

$$\frac{4}{\pi\theta} - |E(\delta_0,\theta)|\delta \le |F_\theta(\delta)| \le \frac{\pi^2}{2\theta} + |E(\delta_0,\theta)|\delta, \tag{3.11}$$

where $|E(\delta_0,\theta)| = \sin\theta\sin\delta_0/(2(\cos\delta_0-\cos\theta)^2)$. But from the fact that

$$|E(\delta_0,\theta)| = \frac{1}{8}\frac{\sin\theta\sin\delta_0}{\left(\sin\left(\frac{\theta+\delta_0}{2}\right)\sin\left(\frac{\theta-\delta_0}{2}\right)\right)^2},$$

and assuming $\delta < 4\theta/\pi^5$, it is not difficult to see that

$$\delta|E(\delta_0,\theta)| < \frac{2}{\pi\theta}.$$

So, from (3.11) we have

$$\frac{C_1}{\theta} = \frac{2}{\pi\theta} \le |\widetilde{f_\delta}(\theta)| \le \frac{1}{\theta}\left(\frac{\pi^2}{2}+\frac{2}{\pi}\right) = \frac{C_2}{\theta} \tag{3.12}$$

for $\delta < 4\theta/\pi^5$. Therefore for $\lambda \in (4/\pi^4, 4/(\pi^6\delta))$ and δ small enough, we obtain

$$\begin{aligned}|\{\theta \in [-\pi,\pi] : |\widetilde{f}_\delta(\theta)| > \lambda\}| &\geq \left|\left\{\frac{\pi^5}{4}\delta < \theta < \frac{\pi}{2} \text{ and } |\widetilde{f}_\delta(\theta)| > \lambda\right\}\right| \\ &\geq \left|\left\{\frac{\pi^5}{4}\delta < \theta < \frac{\pi}{2} \text{ and } \frac{2}{\pi\theta} > \lambda\right\}\right| \\ &= \left|\left\{\frac{\pi^5}{4}\delta < \theta < \frac{2}{\pi\lambda}\right\}\right| \\ &= \frac{2}{\pi\lambda} - \frac{\pi^5}{4}\delta > \frac{1}{\pi\lambda}.\end{aligned}$$

Finally, proceeding as in the proof of (2.7)⇒(2.3) in Theorem (2.2), we get (2.19). ∎

Proof of Theorem (2.22). We first prove that (2.23) implies (2.24). The good-λ inequality for $h(x) = I^\alpha_\Omega f(x)$ and $g(x) = M^\alpha_\Omega f(x)$ follows in the same way as in [MW] for $\Omega = \mathbb{R}^n$. Then, by Lemma (3.8), we have

$$\int_\Omega \phi(|I^\alpha_\Omega f(x)|)\,dx \leq C + C\int_\Omega \phi(CM^\alpha_\Omega f(x))\,dx,$$

which, by Theorem (2.9), implies (2.24). Conversely, easy estimates show that

$$M^\alpha_\Omega f(x) \leq I^\alpha_\Omega f(x) \quad \text{for every } x \in \Omega.$$

Therefore inequality (2.24) allows us to obtain (2.17). Finally, applying Theorem (2.9) again, we get (2.23). ∎

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Instituto de Matemática Aplicada del Litoral
Universidad Nacional del Litoral
Güemes 3450
3000 Santa Fe, Argentina
E-mail: harbour@ceride.gov.ar
salinas@ceride.gov.ar
viviani@ceride.gov.ar

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