CHARACTERIZATIONS OF $BMO_\varphi(w)$

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Abstract. In this paper we give two characterizations of functions with weighted mean oscillation over cubes controlled by a non-negative function φ, that is functions in $BMO_\varphi(w)$. The first one, by conditions on their rearrangements, and the second one, by means of Riesz transforms and φ-Lipschitz functions. These results extend those contained in [S] and [J].

1. Introduction

The aim of this paper is to obtain characterizations of spaces of functions whose oscillation, when averaged over cubes is controlled by means of a weight w and a growth function φ, measuring their degree of smoothness.

The first appearance of this kind of weighted spaces goes back to [MW]. There, the authors introduced $BMO(w)$ ($\varphi \equiv 1$ in our context) as the natural space where weighted L^∞ functions are mapped by $\mathcal{H}$, the Hilbert transform on the line, and generalizing the well known BMO space of John and Niremberg. In the more general context $\varphi(t) = t^\beta, \quad 0 < \beta < 1$, it is shown in [HSV1] that the fractional integral operator I_α applies $L^p(w)$ with $p > n/\alpha$ into these spaces, under suitable conditions on the weight. Later on this result was extended to weighted Orlicz spaces [HSV2] giving rise to the spaces under consideration in their full generality. Finally in [M] it is shown that they are preserved by the Hilbert transform on the line.

We start by giving the precise definition of our spaces and reminding some basic notions about weights.

Let φ be a continuous non-negative and non-decreasing function defined on $[0, \infty)$ with $\varphi(0) = 0$ and satisfying a doubling condition (or a Δ_2-condition), that is there exists a constant C such that

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$$\varphi(2R) \leq C\varphi(R) \tag{1.1}$$

for every $R > 0$. Let w be a weight in the A_∞ Muckenhoupt's class, that is a non-negative a.e. and locally integrable function satisfying

$$\frac{w(E)}{w(Q)} \leq C\left(\frac{|E|}{|Q|}\right)^{\delta} \tag{1.2}$$

for every cube Q in $\mathbb{R}^n$ and every measurable set $E \subset Q$, where C and δ are positive constants depending neither on Q nor on E and $w(E) = \int_E w(x)dx$.

We shall say that a function f in $L^1_{loc}(\mathbb{R}^n)$ has w-mean oscillations over cubes controlled by φ or, shorter, that it belongs to $BMO_\varphi(w)$, if there exists a constant C such that the inequality

$$\frac{1}{w(Q(x,r))}\int_{Q(x,r)} |f(y) - m_{Q(x,r)}|dy \leq C\varphi(r) \tag{1.3}$$

holds for every cube $Q(x,r) = \{y \in \mathbb{R}^n / |x_i - y_i| < r, i = 1,\ldots,n\}$ in $\mathbb{R}^n$, where $m_{Q(x,r)}f = |Q(x,r)|^{-1}\int_{Q(x,r)} f(y)dy$.

The infimum of the constants C satisfying (1.3) will be denoted by $||f||_{BMO_\varphi(w)}$. It is not too hard to see that it is a norm in $BMO_\varphi(w)$ modulo constants. When $w = 1$, we will denote these spaces by BMO_φ. Note that, because of our hypothesis on w and ϕ, we can take balls $B(x,r) = \{y/|x-y| < r\}$ instead of cubes $Q(x,r)$ in (1.3) and obtain and equivalent version of $BMO_\varphi(w)$.

In connection with the above definition, we shall say that a function f belongs to the (w,φ)-Lipschitz space, denoted by $\Lambda_\varphi(w)$, if there exists a constant C such that

$$|f(x) - f(y)| \leq C(w(x) + w(y))\varphi(|x-y|), \tag{1.4}$$

holds for a.e. x and y in $\mathbb{R}^n$. It is easy to prove that $\Lambda_\varphi(w) \subset BMO_\varphi(w)$. For $w = 1$, as before, we write Λ_φ instead of $\Lambda_\varphi(w)$.

Some special cases and, moreover, generalizations of the spaces $BMO_\varphi(w)$ have been studied by several authors (see, for instance, [JN], [J], [S], [F], [FS], [B], [Y], [N]). In particular, in [S], S. Spanne considered the case $w \equiv 1$ and proved a characterization of the functions in BMO_φ by means of rearrangements.

On the other hand, S. Janson, in [J], gave another characterization of BMO_φ, this time in terms of Riesz transforms and Λ_φ, generalizing the well known decomposition of BMO functions in terms of Riesz transforms and L^∞ (see [F] and [FS]). Also, in [MW] such characterization is given for the case $\varphi = 1$) and w belonging to the A_1 class of Muckenhoupt.

In this work we obtain similar characterizations to those in [S] and [J] for more general weighted spaces $BMO_\varphi(w)$. Before stating our results we recall some definitions.

A non negative and measurable function w is in the A_1 class of Muckenhoupt if there exists a constant C such that

$$\frac{1}{|Q(x,r)|}\int_{Q(x,r)} w(y)dy \le C \operatorname{ess}\inf_{Q(x,r)} w \tag{1.5}$$

holds for every cube $Q(x,r)$ in $\mathbb{R}^n$.

A non-negative function ψ is quasi-decreasing when a constant C exists such that

$$\psi(t_1) \le C\psi(t_2) \tag{1.6}$$

is satisfied for every t_1 and t_2 with $0 \le t_2 < t_1$.

Now, we are in position to state our main results.

THEOREM **1.7.** *Let w be in A_1 and φ as in (1.1). Then, a locally integrable function f belongs to $BMO_\varphi(w)$ if and only if there exists a constant C such that*

$$f_Q^*(s) \le C \int_{s^{\frac{1}{n}}\left(\frac{|Q|}{Cw(Q)}\right)^{\frac{1}{n}}}^{2r_Q} \frac{\varphi(t)}{t}dt, \tag{1.8}$$

for every $s \in \mathbb{R}$ and every cube Q in $\mathbb{R}^n$, where f_Q^* means the non increasing rearrangement of $\mathcal{X}_Q|f - m_\varphi f|/w$ with respect to the measure given by w and r_Q denotes the half length edge of Q.

COROLLARY **1.9.** *If w and φ are as in the theorem above and, in addition, $\int_0^1 \frac{\varphi(t)}{t}dt < \infty$, then $BMO_\varphi(w)$ is contained in $\Lambda_\psi(w)$ with $\psi(r) = \int_0^r \frac{\varphi(t)}{t}dt$, so it coincides with $BMO_\varphi(w)$ whenever $\psi(r) \le C\varphi(r)$ for every $r > 0$.*

THEOREM **1.10.** *Let w be in A_1 and φ as in (1.1) such that $\varphi(t)/t$ is quasi-decreasing. Then, given x_0 in $\mathbb{R}^n$, the function*

$$h_{x_0}(x) = \int_{|x-x_0|}^{1} \frac{w(B(x_0,t))}{t^n} \frac{\varphi(t)}{t} dt,$$

with $B(x_0,t) = \{y \in \mathbb{R}^n / |x_0 - y| < t\}$*, belongs to* $BMO_\varphi(w)$*. Moreover, there exist two constants* C_1 *and* C_2*, not depending on* x_0*, such that the inequality*

(1.11)
$$C_1\varphi(r) \le \sup_{\substack{s \le r \\ z \in \mathbb{R}^n}} \frac{1}{w(B(z,s))} \int_{Q(z,s)} |h_{x_0}(y) - m_{Q(z,s)}h_{x_0}| dy \le C_2\varphi(r)$$

holds for every $r > 0$*.*

Corollary **1.12.** *Let* w *and* φ *be as in Theorem 1.10. If* $\int_0^1 \frac{\varphi(t)}{t} dt = \infty$ *then there are functions in* $BMO_\varphi(w)$ *not belonging to* $\Lambda_\varphi(w)$*. In particular we get* $\Lambda_\varphi(w) \subsetneq BMO_\varphi(w)$*.*

Remark **1.13.** *Notice that corollary 1.12 gives the converse of corollary 1.9 above under the additional assumption that* $\varphi(t)/t$ *is quasi decreasing.*

The statement of the next theorem requires to specify some details about the weight w. We know that if w is in A_1, then it satisfies an A_∞ condition (see (1.2)). In general if (1.2) holds for some fixed δ, we are going to say that w belongs to A_∞^δ. Now we get

Theorem **1.14.** *Let* w *be in* $A_1 \cap A_\infty^\delta$*. If* φ *is as in (1.1) and satisfying*

$$r^\delta \int_r^\infty \frac{\varphi(t)}{t^{1+\delta}} dt \le C\varphi(r)$$

for every $r > 0$*, then* $BMO_\varphi(w) = \Lambda_\varphi(w) + \sum_{j=1}^n \mathcal{R}_i(\Lambda_\varphi(w))$*, where* $\mathcal{R}_j$ *denotes the modified Riesz transform of order* j*, defined by*

$$\mathcal{R}_j f(x) = \lim_{\varepsilon\to 0} \int_{|x-y|>\varepsilon} \left(\frac{x_j - y_j}{|x-y|^{n+1}} + \mathcal{X}_{B_1^C}(y) \frac{y_j}{|y|^{n+1}} \right) f(y) dy, \tag{1.15}$$

where B_1 *denotes the unit ball centered at the origin.*

The next section contains the proofs of Theorems 1.7 and 1.10 and their corollaries, while section 3 is devoted to prove Theorem 1.14. We wish to say that our techniques are based on those of S. Spanne and S. Janson.

2. $BMO_\varphi(w)$ in terms of rearrangements

In order to prove Theorem 1.7 we need a result about the behavior of the distribution function of $|f - m_Q f|/w$ over Q for each cube Q. It will be obtained as an easy consequence of the following lemma, whose proof can be found in [M].

Lemma **2.1.** *Let w be in A_1. Then there exist two constants a_1 and a_2 such that, for each cube Q_0 in $\mathbb{R}^n$, the inequality*

$$w(\{x \in Q / \frac{|f(x) - m_Q f|}{w(x)} > \lambda\}) \leq a_1 e^{\frac{a_2}{[f]_{Q_0}}\lambda} w(Q) \tag{2.2}$$

holds for every $\lambda > 0$, every cube $Q \subset Q_0$ and every f in $L^1(Q_0)$ where

$$[f]_{Q_0} = \sup_{Q \subset Q_0} \frac{1}{w(Q)} \int_Q |f(x) - m_Q f| dx.$$

Corollary **2.3.** *Let w be in A_1. Then there exist two constants C_1 and C_2, such that, for each cube $Q = Q(x_Q, r_Q)$ in $\mathbb{R}^n$, the inequality*

$$w(\{x \in Q / \frac{|f(x) - m_Q f|}{w(x)} > C_1 t \varphi(r_Q) ||f||_{BMO_\varphi(w)}\}) \leq C_2 2^{-t} w(Q(x_Q, r_Q))$$

holds for every $t > 0$ and f in $BMO_\varphi(w)$.

Proof: Given a cube $Q = Q(x_Q, r_Q)$, it is clear that

$$\begin{aligned}[f]_Q &\leq \sup_{\substack{z \\ r \leq r_Q}} \frac{1}{w(Q(z,r))} \int_{Q(z,r)} |f(x) - m_{Q(z,r)} f| dx \\ &\leq \varphi(r_Q) ||f||_{BMO_\varphi(w)}\end{aligned}$$

is valid for every f in $BMO_\varphi(w)$. Then, from (2.2) we get

$$\begin{aligned} w(\{x \in Q / \frac{|f(x) - m_Q f|}{w(x)} > \lambda\}) &\leq a_1 e^{-\frac{a_2}{[f]_Q}\lambda} w(Q) \\ &\leq a_1 e^{-\frac{a_2}{\varphi(r_Q)||f||_{BMO_\varphi(w)}}\lambda} w(Q), \end{aligned}$$

Finally, taking $\lambda = t\varphi(r_Q)||f||_{BMO_\varphi(w)} \log 2/a_2$ we obtain the desired result with $C_1 = \log 2/a_2$ and $C_2 = a_1$.□

Now we are able to proceed with the proof of our first theorem.

PROOF OF THEOREM 1.7: First we are going to prove that (1.8) is a necessary condition for f to be in $BMO_\varphi(w)$. Let $Q = Q(x_Q, r_Q)$ be a cube in $\mathbb{R}^n$. Given $r > 0$, we choose j such that $2^{-j}r_Q < r \le 2^{-j+1}r_Q$. Now, by repeated halving all edges, let us divide Q into 2^{jn} subcubes Q_k with lenght edge equal to $r_Q 2^{-j}$. Given k, let $\{I_i^k\}_{i=0}^j$ be the subcubes of the dyadic partition such that $Q = I_0^k \supset \ldots \supset I_j^k = Q_k$ with $|I_i^k| = 2^n|I_{i+1}^k|$. Then, taking y in Q_k and recalling that $w \in A_1$, we get

$$
\begin{aligned}
\frac{|m_{Q_k}f - m_Q f|}{w(y)} &\le \frac{1}{\inf_{Q_k} w}\sum_{i=0}^{j-1} |m_{I_{i+1}^k} f - m_{I_i^k} f| \\
&\le \frac{2^n}{\inf_{Q_k} w}\sum_{i=0}^{j-1} \frac{1}{|I_i^k|}\int_{I_i^k} |f(y) - m_{I_i^k} f|dy \\
&\le \frac{2^n||f||_{BMO_\varphi(w)}}{\inf_{Q_k} w}\sum_{i=0}^{j-1} \frac{w(I_i^k)}{|I_i^k|}\varphi(2^{-i}r_Q) \\
&\le \frac{C_0||f||_{BMO_\varphi(w)}}{\inf_{Q_k} w}\sum_{i=0}^{j-1} \inf_{I_i^k} w\varphi(2^{-i}r_Q) \\
&\le C_0||f||_{BMO_\varphi(w)}\sum_{i=0}^{j-1} \varphi(2^{-i}r_Q).
\end{aligned}
\tag{2.4}
$$

Now, taking $\lambda_0 = (C_0 + C_1 n)||f||_{BMO_\varphi(w)} \sum_{i=0}^{j-1} \varphi(2^{-i}r_Q)$, where C_1 and C_2 are the constants appearing in Corollary 2.3, from (2.2) and (2.4), we have

$$(2.5)\qquad w(\{y \in Q / \frac{|f(y) - m_Q f|}{w(y)} > \lambda_0\})$$
$$\leq \sum_{k=1}^{2^{jn}} w(\{y \in Q_k / \frac{|f(y) - m_Q f|}{w(y)} > \lambda_0\})$$
$$\leq \sum_{k=1}^{2^{jn}} w(\{y \in Q_k / \frac{|f(y) - m_{Q_k} f|}{w(y)}$$
$$> C_1(n + \log 2 \log C_2) j ||f||_{BMO_\varphi(w)} \varphi(2^{-j} r_Q)\})$$
$$\leq C_2 2^{-jn} \sum_{k=1}^{2^{jn}} w(Q_k)$$
$$= C_2 2^{-jn} w(Q) = C_2 (\frac{r_Q}{2^j})^n \frac{w(Q)}{|Q|}$$
$$< C_2 r^n \frac{w(Q)}{|Q|}.$$

On the other hand, we get

$$\begin{aligned} \lambda_0 &\leq \frac{1}{log2}(C_0 + C_1 n)||f||_{BMO_\varphi(w)} \sum_{i=0}^{j-1} \int_{2^{-i} r_Q}^{2^{-i+1} r_Q} \frac{\varphi(t)}{t} dt \\ &\leq C_3 ||f||_{BMO_\varphi(w)} \int_r^{2r_Q} \frac{\varphi(t)}{t} dt. \end{aligned}$$

Then, from (2.5)

$$w(\{y \in Q / \frac{|f(y) - m_Q f|}{w(y)} > C_3 ||f||_{BMO_\varphi(w)} \int_r^{2r_Q} \frac{\varphi(t)}{t} dt\}) < C_2 r^n \frac{w(Q)}{|Q|}.$$

Taking $s = C_2 r^n \frac{w(Q)}{|Q|}$ we have

$$w(\{y \in Q / \frac{|f(y) - m_Q f|}{w(y)} > C ||f||_{BMO_\varphi(w)} \int_{(\frac{s|Q|}{Cw(Q)})^{\frac{1}{n}}}^{2r_Q} \frac{\varphi(t)}{t} dt\}) < s,$$

where $C = \max(C_2, C_3)$, and (1.8) follows easily.

Now, we assume (1.8) holds. Then, given a cube Q in $\mathbb{R}^n$, we have

$$\begin{aligned}
\frac{1}{w(Q)}\int_Q |f(y)-m_Qf|dy &= \frac{1}{w(Q)}\int_Q \frac{|f(y)-m_Qf|}{w(y)}w(y)dy\\
&= \frac{1}{w(Q)}\int_0^{w(Q)} f_Q^*(s)ds\\
&\leq \frac{C}{w(Q)}\int_0^{w(Q)}\Big(\int_{(\frac{s|Q|}{Cw(Q)})^{\frac{1}{n}}}^{2r_Q}\frac{\phi(t)}{t}dt\Big)ds\\
&= \frac{C}{w(Q)}\int_0^{2r_Q}\frac{\varphi(t)}{t}\Big(\int_0^{t^nC\frac{w(Q)}{|Q|}}ds\Big)dt\\
&= \frac{C}{w(Q)}\int_0^{2r_Q}\frac{\varphi(t)}{t}t^n\frac{w(Q)}{|Q|}dt\\
&\leq \frac{C\varphi(2r_Q)}{|Q|}(2r_Q)^n\\
&\leq C\varphi(r_Q).
\end{aligned}$$

Since the above inequality is valid for every Q, we get f is in $BMO_\varphi(w)$.□

PROOF OF COROLLARY 1.9: Let f be in $BMO_\varphi(w)$. Then, given x and y, we have

$$\begin{aligned}
\frac{|f(x)-f(y)|}{w(x)+w(y)} &\leq \frac{|f(x)-m_Qf|}{w(x)+w(y)}+\frac{|f(y)-m_Qf|}{w(x)+w(y)}\\
&\leq \frac{|f(x)-m_Qf|}{w(x)}+\frac{|f(y)-m_Qf|}{w(y)},
\end{aligned} \tag{2.6}$$

where Q is a cube containing x and y with length side $r_Q = |x-y|$. On the other hand, it is clear that

$$\operatorname*{ess\,sup}_{z\in Q}\frac{|f(z)-m_Qf|}{w(z)} = \sup_s f_Q^*(s) = \lim_{s\to 0} f_Q^*(s).$$

Then, from the Theorem, we get

$$\begin{aligned}
\operatorname*{ess\,sup}_{z\in Q}\frac{|f(z)-m_Qf|}{w(z)} &\leq C||f||_{BMO_\varphi(w)}\lim_{s\to 0}\int_{\left(\frac{s|Q|}{Cw(Q)}\right)^{\frac{1}{n}}}^{2r_Q}\frac{\varphi(t)}{t}dt \\
&= C||f||_{BMO_\varphi(w)}\int_0^{2r_Q}\frac{\varphi(t)}{t}dt \\
&= C||f||_{BMO_\varphi(w)}\int_0^{r_Q}\frac{\varphi(2t)}{t}dt \\
&\leq C||f||_{BMO_\varphi(w)}\int_0^{r_Q}\frac{\varphi(t)}{t}dt.
\end{aligned}$$

Finally, combining this inequality with (2.6) we can write

$$|f(x)-f(y)|\leq C||f||_{BMO_\varphi(w)}(w(x)+w(y))\int_0^{|x-y|}\frac{\varphi(t)}{t}dt,$$

for a.e. x and y in $\mathbb{R}^n$, proving that f belongs to $\Lambda_\psi(w)$ with $\psi(r)=\int_0^r\frac{\varphi(t)}{t}dt.\Box$

Proof of Theorem 1.10: First, recall that, because of our hypothesis on w and φ, we can take balls $B(x,r)=\{y\in\mathbb{R}^n/|x-y|<r\}$ instead of cubes $Q(x,r)$ in (1.1) and obtain an equivalent version of $BMO_\varphi(w)$. In this proof, for the sake of simplicity, we consider the version with balls.

Let $x_0\in\mathbb{R}^n$ fixed and let $B(z,r)$ a ball in $\mathbb{R}^n$. Suppose that $|z-x_0|<2r$. Then, using the doubling property of w and φ, we have

(2.7)

$$\begin{aligned}
\int_{B(z,r)} &|h_{x_0}(x) - h_{x_0}(z + r\frac{z - x_0}{|z - x_0|})|dx \\
&= \int_{B(z,r)} (\int_{|x-x_0|}^{|x_0-z|+r} \frac{w(B(x_0,t))}{t^n}\frac{\varphi(t)}{t}dt)dx \\
&\le \int_0^{|x_0-z|+r} \frac{w(B(x_0,t))}{t^n}\frac{\varphi(t)}{t}(\int_{B(x_0,t)\bigcap B(z,r)} dx)dt \\
&\le C\int_0^{|x_0-z|+r} \frac{w(B(x_0,t))}{t^n}\frac{\varphi(t)}{t}t^n dt \\
&\le C\varphi(3r)\int_0^{|x_0-z|+r} \frac{w(B(x_0,t))}{t}dt \\
&= C\varphi(3r)\sum_{i=0}^{\infty}\int_{(|x_0-z|+r)/2^{i+1}}^{(|x_0-z|+r)/2^i} \frac{w(B(x_0,t))}{t}dt \\
&\le C\varphi(3r)\sum_{i=0}^{\infty} w(B(x_0, \frac{|x_0-z|+r}{2^i})) \\
&\le C\varphi(r)\sum_{i=0}^{\infty} w(B(x_0, \frac{|x_0-z|+r}{2^i} - B(x_0, \frac{|x_0-z|+r}{2^{i+1}})) \\
&\le C\varphi(r)w(B(x_0, |x_0 - z| + r)) \\
&\le C\varphi(r)w(B(x_0, r)).
\end{aligned}$$

Now, assuming $|z - x_0| > 2r$ and keeping in mind that w satisfies the doubling condition, we have

$$\begin{aligned}
\int_{B(z,r)} |h_{x_0}(x) \quad &- \quad h_{x_0}(z + r\frac{z - x_0}{|z - x_0|})|dx \\
&\le \int_0^{|x_0-z|+r} \frac{w(B(x_0,t))}{t^n}\frac{\varphi(t)}{t}|B(z,r)\cap B(x_0,t)|dt \\
&\le Cr^n\int_{|x_0-z|-r}^{|x_0-z|+r} \frac{w(B(x_0,t))}{t^n}\frac{\varphi(t)}{t}dt \\
&\le Cr^n\int_{|x_0-z|-r}^{|x_0-z|+r} \frac{w(B(z,t))}{t^n}\frac{\varphi(t)}{t}dt.
\end{aligned}$$

Note that, since $w \in A_1, w(B(z,t))/t^n$ is quasi-decreasing. Then from the above inequality and the fact that $|x_0 - z| - r \geq r$, having in mind that that $\varphi(t)/t$ is quasi-decreasing, we get

$$\begin{aligned} \int_{B(z,r)} |h_{x_0}(x) - h_{x_0}(z + r\frac{z-x_0}{|z-x_0|})|dx \\ \leq Cr^n \frac{w(B(z,r))}{r^n}\frac{\varphi(r)}{r}r \\ = Cw(B(z,r))\varphi(r). \end{aligned} \tag{2.8}$$

So, from (2.7) and (2.8), it is immediate that $h_{x_0} \in BMO_\varphi(w)$. Moreover, the upper bound on (1.11) is clear. To check the lower bound, let us note first that there exists a constant C such that

$$\begin{aligned} &\frac{1}{w(B(z,s)} \int_{B(z,s)} |h_{x_0}(y) - m_{B(z,s)}h_{x_0}|dy \\ &\qquad \geq \frac{1}{2w(B(z,s))}\frac{1}{|B(z,s)|}\int_{B(z,s)}\int_{B(z,s)} |h_{x_0}(x) - h_{x_0}(y)|dydx \end{aligned}$$

for every $z \in \mathbb{R}^n$ and $s > 0$. Then, we can write

$$\begin{aligned} &\sup_{\substack{0<s\leq r \\ z\in\mathbb{R}^n}} \frac{1}{w(B(z,s))}\int_{B(z,s)} |h_{x_0}(y) - m_{B(z,s)}h_{x_0}|dy \\ &\geq \frac{1}{w(B(x_0,r))}\int_{B(x_0,r)} |h_{x_0}(y) - m_{B(x_0,r)}h_{x_0}|dy \\ &\geq \frac{C}{w(B(x_0,r))}\frac{1}{|B(x_0,r)|}\int_{|x-x_0|<\frac{r}{4}}\int_{\frac{r}{2}<|y-x_0|<r} |h_{x_0}(x) - h_{x_0}(y)|dydx \\ &= \frac{C}{w(B(x_0,r))}\frac{1}{|B(x_0,r)|} \\ &\quad \times \int_{|x-x_0|<\frac{r}{4}} dx \int_{\frac{r}{2}<|y-x_0|<r} dy(\int_{|x-x_0|}^{|y-x_0|} \frac{w(B(x_0,t))}{t^n}\frac{\varphi(t)}{t}dt) \\ &\geq \frac{C}{w(B(x_0,r))}\frac{1}{r^n}r^{2n}\int_{\frac{r}{4}}^{\frac{r}{2}} \frac{w(B(x_0,t))}{t^n}\frac{\varphi(t)}{t}dt \\ &\geq \frac{Cr^n}{w(B(x_0,r))}\frac{w(B(x_0,r/4))}{r^{n+1}}\varphi(\frac{r}{4})r \end{aligned}$$

Finally, from the fact that w and φ satisfy a doubling condition we get

$$\sup_{\substack{0<s\leq r\\ z\in\in\mathbb{R}^n}} \frac{1}{w(B(z,s))}\int_{B(z,s)} |h_{x_0}(x) - m_{B(z.s)}h_{x_0}|dx \geq C\varphi(r) \tag{2.9}$$

as we wanted to prove.□

Our proof of Corollary 1.12 requires the following characterization of the functions in $\Lambda_\varphi(w)$ (see (1.4)).

LEMMA **2.10.** *Let w be in A_1 and φ satisfying a doubling condition. Then a function f belongs to $\Lambda_\varphi(w)$ if and only if $f \in L^1_{loc}(\mathbb{R}^n)$ and there exists a constant C such that*

$$ess \sup_{\substack{x\in B(z,r)\\ z\in\mathbb{R}^n}} \frac{|f(x) - m_{B(z,r)}f|}{w(x)} \leq C\varphi(r) \tag{2.11}$$

for every $r > 0$.

PROOF: It is easy to see that functions satisfying (2.11) are in $\Lambda_\varphi(w)$. Actually we do not need w be in A_1 nor the doubling condition on φ for this part. Let us prove the reciprocal. If f is in $\Lambda_\varphi(w)$, then, by (1.2), we get

$$|f(x) - f(y)| \leq C(w(x) + w(y))\varphi(|x-y|) \tag{2.12}$$

for a.e. x and y in $\mathbb{R}^n$. Now, let $B(z,r)$ be a ball in $\mathbb{R}^n$. Taking x and y in $B(z,r)$ and integrating with respects to y both sides of (2.12) we get

$$\begin{aligned} |f(x)|B(z,r)| - \int_{B(z,r)} f(y)dy| &\leq \int_{B(z,r)} |f(x) - f(y)|dy \\ &\leq C(w(x)|B(z,r)| + w(B(z,r)))\varphi(2r). \end{aligned}$$

for a.e. x in $B(z,r)$. From this inequality, using our assumptions on w and φ, we have

$$\begin{aligned} |f(x) - m_{B(z,r)}f| &\leq C(w(x) + \frac{w(B(z,r))}{|B(z,r)|})\varphi(2r) \\ &\leq C(w(x) + C \inf_{B(z,r)} w)\varphi(r) \\ &\leq Cw(x)\varphi(r) \end{aligned}$$

for a.e. x in $B(z,r)$. Now (2.11) is obvious.□

Proof of Corollary 1.12: Let x_0 be a Lebesgue point of w such that $0 < w(x_0) < \infty$. Note that since w is finite a.e., for each ε in $(0,1)$ and we can find $A^\varepsilon \subset B(x_0, \varepsilon)$ such that $|A^\varepsilon| > 0$ and $w(x) \leq 2w(x_0)+1$ for every $x \in A^\varepsilon$. Now, let the function h_{x_0} be defined as in Theorem 1.10. Since $w \in A_1$, for each ε in $(0,1)$, we have

$$\begin{aligned} \frac{h_{x_0}(x)}{w(x)} &= \frac{1}{w(x)} \int_{|x-x_0|}^1 \frac{w(B(x_0,t))}{t^n} \frac{\varphi(t)}{t} dt \\ &\geq \frac{C}{w(x)} w(B(x_0,1)) \int_{|x-x_0|}^1 \frac{\varphi(t)}{t} dt \\ &\geq \frac{C}{2w(x_0)+1} w(B(x_0,1)) \int_\varepsilon^1 \frac{\varphi(t)}{t} dt \end{aligned}$$

for every $x \in A^\varepsilon$. Then, taking ε close enough to zero, it is clear that h_{x_0}/w is not bounded on $B(x_0,1)$ and, consequently, since $w(x) \geq \operatorname{ess\,inf}_{B(x_0,1)} w > 0$ a.e. in $B(x_0,1)$

$$\operatorname{ess} \sup_{x \in B(x_0,1)} \frac{|h_{x_0}(x) - m_{B(x_0,1)} h_{x_0}|}{w(x)} = \infty.$$

So, from Lemma 2.11, h_{x_0} does not belong to $\Lambda_\varphi(w)$. However, from Theorem 1.10, $h_{x_0} \in BMO_\varphi(w)$. This completes the proof of the Corollary.□

3. $BMO_\varphi(w)$ in terms of Riesz transforms

In this section we shall give the proof of Theorem 1.14. We will use some technical lemmas and also an extension to n-dimensions of the following result appearing in $[M]$ for the Hilbert transform.

Proposition 3.1. *Let w be an A_∞ weight and φ a non decreasing function defined on $[0,\infty)$ satisfying a doubling condition. Assume further that there exists a constant C such that*

$$\frac{|B|^{1/n}}{\varphi(|B|^{1/n})}\int_{B^c} w(y)\frac{\varphi(|x_0-y|)}{|x_0-y|^{n+1}}dy \leq C\frac{w(B)}{|B|} \tag{3.2}$$

holds for any ball B, where x_0 denotes the center of B. Then the Riesz-transforms $\mathcal{R}_i$ given by (1.15) are finite almost everywhere for $f \in BMO_\varphi(w)$. Moreover there is a constant C such that

$$||\mathcal{R}_i f||_{BMO_\varphi(w)} \leq C||f||_{BMO_\varphi(w)} \qquad 1 \leq i \leq n. \tag{3.3}$$

The proof follows the same lines of the one-dimensional case with some minor modifications.

Our next result shows that, under the assumptions of theorem 1.14, Proposition 3.1 holds

Lemma 3.4. *Let w be a weight in $A_1 \cap A_\infty^\delta$ and φ as in theorem 1.14, that is, there is a constant C such that*

$$r^\delta \int_r^\infty \frac{\varphi(t)}{t^{1+\delta}}dt \leq C\varphi(r).$$

Then w and φ satisfy (3.2) above.

Proof: For B a ball with center x_0 and radious r, we denote by B_k the ball with the same center and radious $2^k r$. Using that φ is non-increasing and doubling and that w belongs to A_1 we have

$$\begin{aligned}(3.5)\qquad \int_{B^c} w(y)\frac{\varphi(|x_0-y|)}{|x_0-y|^{n+1}}dy &= \sum_{k=1}^{\infty}\int_{B_{k+1}-B_k} w(y)\frac{\varphi(|x_0-y|)}{|x_0-y|^{n+1}}dy \\ &\le C\sum_{k=1}^{\infty}\frac{\varphi(2^k r)}{2^k r}\frac{w(B_k)}{|B_k|} \\ &\le C\frac{w(B)}{|B|}\sum_{k=1}^{\infty}\frac{\varphi(2^k r)}{2^k r} \\ &\le C\frac{w(B)}{|B|}\int_r^{\infty}\frac{\varphi(t)}{t^2}dt \\ &\le C\frac{w(B)}{|B|r^{1-\delta}}\int_r^{\infty}\frac{\varphi(t)}{t^{1+\delta}}dt \\ &\le C\frac{w(B)}{|B|}\frac{\varphi(r)}{r} \\ &= C\frac{\varphi(|B|^{1/n})w(B)}{|B|^{1+1/n}}\end{aligned}$$

as we wished.□

Before stating the next lemma we introduce some notation. Let us denote by $\mathcal{X}_r$ the characteristic function of the ball $B_r = B(0,r)$, and by $\psi_r = r^{-n}\mathcal{X}_r$. With this notation we have

$$\psi_r * f(x) = m_{B(x,r)}f.$$

Also, for a weight w and a locally integrable function f, we set

$$\rho_w(f,r) = \sup_{x,r'\le r}\frac{1}{w(B(x,r'))}\int_{B(x,r')}|f(y)-m_{B(x,r')}f|dy.$$

With this notation we state the following lemma.

Lemma **3.6.** *Let w be a weight and f an integrable function. Then for any $r>0$*

$$||f-\psi_r * f||_{BMO(w)} \le C\rho_w(f,2r).$$

In particular for $f\in BMO_\varphi(w)$,

$$||f-\psi_r * f||_{BMO(w)} \le C\varphi(r)$$

PROOF: We will use the following estimate for the averages:

$$|m_{B_0}f - m_{B_1}f| \leq \left(\frac{w(B_2)}{|B_0|} + \frac{w(B_2)}{|B_1|}\right)\rho_w(f, r_2) \tag{3.7}$$

where B_2 is a ball with radious r_2 and such that $B_0 \subset B_1$ and $B_1 \subset B_2$. This can be easily seen by adding and substracting $m_{B_2}f$.

Let now be $B = B(x_0, s)$ any ball. Then, to prove the lemma we need to estimate

$$\begin{aligned}&\Omega_w(B, f - \psi_r * f)\\ &\quad = \frac{1}{w(B(x_0,s))}\int_{B(x_0,s)} |f(x) - (\psi_r * f)(x) - m_{B(x_0,s)}(f - \psi_r * f)|dx.\end{aligned}$$

Let us suppose first that $s \leq r$. Then

$$\begin{aligned}\Omega_w(B, f - \psi_r * f) &\leq \frac{1}{w(B(x_0,s))}\int_{B(x_0,s)} |f(x) - m_{B(x_0,s)}f|dx\\ &\quad + \frac{1}{w(B(x_0,s))}\int_{B(x_0,s)} |m_{B(x,r)}f - m_{B(x_0,s)}(m_{B(.,r)}f)|dx\\ &= I + II\end{aligned}$$

Since $s \leq r$, the first term is bounded by $\rho_w(f, r)$. As for the second, we have

$$\begin{aligned}II &\leq \frac{1}{w(B(x_0,s))}\frac{1}{|B(x_0,s)|}\int_{B(x_0,s)}\int_{B(x_0,s)} |m_{B(x,r)}f - m_{B(y,r)}f|dxdy\\ &\leq \frac{Cw(B(x_0,2r))}{r^n}\frac{s^n}{w(B(x_0,s))}\rho_w(f, 2r),\end{aligned}$$

where we have used (3.7), since for any $z \in B(x_0, s)$, $B(z, r) \subset B(x_0, 2r)$. Now $w \in A_1$ implies the doubling property and also that the function $w(B(x, t))/t^n$ is almost decreasing with a constant independent of x. Since $s \leq r$ we get the desired estimate.

Next we suppose that $s \geq r$. In this case we observe that

$$\Omega_w(B, f - \psi_r * f) \leq \frac{2}{w(B(x_0,s))}\int_{B(x_0,s)} |f(x) - m_{B(x,r)}f|dx$$

Now we can cover the ball $B(x_0, s)$ by a finite family of balls $B_i = B(x_i, r), i = 1, \ldots, N$ and such that $B(x_i, r/2)$ are mutually disjoint.

The number N of such balls is like $(s/r)^n$. Then the integral above is bounded by

$$\begin{aligned}\sum_{i=1}^N \int_{B(x_i,r)} |f(x)-m_{B(x,r)}f|dx &\le \sum_{i=1}^N \int_{B(x_i,r)} |f(x)-m_{B(x_i,r_i)}f| \\ &\quad + \sum_{i=1}^N \int_{B(x_i,r)} |m_{B(x_i,r_i)}f - m_{B(x,r)}f| \\ &\le \rho_w(f,r)\sum_{i=1}^N w(B(x_i,r)) \\ &\quad + 2\rho_w(f,2r)\sum_{i=1}^N w(B(x_i,2r))\end{aligned}$$

where, for the second sum we use again (3.7) and that $B(x,r) \subset B(x_i,2r)$ for $x \in B(x_i,r)$. Finally, using the doubling property of w and that $B(x_i,r/2)$ are disjoint, we get also the desired estimate in this case.

Therefore, taking the supremum on x_0 and s we get the result for the BMO-norm. To prove the estimate for $f \in BMO_\varphi(w)$ we just use that $\varphi(2r) \le C\varphi(r)$.□

We have defined for functions on $BMO_\varphi(w)$ the modified Riesz transforms $\mathcal{R}_j$. It is not hard to prove that, for good functions with zero average, they are equal to the classical version R_jf. For the latter operators it is known that the following formula holds

$$\int R_jf(x)\eta(x)dx = -\int f(x)R_j\eta(x)dx \tag{3.8}$$

for $f \in L^p(\mathbb{R}^n)$ and η, say, in $C_0^\infty(\mathbb{R}^n)$. In the next lemma we extend this result to $\Lambda_\varphi(w)$.

Lemma **3.9.** *Let η be a $C_0^\infty(\mathbb{R}^n)$ function with zero average and $g \in \Lambda_\varphi(w)$ with w and φ as in Theorem 1.14. Then*

$$\int \mathcal{R}_jg(x)\eta(x)dx = -\int g(x)R_j\eta(x)dx$$

PROOF: First, the integral on the left is absolutely convergent since we know that $\mathcal{R}_j g$ is in $BMO_\varphi(w)$ and hence locally integrable. Moreover $\mathcal{R}_j g$ equals $\mathcal{R}_j(g-C)$, where C is any constant. Therefore

$$\begin{aligned}\int \mathcal{R}_j g(x)\eta(x)dx &= \int \eta(x)\mathcal{R}_j(g-C)(x)dx \\ &= \int \eta(x)\mathcal{R}_j(\mathcal{X}_R(g-C))(x)dx \\ &\quad + \int \eta(x)\mathcal{R}_j(\mathcal{X}'_R(g-C))(x)dx \\ &= I_1 + I_2,\end{aligned}$$

where $\mathcal{X}_R = \mathcal{X}_{B(0,R)}, \mathcal{X}'_R = 1 - \mathcal{X}_R$.

To estimate I_1 we observe that g belongs locally to $L^q(\mathbb{R}^n)$ for some $q > 1$. In fact, it is known that an A_∞ weight satisfies a Reverse-Hölder inequality for some $q > 1$ (see [CF]). Therefore for such q and any ball B with radious r we have

$$\begin{aligned}\int_B |g(x)-g(x_0)|^q dx &\le C\int_B (w(x)+w(x_0))^q \varphi(|x-x_0|)dx \\ &\le C\varphi(r)(w(x_0)|B| + \int_B (w(x))^q dx) < \infty,\end{aligned}$$

where we have chosen $x_0 \in B$ to be a Lebesgue point of w. Therefore $R_j(\mathcal{X}_R(g-C))$ is a function in L^q and, moreover, equals, up to a constant, to $\mathcal{R}_j(\mathcal{X}_R(g-C))$. So, since η has zero average, an application of (3.8) gives

$$I_1 = \int \eta(x)R_j(\mathcal{X}_R(g-C))(x)dx = -\int R_j\eta(y)\mathcal{X}_R(y)(g(y)-C)dy.$$

Now, to estimate I_2 we choose R such that $\text{supp}\eta \subset B(0,R/2)$ and $R > 1$. Then

$$\mathcal{R}_j(\mathcal{X}'_R(g-C))(x) = \lim_{\varepsilon\to 0}\int_{\substack{|x-y|>\varepsilon \\ |y|>R}} \left(\frac{x_j-y_j}{|x-y|^{n+1}} + \frac{y_j}{|y|^{n+1}}\right)(g(y)-C)dy$$

But for $x \in \text{supp}\eta$ and $|y| > R$ we have $|x-y| > R/2$ and, therefore, we may drop the limit above. Moreover taking absolute values inside the integral and applying the mean value theorem we have

$$(3.10)\quad \int_{|y|>R}\left|\frac{x_j-y_j}{|x-y|^{n+1}}+\frac{y_j}{|y|^{n+1}}\right||g(y)-C|dy$$
$$\leq C\int_{|y|>R}\frac{|x|}{|y|^{n+1}}|g(y)-C|$$
$$\leq C|x|\int_{|y|>R}\frac{\varphi(|x_0-y|)}{|y|^{n+1}}(w(x_0)+w(y))dy,$$

where we have chosen $C=g(x_0)$ with $x_0\in B(0,R/2)$ a Lebesgue point of w. Again $|x_0|<R/2$ and $|y|>R$ imply $|x_0-y|<R/2+|y|<2|y|$ so the last integral is bounded by

$$C|x|(R^{\delta-1}\int_R^\infty\frac{\varphi(t)}{t^{1+\delta}}dt+\int_{|y|>R}\frac{w(y)\varphi(|y|)}{|y|^{n+1}}dy)\leq C|x|$$

for $x\in\operatorname{supp}\eta$, since both integrals are finite as a consequence of lemma 3.4. In this way we have proved that the iterated integral

$$\int|\eta(x)|\int|\mathcal{K}_j(x,y)|\mathcal{X}_{R'}(y)|g(y)-C|dydx$$

is finite, where $\mathcal{K}_j(x,y)$ denotes the kernel of $\mathcal{R}_j$. Therefore in I_2 the order of integration can be reversed and hence

$$(3.11)\quad \int\eta(x)\mathcal{R}_j(\mathcal{X}'_R(g-C))(x)dx$$
$$=\int\mathcal{X}'_R(y)(g(y)-C)\int K_j(x,y)\eta(x)dxdy$$
$$=-\int\mathcal{X}'_R(y)(g(y)-C)R_j\eta(y)dy$$

Adding I_1 and I_2 we get

$$I_1+I_2=-\int R_j\eta(y)(g(y)-C)dy=-\int R_j\eta(y)g(y)dy.\square$$

Now we turn into the proof of the last theorem.

Proof of Theorem 1.14: First, if f can be written as

$$(3.12)\qquad f=\sum_0^n\mathcal{R}_j(f_j)$$

with $f_0, \dots, f_n$ in $\Lambda_\varphi(w)$, it follows easily that $f \in BMO_\varphi(w)$. In fact, we noticed that $\Lambda_\varphi(w)$ is continuously embedded in $BMO_\varphi(w)$ so, from Lemma 3.4 and Proposition 3.1, the function on the right hand side of (3.12) belongs to $BMO_\varphi(w)$ and, moreover,

$$||f||_{BMO_w(\varphi)} \le C \sum_0^n ||f_i||_{\Lambda_\varphi(w)}. \tag{3.13}$$

On the other hand, let f belong to $BMO_\varphi(w)$. Following [J], since φ is continuous, there are numbers r_i such that $\varphi(r_i) = 2^i \varphi(r_0)$ for a fixed r_0 with $\varphi(r_0) \neq 0$. The numbers r_i will be defined for $i \in \mathbb{Z}$ and belonging to a certain interval $[-L, M]$ where L and M may be finite or infinite, depending on the boundedness properties of φ. For each r_i, according to Lemma 3.5, the function $f - \psi_{r_i} * f$ belongs to $BMO(w)$ and moreover

$$||f - \psi_{r_i} * f||_{BMO(w)} \le C\varphi(r_i) = C2^i. \tag{3.14}$$

From here we have that

$$||\psi_{r_i} * f - \psi_{r_{i+1}} * f||_{BMO(w)} \le C(\varphi(r_i) + \varphi(r_{i+1})) = C\varphi(r_i). \tag{3.15}$$

Now, we apply the decomposition result of Muckenhoupt and Wheeden (see [MW]), for the space $BMO(w)$ to each of the functions on the left hand side of (3.15). In this way we get

$$\psi_{r_i} * f - \psi_{r_{i+1}} * f = \sum_{j=0}^n \mathcal{R}_j(u_j^i), \tag{3.16}$$

where u_j^i are in $L^\infty(w)$ with

$$||u_j^i||_{L^\infty(w)} \le C\varphi(r_i). \tag{3.17}$$

The tempting idea now is to recover f adding these pieces since, at least when L and M are infinite, the sum of the series will give f back. But, even in that case, the sum of the functions u_j^i will be not smooth enough to provide a $\Lambda_\varphi(w)$-function for each j. To make things work we need to smoother the functions u_j^i. To this end, let us choose a point x_0 such that is a Lebesgue point for the weight w and for the functions $(\psi_{r_i} + \psi_{r_{i+1}}) * u_j^i$ and define

$$v_j^i = (\psi_{r_i} + \psi_{r_{i+1}}) * u_j^i - C_{ij}$$

where $C_{ij} = ((\psi_{r_i} + \psi_{r_{i+1}}) * u_j^i)(x_0)$. Now, we want to prove that v_j^i are functions in $\Lambda_\varphi(w)$, giving an estimate for $||v_j^i||_{\Lambda_\varphi(w)}$. For each i and j fixed, we take x, z two points in $\mathbb{R}^n$ and we consider the two possible cases

Case 1: $|x - z| > r_i$

$$
\begin{aligned}
|v_j^i(x) - v_j^i(z)| &\le \frac{1}{r_i^n}\int_{B(x,r_i)} |u_j^i| + \frac{1}{r_{i+1}^n}\int_{B(x,r_{i+1})} |u_j^{i+1}| \\
&\quad + \frac{1}{r_i^n}\int_{B(z,r_i)} |u_j^i| + \frac{1}{r_{i+1}^n}\int_{B(z,r_{i+1})} |u_j^{i+1}| \\
&\le \frac{1}{r_i^n}||u_j^i||_{L^\infty(w)}(w(B(x,r_i)) + w(B(z,r_i))) \\
&\quad + \frac{1}{r_{i+1}^n}||u_j^{i+1}||_{L^\infty(w)}(w(B(x,r_{i+1})) + w(B(z,r_{i+1})))
\end{aligned}
$$

Using now estimate (3.17) and that $w \in A_1$, we obtain

$$
|v_j^i(x) - v_j^i(z)| \le C(\varphi(r_i) + \varphi(r_{i+1}))(w(x) + w(z)). \tag{3.18}
$$

Case 2: $|x - z| \le r_i$. In this case $B(x, r_i)$ and $B(z, r_i)$ have a thick intersection and, since r_i is increasing, the same happens with $B(x, r_{i+1})$ and $B(z, r_{i+1})$. Let us call $A_i = B(x, r_i)\Delta B(z, r_i), A_{i+1} = B(x, r_{i+1})\Delta B(z, r_{i+1})$, $\widetilde{B}_i = B(x, 3r_i)$ and $\widetilde{B}_{i+1} = B(x, 3r_{i+1})$. Then we have $A_i \subset \widetilde{B}_i$ and $A_{i+1} \subset \widetilde{B}_{i+1}$ and, using the A_∞^δ condition in w, we have for $k = i, i+1$

$$
w(A_k) \le Cw(\widetilde{B}_k)(\frac{|A_k|}{|\widetilde{B}_k|})^\delta \le Cw(B_k)(\frac{|x - z|}{r_k})^\delta,
$$

where, for the last inequality, we have used the estimate $|A_k| \le C|x - z|r_k^{n-1}$. Thus

$$\begin{aligned}|v_j^i(x)-v_j^i(z)| &\le \frac{1}{r_i^n}\int_{A_i}|u_j^i| + \frac{1}{r_{i+1}^n}\int_{A_{i+1}}|u_j^{i+1}| \\ &\le ||u_j^i||_{L^\infty(w)}\frac{w(A_i)}{r_i^n} + ||u_j^{i+1}||_{L^\infty(w)}\frac{w(A_{i+1})}{r_{i+1}^n} \\ &\le C|x-z|^\delta(\frac{\varphi(r_i)}{r_i^\delta}\frac{w(B_i)}{r_i^n} + \frac{\varphi(r_{i+1})}{r_{i+1}^\delta}\frac{w(B_{i+1})}{r_{i+1}^n}) \\ &\le C\varphi(r_i)(\frac{|x-z|}{r_i})^\delta(w(x)+w(z))\end{aligned}$$

where in the last inequality we have used that $w \in A_1$.

Therefore in both cases we have proved the inequality

$$|v_j^i(x)-v_j^i(z)| \le C\varphi(r_i)(\frac{|x-z|}{r_i})^\delta(w(x)+w(z)) \tag{3.19}$$

With (3.18) and (3.19) we are ready to show that the function $g_j = \sum_i v_i^j$ is well defined and, moreover, it belongs to $\Lambda_\varphi(w)$. In fact, using the estimates (3.18) and (3.19) for fixed x and z, we have

$$\begin{aligned}\sum_i |v_j^i(x)-v_j^i(z)| &= (\sum_{r_i<|x-z|} + \sum_{r_i\ge|x-z|})|v_j^i(x)-v_j^i(z)| \\ &\le C(w(x)+w(z))(\sum_{r_i<|x-z|}\varphi(r_i) + |x-z|^\delta\sum_{r_i\ge|x-z|}\frac{\varphi(r_i)}{r_i^\delta}).\end{aligned}$$

But, since $\varphi(r_i) = 2\varphi(r_{i-1})$ and $\{r_i\}$ is non-decreasing, we get

$$\begin{aligned}\sum_k^m \varphi(r_i) &= 2\sum_k^m(\varphi(r_i)-\varphi(r_{i-1})) \\ &= 2(\varphi(r_m)-\varphi(r_{k-1}) \\ &\le 2\varphi(r_m),\end{aligned}$$

and

$$
\begin{aligned}
\sum_k^m \frac{\varphi(r_i)}{r_i^\delta} &= 2\sum_k^m \frac{\varphi(r_i)-\varphi(r_{i-1})}{r_i^\delta} \\
&= 2\sum_k^{m-1} \varphi(r_i)(\frac{1}{r_i^\delta} - \frac{1}{r_{i+1}^\delta}) + 2\frac{\varphi(r_m)}{r_m^\delta} - 2\frac{\varphi(r_{k-1})}{r_k^\delta} \\
&\leq C\left(\sum_k^{m-1} \varphi(r_i)\int_{r_i}^{r_{i+1}} \frac{dt}{t^{1+\delta}} + \varphi(r_m)\int_{r_m}^{\infty} \frac{dt}{t^{1+\delta}}\right) \\
&\leq C\int_{r_k}^{\infty} \frac{\varphi(t)}{t^{1+\delta}} dt.
\end{aligned}
$$

With these estimates we obtain

$$
\sum_i |v_j^i(x) - v_j^i(z)| \leq C(w(x)-w(z))(\varphi|x-z| + |x-z|^\delta \int_{|x-z|}^{\infty} \frac{\varphi(t)}{t^{1+\delta}} dt)
$$

and using the hypothesis on φ we conclude

$$
\sum_i |v_j^i(x) - v_j^i(z)| \leq C(w(x)-w(z))\varphi(|x-z|) \tag{3.20}
$$

Therefore, taking $z = x_0$ in the above inequality, we have

$$
\sum_i |v_j^i(x)| \leq C(w(x)-w(x_0))\varphi(|x-x_0|),
$$

which implies that the series $\sum v_j^i(x)$ converges absolutely for almost every x, in fact for the Lebesgue points of w. Also if we set $g_j = \sum_i v_j^i$, the inequality (3.20) gives

$$
|g_j(x) - g_j(z)| \leq C(w(x)-w(z))\varphi(|x-z|),
$$

proving that g_j is in $\Lambda_\varphi(w)$ and $||g_j||_{\Lambda_\varphi(w)} \leq C$.

Now we would like to show that f and $\sum_{j=0}^n \mathcal{R}_j g_j$ are basically the same, in the sense that their difference is either zero or a function which can be decomposed in the way we want.

First we observe that for each fixed i we have

$$\begin{aligned}(3.21)\qquad \sum_{j=0}^{n}\mathcal{R}_j(v_j^i) &= \sum_{j=0}^{n}\mathcal{R}_j((\psi_{r_i}+\psi_{r_{i+1}})*u_j^i)\\ &= \sum_{j}^{n}(\psi_{r_i}+\psi_{r_{i+1}})*\mathcal{R}_j(u_j^i)\\ &= (\psi_{r_i}+\psi_{r_{i+1}})*(\psi_{r_i}-\psi_{r_{i+1}})*f\\ &= \psi_{r_i}*\psi_{r_i}*f-\psi_{r_{i+1}}*\psi_{r_{i+1}}*f\end{aligned}$$

Since for approximations to the identity, say $\rho_r(x)=r^{-n}\rho(x/r)$, we know that $\lim_{r\to\infty}(\rho_r*f)=0$ and $\lim_{r\to 0}(\rho_r*f)=f$, we may expect to recover f from adding up on i the last equality. But, since the sequence r_i belongs to the range of φ, we have to distinguish whether or not L and M are finite.

In any case, if η is a C_0^∞ function with $\int\eta=0$, according to Lemma 3.9 we have

$$\begin{aligned}(3.22)\qquad \int\mathcal{R}_jg_j\eta &= -\int g_jR_j\eta\\ &= -\sum_i\int v_j^iR_j\eta\\ &= \sum_i\int\mathcal{R}_jv_j^i\eta\end{aligned}$$

where in order to take the sum outside of the integral we have made use of the fact that $\sum_i|v_j^i|$ converges almost everywhere to a function in $\Lambda_\varphi(w)$ and, by Lemma 3.8, the integral of the product of this function by $R_j\eta$ is absolutely convergent. From (3.21) and (3.22) we obtain

(3.23)

$$\begin{aligned}\int(\sum_{j=0}^{n}\mathcal{R}_jg_j)\eta &= \sum_i\int(\sum_{j=0}^{n}\mathcal{R}_jv_j^i)\eta\\ &= \sum_i(\int(\psi_{r_i}*\psi_{r_i}*f)\eta-\int(\psi_{r_{i+1}}*\psi_{r_{i+1}}*f)\eta)\\ &= \lim_{i\to -L}\int(\psi_{r_i}*\psi_{r_i}*f)\eta-\lim_{i\to M}\int(\psi_{r_i}*\psi_{r_i}*f)\eta\end{aligned}$$

where the limit should be understood as the evaluation in $-L$ or M when they are finite. To evaluate each of these terms we consider the different possibilities for L and M. The goal is to prove that the first limit gives either $\int f\eta$ or $\int (f+H)\eta$ where H is a sum of Riesz transforms of $\Lambda_\varphi(w)$-functions; similarly we will prove that the second limit gives either zero or $\int G\eta$ with G satisfying the desired property.

i) $L=\infty$. In this case $r_i \to 0$ for $i \to -L$ and therefore

$$\lim_{i\to -L} \int (\psi_{r_i} * \psi_{r_i} * f)\eta = \lim_{r\to 0} \int f(\psi_r * \psi_r * \eta) = \int f\eta, \tag{3.24}$$

since f es locally integrable, $\eta \in C_0^\infty$ and ψ_{r_i} has compact support.

ii) $L<\infty$. In this case $\varphi(r_{-L}) \le 2\varphi(r)$ for all $r>0$ since otherwise r_{-L-1} could have been constructed. Also, by Lemma 3.6

$$\begin{aligned} ||f-\psi_{r_{-L}} * \psi_{r_{-L}} * f||_{BMO(w)} &\le ||f-\psi_{r_{-L}} f||_{BMO(w)} \\ &\quad + ||\psi_{r_{-L}} * (f-\psi_{r_{-L}} * f)||_{BMO(w)} \\ &\le 2||f-\psi_{r_{-L}} * f||_{BMO(w)} \\ &\le C\varphi(r_{-L}). \end{aligned}$$

Therefore, using again the decomposition result for $BMO(w)$, we get

$$(\psi_{r_{-L}} * \psi_{r_{-L}} * f) - f = \sum_{j=0}^n \mathcal{R}_j(h_j)$$

with $||h_j||_{L^\infty(w)} \le C\varphi(r_{-L})$. Moreover we have

$$|h_j(x)-h_j(y)| \le (w(x)+w(y))||h_j||_{L^\infty(w)} \le C(w(x)+w(y))\varphi(|x-y|)$$

giving that $h_j \in \Lambda_\varphi(w)$. In this way we have shown that

$$\lim_{i\to -L} \int (\psi_{r_i} * \psi_{r_i} * f)\eta = \int f\eta + \sum_{j=0}^n \int \mathcal{R}_j h_j \eta \tag{3.25}$$

with $h_j \in \Lambda_\varphi(w)$.

iii) $M=\infty$. In this case $r_i \to \infty$ for $i \to M$ and therefore supp$\eta \subset B(0,r_i)$ for any i large enough.

Now, as above

$$\int (\psi_{r_i} * f * f)\eta = \int f(\psi_{r_i} * \psi_{r_i} * \eta)$$

But, for i large enough, $\psi_{r_i} * \psi_{r_i} * \eta$ vanishes outside of $\tilde{B} = B(0, 3r_i)$ and has zero average. Thus

$$\begin{aligned} |\int f(\psi_{r_i} * \psi_{r_i} * \eta)| &\leq \int_{B(0,3r_i)} |f - m_{\tilde{B}} f||\psi_{r_i} * \psi_{r_i} * \eta| \\ &\leq Cw(B(0, 3r_i)\varphi(r_i)||\psi_{r_i} * \psi_{r_i} * \eta||_\infty \end{aligned}$$

since $f \in BMO_\varphi(w)$. Also, using again the zero average for η,

$$\begin{aligned} ||\psi_{r_i} * \psi_{r_i} * \eta||_\infty &\leq r_i^{-n} ||\psi_{r_i} * \eta||_1 \\ &\leq r_i^{-n} \int_{B(0,2r_i)} \int_{B(0,r_i)} |\psi_{r_i}(x - y) - \psi_{r_i(x)}||\eta(y)|dydx \\ &\leq r_i^{-n} \int_{B(0,r_i)} |\eta(y)|(\int_{B(0,2r_i)} |\psi_{r_i}(x - y) - \psi_{r_i}(x)|dx)dy \\ &\leq r_i^{-2n} \int_{B(0,r_i)} |\eta(y)||B(0, r_i)\Delta B(y, r_i)|dy \\ &\leq Cr_i^{-n-1} \int_{B(0,r_i)} |y||\eta(y)|dy = Cr_i^{-n-1}. \end{aligned}$$

With this estimate we get for i large enough

$$\int (\psi_{r_i} * \psi_{r_i} * f)\eta) \leq C \frac{w(B(0, 3r_i))}{r_i^n} \frac{\varphi(r_i)}{r_i} \leq C \inf_{x \in B(0,1)} w(x) \frac{\varphi(r_i)}{r_i}$$

Now, using that φ is non-decreasing, we have

$$\frac{\varphi(r)}{r} \leq Cr^{\delta - 1} \int_r^\infty \frac{\varphi(t)}{t^{1+\delta}} dt,$$

where the right side tends to zero when $r \to \infty$, because of $\delta \leq 1$ and $\int_1^\infty (\varphi(t)/t^{1+\delta})dt < \infty$. Hence we get

$$\lim_{i \to M} \int (\psi_{r_i} * \psi_{r_i} * f)\eta = 0. \tag{3.26}$$

iv) $M < \infty$. In this case we have $\varphi(r) \leq 2\varphi(r_M)$ for any $r > 0$ and therefore the given function f belongs to $BMO(w)$ with $||f||_{BMO(w)} \leq C\varphi(r_M)$. Applying the decomposition result for functions in this space we get

$$f = \sum_{j=0}^n \mathcal{R}_j h'_j$$

with $||h'_j||_{L^\infty(w)} \leq C\varphi(r_M)$. Then we have

$$\begin{aligned}\int(\psi_{r_M} * \psi_{r_M} * f)\eta &= \sum_{j=0}^{n}\int(\psi_{r_M} * \psi_{r_M} * \mathcal{R}_j(h'_j)\eta) \\ &= \sum_{j=0}^{n}\int \mathcal{R}_j(\psi_{r_M} * \psi_{r_M} * h'_j)\eta\end{aligned}$$

So, if we are able to prove that the functions $\tilde{h}_j = \psi_{r_M} * \psi_{r_M} * h'_j$ belong to $\Lambda_\varphi(w)$, we would get the desired result, i. e.:

$$\lim_{i\to -M}\int(\psi_{r_i} * \psi_{r_i} * f)\eta = \sum_{j=0}^{n}\int \mathcal{R}_j\tilde{h}_j\eta \tag{3.27}$$

with $\tilde{h}_j \in \Lambda_\varphi(w)$. To do that, we first observe that $\phi_{r_M}(x) = (\psi_{r_M} * \psi_{r_M})(x) = r_M^{-n}(\mathcal{X}_{B_1} * \mathcal{X}_{B_1})(x/r_M)$ and that $\mathcal{X}_{B_1} * \mathcal{X}_{B_1}$ is a Lipschitz function supported in $B(0,3)$. Therefore ϕ_{r_M} is supported in $B(0,3r_M)$ and satisfies

$$|\phi_{r_M}(x)| \leq \frac{C}{r_M^n} \text{ and } |\phi_{r_M}(x) - \phi_{r_M}(y)| \leq \frac{C}{r_M^n}\frac{|x-y|}{r_M} \tag{3.28}$$

Now, for x and y such that $|x-y| < r_M$ we have

$$\begin{aligned}|\tilde{h}_j(x) - \tilde{h}_j(y)| &\leq \int |\phi_{r_M}(x-z) - \phi_{r_M}(y-z)||h'_j(z)|dz \\ &\leq C||h'_j||_{L^\infty(w)}\frac{|x-y|}{r_M}\frac{1}{r_M^n}\int_{B(x,3r_M)\bigcup B(y,3r_M)} w(z)dz \\ &\leq C\varphi(r_M)\frac{|x-y|}{r_M}(w(x)+w(y)) \\ &\leq C\varphi(|x-y|)(w(x)+w(y)),\end{aligned}$$

where in the last inequality we have used that $\varphi(t)/t$ is almost decreasing. Finally for x and y such that $|x-y| \geq r_M$ we have

$$\begin{aligned}|\tilde{h}_j(x) - \tilde{h}_j(y)| &\leq |\tilde{h}_j(x)| + |\tilde{h}_j(y)| \\ &\leq C||h'_j||_{L^\infty(w)}(w(x)+w(y)) \\ &\leq C\varphi(r_M)(w(x)+w(y)).\end{aligned}$$

In this way we proved $\tilde{h}_j \in \Lambda_\varphi(w)$.

The conclusion of the theorem follows now by (3.24), (3.25), (3.26) and (3.27).□

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