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Multiresolution Approximations and Wavelet Bases of Weighted L^p Spaces

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ABSTRACT. *We study boundedness and convergence on $L^p(\mathbb{R}^n, d\mu)$ of the projection operators P_j given by MRA structures with non-necessarily compactly supported scaling function. As a consequence, we prove that if w is a locally integrable function such that $w^{-\frac{1}{p-1}}(x)(1+|x|)^{-N}$ is integrable for some $N > 0$, then the Muckenhoupt A_p condition is necessary and sufficient for the associated wavelet system to be an unconditional basis for the weighted space $L^p(\mathbb{R}^n, w(x)\,dx)$, $1 < p < \infty$.*

1. Introduction

Our main purpose in this article is to solve the following problem which we shall call (P1): search for necessary and sufficient conditions on the Borel measure μ of $\mathbb{R}^n$ for which there is mean and almost everywhere convergence of multiresolution approximations of $L^p(d\mu)$ functions, when the scaling function is continuous and absolutely bounded by an L^1 radial decreasing function. We also aim, as a byproduct of the results for (P1), to study (P2): search for necessary and sufficient conditions on the Borel measure μ for the associated wavelet systems to be unconditional bases for the space $L^p(d\mu)$.

Let us briefly mention some previous articles related to the subject. K.S. Kazarian [10] considers the problem (P2) in the Haar context for the interval. In two consecutive volumes of Studia Mathematica both in 1994, independently, the issue is considered by P.G. Lemariè-Rieusset [12] and by J. García-Cuerva and K.S. Kazarian [5]. Both problems (P1) and (P2) are considered in [12] for the compactly supported Daubechies MRA. Problem (P2) is

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considered in [5], where sufficient conditions on the measure, in terms of the Muckenhoupt classes, are given for the case of splines. Necessary conditions are obtained by García-Cuerva and Kazarian in [6] for the spline wavelet system. For the special case of the Haar system, problem (P2) is also considered in [6] where, modulo the analogy with the Haar system in the segment [0, 1] given in [10], necessary and sufficient conditions, wider than dyadic A_p, on the weight are obtained.

In this article, we accomplish and completely solve the following aspect of (P1): we find necessary and sufficient conditions on the measure μ of $\mathbb{R}^n$ for which the multiresolution approximations of $L^p(d\mu)$ functions are uniformly bounded (see Theorems 2 and 3). These results generalize those in [12] both, because we obtain the results for more general MRA and for dimension bigger than one. Nevertheless our proofs are different from those given in [12]. Our approach relies on a weak positivity property of the kernels of the projection operators (see Theorem 1).

Regarding problem (P2), our results are contained in Theorem 4. Let us point out that we can not solve it completely, and as far as we know, the problem remains open in its full generality. Let us emphasize that two compactly supported cases of (P2) are considered in the literature: the Haar wavelet in [6] and the Daubechies wavelet in [12]. Both of them give necessary and sufficient conditions on the weight function, but the surprising situation is that while in [12] those conditions are given by Muckenhoupt A_p classes, in [6] the class of weights strictly contains dyadic Muckenhoupt classes. At this point we have to say that, as it was pointed out by our referee in a previous version of this article, we are unable to prove, without some extra condition on the weight, the implication (D1) $\Rightarrow$ (D3) in [12] which gives A_p as necessary condition for the fact that the Daubechies wavelet system is an unconditional basis for $L^p(d\mu)$. Actually for the Daubechies case our result is contained in Corollary 2 and we are only able to show that A_p is necessary under the extra assumption of the local integrability of $w^{-\frac{1}{p-1}}$.

The MRA covered by our approach through the weakly positive kernels do not include the Haar case. However we can prove similar results changing the A_p classes by the dyadic A_p classes A_p^{dy} (see Theorems 5 and 6).

The article is organized as follows: In Section 2 we state the results. Section 3 is devoted to prove the basic theorem: Theorem 1. In Section 4 we give the proofs of the results related to problems (P1) and (P2) when the scaling function is continuous. Finally, in Section 5 we give the proofs of the results for the Haar case.

2. Statement of the Results

In order to state the results, let us recall that a weight w belongs to the class A_p with $1 < p < \infty$ if there exists a constant C such that for all cube Q

$$\left(\int_Q w\right)^{1/p} \left(\int_Q w^{-\frac{1}{p-1}}\right)^{1/p'} \leq C|Q| ,$$

where $|Q|$ is the Lebesgue measure of Q and p' is the conjugate exponent of p. We say that $w \in A_1$ if there exists a constant C such that for all cube Q and almost every $x \in Q$

$$\frac{1}{|Q|}\int_Q w \leq Cw(x) .$$

Let us also recall that if the cubes Q in the definitions of A_p are dyadic cubes, the classes of weights are the dyadic A_p classes and we shall denote them by A_p^{dy}.

In the search of necessary conditions, the next result will be the key argument. To state it we introduce the concept of a family of weakly positive kernels. Let n be a positive integer and let $\{\ell_j : j \in \mathbb{Z}\}$ be a decreasing sequence of positive real numbers ($\ell_{j+1} < \ell_j$) with $\ell_j \to 0$ when $j \to \infty$ and $\ell_j \to \infty$ when $j \to -\infty$. A family $\{K_j : j \in \mathbb{Z}\}$ of measurable real valued kernels defined on $\mathbb{R}^{2n} = \mathbb{R}^n \times \mathbb{R}^n$ is said to be "*weakly positive*" if there exist a sequence $\{\ell_j\}$ as above and a positive constant $\mathcal{C}$ such that the set inclusion

$$\left\{(x, y) \in \mathbb{R}^{2n} : |x - y| < \ell_j\right\} \subset \left\{(x, y) \in \mathbb{R}^{2n} : K_j(x, y) > \mathcal{C}(\ell_{j+1})^{-n}\right\}$$

holds true for every $j \in \mathbb{Z}$.

Theorem 1.

Let $1 \leq p < \infty$. Let L_c^∞ be the set of the bounded functions with compact support and let μ be a positive Borel measure on $\mathbb{R}^n$ finite on compact sets. Assume that $T_j f(x) = \int K_j(x, y) f(y)\, dy$, $j \in \mathbb{Z}$ and $f \in L_c^\infty$, where $\{K_j : j \in \mathbb{Z}\}$ is a family of weakly positive kernels. If there exists $C > 0$ such that

$$\mu(\{x : |T_j f(x)| > \lambda\}) \leq C\lambda^{-p} ||f||^p_{L^p(d\mu)} = C\lambda^{-p}||f||^p_{p,\mu} ,$$

for all $\lambda > 0$, all $j \in \mathbb{Z}$ and all $f \in L_c^\infty$, then μ is absolutely continuous with density $w \in A_p$.

Remark 1. We would like to point out that the finiteness of μ on compact sets, under the weak type assumption on T_j, is implied by the existence of a Borel set A with $|A| > 0$ and $\mu(A) < \infty$.

The above theorem together with Lemma 2.8 in [11] allow us to get boundedness and convergence results of multiresolution approximations. Following Meyer [13], by a multiresolution analysis (MRA) on $\mathbb{R}^n$ ($n \geq 1$) we mean an approximation of the space $L^2(\mathbb{R}^n)$ through an increasing sequence of closed subspaces V_j, i.e.,

$$\cdots V_{-2} \subset V_{-1} \subset V_0 \subset V_1 \subset V_2 \cdots ,$$

with the following properties

(i) $\cap_{j\in\mathbb{Z}} V_j = \{0\}$ and $\overline{\cup_{j\in\mathbb{Z}} V_j} = L^2(\mathbb{R}^n)$;

(ii) for every $f \in L^2(\mathbb{R}^n)$ and every $j \in \mathbb{Z}$, $f(x) \in V_j$ if and only if $f(2x) \in V_{j+1}$;

(iii) for every $f \in L^2(\mathbb{R}^n)$ and every $k \in \mathbb{Z}^n$, $f(x) \in V_0$ if and only if $f(x - k) \in V_0$;

(iv) there exists a function $\phi \in L^2(\mathbb{R}^n)$ such that the family $\{\phi_k(x) = \phi(x - k),\ k \in \mathbb{Z}^n\}$ is an orthonormal basis for V_0.

Such a function ϕ is called a scaling function. The family $\{\phi_{j,k}(x) = 2^{nj/2}\phi(2^j x - k),\ k \in \mathbb{Z}^n\}$ is an orthonormal basis for V_j. Then, associated with the increasing sequence of subspaces $\{V_j\}_{j\in\mathbb{Z}}$ we have the orthogonal projections of $L^2(\mathbb{R}^n)$ onto V_j given by

$$P_j f = \sum_{k\in\mathbb{Z}^n} < f, \phi_{j,k} > \phi_{j,k} , \quad \text{for } f \in L^2\left(\mathbb{R}^n\right) .$$

From now on, we shall work with a scaling function ϕ in the class $\mathcal{RB}$; that means that ϕ is absolutely bounded by an L^1 radial decreasing function η, i.e., $|\phi(x)| \leq \eta(x)$ with

$\eta(0) < \infty$, $\eta(x_1) = \eta(x_2)$ whenever $|x_1| = |x_2|$, $\eta(x_1) \le \eta(x_2)$ whenever $|x_1| \ge |x_2|$ and $\eta \in L^1(\mathbb{R}^n)$. Under the assumption $\phi \in \mathcal{RB}$ we get that the kernel $P_j(x, y)$ of P_j is given by $2^{nj} P_0(2^j x, 2^j y)$ with $P_0(x, y) = \sum_{k\in\mathbb{Z}^n} \phi(x-k)\overline{\phi(y-k)}$, in the sense that, for $f \in L^2(\mathbb{R}^n)$,

$$P_j f(x) = \int_{\mathbb{R}^n} P_j(x, y) f(y)\, dy . \tag{2.1}$$

In [11], Kelly, Kon and Raphael proved that if $\phi \in \mathcal{RB}$ then the kernel $P_0(x, y)$ satisfies

$$|P_0(x, y)| \le H(|x - y|) \tag{2.2}$$

where $H(|x|)$ is a bounded radial decreasing $L^1(\mathbb{R}^n)$ function (see Lemma 2.8 in [11]). From the estimate (2.2), in Theorem 2.6 [11], the operators $P_j f(x) = \int_{\mathbb{R}^n} P_j(x, y) f(y)\, dy$ are well defined for $f \in L^p(\mathbb{R}^n)$, $1 \le p < \infty$, and $P_j f$ converge to f almost everywhere and in the L^p norm, when $j \to \infty$.

The following two theorems are related to problem (P1). Notice that in these statements and in the further development, $P_j f(x)$ is understood in the pointwise sense given by (2.1).

Theorem 2 ($p > 1$).
Let $\{V_j\}_{j\in\mathbb{Z}}$ be a MRA on $\mathbb{R}^n$ with a continuous scaling function $\phi \in \mathcal{RB}$. Let μ be a positive Borel measure on $\mathbb{R}^n$ finite on compact sets and $p \in (1, \infty)$. Then, the following statements are equivalent:

- **(A1)** *the operators P_j are continuous on $L^p(d\mu)$ and for all $f \in L^p(d\mu)$,*

$$\lim_{j\to\infty} ||f - P_j f||_{p,\mu} = 0 \quad \textit{and} \quad \lim_{j\to-\infty} ||P_j f||_{p,\mu} = 0 \,;$$

- **(A2)** *the operators P_j are uniformly bounded on $L^p(d\mu)$;*
- **(A3)** *the operators P_j are uniformly of weak type (p, p) with respect to μ;*
- **(A4)** *the operator $P^* f = \sup_{j\in\mathbb{Z}} |P_j f|$ is of weak type (p, p) with respect to μ;*
- **(A5)** *the operator $P^* f$ is of strong type (p, p) with respect to μ;*
- **(A6)** *μ is absolutely continuous and $d\mu = w\, dx$ with $w \in A_p$.*

Further, each one of the above statement implies

- **(A)** *for $f \in L^p(d\mu)$, $P_j f \to f$ and $P_{-j} f \to 0$ almost everywhere when $j \to \infty$.*

Theorem 3 ($p = 1$).
Let $\{V_j\}_{j\in\mathbb{Z}}$ be a MRA on $\mathbb{R}^n$ with a continuous scaling function $\phi \in \mathcal{RB}$ and let μ be a positive Borel measure on $\mathbb{R}^n$ finite on compact sets. Then, the following statements are equivalent:

- **(B1)** *the operators P_j are uniformly bounded on $L^1(d\mu)$;*
- **(B2)** *the operators P_j are uniformly of weak type $(1, 1)$ with respect to μ;*
- **(B3)** *the operator $P^* f$ is of weak type $(1, 1)$ with respect to μ;*
- **(B4)** *μ is absolutely continuous and $d\mu = w\, dx$ with $w \in A_1$.*

Further, each one of the above statement implies

(B) *for $f \in L^1(d\mu)$, $P_j f \to f$ almost everywhere and in L^1 norm, when $j \to \infty$ and $P_{-j} f \to 0$ a.e. when $j \to \infty$.*

As an application of Theorem 2 we obtain results about the wavelet expansions. Now we shall work with MRA with more regular scaling function.

We shall say that a MRA $\{V_j\}_{j\in\mathbb{Z}}$ on $\mathbb{R}^n$ is r-regular ($r \in \mathbb{N}$) if the scaling function ϕ verifies that $|\partial^\alpha \phi(x)| \le C_m(1+|x|)^{-m}$ for all $\alpha = (\alpha_1, \dots, \alpha_n)$ such that $|\alpha| \le r$ and for all $m \in \mathbb{N}$, where $\partial^\alpha = (\partial/\partial x_1)^{\alpha_1} \dots (\partial/\partial x_n)^{\alpha_n}$ and $|\alpha| = \alpha_1 + \dots + \alpha_n$.

Let $\{V_j\}_{j\in\mathbb{Z}}$ be an r-regular MRA on $\mathbb{R}^n$ and let W_j denote the orthogonal complement of V_j in V_{j+1}. From the existence of ϕ it follows (see, e.g., [13]) that there exist $2^n - 1$ functions $\psi^1, \dots, \psi^{2^n-1}$ such that $\{\psi^\lambda_{j,k}(x) = 2^{jn/2}\psi^\lambda(2^j x - k),\ j \in \mathbb{Z}, k \in \mathbb{Z}^n, 1 \le \lambda \le 2^n - 1\}$ form an orthonormal basis for W_j for fixed j, and form an orthonormal basis for $L^2(\mathbb{R}^n)$ as j, k vary. Moreover, the functions ψ^λ also verify that $|\partial^\alpha \psi^\lambda(x)| \le C_\ell(1+|x|)^{-\ell}$ for all $\alpha \in \mathbb{N}^n$ such that $|\alpha| \le r$ and for all $\ell \in \mathbb{N}$.

In the following theorem we give our partial answer to problem (P2), where we also prove a result concerning the characterization through wavelet coefficients of the spaces $L^p(d\mu)$.

Theorem 4.

Let $\{V_j\}_{j\in\mathbb{Z}}$ be an 1-regular MRA on $\mathbb{R}^n$, let μ be a positive Borel measure on $\mathbb{R}^n$ finite on compact sets and $p \in (1, \infty)$. Then, the following statements are equivalent:

(D1) *the sequence $\{\psi^\lambda_{j,k} : j \in \mathbb{Z}, k \in \mathbb{Z}^n, 1 \le \lambda \le 2^n - 1\}$ forms an unconditional basis for $L^p(d\mu)$ and the functionals $(\psi^\lambda_{j,k})^*(f) = < f, \psi^\lambda_{j,k} >$ belong to the dual space of $L^p(d\mu)$;*

(D2) *μ is absolutely continuous and $d\mu = w\,dx$ with $w \in A_p$;*

(D3) *$||\psi^\lambda_{j,k}||_{p,\mu} > 0$ for all $j \in \mathbb{Z}$, $k \in \mathbb{Z}^n$ and $1 \le \lambda \le 2^n - 1$ and there exist two constant C_1 and C_2 such that for all $f \in L^p(d\mu)$,*

$$C_1||f||_{p,\mu} \le \left\| \left(\sum_{j\in\mathbb{Z}} \sum_{k\in\mathbb{Z}^n} \sum_{\lambda=1}^{2^n-1} \left|< f, \psi^\lambda_{j,k} >\right|^2 \left|\psi^\lambda_{j,k}\right|^2 \right)^{1/2} \right\|_{p,\mu} \le C_2||f||_{p,\mu} .$$

(D4) *$\mu(Q_{j,k}) > 0$ for all $j \in \mathbb{Z}$, $k \in \mathbb{Z}^n$, where $Q_{j,k} = \{x \in \mathbb{R}^n : 2^j x - k \in [0,1)^n\}$, and there exist two constant C_1 and C_2 such that for all $f \in L^p(d\mu)$,*

$$C_1||f||_{p,\mu} \le \left\| \left(\sum_{j\in\mathbb{Z}} \sum_{k\in\mathbb{Z}^n} \sum_{\lambda=1}^{2^n-1} \left|< f, \psi^\lambda_{j,k} >\right|^2 \chi_{j,k} \right)^{1/2} \right\|_{p,\mu} \le C_2||f||_{p,\mu} ,$$

where $\chi_{j,k}(x) = 2^{nj/2}\chi(2^j x - k)$ and χ is the characteristic function $\chi_{[0,1)^n}$.

If we start with a weighted measure, $d\mu(x) = w(x)\,dx$, the continuity of the functionals in (D1) is guaranteed *a priori* as soon as some integrability condition on $w^{-\frac{1}{p-1}}$ is assumed. This is stated in the following corollary.

Corollary 1.

Let $\{V_j\}_{j\in\mathbb{Z}}$ be an 1-regular MRA on $\mathbb{R}^n$, let $w \ge 0$ be a locally integrable function such that $\int w^{-\frac{1}{p-1}}(x)(1+|x|)^{-N}\,dx < \infty$, for some $N > 0$ and let $p \in (1, \infty)$. Then, *the sequence* $\{\psi^{\lambda}_{j,k} : j \in \mathbb{Z}, k \in \mathbb{Z}^n, 1 \le \lambda \le 2^n - 1\}$ *forms an unconditional basis for* $L^p(w\,dx)$ *if and only if* $w \in A_p$.

The proof of the corollary is reduced to check that $\psi^{\lambda}_{j,k} \in L^{p'}(w^{-\frac{1}{p-1}})$. This follows from the regularity of the MRA. We notice that in the case of the Daubechies' wavelets, the assumption on $w^{-\frac{1}{p-1}}$ can be weakened.

Corollary 2.

Let $\{V_j\}_{j\in\mathbb{Z}}$ *be a MRA of I. Daubechies on* $\mathbb{R}^n$*, let* $w \ge 0$ *such that* w *and* $w^{-\frac{1}{p-1}}$ *are locally integrable functions and let* $p \in (1, \infty)$*. Then, the sequence* $\{\psi^{\lambda}_{j,k} : j \in \mathbb{Z}, k \in \mathbb{Z}^n, 1 \le \lambda \le 2^n - 1\}$ *forms an unconditional basis for* $L^p(w\,dx)$ *if and only if* $w \in A_p$.

The Haar MRA is generated by the scaling function $\chi = \chi_{[0,1)^n}$. In this case the projection operators P_j are

$$P_j f(x) = \sum_{k\in\mathbb{Z}^n} < f, \chi_{j,k} > \chi_{j,k}(x) = \int_{\mathbb{R}^n} P_j(x, y) f(y)\, dy\ , \tag{2.3}$$

where $P_j(x, y) = 2^{nj} \sum_{k\in\mathbb{Z}^n} \chi_{Q_{j,k}}(x)\chi_{Q_{j,k}}(y)$, with $Q_{j,k}$ the dyadic cube defined in (D4) of Theorem 4.

Theorem 5.

Let $\{V_j\}_{j\in\mathbb{Z}}$ *be the Haar MRA on* $\mathbb{R}^n$ *and* μ *a positive Borel measure on* $\mathbb{R}^n$ *finite on compact sets. Then, the conclusions of Theorems 2 and 3 hold for the operators* P_j *defined in* (2.3)*, changing the classes* A_p *to* A^{dy}_p.

Now, we establish the result for the Haar wavelets. The n-dimensional Haar system is generated by dilation and translation of $2^n - 1$ wavelet basics. In fact, let Λ be the set of $2^n - 1$ elements $\lambda = (\lambda_1, \dots, \lambda_n)$ where $\lambda_i = 0$ or $\lambda_i = 1$ except $(0, \dots, 0)$. The functions

$$h^{\lambda}_{j,k}(x_1 \dots x_n) = 2^{nj/2} h^{\lambda_1}\left(2^j x_1 - k_1\right) \cdots h^{\lambda_n}\left(2^j x_n - k_n\right)\ ,$$

where $h^0 = \chi_{[0,1)}$ and $h^1 = \chi_{[0,1/2)} - \chi_{[1/2,1)}$, form an orthonormal basis of $L^2(\mathbb{R}^n)$. For this type of wavelet we can prove the following result.

Theorem 6.

Let $\{V_j\}_{j\in\mathbb{Z}}$ *be the Haar MRA on* $\mathbb{R}^n$*,* μ *a positive Borel measure on* $\mathbb{R}^n$ *finite on compact sets and* $p \in (1, \infty)$*. Then, the following statements are equivalent:*

- **(H1)** *the sequence* $\{h^{\lambda}_{j,k} : j \in \mathbb{Z}, k \in \mathbb{Z}^n, \lambda \in \Lambda\}$ *forms an unconditional basis for* $L^p(d\mu)$ *and the functionals* $(h^{\lambda}_{j,k})^*(f) = < f, h^{\lambda}_{j,k} >$ *belong to the dual space of* $L^p(d\mu)$*;*
- **(H2)** μ *is absolutely continuous and* $d\mu = w\,dx$ *with* $w \in A^{dy}_p$*;*
- **(H3)** $||\chi_{j,k}||_{p,\mu} > 0$ *for all* $j \in \mathbb{Z}$*,* $k \in \mathbb{Z}^n$ *and there exist two constants* C_1 *and* C_2 *such that for all* $f \in L^p(d\mu)$*,*

$$C_1||f||_{p,\mu} \le \left\| \left(\sum_{j\in\mathbb{Z}} \sum_{k\in\mathbb{Z}^n} \sum_{\lambda\in\Lambda} \left| < f, h^{\lambda}_{j,k} > \right|^2 \chi_{j,k} \right)^{1/2} \right\|_{p,\mu} \le C_2||f||_{p,\mu}\ .$$

Finally, for weighted measures we have the following result which is similar to Corollary 2.

Corollary 3.

Let $\{V_j\}_{j\in\mathbb{Z}}$ be the Haar MRA on $\mathbb{R}^n$, let $w \geq 0$ such that w and $w^{-\frac{1}{p-1}}$ are locally integrable functions and let $p \in (1,\infty)$. Then, the sequence $\{h^{\lambda}_{j,k} : j \in \mathbb{Z}, k \in \mathbb{Z}^n, \lambda \in \Lambda\}$ forms an unconditional basis for $L^p(d\mu)$ if and only if $w \in A^{dy}_p$.

3. Proof of Theorem 1

First, we prove that μ is absolutely continuous. Let E be a set such that $|E| = 0$. Since, being finite on compact sets, μ is regular, then for each $\varepsilon > 0$ there exists an open set G such that $E \subset G$ and $\mu(G \setminus E) < \varepsilon$. The open set G can be written as a countable union of disjoint and dyadic cubes Q_i. We denote with $d(Q)$ the diameter of the cube Q. Let $\{\ell_j\}$ be the sequence associated to the family $\{K_j\}$. For fixed i, let j_0 be the integer such that $\ell_{j_0+1} \leq d(Q_i) < \ell_{j_0}$. If $x, y \in Q_i$ we get that $|x-y| < \ell_{j_0}$ and $K_{j_0}(x,y) > \mathcal{C}(\ell_{j_0+1})^{-n}$. Therefore, for all $x \in Q_i$

$$|T_{j_0}(\chi_{Q_i\setminus E})(x)| = \left|\int_{Q_i\setminus E} K_{j_0}(x,y)\,dy\right| \geq \mathcal{C}(\ell_{j_0+1})^{-n}|Q_i\setminus E| .$$

Since $|E| = 0$ we get that $|T_{j_0}(\chi_{Q_i\setminus E})(x)| \geq c_n$, where c_n is a constant depending only on the constant $\mathcal{C}$ and on the dimension n. So that, from the weak type inequality we have

$$\mu(Q_i) \leq \mu(\{x : |T_{j_0}(\chi_{Q_i\setminus E})(x)| > c_n\}) \leq Cc_n^{-p}\mu(Q_i\setminus E) .$$

Summing in i we get

$$\mu(G) = \sum_i \mu(Q_i) \leq Cc_n^{-p}\sum_i \mu(Q_i\setminus E) = Cc_n^{-p}\mu(G\setminus E) < Cc_n^{-p}\varepsilon .$$

for all $\varepsilon > 0$. Then, $\mu(E) = 0$. Now, by the Radon–Nikodym Theorem, we get a locally integrable function w such that $d\mu = w(x)\,dx$. We shall prove that $w \in A_p$. Let us first assume that $1 < p < \infty$. Let Q be a cube on $\mathbb{R}^n$ and $j_0 \in \mathbb{Z}$ such that $\ell_{j_0+1} \leq d(Q) < \ell_{j_0}$. With $\sigma_\varepsilon = (w+\varepsilon)^{-\frac{1}{p-1}}$, $\varepsilon > 0$, the inequalities

$$|T_{j_0}(\sigma_\varepsilon\chi_Q)(x)| = \left|\int_Q K_{j_0}(x,y)\sigma_\varepsilon(y)\,dy\right| > \mathcal{C}(\ell_{j_0+1})^{-n}\int_Q \sigma_\varepsilon \geq c_n|Q|^{-1}\int_Q \sigma_\varepsilon \equiv \lambda ,$$

hold for all $x \in Q$. From the uniformly weak type (p,p) of the operators T_j we get

$$w(Q) \leq w(\{x : |T_{j_0}(\sigma_\varepsilon\chi_Q)(x)| > \lambda\}) \leq Cc_n^{-p}|Q|^p\left(\int_Q \sigma_\varepsilon\right)^{1-p} .$$

Multiplying by $(\int_Q \sigma_\varepsilon)^{p-1}$ and letting ε go to zero we obtain that w belongs to A_p. Now, we shall consider the case $p = 1$. Let x_0 be a Lebesgue point of w. Take Q and $\tilde{Q}$ cubes such that $x_0 \in \tilde{Q} \subset Q$. Pick $j_0 \in \mathbb{Z}$ such that $\ell_{j_0+1} \leq d(Q) < \ell_{j_0}$. Then, for all $x \in Q$

$$\left|T_{j_0}\left(\chi_{\tilde{Q}}\right)(x)\right| > \mathcal{C}(\ell_{j_0+1})^{-n}\left|\tilde{Q}\right| \geq c_n|Q|^{-1}\left|\tilde{Q}\right| .$$

From the weak type (1, 1) inequality we obtain that $w(Q) \le w(\{|T_{j_0}(\chi_{\tilde{Q}})(x)| > c_n \frac{|\tilde{Q}|}{|Q|}\}) \le C c_n^{-1} \frac{|Q|}{|\tilde{Q}|} \int_{\tilde{Q}} w$. Therefore,

$$\frac{1}{|Q|} \int_Q w \le C c_n^{-1} \frac{1}{\left|\tilde{Q}\right|} \int_{\tilde{Q}} w \, .$$

Letting $\tilde{Q} \to x_0$ we get that $\frac{1}{|Q|} \int_Q w \le C c_n^{-1} w(x_0)$ and we are done.

4. Proofs of Theorems 2, 3, and 4

The proofs of (A5) $\Rightarrow$ (A4) $\Rightarrow$ (A3), (A5) $\Rightarrow$ (A2) $\Rightarrow$ (A3), (B1) $\Rightarrow$ (B2) and (B3) $\Rightarrow$ (B2) are obvious. We notice also that (A1) $\Rightarrow$ (A2) is an immediate consequence of the Uniform Boundedness Principle while (B4) $\Rightarrow$ (B1) follows from (2.2) and the characterization of the weighted inequalities for the convolution operators associated to dilations of an L^1 radially decreasing function (see e.g., [14]). Therefore, to complete the proofs of Theorems 2 and 3, we only have to show the following implications: (A3) $\Rightarrow$ (A6) $\Rightarrow$ (A5) $\Rightarrow$ (A1), (A5) $\Rightarrow$ (A), (B2) $\Rightarrow$(B4) $\Rightarrow$ (B3) and (Bi) $\Rightarrow$ (B), $i = 1, 2, 3, 4$.

(A3) $\Rightarrow$ (A6) and (B2) $\Rightarrow$ (B4). We start proving the following lemma.

Lemma 1.

Let φ be a continuous function in $\mathcal{RB}$ such that $\sum_{k\in\mathbb{Z}^n} \varphi(x-k) \neq 0$ for all $x \in \mathbb{R}^n$. Then $F(x, y) = \sum_{k\in\mathbb{Z}^n} \varphi(x-k)\overline{\varphi(y-k)}$ satisfies

$$\left\{(x, y) \in \mathbb{R}^{2n} : |x-y| < \ell\right\} \subset \left\{(x, y) \in \mathbb{R}^{2n} : F(x, y) > \delta\right\} , \tag{4.1}$$

for some positive real numbers ℓ and δ.

Proof of the lemma. Since $\varphi \in \mathcal{RB}$ the series $F(x, y) = \sum_{k\in\mathbb{Z}^n} \varphi(x-k)\overline{\varphi(y-k)}$ converges uniformly on each compact set of $\mathbb{R}^{2n}$ and $F(x, y)$ results a continuous function. On the other hand, $F(x+k, y+k) = F(x, y)$ for all $k \in \mathbb{Z}^n$. Then, we have that $F(x, x) \ge \alpha$ for some $\alpha > 0$. Let $0 < \delta < \alpha$ and $E_\delta = \{(x, y) \in \mathbb{R}^{2n} : F(x, y) > \delta\}$. Notice that E_δ is open, it contains the diagonal $\Delta = \{(x, x) : x \in \mathbb{R}^n\}$ and it is periodic: $(x, y) \in E_\delta$ if and only if $(x+k, y+k) \in E_\delta$. So, we have that the distance ℓ from Δ to the complement of E_δ is positive. Thus (4.1) holds. □

Notice that if ϕ is a scaling function, then $\{\phi(x-k), k \in \mathbb{Z}^n\}$ is an orthonormal basis and we get that $\sum_{k\in\mathbb{Z}^n} \phi(x-k) = 1$ for almost every $x \in \mathbb{R}^n$ (see e.g., [13] or [15]). On the other hand, since $\phi \in \mathcal{RB}$ the series $\sum_{k\in\mathbb{Z}^n} \phi(x-k)$ converges uniformly on each compact set of $\mathbb{R}^n$ and since ϕ is continuous we get that $\sum_{k\in\mathbb{Z}^n} \phi(x-k) = 1$ for every $x \in \mathbb{R}^n$. This fact allows us to apply the above lemma to the function $P_0(x, y) = \sum_{k\in\mathbb{Z}^n} \phi(x-k)\overline{\phi(y-k)}$ and easily we obtain that the family $\{P_j(x, y)\} = \{2^{nj} P_0(2^j x, 2^j y)\}$ is weakly positive. Then, applying Theorem 1 we get the desired implications.

(A6) $\Rightarrow$ (A5) and (B4) $\Rightarrow$ (B3). By using (2.2) and standard arguments we can prove that

$$P^* f(x) = \sup_{j\in\mathbb{Z}} |P_j f(x)| \le C \left(\int_{\mathbb{R}^n} H(|x|)\, dx\right) M f(x) \, ,$$

where M is the Hardy–Littlewood maximal function. From the corresponding weighted boundedness of M (see e.g., [8]) we get the desired implications.

(A5) $\Rightarrow$ (A) and (A5) $\Rightarrow$ (A1). By the results in [11] we have that $P_j f$ converge to f almost everywhere when $j \to \infty$ for all $f \in L^p(dx) \cap L^p(d\mu)$ which is a dense subset of $L^p(d\mu)$. This fact together with (A5) implies that the convergence holds for all $f \in L^p(d\mu)$. On the other hand, by the Hölder inequality,

$$|P_j f(x)| \leq 2^{nj/p} \|H(|\cdot|)\|_{p'} \|f\|_p .$$

Since $H(|x|)$ belongs to $L^{p'}(\mathbb{R}^n)$, $1 < p' < \infty$, we get that $P_j f$ converges to 0 almost everywhere when $j \to -\infty$ for all $f \in L^p(dx)$, and, as before, the convergence holds for all $f \in L^p(d\mu)$. This finishes the proof of (A5) $\Rightarrow$ (A). To prove that (A5) $\Rightarrow$ (A1) we observe that (A5) implies obviously that the operators P_j are continuous on $L^p(d\mu)$ and the limits in (A1) hold by (A5) $\Rightarrow$ (A) and the dominated convergence theorem.

(Bi) $\Rightarrow$ (B), $i = 1, 2, 3, 4$. The proof of this implication is similar to (A5) $\Rightarrow$ (A).

Now we shall prove Theorem 4.

(D1) $\Rightarrow$ (D2). We shall divide the proof into several steps.

Step a: For every function $f \in L^p(d\mu)$ we have that (D1) implies that

$$f = \sum_{j,k,\lambda} < f, \psi^\lambda_{j,k} > \psi^\lambda_{j,k} ,$$

where the sum is understood in the $L^p(d\mu)$-sense. In fact, since $\{\psi^\lambda_{j,k}\}$ is an unconditional basis for $L^p(d\mu)$ we have that for all $f \in L^p(d\mu)$

$$f = \sum_{j,k,\lambda} \alpha^\lambda_{j,k} \psi^\lambda_{j,k} ,$$

where the sum is understood in the $L^p(d\mu)$-sense. Now, the continuity and the linearity of $(\psi^\lambda_{j,k})^*$ gives that $\alpha^\lambda_{j,k} = (\psi^\lambda_{j,k})^*(f) =< f, \psi^\lambda_{j,k} >$.

Step b: (D1) implies that the measure μ is equivalent to the Lebesgue measure, i.e., $\mu(A) = 0$ if and only if $|A| = 0$. To prove it we may assume that A is a bounded Borel set. If $\mu(A) = 0$ then, for every j, k, λ we get that $(\psi^\lambda_{j,k})^*(\chi_A) = 0$ since $(\psi^\lambda_{j,k})^*$ are bounded functionals on $L^p(\mu)$. On the other hand,

$$\chi_A = \sum_{j,k,\lambda} \left(\psi^\lambda_{j,k}\right)^* (\chi_A) \psi^\lambda_{j,k} = 0 ,$$

where the sum is understood in the $L^2(dx)$-sense. Then, $\chi_A = 0$ in $L^2(dx)$ which implies that $|A| = 0$. Conversely, assume that $|A| = 0$. Then, $(\psi^\lambda_{j,k})^*(\chi_A) = \int \psi^\lambda_{j,k}(x)\chi_A(x)\, dx = 0$ for every j, k, λ. Since $\chi_A \in L^p(\mu)$, Step (a) gives that

$$\chi_A = \sum_{j,k,\lambda} \left(\psi^\lambda_{j,k}\right)^* (\chi_A) \psi^\lambda_{j,k} = 0,$$

in the $L^p(\mu)$-sense. Therefore $\mu(A) = 0$.

Step c: (D1) implies that the operators

$$S_m f = \sum_{j \leq m, k, \lambda} < f, \psi^\lambda_{j,k} > \psi^\lambda_{j,k} \tag{4.2}$$

are uniformly bounded on $L^p(d\mu)$, where the sum is understood in the $L^p(d\mu)$-sense. This statement follows from the fact that $\{\psi_{j,k}^\lambda\}$ is an unconditional basis for $L^p(d\mu)$ and

$$S_m f = \sum_{j,k,\lambda} \beta_{j,k}^\lambda < f, \psi_{j,k}^\lambda > \psi_{j,k}^\lambda \ ,$$

with $\beta_{j,k}^\lambda = 1$ for $j \leq m$ and 0 elsewhere (see [9], p. 213, Chapter 5, Lemma 2.7).

Step d: (D1) implies that if $f \in L_c^\infty$ then $S_m f(x) = P_m f(x)$ a.e., where $S_m f$ is understood as the sum in (4.2) in the $L^p(d\mu)$-sense and $P_m f$ as given by (2.1). We first observe that if $f \in L^2(dx)$ then the sum in (4.2) is well defined with convergence in $L^2(dx)$ and the sum is $P_m f$, the projection given by (2.1), since $\{\psi_{j,k}^\lambda\}$ comes from a MRA. On the other hand, if $f \in L_c^\infty$ then $f \in L^p(d\mu) \cap L^2(dx)$ and the sum in (4.2) can be understood in the $L^p(d\mu)$-sense, $S_m f$, and in the $L^2(dx)$-sense, $P_m f$. Therefore, since the measure μ is equivalent to the Lebesgue measure, $S_m f(x) = P_m f(x)$ a.e..

Now, the proof of (D1) $\Rightarrow$ (D2) is easy. In fact, by Step c and Step d we have that

$$\sup_m ||P_m f||_{p,\mu} \leq C||f||_{p,\mu} \ ,$$

for all $f \in L_c^\infty$. Then, (D2) follows from Theorem 1.

(D2) $\Rightarrow$ (D3). Since $w \in A_p$ we have that $||\psi_{j,k}^\lambda||_{p,\mu} > 0$ for all $j \in \mathbb{Z}$, $k \in \mathbb{Z}^n$ and $1 \leq \lambda \leq 2^n - 1$. Now, let A be a finite subset of $\mathbb{Z}^{n+1}$. For $j \in \mathbb{Z}$ and $k \in \mathbb{Z}^n$ we shall write $(j, k) = (j, k_1, \dots, k_n)$. Let us define the operators

$$T_{A,\varepsilon} f(x) = \sum_{(j,k)\in A} \sum_{\lambda=1}^{2^n-1} \varepsilon_{j,k}^\lambda < f, \psi_{j,k}^\lambda > \psi_{j,k}^\lambda(x) \ ,$$

where $\varepsilon_{j,k}^\lambda = \pm 1$. The operators $T_{A,\varepsilon}$ can be written as integral operators $T_{A,\varepsilon} f = \int_{\mathbb{R}^n} K_{A,\varepsilon}(x, y) f(y)\, dy$ with kernel

$$K_{A,\varepsilon}(x, y) = \sum_{(j,k)\in A} \sum_{\lambda=1}^{2^n-1} \varepsilon_{j,k}^\lambda \psi_{j,k}^\lambda(x) \overline{\psi_{j,k}^\lambda}(y) \ .$$

Since the MRA is 1-regular we get that $T_{A,\varepsilon}$ is a family of Calderón–Zygmund operators (see [13]). Then, by a classical result of harmonic analysis (see [1]) we get that if $w \in A_p$ then

$$||T_{A,\varepsilon} f||_{L^p(w)} \leq C||f||_{L^p(w)}$$

for all $f \in L^p(w)$ and where C does not depend on A and ε. Then, by using Khintchine inequality we get the right hand side inequality in (D3) as in the non-weighted case (see [13]). The left hand side inequality follows by using a duality argument (see e.g., the proof of Theorem 4.16 in [7]).

To prove (D3) $\Rightarrow$ (D1) we shall follow the lines of [7]. First we shall prove that $\psi_{j,k}^\lambda \in L^p(d\mu)$. Let $N > 0$ be such that $\int_{B(0,N)} |\psi_{j,k}^\lambda(x)|\, dx > 0$. Then, by using (D3) with $f = \operatorname{sgn}(\psi_{j,k}^\lambda)\chi_{B(0,N)}$, where sgn is the sign function, we get that $\int |\psi_{j,k}^\lambda|^p d\mu \leq C_2 \mu(B(0, N)) \left(\int_{B(0,N)} |\psi_{j,k}^\lambda(x)|\, dx\right)^{-p} < \infty$. By (D3) we can also prove that the functionals $(\psi_{j,k}^\lambda)^*(f) =< f, \psi_{j,k}^\lambda >$ belong to the dual of $L^p(d\mu)$. In fact, $| < f, \psi_{j,k}^\lambda > | \leq$

$C_2||\psi_{j,k}^{\lambda}||_{L^p(d\mu)}^{-1}||f||_{L^p(d\mu)}$. On the other hand, for the operators $T_{A,\varepsilon}$ defined above we get that

$$||T_{A,\varepsilon} f||_{p,\mu} \le \frac{1}{C_1} \left\| \left(\sum_{(j,k)\in A} \sum_{\lambda=1}^{2^n-1} \left|\varepsilon_{j,k}^{\lambda} < f, \psi_{j,k}^{\lambda} >\right|^2 \left|\psi_{j,k}^{\lambda}\right|^2 \right)^{1/2} \right\|_{p,\mu} \le \frac{C_2}{C_1} ||f||_{p,\mu} .$$

So the partial sum operators and the modified partial sum operators with any sequence of signs are uniformly bounded in $L^p(d\mu)$. Finally, we shall prove that, for every $f \in L^p(d\mu)$, its wavelet expansion converges. By the hypothesis the series of positive terms $\sum_{(j,k)\in A} \sum_{\lambda=1}^{2^n-1} | < f, \psi_{j,k}^{\lambda} > |^2 |\psi_{j,k}^{\lambda}(x)|^2$ converges almost everywhere and by using the dominated convergence theorem we get that

$$\left\| \left(\sum_{j\in\mathbb{Z}} \sum_{k\in\mathbb{Z}^n} \sum_{\lambda=1}^{2^n-1} \left| < f - T_A f, \psi_{j,k}^{\lambda} >\right|^2 \left|\psi_{j,k}^{\lambda}\right|^2 \right)^{1/2} \right\|_{p,\mu} \to 0 ,$$

when $A \nearrow \mathbb{Z}^{n+1}$ and by using (D3) again we get that

$$f = \sum_{j\in\mathbb{Z}} \sum_{k\in\mathbb{Z}^n} \sum_{\lambda=1}^{2^n-1} < f, \psi_{j,k}^{\lambda} > \psi_{j,k}^{\lambda} ,$$

in $L^p(d\mu)$.

(D4) $\Rightarrow$ (D1). It follows the same lines of (D3) $\Rightarrow$ (D1) (see also [7]).

(D2) $\Rightarrow$ (D4). As in [12], in order to apply the weighted inequalities for Calderón–Zygmund operators in their standard form, one can take a smooth function φ with certain decay such that $|\varphi| \ge \chi_{[0,1]^n}$ and, at the same time, $\{\varphi_{j,k} = 2^{nj/2}\varphi(2^j x - k),\ j \in \mathbb{Z}, k \in \mathbb{Z}^n\}$ is a Bessel sequence (see [2]). In one dimension we can take, for instance, $\varphi(x) = C\psi(ax + b)$, where ψ is a C^1 Daubechies wavelet and the constants a, b and C are chosen in such a way that $|\varphi| \ge \chi_{[0,1]}$. We can apply Lemma 1 (p. 31) in [2] in order to obtain that $\{\varphi_{j,k},\ j \in \mathbb{Z}, k \in \mathbb{Z}^n\}$ is a Bessel sequence. For dimension n we can take $\tilde{\varphi}(x) = \varphi(x_1)\varphi(x_2)\ldots\varphi(x_n)$. Now the operators

$$\tilde{T}_{A,\varepsilon} f(x) = \sum_{(j,k)\in A} \sum_{\lambda=1}^{2^n-1} \varepsilon_{j,k}^{\lambda} < f, \psi_{j,k}^{\lambda} > \tilde{\varphi}_{j,k}$$

constitute a family of Calderón–Zygmund operators. Using the uniform boundedness of $\tilde{T}_{A,\varepsilon}$ in $L^p(w)$ with $w \in A_p$, Khintchine inequality and

$$\left\| \left(\sum_{(j,k)\in A} \sum_{\lambda=1}^{2^n-1} \left| < f, \psi_{j,k}^{\lambda} >\right|^2 \chi_{j,k} \right)^{1/2} \right\|_{p,w} \le \left\| \left(\sum_{(j,k)\in A} \sum_{\lambda=1}^{2^n-1} \left| < f, \psi_{j,k}^{\lambda} >\right|^2 |\tilde{\varphi}_{j,k}|^2 \right)^{1/2} \right\|_{p,w}$$

we get the right hand side inequality in (D4). The left hand side inequality follows as usual by a duality argument.

5. Proofs of Theorems 5 and 6

Proof of Theorem 5. We shall use the same notation that in Theorems 2 and 3. As in the proofs of Theorems 2 and 3, the implications (A5) $\Rightarrow$ (A4) $\Rightarrow$ (A3), (A5) $\Rightarrow$ (A2) $\Rightarrow$ (A3), (B1) $\Rightarrow$ (B2) and (B3) $\Rightarrow$ (B2) are obvious. On the other hand, the implications (A1) $\Rightarrow$ (A2), (A5) $\Rightarrow$ (A), (A5) $\Rightarrow$ (A1) and (Bi) $\Rightarrow$ (B), $i = 1, 2, 3, 4$, follow as in the proof of the corresponding results in Theorems 2 and 3. Therefore, we only have to show that (A3) $\Rightarrow$ (A6) $\Rightarrow$ (A5), (B2) $\Rightarrow$ (B4) $\Rightarrow$ (B1) and (B4) $\Rightarrow$ (B3).

(A3) $\Rightarrow$ (A6) and (B2) $\Rightarrow$ (B4). First we shall prove that μ is absolutely continuous. Given a set E such that $|E| = 0$ and $\varepsilon > 0$ there exists an open set G such that $E \subset G$, $\mu(G \setminus E) < \varepsilon$. For each $j \in \mathbb{Z}$ large enough, take G_j the union of the dyadic cubes $Q_{j,k}$ of side length 2^{-j} and such that $Q_{j,k} \subset G$. Now, for each $x \in G_j$ there is $k_0 \in \mathbb{Z}^n$ such that $x \in Q_{j,k_0}$ and

$$P_j(\chi_{G\setminus E})(x) = 2^{nj}|Q_{j,k_0}\setminus E| = 1 ,$$

since $|E| = 0$. So, because of the uniform weak type (p, p) with respect to μ of P_j, we get that $\mu(G_j) \leq C\mu(G\setminus E)$. On the other hand, since the family of sets G_j is increasing ($G_j \subset G_{j+1}$) and $\cup_j G_j = G$ we get

$$\mu(E) \leq \mu(G) = \lim_{j\to\infty} \mu(G_j) \leq C\mu(G\setminus E) < C\varepsilon ,$$

for all $\varepsilon > 0$. Therefore $\mu(E) = 0$. It follows, by Radon–Nikodym Theorem that there exists a locally integrable function w such that $d\mu = w(x)\,dx$. Now we shall prove that $w \in A_p^{dy}$. Let us suppose that $1 < p < \infty$, let Q be a dyadic cube with side length 2^{-j} and $\sigma_\varepsilon = (w + \varepsilon)^{-\frac{1}{p-1}}$, $\varepsilon > 0$. Then, for all $x \in Q$ we get that

$$P_j(\sigma_\varepsilon\chi_Q)(x) = 2^{nj}\int_Q \sigma_\varepsilon(y)\,dy = |Q|^{-1}\sigma_\varepsilon(Q) .$$

By using the weak type inequality we obtain that $w(Q) \leq C|Q|^p\sigma_\varepsilon(Q)^{1-p}$ and letting ε go to zero we obtain that $w \in A_p^{dy}$.
Consider now the case $p = 1$. Let x_0 be a Lebesgue point of w, Q a dyadic cube of side length 2^{-j} and $\tilde{Q}$ a cube such that $x_0 \in \tilde{Q} \subset Q$. Then, for all $x \in Q$

$$\left|P_j\left(\chi_{\tilde{Q}}\right)(x)\right| = 2^{nj}\left|\tilde{Q}\right| = |Q|^{-1}\left|\tilde{Q}\right| ,$$

and by the hypothesis we obtain that

$$\frac{1}{|Q|}\int_Q w \leq C\frac{1}{\left|\tilde{Q}\right|}\int_{\tilde{Q}} w ,$$

and letting $\tilde{Q} \to x_0$ we get that $\frac{1}{|Q|}\int_Q w \leq Cw(x_0)$ and we are done.

(B4) $\Rightarrow$ (B1). This implication follows from the inequalities

$$\begin{aligned} ||P_j f||_{L^1(w)} &\leq \sum_{k\in\mathbb{Z}^n}\left(\int_{Q_{j,k}}|f(y)|\left(\frac{1}{|Q_{j,k}|}\int_{Q_{j,k}} w(x)\,dx\right)dy\right) \\ &\leq C\sum_{k\in\mathbb{Z}^n}\left(\int_{Q_{j,k}}|f(y)|w(y)\,dy\right) = C||f||_{L^1(w)} , \end{aligned}$$

where in the last inequality we have used the condition A_1^{dy}.

(A6) ⇒ (A5) and (B4) ⇒ (B3). These implications follow easily from the pointwise estimate $P^* f(x) \leq M^{dy} f(x)$, where M^{dy} is the Hardy–Littlewood maximal function over dyadic cubes, and the fact that, for $1 < p < \infty$, the dyadic condition A_p^{dy} characterize the boundedness of M^{dy} on the spaces $L^p(w)$ (see Corollary 4.5 in [8]). On the other hand, when $p = 1$ it easy to prove that condition A_1^{dy} implies that M^{dy} is of weighted weak type $(1, 1)$. □

Proof of Theorem 6. The implications (H1) ⇒ (H2) and (H3) ⇒ (H1) can be proved as the corresponding implications in Theorem 4. In order to prove (H2) ⇒ (H3), we can argue again as in Theorem 4. Let $T_{A,\varepsilon}$ denote the operators defined in the proof of Theorem 4 with $h_{j,k}^{\lambda}$ instead of $\psi_{j,k}^{\lambda}$. Even when the operators $T_{A,\varepsilon}$ are no longer standard singular integrals because of the lack of regularity, in [6] it was proved that they are uniformly bounded on $L^p(w)$, $1 < p < \infty$, if $w \in A_p^{dy}$. This is actually the only point needed to finish the proof of the theorem. □

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