VECTOR-VALUED EXTENSIONS OF OPERATORS RELATED TO THE ORNSTEIN–UHLENBECK SEMIGROUP

By

E. HARBOURE, J. L. TORREA AND B. VIVIANI*

Abstract. We find necessary and sufficient conditions on a Banach space X in order for the vector-valued extensions of several operators associated to the Ornstein–Uhlenbeck semigroup to be of weak type $(1,1)$ or strong type (p,p) in the range $1 < p < \infty$. In this setting, we consider the Riesz transforms and the Littlewood–Paley g-functions. We also deal with vector-valued extensions of some maximal operators like the maximal operators of the Ornstein–Uhlenbeck and the corresponding Poisson semigroups and the maximal function with respect to the gaussian measure.

In all cases, we show that the condition on X is the same as that required for the corresponding harmonic operator: UMD, Lusin cotype 2 and Hardy–Littlewood property. In doing so, we also find some new equivalences even for the harmonic case.

Introduction

The purpose of this work is to characterize those Banach spaces X for which the Riesz transforms, Littlewood–Paley g-functions and maximal operators related with the Ornstein–Uhlenbeck semigroup are bounded when acting on X-valued functions.

The first reference to vector-valued extensions of operators associated with the Ornstein–Uhlenbeck semigroup appears in Pisier [P] as part of an effort to prove a dimensional-free $L^p(d\gamma)$-estimate for the Riesz transforms. The transference method he used there gives the boundedness on $L^p_X(d\gamma)$, $p > 1$ when X is UMD. Unfortunately, this technique does not allow us to deal with functions in $L^1_X(d\gamma)$, nor does it seem to be applicable to other operators, as is our intention. We recall that the UMD property for a Banach space X was first introduced by Burkholder in a probability setting. However, it was shown to be equivalent to the fact that the

*The first and third authors were partially supported by CONICET (Argentina) and Convenio Universidad Autónoma de Madrid-Universidad Nacional del Litoral.

The second author was partially supported by the European Commission via the TMR network "Harmonic Analysis".

Hilbert transform maps L^p_X into itself for some p in the range $1 < p < \infty$; see [Bu] and [Bo].

In recent years, there has been considerable activity in the study of the operators associated to the Ornstein–Uhlenbeck semigroup following the original one-dimensional approach of Muckenhoupt [Mu] of dividing $\mathbb{R}$ into two regions: one where the gaussian and Lebesgue measures are equivalent and the corresponding operators comparable, and the other where the kernels of the operators can be estimated by a well-behaved positive kernel. These regions and the corresponding parts of the operators are nowadays referred to as local and global, respectively. Techniques to get the right bounds for the global part of the kernels in any finite dimension were first developed by Sjögren [Sj] for the maximal Ornstein–Uhlenbeck semigroup operator and later extended to other operators by various authors; see [FGS], [U], [PS], [GMST1].

This suggests that the gaussian operators might be extended to those Banach spaces where their corresponding harmonic versions are well-defined. In fact, using these techniques, we are able to prove this type of result. More precisely, we show that a Banach space is UMD if and only if the gaussian Riesz transforms (and their maximal operators) have all the expected boundedness properties (see Theorem 1.10). Similarly, for the g function, we find that the Banach spaces are the same as those in the euclidean case (see Theorem 1.12) and, following [X], we say that these spaces have the Lusin cotype 2 property. Finally, we prove a similar result for other maximal operators as well (see Theorem 1.13).

In proving these theorems, we need a deeper understanding of the local parts in order to compare each gaussian operator with its precise harmonic version. For example, in [FGS], [PS], [U], [GMST1], the local parts of the Riesz transforms were compared to certain singular integral operators but not precisely to the harmonic Riesz transforms. To make the right comparison, we follow the pattern suggested by the definitions of the operators given through the spectral theory. Following this thought, it is not difficult to see that the right choice is to start with the Gauss–Weierstrass semigroup associated to $\frac{1}{2}\Delta$ rather than to Δ. This is the main content of Section 3.

For the harmonic Riesz transforms, it was already known that the UMD condition on X is equivalent to the almost everywhere finiteness of the associated maximal operators for functions in $L^1_X(dx)$. However, there does not seem to a be similar result for the Littlewood–Paley g-function or the maximal Hardy–Littlewood operator in the literature. We prove the corresponding statements to be true in Propositions 4.5 and 4.12.

1 Preliminaries and statement of results

Let $\{T_t\}$ be a symmetric diffusion semigroup of operators acting on measurable functions on $\mathbb{R}^n$, with a second order differential operator $-L$ as its infinitesimal generator. In this context, the following operators can be considered; see [St1].

(1) Maximal operator: $T^*f(x) = \sup_{t>0} |T_t f(x)|$.

(2) Maximal operator of the subordinated Poisson semigroup:

$P^*f(x) = \sup_{t>0} |P_t f(x)|$, where P_t is defined by the following subordination formula,

$$P_t f(x) = \frac{1}{\sqrt{2\pi}} \int_0^\infty t e^{-t^2/4s} T_s f(x) s^{-3/2} ds. \tag{1.1}$$

(3) Riesz potentials: For $0 < \alpha$, $L^{-\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_0^\infty t^{\alpha-1} T_t f(x) dt$, which can be derived from the identity $s^{-\alpha} = \frac{1}{\Gamma(\alpha)} \int_0^\infty t^{\alpha-1} e^{-ts} dt$.

(4) Riesz transforms: For $1 \le i \le n$, $R_i f(x) = \frac{\partial}{\partial x_i} L^{-1/2} f(x)$.

(5) Littlewood–Paley g-functions:

$$g_0(f)(x) = \Big(\int_0^\infty \Big| t\frac{\partial}{\partial t} P_t f(x) \Big|^2 \frac{dt}{t} \Big)^{1/2}$$

and

$$g_{x_i}(f)(x) = \Big(\int_0^\infty \Big| t\frac{\partial}{\partial x_i} P_t f(x) \Big|^2 \frac{dt}{t} \Big)^{1/2}, \quad i = 1, \ldots, n.$$

Here $\partial/\partial x_i$, $i = 1, \ldots, n$ are the partial derivatives associated to the operator L, that is, if ∇_x is the vector $(\partial/\partial x_1, \ldots, \partial/\partial x_n)$ it satisfies

$$(-L)(u^p) = p(p-1)u^{p-2} |\nabla_x u|^2$$

for functions $u \ge 0$ solutions of the equation $Lu = 0$.

In the classical case $L = -\Delta$, that is, T_t is the Gauss–Weierstrass semigroup, all of these operators are well-known. We refer to [St2] for their $L^p(dx)$-boundedness properties, where dx is the Lebesgue measure in $\mathbb{R}^n$. However, for our purposes it will be best to deal with the semigroup W_t whose infinitesimal infinitesimal generator is $-\frac{1}{2}\Delta$, that is,

$$W_t f(x) = (2\pi t)^{-n/2} \int_{\mathbb{R}^n} e^{-|x-y|^2/2t} f(y) dy. \tag{1.2}$$

Anyway, we notice that the operators (1) to (5) defined above differ only in a constant after this change in the infinitesimal generator.

Let us recall that in this case the Riesz transforms can be viewed as principal values of the integrals against the kernels

$$K_i(x-y) = c_n \frac{x_i - y_i}{|x-y|^{n+1}},$$

which appear as the corresponding partial derivatives of the kernel associated to the $(-\frac{1}{2}\Delta)^{-1/2}$ operator. Moreover, the maximal operators

$$K_i^* f(x) = \sup_{\varepsilon>0} K_{i,\varepsilon} f(x) = \sup_{\varepsilon>0} \left| \int_{\mathbf{R}^n} K_{i,\varepsilon}(x-y) f(y) dy \right|, \quad 1 \leq i \leq n,$$

where $K_{i,\varepsilon}(x-y) = K_i(x-y)\chi_{\{|x-y|>\varepsilon\}}$, are bounded on $L^p(dx)$, $1 < p < \infty$, and of weak type $(1,1)$. This implies both the almost everywhere convergence of $K_{i,\varepsilon} f$ for $f \in L^p(dx)$, $1 \leq p < \infty$ and the convergence in $L^p(dx), 1 < p < \infty$ or the weak-$L^1(dx)$ convergence; see [St2].

In the case $L = -\frac{1}{2}\Delta + x.\nabla$, that is, T_t is the Ornstein–Uhlenbeck semigroup O_t given by

$$O_t f(x) = (\pi(1-e^{-2t}))^{-n/2} \int_{\mathbf{R}^n} e^{-|e^{-t}x-y|^2/(1-e^{-2t})} f(y) dy,$$

the above operators have been intensively studied over the last twenty years by several authors. In this setting, the natural measure is the gaussian measure $d\gamma(x) = e^{-|x|^2} dx$. For the $L^p(d\gamma)$-boundedeness of the maximal operator O^*, we refer to [St1] and [Sj]; for the Riesz transforms R_i, see [Mu], [Gu], [Me], [P], [FGS], [Gt], [U], [GMST2]. As in the classical case, the Riesz transforms can be viewed as principal values of integrals against the kernels

$$R_i(x,y) = c_n \int_0^\infty t^{-1/2} e^{-t} \frac{e^{-t}x_i - y_i}{(1-e^{-2t})^{n/2+1}} \exp\Big(-\frac{|e^{-t}x-y|^2}{1-e^{-2t}}\Big) dt,$$

which appear as the corresponding derivatives of the kernel associated to the $(-\frac{1}{2}\Delta + x.\nabla)^{-1/2}$ operator. Namely, for functions f good enough, we have

$$R_i f(x) = \lim_{\varepsilon \to 0} R_{i,\varepsilon} f(x), \quad \text{a.e. } x,$$

where $R_{i,\varepsilon}(x,y) = R_i(x,y)\chi_{\{|x-y|>\varepsilon\}}$.

The Littlewood–Paley g-functions studied in [PS] and [Gt] can be explicitly expressed by taking the corresponding derivatives of the associated Poisson kernel given by

$$(1.3) \quad P_t(x,y) = \frac{1}{\sqrt{2\pi}} \int_0^\infty t e^{-t^2/4s} (\pi(1-e^{-2s}))^{-n/2} e^{-|e^{-s}x-y|^2/(1-e^{-2s})} s^{-3/2} ds.$$

In the classical context, the study of the behaviour of the above operators is closely related to a variant of the Hardy–Littlewood maximal operator

$$Mf(x) = \sup_{r>0} \left| \frac{1}{|B(x,r)|} \int_{B(x,r)} f(y) dy \right|.$$

It is well-known that this operator is one of the fundamentals in real analysis. For that reason, we are led to consider the gaussian Hardy–Littlewood maximal operator, that is,

$$M_\gamma f(x) = \sup_{r>0} \left| \frac{1}{\gamma(B(x,r))} \int_{B(x,r)} f(y) d\gamma(y) \right|.$$

Both operators share the same boundedness properties on $L^p(d\mu)$, where μ denotes either Lebesgue or gaussian measure; this is a consequence of a general theory for centered Hardy–Littlewood maximal functions.

Next, we consider extensions of these operators to functions taking values on a Banach space X, in both the Gauss–Weierstrass and the Ornstein–Uhlenbeck semigroups.

Since the Riesz transforms R_i are linear, they extend in a natural way to the tensor product $L^p(d\mu) \otimes X$ as $R_i(\sum_{k=1}^n \varphi_k v_k) = \sum_{k=1}^n R_i(\varphi_k) v_k$, $1 \leq p < \infty$. In the harmonic case, it is known that the extensions of K_i are bounded from $L^p_X(dx)$, $1 < p < \infty$, into itself or from $L^1_X(dx)$ into weak-$L^1_X(dx)$ if and only if X satisfies the so-called UMD property; see [Bu] and [Bo]. Moreover, in a UMD space, one has the same result for the associated maximal operator K_i^* and the almost everywhere convergence of $K_{i,\varepsilon} f$ to $K_i f$, as in the scalar case.

Concerning the Littlewood–Paley g-functions, we extend their definitions to X-valued functions f by

$$g_0(f)(x) = \left(\int_0^\infty \left\| t \frac{\partial}{\partial t} P_t f(x) \right\|_X^2 \frac{dt}{t} \right)^{1/2}$$

and

$$g_{x_i}(f)(x) = \left(\int_0^\infty \left\| t \frac{\partial}{\partial x_i} P_t f(x) \right\|_X^2 \frac{dt}{t} \right)^{1/2}, \quad i = 1, \ldots, n,$$

where it is understood that P_t, being linear, has been extended to functions taking values in X as above.

In the classical case, since these g-functions can be seen as vector-valued Caderón–Zygmund operators (see [RRT]), their boundedness in some $L^{p_0}_X(dx)$, $1 < p_0 < \infty$, is equivalent to the boundedness for all $p, 1 < p < \infty$, or even to the boundedeness from $L^1_X(dx)$ into weak-$L^1(dx)$. This remains true for functions defined in the torus, and the corresponding Banach spaces X have been called in [X] of **Lusin cotype** 2. For functions defined on $\mathbb{R}^n$, we shall adopt this terminology.

Finally, for the maximal operators defined in (1) and (2) and for the Hardy–Littlewood maximal operator, an extension of the type $T^* f(x) = \sup_{t>0} \|T_t f(x)\|_X$ gives rise to a trivial problem, since in this case $T^* f(x) \leq T^*(\|f\|_X)(x)$, and therefore the properties of the scalar version of the operator are automatically true for every Banach space X. However, when X is a Banach lattice, following [GMT], a nontrivial extension can be given by

$$T^*f(x) = \sup_{t>0} |T_t f(x)|_X, \tag{1.4}$$

$$P^*f(x) = \sup_{t>0} |P_t f(x)|_X \quad \text{and} \tag{1.5}$$

$$M_\mu f(x) = \sup_{r>0} \left| \frac{1}{\mu(B(x,r))} \int_{B(x,r)} f(y) d\mu(y) \right|_X, \tag{1.6}$$

where $|.|_X$ is the absolute value of the Banach lattice X. The problem now is that for a general Banach lattice we cannot guarantee that the supremum exists as an element in X for a.e. x. However, these operators were considered by Rubio de Francia [R] in the case that X is a Banach lattice of measurable functions in a σ-finite measure space $(\Omega, d\omega)$, often referred to as Köthe function spaces, see [LT]. In this context, the above operators are well-defined and take the form

$$T^*f(x,\omega) = \sup_{t>0} |T_t f(x,\omega)|, \quad x \in \mathbb{R}^n, \quad \omega \in \Omega, \tag{1.7}$$

$$P^*f(x,\omega) = \sup_{t>0} |P_t f(x,\omega)|, \quad x \in \mathbb{R}^n, \quad \omega \in \Omega \quad \text{and} \tag{1.8}$$

$$M_\mu f(x,\omega) = \sup_{r>0} \left| \frac{1}{\mu(B(x,r))} \int_{B(x,r)} f(y,\omega) d\mu(y) \right|, \quad x \in \mathbb{R}^n, \quad \omega \in \Omega, \tag{1.9}$$

where $|\cdot|$ denotes the absolute value in X with the obvious definition $|v|(\omega) = |v(\omega)|$, $v \in X$. The results in [GMT] can be applied to this context, yielding that the boundedness of the classical Hardy–Littlewood operator in some $L^{p_0}_X(dx)$, $1 < p_0 < \infty$, is equivalent to the boundedness for all $p, 1 < p < \infty$, or even to the boundedness from $L^1_X(dx)$ into weak-$L^1_X(dx)$. Following these authors the Köthe function spaces for which one of these boundedness holds are said to satisfy the **Hardy–Littlewood (H.L.) property**.

We point out that the UMD property has been given a geometric description; see [Bu] and [Bo]. There is also an equivalent geometric description for Lusin cotype for functions in the torus; see [X]. For the Hardy–Littlewood property, geometric conditions that are either sufficient or necessary were given in [GMT].

We are now in a position to state the main results of our paper.

Theorem 1.10. *Given a Banach space X, let R_i, $1 \leq i \leq n$, be the X-valued extended Riesz transforms associated to the Orsntein–Uhlenbeck semigroup. Then the following conditions are equivalent:*

(i) *X has the UMD property.*

(ii) $\gamma\{x \in \mathbb{R}^n : \|R_i f(x)\|_X > \lambda\} \leq \frac{C}{\lambda} \int_{\mathbf{R}^n} \|f(x)\|_X d\gamma(x)$, $1 \leq i \leq n$.

(iii) *For every p, $1 < p < \infty$ (and equivalently for some $1 < p < \infty$),*

$$\|R_i f\|_{L^p_X(d\gamma)} \leq C_p \|f\|_{L^p_X(d\gamma)}, \quad 1 \leq i \leq n.$$

(iv) $\gamma\{x \in \mathbb{R}^n : R_i^* f(x) > \lambda\} \leq \frac{C}{\lambda} \int_{\mathbf{R}^n} \|f(x)\|_X d\gamma(x)$, $1 \leq i \leq n$.

(v) *For every p, $1 < p < \infty$ (and, equivalently, for some $1 < p < \infty$),*

$$\|R_i^* f\|_{L^p(d\gamma)} \leq C_p \|f\|_{L^p_X(d\gamma)}, \quad 1 \leq i \leq n.$$

(vi) *For any $f \in L^1_X(d\gamma)$, $R_{i,\varepsilon} f(x)$ converges a.e. x, $1 \leq i \leq n$.*

(vii) *For every $f \in L^1_X(d\gamma)$, then $R_i^* f(x) < \infty$ a.e. x, $1 \leq i \leq n$.*

Here, the constants C and C_p are independent of f but they may depend on the Banach space X.

Remark 1.11. All the statements (ii) through (vii) hold true for the scalar case. In particular, (vi) implies that the gaussian Riesz transforms exist in the principal value sense for $f \in L^1(d\gamma)$, a fact that does not appear explicitly in the literature.

In what follows, given an X-valued function f, we shall consider a Littlewood–Paley square function, $g(f)$, involving both t and x derivatives, namely

$$g(f)(x) = \left(g_0(f)^2(x) + \sum_{i=1}^{n} g_{x_i}(f)^2(x) \right)^{1/2},$$

and we shall use lower-case letters g for the Orsntein–Uhlenbeck semigroup while we shall denote with capital letters G the corresponding functions for the Gauss–Weierstrass semigroup. With this notation we have the following theorem in the context of the Ornstein–Uhlenbeck semigroup.

Theorem 1.12. *The following conditions are equivalent:*

(i) *X has Lusin cotype 2.*

(ii) $\gamma\{x \in \mathbb{R}^n : g(f)(x) > \lambda\} \leq \frac{C}{\lambda} \int_{\mathbf{R}^n} \|f(x)\|_X d\gamma(x)$.

(iii) *For every p, $1 < p < \infty$ (and, equivalently, for some $1 < p < \infty$),*

$$\|g(f)\|_{L^p(d\gamma)} \leq C_p \|f\|_{L^p_X(d\gamma)}.$$

(iv) *For every* $f \in L^1_X(d\gamma)$, *then* $g(f)(x) < \infty$ *a.e.* x.
Here, the constants C *and* C_p *are independent of* f *but they may depend on the Banach space* X.

For the vector-valued extensions of the maximal operators given in (1.7), (1.8) and (1.9), we introduce the following notation. In the case of the Ornstein–Uhlenbeck semigroup, the operator T_t will be called O_t and O^* will denote the corresponding maximal operator, while we keep the notation P_t and P^* for the subordinated Poisson semigroup. For the Gauss–Weierstrass semigroup, we use the letters W_t, W^*, U_t and U^*, respectively.

Theorem 1.13. *Given a Köthe function space* X, *let* $\mathcal{N}$ *denote the maximal operator of the Ornstein–Uhlenbeck semigroup,* O^*, *or the maximal operator of the subordinated Poisson semigroup,* P^*, *or the Hardy–Littlewood maximal operator with respect to the gaussian measure,* M_γ. *Then the following statements are equivalent:*

(i) X *satisfies the Hardy–Littlewood property.*

(ii) $\gamma\{x \in \mathbb{R}^n : \|\mathcal{N}f(x)\|_X > \lambda\} \leq \frac{C}{\lambda} \int_{\mathbf{R}^n} \|f(x)\|_X \, d\gamma(x)$.

(iii) *For every* $p, 1 < p < \infty$ *(and, equivalently, for some* $1 < p < \infty$ *),*

$$\|\mathcal{N}f\|_{L^p_X(d\gamma)} \leq C_p \|f\|_{L^p_X(d\gamma)}.$$

(iv) *For every* $f \in L^1_X(d\gamma)$, $\mathcal{N}f(x) \in X$ *a.e.* x.
Here, the constants C *and* C_p *are independent of* f *but they may depend on the Banach space* X.

2 Previous results

In proving our theorems, following the technique initiated in [Mu], we make systematic use of a partition of the operators into their local and global parts, according to some particular region, where Lebesgue and gaussian measure are equivalent, and its complement. In the literature, depending on the operator under consideration, two kind of regions have been used: either

$$N_t = \left\{(x,y) : |x-y| < \frac{t}{1+|x|+|y|}\right\}, \text{ for some fixed } t > 0$$

or

$$\mathcal{N}_R = \left\{(x,y) : |x-y| < \min\left(R, \frac{R}{|x|}\right)\right\}, \text{ for some fixed } R > 0.$$

See [Sj], [FGS], [PS], [GMST1].

For the global part of the operators, we want to make use of estimates given by different authors for the kernels in the complement of regions like $\mathcal{N}_R$, while for the local part, we will refer to the technique developed in [GMST1] in terms of N_t-regions. However, it is immediate that we have the relationship

$$N_t \subset \mathcal{N}_t \subset N_{t(t+3)}. \tag{2.1}$$

In what follows, we shall make a partition using the region N_t for $t = n(n+3)$. This choice of t together with (2.1) allows us to handle both parts for all the operators involved. We denote this particular region simply by N.

Also, since some of the operators we deal with, even not linear, can be viewed as linear operators taking values in some Banach spaces, we need to take in consideration operators T as follows.

Given Banach spaces B_1, B_2, let $d\mu$ denote either the Lebesgue or the Gauss measure on $\mathbb{R}^n$. Let T be a linear operator defined in L^∞_{0,B_1}, the space of B_1-valued, compactly supported and essentially bounded functions, into the space of B_2-valued and strongly measurable functions on $\mathbb{R}^n$, satisfying the following conditions.

(a) T extends to a bounded operator either from $L^q_{B_1}(d\mu)$ into $L^q_{B_2}(d\mu)$ for some q, $1 < q < \infty$, or from $L^1_{B_1}(d\mu)$ into weak-$L^1_{B_2}(d\mu)$.

(b) There exists a $\mathcal{L}(B_1, B_2)$-valued measurable function K defined on the complement of the diagonal in $\mathbb{R}^n \times \mathbb{R}^n$ such that for every function f in L^∞_{0,B_1},

$$Tf(x) = \int K(x,y)\, f(y)dy,$$

for all x outside the support of f.

(c) The kernel K mentioned above satisfies the estimates

$$\|K(x,y)\| \le \frac{C}{|x-y|^n}, \qquad \|\partial_x K(x,y)\| + \|\partial_y K(x,y)\| \le \frac{C}{|x-y|^{n+1}},$$

for all (x,y) in the local region $\tilde{N} = N_{2n(n+3)}$, $x \ne y$.

As in [GMST1], we introduce the following definitions.

For an operator T as above, given a smooth function φ on $\mathbb{R}^n \times \mathbb{R}^n$ such that $\varphi(x,y) = 1$ if $(x,y) \in N$, $\varphi(x,y) = 0$ for $(x,y) \notin \tilde{N}$ and

$$|\partial_x \varphi(x,y)| + |\partial_y \varphi(x,y)| \le C\,|x-y|^{-1} \quad \text{if } x \ne y, \tag{2.2}$$

we define the **global** and the **local** parts of the operator T by

$$T_{glob}f(x) = \int K(x,y)(1-\varphi(x,y))f(y)dy,$$
$$T_{loc}f(x) = Tf(x) - T_{glob}f(x),$$

and accordingly call their kernels K_{glob} and K_{loc}, respectively.

We say that an operator T defined on L^{∞}_{0,B_1} into the space of B_2-valued strongly measurable functions is **local** if its associated kernel in the sense of (b) is supported in $\tilde{N}$.

We make use of the following results; cf. [GMST1].

Proposition 2.3. *I. If the operator T satisfies assumptions (a), (b) and (c) as above, then the operator T_{loc} is bounded from $L^p_{B_1}(d\gamma)$ into $L^p_{B_2}(d\gamma)$ and from $L^p_{B_1}(dx)$ into $L^p_{B_2}(dx)$, for $1 < p < \infty$. Moreover, T_{loc} is bounded from $L^1_{B_1}(d\mu)$ into weak-$L^1_{B_2}(d\mu)$ with respect to the Lebesgue and the Gauss measure.*

II. If T is an operator satisfying condition (a), (b) and only the size condition of the kernel stated in (c), then T_{loc} inherits from T either the L^q-boundedness or the weak type (1,1), as the case may be. Moreover, the corresponding boundedness holds for both Lebesgue and Gauss measure.

Proposition 2.4. *If S is a local operator, then strong type (p,p) for Lebesgue and gaussian measure are equivalent. The same holds for weak type (p,p), $1 \le p < \infty$.*

Next, we review some known results for the scalar versions of the operators associated to the Ornstein–Uhlenbeck semigroup. We state them as lemmas for future reference.

Lemma 2.5. *The global part of the Riesz transform kernels satisfies*

$$|R_{i,\varepsilon,glob}(x,y)| \le |R_{i,glob}(x,y)| \le Q(x,y), \quad i = 1,\dots,n$$

for some nonnegative kernel $Q(x,y)$, independent of ε, supported in N^c and such that its associated integral operator is of weak type $(1,1)$ and strong type (p,p), $1 < p < \infty$, with respect to the gaussian measure.

For a proof, see [FGS] and [PS].

Remark 2.6. These results imply in particular that the integrals $\int R_{i,\varepsilon,glob}(x,y)f(y)dy$ and $\int R_{i,glob}(x,y)f(y)dy$ are finite almost everywhere for every $f \in L^1(d\gamma)$.

For the nonlinear operators under consideration, we use the following notation:

$$O^*_{glob}(f)(x) = O^*((1-\varphi(x,.))f)(x),$$

$$g_{glob}(f)(x) = g((1-\varphi(x,.))f)(x)$$

and

$$M_{\gamma,glob}(f)(x) = M_\gamma((1-\varphi(x,.))f)(x).$$

Then we have

Lemma 2.7. *The global parts of the operators defined above satisfy*

(i) $O^*_{glob}f(x) \leq \int_{\mathbb{R}^n} Q(x,y)|f(y)|dy,$

(ii) $g_{glob}(f)(x) \leq \int_{\mathbb{R}^n} Q(x,y)|f(y)|dy,$

(iii) $M_{\gamma,glob}f(x) \leq \int_{\mathbb{R}^n} S(x,y)|f(y)|dy$

for some nonnegative kernels $Q(x,y)$ *and* $S(x,y)$*, supported on* N^c *such that the associated integral operators are of weak type* $(1,1)$ *and strong type* (p,p)*,* $1 < p < \infty$*, and of strong type* (p,p)*,* $1 \leq p < \infty$*, respectively.*

For a proof of these facts, see [Sj], [PS], [HVT], [FSU].

Remark 2.8. In fact, from the proof given in [PS], $Q(x,y)$ is a bound for $||t\frac{\partial}{\partial t}P_t(x,y)||_{L^2((0,\infty),dt/t)} + \sum_{i=1}^n ||t\frac{\partial}{\partial x_i}P_t(x,y)||_{L^2((0,\infty),dt/t)}$, whenever $(x,y) \in N^c$.

3 Comparison of the operators in the local region

The aim of this section is to show that in the local region, the difference between the corresponding classical and Ornstein–Uhlenbeck operators, acting on scalar functions, behaves nicely; more precisely, they define bounded operators on all $L^p(d\mu)$, for $1 \leq p \leq \infty$, where $d\mu$ is either Lebesgue or the gaussian measure.

The local part of the Riesz transforms has already been defined in the previous section. As for the the nonlinear operators under consideration, we introduce the following notation:

$$O^*_{loc}(f)(x) = O^*((\varphi(x,.))f)(x) \quad \text{and} \quad g_{loc}(f)(x) = g((\varphi(x,.))f)(x).$$

Similarly, we define the corresponding local parts of the classical operators W^*, G, and M; they will be denoted by W^*_{loc}, G_{loc}, and M_{loc}.

Now we are in position to state the main result of this section.

Lemma 3.1. *The difference between the local parts defined above satisfy*

(i) $|R_{i,\varepsilon,loc}(x,y) - K_{i,\varepsilon,loc}(x,y)| \leq L_1(x,y),$ *uniformly in* $\varepsilon > 0$,

(ii) $|g_{loc}f(x) - G_{loc}f(x)| \leq \int_{\mathbb{R}^n} L_2(x,y)|f(y)|dy$,

(iii) $|O^*_{loc}f(x) - W^*_{loc}f(x)| \leq \int_{\mathbb{R}^n} L_3(x,y)|f(y)|dy$,

where L_i, $i = 1,2,3$ *are nonnegative kernels supported on* N *and satisfying*

$$\sup_x \int_{\mathbb{R}^n} L_i(x,y)dy < \infty \quad \text{and} \quad \sup_y \int_{\mathbb{R}^n} L_i(x,y)dx < \infty, \quad i = 1,2,3. \tag{3.2}$$

In particular, from (i) we have that the linear operator $R_{i,loc} - K_{i,loc}$ *satisfies*

$$|R_{i,loc}f(x) - K_{i,loc}f(x)| \leq \int_{\mathbb{R}^n} L_1(x,y)|f(y)|dy.$$

Consequently, all the integral operators associated to $L_i, i = 1,2,3$, *are of strong type* $(p,p), 1 \leq p < \infty$, *with respect to either Lebesgue or gaussian measure.*

Remark 3.3. This lemma together with the estimates for $K_{i,glob}$ imply that for good enough functions, say C^1 with compact support, the limit $\lim_{\varepsilon\to 0} R_{i,\varepsilon}f(x)$ exists for every x.

In fact, writing

$$R_{i,\varepsilon}f(x) = (R_{i,\varepsilon,loc} - K_{i,\varepsilon,loc})f(x) + K_{i,\varepsilon,loc}f(x) + R_{i,\varepsilon,glob}f(x),$$

we see that the first and third integrals, taking absolute values inside, are bounded independently of ε for any value of x. This is a consequence of the previous lemma and the precise estimate of the global part of the kernel given in [PS]. As for the second term, the limit for $\varepsilon \to 0$ exists for all x since the same is true for the harmonic Riesz transform whenever f is smooth and boundedly supported.

In order to prove the result above, we need the following technical lemma.

Lemma 3.4. *For* $(x,y) \in N$, *we have the estimates*

$$D_i(x,y) = \int_0^\infty \left|\frac{\partial}{\partial x_i}(O_s(x,y) - W_s(x-y))\right|\frac{ds}{s^{1/2}} \leq C\frac{1+|x|}{|x-y|^{n-1}},$$

$$\begin{aligned} D(x,y) &= \int_0^\infty \left|O_s(x,y) - \chi_{(1,\infty)}(s)\,\frac{e^{-|y|^2}}{\pi^{n/2}} - W_s(x-y)\right|\frac{ds}{s} \\ &\leq C\Big(\frac{1+|x|^{\frac{1}{2}}}{|x-y|^{n-\frac{1}{2}}} + \log\frac{1}{|x-y|}\Big), \end{aligned}$$

$$E_1(x,y) = \sup_{u \geq \frac{1}{2}} W_u(x-y) \leq \frac{C}{|x-y|^{n-1}}$$

and

$$E_2(x,y) = \sup_{0<u\leq 1} |W_{(1-u^2)/2}(x-y) - O_{\log(1/u)}(x,y)| \leq \frac{C|x|}{|x-y|^{n-1}}.$$

In particular, all of the above kernels when truncated by $\chi_N(x,y)$ satisfy conditions like (3.2).

We prove Lemma 3.1 first, assuming that the above estimates hold, and then return to the proof of Lemma 3.4.

Proof of Lemma 3.1. (i) The kernel for $R_{i,\varepsilon,loc} - K_{i,\varepsilon,loc}$ is given by

$$c_n \chi_{\{|x-y|>\varepsilon\}}(x,y)\, \varphi(x,y) \int_0^\infty \frac{\partial}{\partial x_i}(O_t - W_t)(x,y) \frac{dt}{t^{1/2}}.$$

Taking absolute values, we see that it is enough to bound the kernel

$$D_i(x,y) = \int_0^\infty \left|\frac{\partial}{\partial x_i}(O_t - W_t)(x,y)\right| \frac{dt}{t^{1/2}},$$

for $(x,y) \in N$. Then (i) follows from the estimates given in Lemma 3.4.

(ii) First we consider $g_{x_i,loc} - G_{x_i,loc}$, $1 \leq i \leq n$. Observe that if P_t and $\mathcal{P}_t$ denote the corresponding Poisson operators associated with the Ornstein–Ulhenbeck and Gauss–Weierstrass semigroups, respectively, and their kernels as well, we have

$$\begin{aligned} |g_{x_i,loc}f(x) - G_{x_i,loc}f(x)| &\leq \left|\left|t\frac{\partial}{\partial x_i}(P_t - \mathcal{P}_t)(\varphi(x,\cdot)f)(x)\right|\right|_{L^2((0,\infty),dt/t)} \\ &\leq \int_{\mathbf{R}^n} \left|\left|t\frac{\partial}{\partial x_i}(P_t - \mathcal{P}_t)(x,y)\varphi(x,y)\right|\right|_{L^2((0,\infty),dt/t)} |f(y)|dy, \end{aligned} \tag{3.5}$$

where we have used Minkowski's integral inequality. Therefore, we only need to get a bound for $||t\frac{\partial}{\partial x_i}(P_t - \mathcal{P}_t)(x,y)||_{L^2((0,\infty),dt/t)}$ for $(x,y) \in N$. Using the expression given in (1.3) for P_t and the corresponding one for $\mathcal{P}_t$, that is, replacing O_t by W_t as given in (1.2), after taking derivatives with respect to x_i, we get

$$\begin{aligned} &\left|\left|t\frac{\partial}{\partial x_i}(P_t - \mathcal{P}_t)(x,y)\right|\right|_{L^2((0,\infty),dt/t)} \\ &= \left|\left|c_n t \int_0^\infty t e^{-t^2/4s} \frac{\partial}{\partial x_i}(O_s - W_s)(x,y) s^{-3/2} ds\right|\right|_{L^2((0,\infty),dt/t)} \\ &\leq c_n \int_0^\infty \left|\frac{\partial}{\partial x_i}(O_s - W_s)(x,y)\right| \left(\int_0^\infty t^3 e^{t^2/2s} dt\right)^{1/2} s^{-3/2} ds, \end{aligned}$$

where we have again used Minkowski's inequality. After a change of variables, the inner integral can be estimated by Cs^2. Therefore, to prove our claim we only need to estimate

$$D_i(x,y) = \int_0^\infty \left|\frac{\partial}{\partial x_i}(O_s - W_s)(x,y)\right| \frac{ds}{s^{1/2}}$$

for $(x,y) \in N$. But from Lemma 3.4, it follows easily that (3.2) holds for these kernels when restricted to the region N.

Next, for $g_{0,loc} - G_{0,loc}$, proceeding as above, we reduce the problem to estimate $\|t\frac{\partial}{\partial t}(P_t - \mathcal{P}_t)(x,y)\|_{L^2((0,\infty),dt/t)}$ for $(x,y) \in N$. Differentiation of the Poisson kernels with respect to t leads to the integral

$$c_n \int_0^\infty (1 - t^2/2s)e^{-t^2/4s}(O_s - W_s)(x,y)s^{-3/2}ds,$$

which can be rewritten replacing $O_s(x,y)$ by $O_s(x,y) - e^{-|y|^2}/\pi^{n/2}$ since

$$\int_0^\infty (1 - t^2/2s)e^{-t^2/4s}s^{-3/2}ds = 0. \tag{3.6}$$

Therefore

$$\begin{aligned}
&\left\|t\frac{\partial}{\partial t}(P_t - \mathcal{P}_t)(x,y)\right\|_{L^2((0,\infty),dt/t)} \\
&\leq C\left\|t\int_0^1 \left(1 - \frac{t^2}{2s}\right)e^{-t^2/4s}e^{-|y|^2}s^{-3/2}ds\right\|_{L^2((0,\infty),dt/t)} \\
&\quad + C\left\|t\int_0^\infty \left(1 - \frac{t^2}{2s}\right)e^{-t^2/4s}(O_s(x,y) \right. \\
&\quad \left. - \chi_{(1,\infty)}(s)\frac{e^{-|y|^2}}{\pi^{n/2}} - W_s(x,y))s^{-3/2}ds\right\|_{L^2((0,\infty),dt/t)} \\
&= J_1 + J_2.
\end{aligned}$$

To estimate J_1, we observe that the integral

$$\int_0^\infty \left(\int_0^1 \left(1 - \frac{t^2}{2s}\right)e^{-t^2/4s}e^{-|y|^2}s^{-3/2}ds\right)^2 t\,dt$$

is finite. This follows by splitting the outside integral into the intervals $(0, \sqrt{2})$ and $(\sqrt{2}, \infty)$. For the first piece, we use (3.6) to change the integration on the variable s over the interval $(0, 1)$ to $(1, \infty)$. For the second piece, we bound the exponential by a convenient negative power. Therefore J_1 is bounded by $Ce^{-|y|^2}$, which in the local region has the desired properties.

To estimate J_2 we apply Minkowski's inequality to get

$$J_2 \le C \int_0^\infty \Big| O_s(x,y) - \chi_{(1,\infty)}(s) \frac{e^{-|y|^2}}{\pi^{n/2}} - W_s(x,y) \Big| \times \Big(\int_0^\infty t \Big(1 - \frac{t^2}{2s}\Big)^2 e^{-t^2/2s} dt \Big)^{1/2} s^{-3/2} ds.$$

After changing variables, the inner integral in t is bounded by Cs. Hence we only need to estimate

$$D(x,y) = \int_0^\infty \Big| O_s(x,y) - \chi_{(1,\infty)}(s) \frac{e^{-|y|^2}}{\pi^{n/2}} - W_s(x,y) \Big| \frac{ds}{s}.$$

That this kernel satisfies (3.2) on the region N is a consequence of Lemma 3.4.

(iii) We observe that

$$\begin{aligned} W^*_{loc} f(x) - O^*_{loc} f(x) \le & \sup_{u \ge 1/2} \Big| \int W_u(x-y) \varphi(x,y) f(y) dy \Big| \\ & + \sup_{0 \le u \le 1} \Big| \int W_{(1-u^2)/2}(x-y) \varphi(x,y) f(y) dy \Big| \\ & - \sup_{0 \le u \le 1} \Big| \int O_{\log(1/u)}(x,y) \varphi(x,y) f(y) dy \Big|. \end{aligned}$$

Therefore,

$$\begin{aligned} |W^*_{loc} f(x) - O^*_{loc} f(x)| \le & \int \sup_{u \ge 1/2} |W_u(x-y)\varphi(x,y)|\, |f(y)| dy \\ & + \int \sup_{0 \le u \le 1} |(W_{(1-u^2)/2} - O_{\log(1/u)})(x,y)| \varphi(x,y) | f(y)| dy. \end{aligned}$$

Then the result follows by the estimates given in Lemma 3.4. □

Proof of Lemma 3.4. We begin by stating some simple inequalities that will be used frequently.

(A) Given c, k nonnegative numbers and some $0 < \varepsilon < 1$, there exists a constant c_ε such that

$$t^k e^{-ct^2} \le c_\varepsilon e^{-\varepsilon c t^2}, \quad t > 0.$$

(B) For every given $\varepsilon > 0$ and $(x,y) \in N$, there exists a constant C such that

$$e^{-\varepsilon |tx - y|^2/(1-u^2)} \le C_\varepsilon e^{-\varepsilon |x-y|^2/(1-u^2)}, \quad t > 0, \quad 0 \le u \le t \le 1.$$

To prove the lemma, we first estimate $D_i(x,y)$ for $i = 1, \dots, n$. Differentiating the expressions of $O_s(x,y)$ and $W_s(x-y)$ with respect to x_i, we get

$$\frac{\partial}{\partial x_i} O_s(x,y) = e^{-s} \frac{e^{-s} x_i - y_i}{\pi^{n/2}(1-e^{-2s})^{n/2+1}} \exp\Big(- \frac{|e^{-s}x - y|^2}{1 - e^{-2s}} \Big)$$

and

$$\frac{\partial}{\partial x_i}W_s(x-y)=\frac{x_i-y_i}{\pi^{n/2}(2s)^{n/2+1}}e^{-|x-y|^2/2s}.$$

For fixed $(x,y)\in N$, set $\alpha=\min\{1,1/|x|^2\}$. Then split the integral defining D_i to get

$$\begin{aligned}D_i(x,y)&\le\int_0^\alpha\Big|\frac{\partial}{\partial x_i}O_s(x,y)-\frac{\partial}{\partial x_i}W_s(x-y)\Big|\frac{ds}{s^{1/2}}\\&\quad+\int_\alpha^\infty\Big|\frac{\partial}{\partial x_i}O_s(x,y)\Big|\frac{ds}{s^{1/2}}+\int_\alpha^\infty\Big|\frac{\partial}{\partial x_i}W_s(x-y)\Big|\frac{ds}{s^{1/2}}\\&=I_1+I_2+I_3.\end{aligned}$$

For I_3, using (A) for $k=1$, we have for $(x,y)\in N$,

$$I_3\le c\int_\alpha^\infty s^{-(n+2)/2}ds\le c(1+|x|)^n\le c\frac{|x|}{|x-y|^{n-1}}.$$

As for I_2, we first use (A) with $k=1$ and then apply (B), to get

$$\begin{aligned}I_2&\le C\int_\alpha^\infty\frac{e^{-\varepsilon|x-y|^2/(1-e^{-2s})}}{(1-e^{-2s})^{(n+1)/2}}e^{-s}\frac{ds}{s^{1/2}}\le C\Big(\int_\alpha^1\frac{e^{-\varepsilon'|x-y|^2/s}}{s^{(n+2)/2}}ds+\int_1^\infty e^{-s}\frac{ds}{s^{1/2}}\Big)\\&\le\frac{C}{|x-y|^{n-1/2}}\int_\alpha^1 s^{-5/4}ds+C\le C\frac{(1+|x|)^{1/2}}{|x-y|^{n-1/2}},\end{aligned}$$

where we have again used (A) with $k=n-\frac{1}{2}$ and that $(x,y)\in N$.

To estimate I_1, we set

$$f_1(u)=\frac{u(ux_i-y_i)}{(1-e^{-2s})^{n/2+1}}e^{-|ux-y|^2/(1-e^{-2s})}\quad\text{and}\quad f_2(u)=\frac{x_i-y_i}{u^{n/2+1}}e^{-|x-y|^2/u};$$

then

$$I_1\le C\int_0^\alpha|f_1(e^{-s})-f_1(1)|\frac{ds}{s^{1/2}}+C\int_0^\alpha|f_2(1-e^{-2s})-f_2(2s)|\frac{ds}{s^{1/2}}=I_1^1+I_1^2.$$

For the derivatives of f_1 and f_2 we obtain, after using (B),

$$|f_1'(u)|\le\frac{Ce^{-\varepsilon|x-y|^2/s}}{s^{(n+1)/2}}\Big(1+\frac{|x|}{s^{1/2}}\Big)\quad\text{for } e^{-\alpha}\le e^{-s}\le u\le1$$

and

$$|f_2'(v)|\le\frac{Ce^{-\varepsilon|x-y|^2/s}}{s^{n/2+2}}|x-y|\quad\text{for } 1-e^{-2s}\le v\le 2s.$$

Therefore,

$$I_1^1\le c\int_0^\alpha\frac{e^{-\varepsilon|x-y|^2/s}}{s^{n/2}}\Big(1+\frac{|x|}{s^{1/2}}\Big)ds\le\frac{C}{|x-y|^{n-1}}+\frac{C|x|^{1/2}}{|x-y|^{n-1/2}},$$

where we have used (A) with $k = n - 1$ and $k = n - 1/2$.

Finally, since $|1 - e^{-2s} - 2s| \le Cs^2$, using (A) again, this time for $k = n$, we get

$$I_1^2 \le \frac{C}{|x-y|^{n-1}}.$$

Collecting the estimates for I_1^1, I_1^2, I_2 and I_3, we can conclude that in all cases they are bounded by

$$C\frac{1+|x|}{|x-y|^{(n-1)}}.$$

Let us estimate $D(x,y)$. Recall that

$$O_s(x,y) = (\pi(1-e^{-2s}))^{-n/2} e^{-|e^{-s}x-y|^2/(1-e^{-2s})}$$

and

$$W_s(x-y) = (2\pi s)^{-n/2} e^{-|x-y|^2/2s}.$$

As above, for fixed $(x,y) \in N$, we set $\alpha = \min\{1, 1/|x|^2\}$ and split the integral defining $D(x,y)$ as follows:

$$\begin{aligned} D(x,y) \le& \int_0^\alpha |O_s(x,y) - W_s(x,y)|\frac{ds}{s} + \int_\alpha^1 O_s(x,y)\frac{ds}{s} \\ &+ \int_1^\infty \left|O_s(x,y) - \frac{e^{-|y|^2}}{\pi^{n/2}}\right|\frac{ds}{s} + \int_\alpha^\infty W_s(x-y)\frac{ds}{s} \\ =& I_1 + I_2 + I_3 + I_4. \end{aligned}$$

For I_2, we get

$$I_2 \le C\int_\alpha^1 \frac{ds}{s^{n/2+1}} \le C|x|^n \le C\frac{|x|}{|x-y|^{n-1}}$$

since $(x,y) \in N$. Now, to estimate I_3, set

$$f(u) = (1-u^2)^{-n/2} e^{-|ux-y|^2/(1-u^2)}.$$

It is easy to check, after using (A) and (B), that

$$|f'(\tau)| \le \frac{e^{-\varepsilon|x-y|^2/(1-\tau^2)}}{(1-\tau^2)^{n/2+1}}(c\tau + |x|(1-\tau^2)^{1/2})$$

as long as $(x,y) \in N$. Therefore, an application of the mean value theorem and the estimate above for $0 \le \tau \le e, 0 \le c \le 1$ gives

$$\begin{aligned} I_3 =& C\int_1^\infty |f(e^{-s}) - f(0)|\frac{ds}{s} \le C(1+|x|)\int_1^\infty e^{-s}\frac{ds}{s} \\ &\le C(1+|x|)^n \le C\frac{1+|x|}{|x-y|^{n-1}}. \end{aligned}$$

For I_4, we apply directly inequality (A) with $k = n - 1$, leading to the bound $C(1 + |x|)/|x - y|^{n-1}$.

Finally, for I_1, we consider the functions

$$h_1(u) = \frac{e^{-|x-y|^2/u}}{u^{n/2}}, \qquad h_2(u) = \frac{e^{-|ux-y|^2/(1-e^{-2s})}}{(1 - e^{-2s})^{n/2}}.$$

Then I_1 can be estimated by

$$I_1 \le C \int_0^\alpha |h_2(e^{-s}) - h_2(1)| \frac{ds}{s} + C \int_0^\alpha |h_1(1 - e^{-2s}) - h_1(2s)| \frac{ds}{s} = I_1^1 + I_1^2.$$

Next, we observe that we have, using (A),

$$|h_1'(\tau)| \le C \frac{e^{-\varepsilon|x-y|^2/\tau}}{\tau^{n/2+1}}.$$

Now for τ such that $1 - e^{-2s} \le \tau \le 2s$ with $0 \le s \le 1$, we have $\tau \simeq s$ and consequently the mean value theorem applied to h_1 gives

$$\begin{aligned} I_2^1 &\le C \int_0^\alpha (2s - 1 + e^{-2s}) \frac{e^{-\varepsilon'|x-y|^2/s}}{s^{n/2+1}} \frac{ds}{s} \le C \int_0^\alpha \frac{e^{-\varepsilon'|x-y|^2/s}}{s^{(n-1)/2}} \frac{ds}{s^{1/2}} \\ &\le \frac{C}{|x - y|^{n-1}}, \end{aligned}$$

where we have again used (A) with $k = n - 1$.

Similarly, for $e^{-1} \le \tau \le e^{-s}$, using (A) and (B) we get

$$|h_2'(\tau)| \le C|x| \frac{e^{-\varepsilon|x-y|^2/(1-e^{-2s})}}{(1 - e^{-2s})^{(n+1)/2}}.$$

Since for $0 \le s \le 1$, $(1 - e^{-2s}) \simeq s$, the mean value theorem applied to h_2 gives

$$I_1^1 \le C|x| \int_0^\alpha \frac{e^{-\varepsilon'|x-y|^2/s}}{s^{(n+1)/2}} ds \le C \frac{|x|}{|x - y|^{n-1/2}} \int_0^{1/|x|^2} \frac{ds}{s^{3/4}} \le \frac{C|x|^{1/2}}{|x - y|^{n-1/2}},$$

where we have used (A) with $k = n - \frac{1}{2}$ and that $(x, y) \in N$. Collecting all the estimates, we get the desired conclusion for $D(x, y)$.

It remains to take care of E_1 and E_2. By using (A) with $k = n - 1$, we get

$$E_1(x, y) \le \sup_{u \ge 1/2} \frac{Ce^{-|x-y|^2/u}}{u^{(n-1)/2}} \le \frac{C}{|x - y|^{n-1}}.$$

As for E_2, we have

$$\begin{aligned} |W_{(1-u^2)/2}(x - y) - O_{\log 1/u}(x, y)| &= C_n \left| \frac{e^{-|x-y|^2/(1-u^2)}}{(1 - u^2)^{n/2}} - \frac{e^{-|ux-y|^2/(1-u^2)}}{(1 - u^2)^{n/2}} \right| \\ &= C_n \frac{|h(1) - h(u)|}{(1 - u^2)^{n/2}}. \end{aligned}$$

For the derivative of $h(t) = e^{-|tx-y|^2/(1-u^2)}$, we have the estimate

$$|h'(t)| \leq C|x| \frac{e^{-\varepsilon|x-y|^2/(1-u^2)}}{(1-u^2)^{1/2}} \quad \text{for } u \leq t \leq 1,$$

where we make use of (B) and (A) with $k = 1$. Therefore, an application of the mean value theorem to h gives

$$E_2(x,y) \leq C \sup_{0<u\leq 1} |x| \frac{e^{-\varepsilon|x-y|^2/(1-u^2)}}{(1-u^2)^{(n-1)/2}} \leq C \frac{|x|}{|x-y|^{n-1}},$$

again using (A) with $k = n-1$.

Finally, observe that all the bounds given for D_i, D, E_i are uniformly bounded in x when integrated in y over the larger region $\{y : |x-y| \leq C/(1+|x|)\}$. But it is easy to see that $1+|x| \simeq 1+|y|$ for $(x,y) \in N$ and hence the same argument may be applied to show the uniform boundedness of the integrals in x. This ends the proof of the lemma.

Remark 3.7. In fact, from the proof of (ii) in Lemma 3.1, we see that $L_2(x,y)$ is a bound for $||t\frac{\partial}{\partial t}(P_t - \mathcal{P}_t)(x,y)||_{L^2((0,\infty),dt/t)} + \sum_{i=1}^n ||t\frac{\partial}{\partial x_i}(P_t - \mathcal{P}_t)(x,y)||_{L^2((0,\infty),dt/t)}$ whenever (x,y) belongs to N.

4 Proofs of the Theorems

We begin by recording the following useful observation for future reference.

Remark 4.1. Let B_1, B_2 be Banach spaces, and V an operator mapping B_1-valued functions into B_2-valued functions such that

$$||Vf(x)||_{B_2} \leq L(||f||_{B_1})(x) \quad \text{for a.e. } x,$$

where L is a positive linear operator that is either of strong type or of weak type (p,p) for $1 \leq p < \infty$. Then V maps either $L^p_{B_1}$ into $L^p_{B_2}$ or $L^p_{B_1}$ into weak-$L^p_{B_2}$.

Next we point out that the statements (ii) through (vii) of Theorem 1.10 are known to be equivalent to the UMD property for the vector-valued extensions of the harmonic Riesz transforms. Therefore, in the process of proving Theorem 1.10, it is enough to show the equivalence between the corresponding gaussian and harmonic statements.

Proof of Theorem 1.10. (i) $\Rightarrow$ (ii). By Lemma 2.5 and Remark 4.1, we get that $R_{i,glob}$ extended to X-valued functions maps $L^1_X(d\gamma)$ into weak-$L^1_X(d\gamma)$ no matter what the Banach space X is. Now we want to apply part II of Proposition 2.3

to the operators K_i, $i = 1, \dots, n$. By hypothesis, they satisfy assumption (a); and clearly their kernels satisfy assumptions (b) and (c). Therefore we may conclude that

$$\gamma\{x \in \mathbb{R}^n : \|K_{i,loc}f(x)\|_X > \lambda\} \leq \frac{C}{\lambda}\int_{\mathbf{R}^n} \|f(x)\|_X d\gamma(x).$$

By Lemma 3.1 and Remark 4.1, we get

$$\gamma\{x \in \mathbb{R}^n : \|R_{i,loc}f(x)\|_X > \lambda\} \leq \frac{C}{\lambda}\int_{\mathbf{R}^n} \|f(x)\|_X d\gamma(x).$$

(i) ⇒ (iii). We proceed as above, changing the weak type $(1,1)$ estimates to the corresponding strong type (p,p) each time.

(i) ⇒ (iv). We consider the linear $L^\infty_X(\mathbb{R}^+)$-valued operator U given by

$$U_i f(x) = \left\{ \int R_i(x,y)\chi_{\{|x-y|>\varepsilon\}} f(y)dy \right\}_{\varepsilon>0}.$$

By using Lemma 2.5 and Remark 4.1 with $B_1 = X$ and $B_2 = L^\infty_X(\mathbb{R}^+)$, no matter what the Banach space X is, the global parts

$$U_{i,glob}f(x) = \{ \int R_{i,glob}(x,y)\chi_{\{|x-y|>\varepsilon\}} f(y)dy\}_{\varepsilon>0}$$

are bounded from $L^1_X(d\gamma)$ into weak-$L^1(d\gamma)_{L^\infty_X}$. Again we want to apply part II of Proposition 2.3 to the corresponding harmonic operator, that is,

$$S_i f(x) = \left\{ \int K_i(x-y)\chi_{\{|x-y|>\varepsilon\}} f(y)dy \right\}_{\varepsilon>0},$$

which by the UMD hypothesis is bounded from $L^1_X(dx)$ into weak-$L^1_{L^\infty_X(\mathbf{R}^+)}(dx)$. Since it is also clear that the size condition on the kernel is satisfied, we get for the local part that

$$\gamma\{x \in \mathbb{R}^n : \|S_{i,loc}f(x)\|_{L^\infty_X(\mathbf{R}^+)} > \lambda\} \leq \frac{C}{\lambda}\int_{\mathbf{R}^n} \|f(x)\|_X d\gamma(x).$$

By using Lemma 3.1 and Remark 4.1 for B_1 and B_2 as in the global part and $V = S_{i,loc} - U_{i,loc}$, we obtain

$$\gamma\{x \in \mathbb{R}^n : \|U_{i,loc}f(x)\|_{L^\infty_X(\mathbf{R}^+)} > \lambda\} \leq \frac{C}{\lambda}\int_{\mathbf{R}^n} \|f(x)\|_X d\gamma(x).$$

Putting together the estimates for the local and global parts of U_i and the fact that $\|U_i f(x)\|_{L^\infty_X(\mathbf{R}^+)} = R_i^* f(x)$, we get (iv).

(i) ⇒ (v). We proceed as above changing the weak type $(1,1)$ estimates to the corresponding strong type (p,p) each time.

(iv) $\Rightarrow$ (vi). We observe that from Remark 3.3 we obtain that $R_{i,\varepsilon}f(x)$ converges for every x whenever $f \in C_0^1 \otimes X$. This together with the weak-$(1,1)$ type of R_i^* gives the a.e. convergence of $R_{i,\varepsilon}$ for every $f \in L_X^1(d\gamma)$

(iv) $\Rightarrow$ (vii). Obvious.

(ii) $\Rightarrow$ (i). First we make use of Proposition 2.3 applied this time to the operator R_i. Clearly, by assumption R_i is of weak type $(1,1)$; and, in view of Lemma 3.1 and Lemma 3.4, its kernel has the right size on the local region N. Hence we may conclude that $R_{i,loc}$ is of weak type $(1,1)$ with respect to the Lebesgue measure. Using again Lemma 3.1, we get

$$|\{x \in \mathbb{R}^n : \|K_{i,loc}f(x)\|_X > \lambda\}| \leq \frac{C}{\lambda}\int_{\mathbb{R}^n} \|f(x)\|_X dx.$$

Next we want to see that this inequality can be extended to the whole Riesz transform K_i. To this end, let us take $f \in L_X^1(dx)$ with compact support and denote by f^R the dilation of f defined by $f^R(x) = f(Rx)$. By the homogeneity of the Riesz kernel, we have $K_if(x) = (K_if^R)(x/R)$. We claim that for any fixed $\rho > 0$, we may take R large enough (depending on ρ and the support of f) such that

$$(K_if^R)(x/R) = (K_{i,loc}f^R)(x/R) \tag{4.2}$$

for any x with $|x| < \rho$. In fact, it is easy to check that for such x

$$\text{supp} f^R \subset N_{x/R} = \{y : (x/R, y) \in N\}$$

taking R large enough.

Therefore, using the weak type estimate for $K_{i,loc}$, we get

$$\begin{aligned}|\{x : |x| < \rho, \ \text{and}\, \|K_if(x)\|_X \geq \lambda\}| &\leq R^n|\{z : \|(K_{i,loc}f^R)(z)\|_X > \lambda\}|\\ &\leq \frac{C\,R^n}{\lambda}\int_{\mathbb{R}^n} \|f^R(x)\|_X\, dx\\ &= \frac{C}{\lambda}\int_{\mathbb{R}^n} \|f(x)\|_X dx,\end{aligned}$$

with C independent of ρ. Taking $\rho \to \infty$, we obtain the desired estimate.

By the way, we remark that in proving (4.2) we made no use of any special property of the Riesz transforms other than their invariance under dilations.

(iii) $\Rightarrow$ (i). For the local part, we proceed as above, changing the weak type $(1,1)$ estimates to the strong type (p,p) estimates, to get

$$\int_{\mathbb{R}^n} \|K_{i,loc}f(y)\|_X^p dy \leq C_n\|f\|_{L_X^p}^p.$$

By taking f with compact support, arguing as above and using Fatou's lemma, we get the $L_X^p(dx)$-boundedness of the whole K_i.

(v) $\Rightarrow$ (iii). Obvious.

(vi) $\Rightarrow$ (i). For a function $f \in L^1_X(dx) \subset L^1_X(d\gamma)$, we write

$$K_{i,\varepsilon}f(x) = (K_{i,\varepsilon,loc}f(x) - R_{i,\varepsilon,loc}f(x)) + R_{i,\varepsilon,loc}f(x) + K_{i,\varepsilon,glob}f(x). \tag{4.3}$$

As a consequence of Lemma 3.1, we know that the first term converges a.e. x no matter what the Banach space X is. The estimate

$$\begin{aligned} \sup_{\varepsilon>0} |K_{i,\varepsilon,glob}(x,y)| \|f(y)\|_X &\le C\chi_{N^c}(x,y)\frac{1}{|x-y|^n}\|f(y)\|_X \\ &\le C(1+|x|)^n \|f(y)\|_X \end{aligned} \tag{4.4}$$

allows us to derive the a.e. convergence of the third term.

A similar argument, but using the precise estimates obtained in [PS], gives the almost everywhere convergence for $R_{i,\varepsilon,glob}f(x)$ for $f \in L^1_X(d\gamma)$ and hence for $f \in L^1_X(dx)$. This, together with the hypothesis on $R_i f$, clearly imply the almost everywhere convergence of the second term.

(vii) $\Rightarrow$ (i). By taking norms and suprema in (4.3), we get that $K^*f(x)$ is bounded by three terms. Proceeding as above, using Lemma 3.1 and estimate (4.4), we see that the first and third terms are finite a.e. for $f \in L^1_X(dx)$ no matter what the Banach space X is. Again we use the estimates in [PS] to get $\sup_{\varepsilon>0} \|R_{i,\varepsilon,glob}f(x)\| < \infty$ a.e. This together with the hypothesis imply the a.e. finiteness of the second term. □

Before turning to the proof of Theorem 1.12, we recall that given a Banach space X, the statements (ii) and (iii) are known to be equivalent in the harmonic case as a consequence of the vector-valued Calderón–Zygmund theory (see [RRT] and [X]). Recall also that if any of these conditions is satisfied, X is said to be of Lusin cotype 2.

Since, to our knowledge, the almost everywhere finiteness of the G function has not been proved to be equivalent to the other statements, we give a proof of this fact in the following

Proposition 4.5. *Let X be a Banach space and denote by G the X-valued extended Littlewood–Paley function for the Gauss–Weierstrass semigroup. Then X is of Lusin cotype 2 if and only if $G(f)(x) < \infty$ a.e. x for every $f \in L^1_X(dx)$.*

Proof. From the (1,1)-weak type of G, it is easily derived that $G(f)(x) < \infty$ a.e. x, for every $f \in L^1_X(dx)$. For the converse, we introduce the linear operators

$$S_n f(x) = \chi_{\{1/n<t<n\}}(t)\; t\frac{\partial}{\partial t}P_t f(x).$$

Observe that the maximal operator $S^*f(x) = \sup_n \|S_n f(x)\|_{L^2_X(dt/t)}$ coincides pointwise with $G_0 f(x)$ and therefore by hypothesis $S^*f(x) < \infty$ a.e. x. Also, for all n, the operators S_n are continuous in measure from $L^1_X(dx)$ into $L^0_{L^2_X(dt/t)}(dx)$ since they are integral operators against integrable kernels. Arguing as in Proposition VI.1.4 of [GR], we may conclude that $S^* = G_0$ is continuous in measure from $L^1_X(dx)$ into $L^0(dx)$. Now, as G_0 is also invariant under translations and dilations, applying the vector-valued version of Corollary VI.2.9 of [GR], we obtain the weak type $(1,1)$ for the operator G_0. The same argument can be applied to the operators G_{x_i}. □

In view of the latter result, in proving Theorem 1.12 we can make use of any of the statements (ii) thru (iv) for the harmonic G function as an alternative definition that X is of Lusin cotype 2.

Proof of Theorem 1.12. First we observe that in order to prove the boundedness of g (respectively, G), it will be enough to prove it for g_0 and each g_{x_i} (respectively, G_0 and each G_{x_i}). Conversely, g_0 and g_{x_i} (respectively, G_0 and G_{x_i}) inherit the boundedness property of g (respectively, of G)

(i) ⇒ (ii). For any Banach space X, we consider the linearization of the g_0-function given by

$$Hf(x)(t) = \int_{\mathbf{R}^n} t\frac{\partial}{\partial t}P_t(x,y)f(y)dy. \tag{4.6}$$

Then $g_0 f(x) = \|Hf(x)(\cdot)\|_{L^2_X((0,\infty),dt/t)}$ and also

$$g_{0,loc}f(x) = \|H_{loc}f(x)(\cdot)\|_{L^2_X((0,\infty),dt/t)}, \tag{4.7}$$

$$g_{0,glob}f(x) = \|H_{glob}f(x)(\cdot)\|_{L^2_X((0,\infty),dt/t)}. \tag{4.8}$$

From Lemma 2.7 and Remark 2.8,

$$\begin{aligned}\|H_{glob}f(x)(\cdot)\|_{L^2_X((0,\infty),\frac{dt}{t})} &\le \int_{\mathbf{R}^n} \left\|t\frac{\partial}{\partial t}P_t(x,y)\right\|_{L^2_X((0,\infty),dt/t)}(1-\varphi(x,y))\|f(y\|_X dy \\ &\le \int_{\mathbf{R}^n} Q(x,y))\|f(y\|_X dy,\end{aligned} \tag{4.9}$$

where the last integral operator is of weak type $(1,1)$ on $L^1(d\gamma)$. Hence H_{glob} is bounded from $L^1_X(d\gamma)$ into weak-$L^1_{L^2_X((0,\infty),dt/t)}(d\gamma)$, which gives in turn the boundedness of $g_{0,glob}$ from $L^1_X(d\gamma)$ into weak-$L^1(d\gamma)$.

It remains to take care of $g_{0,loc}$. We consider now the linearization of the G_0-function given by

$$Jf(x)(t) = \int_{\mathbf{R}^n} t\frac{\partial}{\partial t}\mathcal{P}_t(x,y)f(y)dy.$$

To apply Proposition 2.3 to this operator, we observe that, by hypothesis, J is bounded from $L^1_X(dx)$ into weak-$L^1_{L^2_X((0,\infty),dt/t)}(dx)$ and that its kernel has the right size, being a vector-valued singular integral (see [St2]). Therefore, we may conclude that J_{loc} has the same boundedness as J but also with respect to the gaussian measure. Now, since a similar inequality to (4.9) holds for $H_{loc} - J_{loc}$, by using Lemma 3.1 and Remark 3.7, we obtain the weak type $(1,1)$ for the difference and hence for H_{loc}. Combining the results for the local and global parts, we see that the same holds for H. Consequently, by (4.6), g_0 has the desired boundedness. Finally, the same argument can be applied to each g_{x_i} for $i = 1, \dots, n$.

(i) $\Rightarrow$ (iii). We proceed as above, changing the weak type $(1,1)$ estimates to the corresponding strong type (p,p) each time.

(ii) $\Rightarrow$ (iv). Obvious.

(ii) $\Rightarrow$ (i). Again we deal only with G_0 and g_0. As was shown in (i) $\Rightarrow$ (ii), for any Banach space X, the operator H_{glob} is bounded from $L^1_X(d\gamma)$ into weak-$L^1_{L^2_X((0,\infty),dt/t)}(d\gamma)$. Since by hypothesis H satisfies the same boundedness, H_{loc} is of weak type $(1,1)$ with respect to the gaussian measure and hence, by Proposition 2.4, also with respect to Lebesgue measure. With the same notation as above, we set

$$J_{loc}f = (J_{loc}f - H_{loc}f) + H_{loc}f.$$

Applying an inequality like (4.9) for the difference $J_{loc}f - H_{loc}f$ and Lemma 3.1 together with Remark 3.7, we get the weak type $(1,1)$ for this operator and hence for J_{loc}.

Next, as in the proof of (ii) $\Rightarrow$ (i) of Theorem 1.10, we extend the weak type $(1,1)$ to the whole operator J. As was pointed out there, we only need the invariance under dilations of the operator, which is certainly true for J.

(iii) $\Rightarrow$ (i). For the local part, we proceed as above changing the weak type $(1,1)$ estimates to strong type (p,p) estimates. Hence

$$\int_{\mathbf{R}^n} G_{0,loc}f(y)^p dy \leq C_n \|f\|^p_{L^p_X}.$$

We conclude the argument as in the preceding proof with the obvious changes.

(iv) $\Rightarrow$ (i). Again we argue just for G_0. We have

$$G_0 f(x) \leq G_{0,loc}f(x) + G_{0,glob}f(x).$$

According to the inequality

$$G_{0,glob}f(x) = \|J_{glob}f(x)\|_{L^2_X((0,\infty),dt/t)} \leq C(1+|x|)^n \|f\|_{L^1_X(dx)},$$

the second term is finite for any x as long as $f \in L^1_X(dx)$. For the first term, we write

$$G_{0,loc}f(x) \leq |G_{0,loc}f(x) - g_{0,loc}f(x)| + g_{0,loc}f(x).$$

Since $|G_{0,loc}f(x) - g_{0,loc}f(x)| \leq \|H_{loc}f(x) - J_{loc}f(x)\|_{L^2_X((0,\infty),dt/t)}$, arguing as in (4.9) and using Lemma 3.1 together with Remark 3.7, we get the pointwise finiteness of the difference for $f \in L^1_X(dx)$. On the other hand, by (4.8) and (4.9), $g_{0,glob}f(x)$ is finite a.e. x for $f \in L^1_X(d\gamma)$ and hence for $f \in L^1_X(dx)$. Now the hypothesis together with the inequality $g_{0,loc}f(x) \leq g_0f(x) + g_{0,glob}f(x)$ gives us the result. □

Before turning to the proof of Theorem 1.13 we recall that, given a Köthe function space X, statements (ii) and (iii) for the Hardy–Littlewood maximal operator are known to be equivalent in the harmonic case; see [GMT]. Although in that paper they consider, for a general Banach lattice X, maximal functions taking suprema over finite sets of averages, their results can be extended in our setting to the whole maximal function. In fact, for a Köthe function space, we have

(4.10)
$$\sup_{\varepsilon>0} \frac{1}{|B(x,\varepsilon)|}\left|\int_{B(x,\varepsilon)} f(y,\omega)dy\right| = \sup_{r\in\mathbb{Q}^+} \frac{1}{|B(x,r)|}\left|\int_{B(x,r)} f(y,\omega)dy\right|, \; x \in \mathbb{R}^n, \; \omega \in \Omega.$$

Since the statements referred to above involve boundedness properties with a constant independent of the finite set of averages, our claim follows.

Moreover, it is also true that each of the statements (ii) and (iii) is equivalent for all the three harmonic maximal operators: Hardy–Littlewood, Gauss–Weierstrass and Poisson. This is an easy consequence of the pointwise inequalities valid for nonnegative functions f,

$$Mf(x,\omega) \leq C_1P^*f(x,\omega) \leq C_2W^*f(x,\omega) \leq C_3Mf(x,\omega), \tag{4.11}$$

see [St2]. Therefore, all of the statements (i), (ii) and (iii) are equivalent for these operators. Since, to our knowledege, the facts that these harmonic operators applied to $L^1_X(dx)$-functions belong to X for almost all x have not been proved to be equivalent to the Hardy–Littlewood property, we prove them in the following

Proposition 4.12. *Let X be a Köthe function space. If $\mathcal{M}$ denotes any of the three harmonic operators above, then X satisfies the Hardy–Littlewood property if and only if for every $f \in L^1_X(dx)$, $\mathcal{M}f(x)$ belongs to X for a.e. $x \in \mathbb{R}^n$.*

Proof. If X satisfies the Hardy–Littlewood property, then $\mathcal{M}$ maps $L^1_X(dx)$ into weak-$L^1_X(dx)$ and hence $\mathcal{M}f(x)$ is well-defined for functions in $L^1_X(dx)$.

Conversely, consider the linear operators M_r, $r \in \mathbb{Q}^+$, defined by

$$M_r(f)(x,\omega) = \frac{1}{|B(x,r)|} \int_{B(x,r)} f(y,\omega)dy.$$

Clearly, these operators are bounded from $L^1_X(dx)$ into itself and therefore continuous in measure from $L^1_X(dx)$ into $L^0_X(dx)$. By hypothesis, the operator $Uf(x,\omega) = \{M_r(f)(x,\omega)\}_{r\in\mathbb{Q}^+}$ belongs to $L^0_{X(\ell^\infty)}(dx)$. Arguing as in Proposition VI.1.4 of [GR], we may conclude that the operator $\mathcal{M}$ is continuous in measure from $L^1_X(dx)$ into $L^0_X(dx)$. Now as $\mathcal{M}$ is also invariant under translations and dilations, applying the vector valued version of Corollary VI.2.9 of [GR], we obtain the weak type $(1,1)$ for the operator $\mathcal{M}$, and this implies that X satisfies the Hardy–Littlewood property. The equivalence for the other two operators follows now from inequalities (4.11). □

Proof of Theorem 1.13 for $\mathcal{N} = M_\gamma$. Clearly, it is enough to deal with nonnegative functions.

(i) ⇒ (ii). First we observe that for any Köthe function space, we get from (iii) in Lemma 2.7 that

$$\|M_{\gamma,glob}f(x,\cdot)\|_X \leq \int_{\mathbb{R}^n} S(x,y)\,\|f(y,\cdot)\|_X dy, \tag{4.13}$$

where the last integral operator is of strong $(1,1)$ in $L^1(d\gamma)$. Hence $M_{\gamma,glob}$ is bounded from $L^1_X(d\gamma)$ into $L^1_X(d\gamma)$. It remains to take care of $M_{\gamma,loc}$. It is a well-known fact and goes back to Muckenhoupt (see [Mu] and also [HVT]) that $M_{\gamma,loc}f(x,\cdot) \sim M_{loc}f(x,\cdot)$ and also $M_{loc}f(x,\cdot) \leq Mf(x,\cdot)$; then, using the hypothesis, $M_{\gamma,loc}$ is of weak type $(1,1)$ with respect to Lebesgue measure. Next, we consider the linearization of $M_{\gamma,loc}f(x,\cdot)$ given by

$$V_{loc}f(x,\omega) = \left\{\frac{1}{\gamma(B(x,r))}\int_{B(x,r)} \varphi(x,y)f(y,\omega)d\gamma(y)\right\}_r.$$

Since

$$\|M_{\gamma,loc}f(x,\cdot)\|_X = \|V_{loc}f(x,\omega)\|_{X(L^\infty((0,\infty))}, \tag{4.14}$$

by using Proposition 2.4, it follows that V_{loc} is bounded from $L^1_X(d\gamma)$ into weak-$L^1_{X(L^\infty(0,\infty))}(d\gamma)$; therefore, $M_{\gamma,loc}$ is of weak type $(1,1)$ with respect to the gaussian measure.

(i) ⇒ (iii). We proceed as above changing the weak type $(1,1)$ estimates to the corresponding strong type (p,p) each time.

(ii) ⇒ (iv). Obvious.

(ii) $\Rightarrow$ (i). First, for the local part, we proceed as in the proof of (i) $\Rightarrow$ (ii), that is, we use

$$M_{loc}f(x,\cdot) \sim M_{\gamma,loc}f(x,\cdot) \leq M_{\gamma}f(x,\cdot). \tag{4.15}$$

Again Proposition 2.4, applied this time to the corresponding linearization of M_{loc}, gives the weak type $(1,1)$ with respect to Lebesgue measure. Now we proceed as in the proof of (ii) $\Rightarrow$ (i) of Theorem 1.10. Clearly, our operator is invariant under dilations; therefore, the argument given there allows us to extend the estimate we just proved for M_{loc} to the whole of M.

(iii) $\Rightarrow$ (i). We proceed as in the previous proof. For the local part, replace the weak type $(1,1)$ estimate by the strong type (p,p) estimates; then, by the invariance under dilations argument, we extend the strong type (p,p) estimate to the entire operator M.

(iv) $\Rightarrow$ (i). By using the hypothesis and (4.15), it is clear that $M_{loc}f(x,\cdot)$ belongs to X for functions f in $L^1_X(d\gamma)$ and hence for $f \in L^1_X(dx)$. On the other hand, the inequality

$$\begin{aligned} \|M_{glob}f(x,\cdot)\|_X &\leq C\Big\| \int_{\mathbb{R}^n} \chi_{N^c}(x,y)\frac{1}{|x-y|^n}\,|f(y,\cdot)|dy\,\Big\|_X \\ &\leq C\int_{\mathbb{R}^n} \chi_{N^c}(x,y)\frac{1}{|x-y|^n}\,\|f(y,\cdot)\|_X dy \leq C(1+|x|)^n\|f\|_{L^1_X(dx)} \end{aligned}$$

guarantees that for any x, $M_{glob}f(x,\cdot) \in X$ as long as f belongs to $L^1_X(dx)$. □

To prove the equivalence of the statements for the Poisson and Ornstein–Uhlenbeck maximal operators, we proceed similarly with some minor changes that we sketch in what follows. The global parts of the gaussian operators P^* and O^* always satisfy all the required boundedness properties; this follows from $P^*_{glob}f(x,\omega) \leq O^*_{glob}f(x,\omega)$ and Lemma 2.7. The same can be said about the corresponding harmonic operators in view of inequalities (4.11). As for the local parts, (iii) in Lemma 3.1, together with the inequality

$$|P^*_{loc}f(x,\omega) - \mathcal{P}^*_{loc}f(x,\omega)| \leq |O^*_{loc}f(x,\omega) - W^*_{loc}f(x,\omega)|$$

allow us to go back and forth from Lebesgue boundedness to gaussian boundedness, by means of Proposition 2.4 applied to the appropiate linearization of each of the maximal operators, as we did for the Hardy–Littlewood maximal operator. □

We finish by offering some comments on a somewhat more general setting for Theorem 1.13.

For a general lattice X, the maximal Hardy–Littlewood operator can only be defined by taking the supremum over a finite number of averages, as was done in [GMT]. If we further assume that X is σ-order complete we may define the maximal function considering now a countable number of averages, say, for example, averages over balls centered at x and with rational radius, whenever $f \in L^1(dx) \otimes X$. In fact, for $f = \sum_{i=1}^n c_i\varphi_i$, we have

$$\begin{aligned}\mathcal{M}f(x) &= \sup_{r\in\mathbb{Q}} \frac{1}{|B(x,r)|}\left|\int_{\mathbf{R}^n} f(y)\,dy\right| \le \sum_{i=1}^n |c_i| \sup_{r\in\mathbb{Q}} \frac{1}{|B(x,r)|}\left|\int_{B(x,r)} |\varphi_i(y)|\,dy\right| \\ &\le \sum_{i=1}^n |c_i| M\varphi_i(x),\end{aligned}$$

where M is the scalar maximal function and hence finite a.e. The right hand size gives an element in X proving the existence of the supremum. With this observation, Proposition 4.12 still holds true. Similar considerations can be made for the other maximal operators.

Therefore, we can restate Theorem 1.13, this time in terms of the gaussian maximal operators obtained by taking suprema over $\varepsilon \in \mathbb{Q}^+$ and applied to functions valued in X, for X a σ-order complete Banach lattice.

Acknowledgement

We are very grateful to the referee for his careful review and helpful comments.

REFERENCES

[Bo] J. Bourgain, *Some remarks on Banach spaces in which martingale differences are unconditional*, Arkiv Mat. **21** (1983), 163–168.

[Bu] D. Burkholder, *A geometric condition that implies the existence of certain singular integrals of Banch-space-valued functions*, in *Conference Harmonic Analysis in Honor of A. Zygmund* (W. Beckner, A. P. Calderón, R. Fefferman and P. W. Jones, eds.), Wadsworth Inc., Belmont, CA, 1981, pp. 270–286.

[FGS] E. Fabes, C. Gutiérrez and R. Scotto, *Weak-type estimates for the Riesz transforms associated with the gaussian measure*, Rev. Mat. Iberoamericana **10** (1994), 229–281.

[FSU] L. Forzani, R. Scotto and W. Urbina, *The area function, the g^k functions and the Riesz potentials for the gaussian measure γ*, Rev. Un. Mat. Argentina **42** (2001), 17–31.

[GMST1] J. García-Cuerva, G. Mauceri, P. Sögren and J. L. Torrea, *Spectral multipliers for the Ornstein–Uhlenbeck semigroup*, J. Analyse Math. **78** (1999), 281–305.

[GMST2] J. García-Cuerva, G. Mauceri, P. Sögren and J. L. Torrea, *Higher-order Riesz operators for the Ornstein–Uhlenbeck semigroup*, Potential Anal. **10** (1999), 379–407.

[GMT] J. García-Cuerva, R. Macías and J. L. Torrea, *The Hardy–Littlewood property of Banach lattices*, Israel J. Math. **83** (1993), 177–201.

[GR] J. García-Cuerva and J. L. Rubio de Francia, *Weighted Norm Inequalities and Related Topics*, North-Holland Mathematical Studies, No. 116, 1985.

[Gt] C. E. Gutiérrez, *On the Riesz transforms for Gaussian measures*, J. Funct. Anal. **120** (1994), 107–134.

[Gu] R. Gundy, *Some Topics in Probability and Analysis*, CBMS Regional Conf. Ser. in Math. 70, American Mathematical Society, Providence, RI, 1989.

[HVT] E. Harboure, J. L. Torrea and B. Viviani, *On the search for weighted inequalities for operators related to the Ornstein–Uhlenbeck semigroup*, Math. Ann. **318** (2000), 341–353.

[LT] J. Lindenstrauss and L. Tzafriri, *Classical Banach Spaces II, Function Spaces*, Springer Ser. Modern Surveys Math. 97, Springer-Verlag, Berlin, 1979.

[Me] P. A. Meyer, *Transformations de Riesz pour le lois Gaussienes*, Lecture Notes in Math. **1059**, Springer-Verlag, New York, 1984, pp. 179–193.

[Mu] B. Muckenhoupt, *Poisson integrals for Hermite and Laguerre expansions*, Trans. Amer. Math. Soc. **139** (1969), 231–242.

[P] G. Pisier, *Riesz transforms: a simpler proof of P.A. Meyer's inequality*, Lecture Notes in Math. **1321**, Springer-Verlag, Berlin, 1988, pp. 485–501.

[PS] S. Pérez and F. Soria, *Operators associated with the Ornstein–Uhlenbeck semigroup*, J. London Math. Soc. **61** (2000), 857–871.

[R] J. L. Rubio de Francia, *Martingale and integral transforms of Banach space valued functions*, in *Conference on Probability and Banach Spaces, Zaragoza, 1985*, Lecture Notes in Math. **1221**, Springer-Verlag, Berlin–New York, 1986, pp. 195–222.

[RRT] J. L. Rubio de Francia, F. J. Ruiz and J. L. Torrea, *Calderón–Zygmund kernels for operator-valued kernels*, Adv. Math. **62** (1986), 7–48.

[Sj] P. Sjögren, *On the maximal function for the Mehler kernel*, in *Harmonic Analysis Cortona 1982*, Lecture Notes in Math. **992**, Springer-Verlag, Berlin, 1983, pp. 73–82.

[St1] E. M. Stein, *Topics in Harmonic Analysis Related to the Littlewood–Paley theory*, Ann. of Math. Stud., Vol. 63, Princeton University Press, Princeton, NJ, 1970.

[St2] E. M. Stein, *On Singular Integrals and Differentiability Properties of Functions*, Princeton University Press, Princeton, NJ, 1971.

[U] W. Urbina, *On singular integrals with respect to the Gaussian measure*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. **17** (1990), 531–567.

[X] Q. Xu, *Littlewood–Paley theory for functions with values in uniformly convex spaces*, J. Reine Angew. Math. **504** (1998), 195–226.

E. Harboure and B. Viviani
I.M.A.L.–UNIVERSIDAD NACIONAL DEL LITORAL
GÜEMES 3450, 3000 SANTA FE, ARGENTINA
email: harbour@ceride.gov.ar, viviani@ceride.gov.ar

J. L. Torrea
DEPARTAMENTO DE MATEMÁTICAS
FACULTDAD DE CIENCIAS
UNIVERSIDAD AUTÓNOMA DE MADRID
28049 MADRID, SPAIN
email: Joseluis.torrea@uam.es

(Received September 9, 2001 and in revised form March 20, 2002)