PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 129, Number 12, Pages 3619–3621
S 0002-9939(01)05939-1
Article electronically published on April 24, 2001

# PERTURBATIONS OF THE HAAR WAVELET BY CONVOLUTION

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(Communicated by Christopher D. Sogge)

Abstract. In this note we show that the standard convolution regularization of the Haar system generates Riesz bases of smooth functions for $L^2(\mathbb{R})$, providing in this way an alternative to the approach given by Govil and Zalik [Proc. Amer. Math. Soc. **125** (1997), 3363–3370].

The simplest compactly supported wavelet, the Haar function given by $h := \chi_{[0,1/2)} - \chi_{[1/2,1)}$, generates by integer translations and dyadic dilations an orthonormal basis for the space $L^2(\mathbb{R})$. In a recent paper Govil and Zalik [2] gave an *ad hoc* spline type regularization $h^\varepsilon$, $\varepsilon > 0$, of the Haar wavelet $h$, in such a way that $h^\varepsilon$ produces by integer translations and dyadic dilations a Riesz basis for $L^2(\mathbb{R})$ with bounds approaching 1 for $\varepsilon \to 0$. A basis $\{f_n, n \in \mathbb{Z}^+\}$ of $L^2(\mathbb{R})$ is called a Riesz basis with bounds $A$ and $B$ if $A\sum_{n\in\mathbb{Z}^+} |c_n|^2 \le || \sum_{n\in\mathbb{Z}^+} c_n f_n||^2 \le B \sum_{n\in\mathbb{Z}^+} |c_n|^2$ for every numerical sequence $\{c_n, n \in \mathbb{Z}^+\} \in \ell^2$.

The aim of this note is to show that standard approximations of the identity provide good Riesz bases as regularizations of the Haar system. We shall denote by $f^\varepsilon$ the convolution of $f$ with $\phi_\varepsilon(x) = 1/\varepsilon\phi(x/\varepsilon)$, where $\phi$ is an appropriate integrable function. Concretely, we shall prove the following theorem.

**Theorem 1.** *Let $m$ be a nonnegative integer and let $\phi$ be an even function with support in $[-1,1]$, $\phi \in W^{1,m}(\mathbb{R}) = \{\phi \in L^1(\mathbb{R}) :$ the $m^{th}$ derivative $\phi^{(m)}$ of $\phi$ belongs to $L^1(\mathbb{R})\}$ and $\int \phi = 1$. Let $\varepsilon > 0$ be such that*

$$M_\varepsilon := 110\ \sqrt{3}\ (1 + ||\phi||_1)^2\ \varepsilon < 1.$$

*Then, $h^\varepsilon := h * \phi_\varepsilon$ belongs to $C^m(\mathbb{R})$, has support in $[-\varepsilon, 1+\varepsilon]$, and $\{h^\varepsilon_{j,k}(x) := 2^{j/2}h^\varepsilon(2^jx - k) :\ j,k \in \mathbb{Z}\}$ is a Riesz basis of $L^2(\mathbb{R})$ with bounds $(1-\sqrt{M_\varepsilon})^2$ and $(1+\sqrt{M_\varepsilon})^2$.*

Notice that spline type regularizations can be obtained from adequate choices of $\phi$, for example $h^\varepsilon$ is piecewise linear if we take $\phi = \frac{1}{2}\chi_{[-1,1]}$.

Received by the editors January 11, 2000 and, in revised form, April 20, 2000.
2000 *Mathematics Subject Classification.* Primary 42C40.
*Key words and phrases.* Riesz bases, Haar wavelets, basis perturbations.
This research was supported by CONICET and Proy. CAI+D - UNL.

Recall that a sequence $\{f_n, n \in \mathbb{Z}^+\}$ of $L^2(\mathbb{R})$ is called a Bessel sequence with bound $M$ if, for every $f \in L^2(\mathbb{R})$, $\sum_{n\in\mathbb{Z}^+} |\langle f, f_n\rangle|^2 \leq M||f||^2$. We need the following particular case of [2, Lemma 2], obtained by setting $a = 2$, $b = 1$ and $|I| < 1$.

**Lemma 2** (page 3364 in [2])**.** *Let $g$ be a bounded variation function with total variation $V(g)$ such that* supp $g \subseteq I$*, where $I$ is an interval of length less than 1, and $\int g(t)\,dt = 0$. Then, $\{g_{j,k}(x) := 2^{j/2} g(2^j x - k) : \ j, k \in \mathbb{Z}\}$ is a Bessel sequence with bound*

$$M_g := 11\ ||g||_\infty \left(V(g) + ||g||_\infty\right)\ |I|.$$

The basic result used in [2] which will also be used here is the following corollary of Theorem 5 in [1]:

**Theorem 3** (page 164 in [1])**.** *If $\{f_n, n \in \mathbb{Z}^+\}$ is an orthonormal basis for $L^2(\mathbb{R})$ and $\{f_n - g_n, n \in \mathbb{Z}^+\}$ is a Bessel sequence with bound $M < 1$, then $\{g_n, n \in \mathbb{Z}^+\}$ is a Riesz basis with bounds $(1 - \sqrt{M})^2$ and $(1 + \sqrt{M})^2$.*

*Proof of Theorem* 1. We start by proving that $h^\varepsilon \in C^m(\mathbb{R})$. Let us first notice that, since $h^\varepsilon$ is the convolution of an $L^\infty$ function $h$ with an $L^1$ function $\phi_\varepsilon$, from the continuity in $L^1$ of every integrable function, we have that $h^\varepsilon \in C^0(\mathbb{R})$. Now, if $m > 0$, since $[h^\varepsilon]^{(m)} = h*[\phi_\varepsilon]^{(m)} = \varepsilon^{-m} h*(\phi^{(m)})_\varepsilon$ and $\phi^{(m)} \in L^1(\mathbb{R})$, the argument given for $m = 0$ shows that $h^\varepsilon \in C^m(\mathbb{R})$. On the other hand, since the support of the convolution of two functions is contained in the sum of the supports of the functions being convolved, we have that *supp* $h^\varepsilon \subseteq [-\varepsilon, 1 + \varepsilon]$. We shall now prove that $\{h^\varepsilon_{j,k}\}$ is a Riesz basis with the desired bounds. Let us first observe that the convolution of two compactly supported functions, $f_1$ in $L^1(\mathbb{R})$ and $f_2$ of bounded variation, has total variation bounded by $V(f_2)||f_1||_1$. Then $V(h^\varepsilon) \leq 4||\phi||_1$. The difference $d^\varepsilon = h - h^\varepsilon$ can be written as the sum of three functions: $d^{\varepsilon,1} = d^\varepsilon \chi_{[-\varepsilon,\varepsilon]}$, $d^{\varepsilon,2} = d^\varepsilon \chi_{[1/2-\varepsilon,1/2+\varepsilon]}$ and $d^{\varepsilon,3} = d^\varepsilon \chi_{[1-\varepsilon,1+\varepsilon]}$. Each function $d^{\varepsilon,i}$, $i = 1, 2, 3$, is of bounded variation over $\mathbb{R}$ with $V(d^{\varepsilon,i}) \leq V(d^\varepsilon) \leq 4(1 + ||\phi||_1)$. On the other hand, each $d^{\varepsilon,i}$ has zero integral since, in fact, $d^{\varepsilon,1}(x)$, $d^{\varepsilon,2}(x+1/2)$ and $d^{\varepsilon,3}(x+1)$ are odd functions. Let us show, for example, that $d^{\varepsilon,1}(x) = -d^{\varepsilon,1}(-x)$. For $0 < \varepsilon < 1/4$ and $x \in [-\varepsilon, \varepsilon]$, since $\int [h(x + y) + h(-x - y)]\phi_\varepsilon(y)\,dy = \int \phi_\varepsilon(y)\,dy = 1$ and since $\phi$ is even, we have that

$$\begin{aligned}
d^{\varepsilon,1}(x) &= h(x) - \int h(x - y)\phi_\varepsilon(y)\,dy \\
&= (h(x) - 1) + (1 - \int h(x + y)\phi_\varepsilon(y)\,dy) \\
&= -h(-x) + \int h(-x - y)\phi_\varepsilon(y)\,dy \\
&= -(h(-x) - h^\varepsilon(-x)) \\
&= -d^{\varepsilon,1}(-x).
\end{aligned}$$

These properties allow us to apply Lemma 2 to each function $d^{\varepsilon,i}$, $i = 1, 2, 3$, obtaining that the sequence $\{d^{\varepsilon,i}_{j,k} : \ j, k \in \mathbb{Z}\}$ is a Bessel sequence with bound $110\ (1 + ||\phi||_1)^2\varepsilon$. Hence the theorem follows from Minkowski's inequality and Theorem 3. □

## References

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