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LOCAL MAXIMAL FUNCTIONS AND OPERATORS ASSOCIATED TO LAGUERRE EXPANSIONS

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Abstract. In this paper we get sharp conditions on a weight v which allow us to obtain some weighted inequalities for a local Hardy-Littlewood Maximal operator defined on an open set in the Euclidean n-space. This result is applied to assure a pointwise convergence of the Laguerre heat-diffusion semigroup $u(x,t) = (T(t)f)(x)$ to f when t tends to zero for all functions f in $L^p(v(x)dx)$ for p greater than or equal to 1 and a weight v. In proving this we obtain weighted inequalities for the maximal operator associated to the Laguerre diffusion semigroup of the Laguerre differential operator of order greater than or equal to 0. Finally, as a by-product, we obtain weighted inequalities for the Riesz-Laguerre operators.

1. Introduction and main results. In this paper we study *weighted problem* for local maximal operators defined on an open set Ω in $\boldsymbol{R}^n$. The weighted problem is the following:

For a linear or sublinear operator T, find conditions on a weight v which assure the existence of a weight u such that T maps $L^p(\Omega, v(x)dx)$ into $L^p(\Omega, u(x)dx)$.

First, we establish the notation and the general background. Let us consider the setting of $\boldsymbol{R}^n$ with Lebesgue measure m. If A is a measurable set, we write $|A|$ for $m(A)$. The cube with *center* x and *semidiagonal* r will be denoted by $Q(x,r)$. In such a case, its *diameter* will be $\operatorname{diam}(Q(x,r)) = 2r$. We have $|Q(x,r)| = a_n r^n$, where $a_n := (2n^{-1/2})^n$. Let Ω be an open set in $\boldsymbol{R}^n$ and suppose the complementary set $F = \boldsymbol{R}^n \setminus \Omega$ is nonempty. For a positive constant $\gamma < 1/4$, consider the family of cubes

$$\mathcal{F}_\gamma = \left\{ Q(x,r);\, x \in \Omega, r > 0 \text{ with } \frac{\gamma \operatorname{dist}(Q(x,r), \partial F)}{r} > 1 \right\}, \tag{1}$$

where ∂F is the common boundary of F and Ω, and $\operatorname{dist}(Q, \partial F)$ denotes the distance between Q and ∂F. Now, we define the *local maximal Hardy-Littlewood function* $\mathcal{M}_{\mathcal{F}_\gamma}$ as

$$\mathcal{M}_{\mathcal{F}_\gamma} f(x) := \sup_{Q \in \mathcal{F}_\gamma, x \in Q} \frac{1}{|Q|} \int_Q |f(y)| dy \tag{2}$$

for any $f \in L^1_{\mathrm{loc}}(\Omega)$ and $x \in \Omega$. This local maximal operator generalizes that contained in [1] when $\Omega = (0,\infty)$ and $\kappa \le 3/2$, that is, $M^\kappa_{\mathrm{loc}} f(x)$ defined in (16). For the (global) Hardy-Littlewood maximal function M, some class of weights for the weighted problem were

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obtained by Carleson and Jones [3], Rubio de Francia [15] and Gatto and Gutiérrez [5], independently. It is worth mentioning that the characterization of the weights v such that the Hardy-Littlewood maximal function is bounded on $L^p(\boldsymbol{R}^n, v(x)dx)$, $1 < p < \infty$, as much as the weak-(1, 1) boundedness, was done by Muckenhoupt in the celebrated paper [13]. The problem of characterization of the pairs (u, v) for which the Hardy-Littlewood function maps $L^p(\boldsymbol{R}^n, v(x)dx)$ into $L^p(\boldsymbol{R}^n, u(x)dx)$ was solved by Sawyer in [16]. Now we can state the main theorem

THEOREM 1.1. *Let v be a weight in Ω. Let $1 \le p < \infty$. The following statements are equivalent:*

(i) *There exists a weight u such that $\mathcal{M}_{\mathcal{F}_\gamma} f(x)$ is bounded from $L^p(\Omega, v(x)dx)$ to $L^p(\Omega, u(x)dx)$ for $p > 1$, and from $L^1(\Omega, v(x)dx)$ to weak-$L^1(\Omega, u(x)dx)$ for $p = 1$.*

(ii) *There exists a weight u such that $\mathcal{M}_{\mathcal{F}_\gamma} f(x)$ is bounded from $L^p(\Omega, v(x)dx)$ to weak-$L^p(\Omega, u(x)dx)$.*

(iii) *$v^{-1/p} \in L^{p'}_{\mathrm{loc}}(\Omega)$ for $p > 1$, and $v^{-1} \in L^\infty_{\mathrm{loc}}(\Omega)$ for $p = 1$ (that is, v is locally bounded from below).*

This theorem is applied to study the convergence of the Laguerre Heat diffusion semigroup. More precisely, for $\alpha > -1$, let us consider the problem

$$\begin{cases} \dfrac{\partial u_\alpha}{\partial t} = -Lu_\alpha\,, \\ u_\alpha(x,0) = f(x) \qquad x > 0\,, \end{cases} \tag{3}$$

where $L = L_\alpha$ is the second order differential Laguerre operator defined by

$$L_\alpha = -x\frac{d^2}{dx^2} - \frac{d}{dx} + \frac{x}{4} + \frac{\alpha^2}{4x} \tag{4}$$

and, at least formally, $u_\alpha(x,t) = T_t f(x) = e^{-tL_\alpha} f(x)$ for $t > 0$.

Let $\mathcal{L}^\alpha_n(x)$ be the Laguerre function

$$\mathcal{L}^\alpha_n(x) = \left(\frac{n!}{n+\alpha+1}\right)^{1/2} L^\alpha_n(x) e^{-x/2} x^{\alpha/2}\,, \tag{5}$$

where $L^\alpha_n(x)$ is the Laguerre polynomial of degree n given by

$$L^\alpha_n(x) = \frac{1}{n!} x^{-\alpha} e^x \left(\frac{d^n}{dx^n}(x^{\alpha+n} e^{-x})\right), \quad x > 0\,, \quad n = 0, 1, 2, \ldots\,.$$

As is well known, the Laguerre functions $\mathcal{L}^\alpha_n(x)$ are eigenfuncions of the differential operator L_α with eigenvalues $\lambda_{\alpha,n} = -n - (\alpha+1)/2$. Also they form an orthogonal basis on $L^2(0, \infty)$. The operator L_α is positive and symmetric on $L^2(0, \infty)$.

One of the aim of this paper is to get conditions on a weight v such that

$$\lim_{t\to 0} T_t f(x) = f(x) \quad \text{for almost all } \ x > 0 \ \text{ and for all } \ f \in L^p((0,\infty), v(x)dx)\,. \tag{6}$$

This problem was studied in the classical case, that is when L is the Laplacian operator, in [10] and for $L = \frac{1}{2}(-\Delta + 2x \cdot \nabla)$, the Ohrstein-Uhlenbeck operator, in [8]. As is well known, the a.e. convergence leads to study the behaviour of the maximal operator associated to the heat difussion semigroup T_t of L_α defined by

$$W_\alpha^* f(x) := \sup_{t>0} |T_t f(x)| \qquad \text{for all } x > 0\,. \tag{7}$$

The maximal operator W_α^* was studied by Stempak in 1994 for $\alpha \geq 0$ (see [19]), and later on by Macías, Segovia and Torrea for negative values of α in [11] (unweighted case) and in [12] (with power weights). Recently the family of weights was enlarged in [2]. We solved this problem by using a non-constructive method due to J. L. Rubio de Francia. We state it in Theorem 2.5. By using Theorem 1.1 we can show the following.

THEOREM 1.2. *Let $1 \leq p < \infty$ and $\alpha \geq 0$. Assume that the weight v satisfies the condition*

$$\|v^{-1/p}(y)\,(\chi_{(0,1)}(y) + e^{-c_0 y}\, y^{\alpha/2}\chi_{[1,\infty)}(y))\|_{L^{p'}((0,\infty),dy)} = C_v < \infty\,, \tag{8}$$

where $p' = p/(p-1)$ and c_0 is any positive constant less than $1/4$. (The constant c_0 can be choosed equal to $1/16$.) Then there exists a weight u such that $W_\alpha^* f$ maps $L^p((0,\infty), v(x)dx)$ into $L^p((0,\infty), u(x)dx)$ for $p > 1$, and $L^1((0,\infty), v(x)dx)$ into weak-$L^1((0,\infty), u(x)dx)$ for $p = 1$.

As a consequence, the following result can be proved.

COROLLARY 1.3. *Let $1 \leq p < \infty$ and $\alpha \geq 0$. If v satisfies* (8), *then*

$$T_t f(x) \to f(x)\,, \quad \text{as } t \to 0^+\,,$$

for almost every $x \in (0,\infty)$ and every $f \in L^p((0,\infty), v(x)dx)$.

Finally, we apply Theorem 1.2 to get the weighted problem for the Riesz-Laguerre operators $\mathcal{R}_+^\alpha$, $\alpha \geq 0$ (see (46)).

THEOREM 1.4. *Given $\alpha \geq 0$ and $1 \leq p < \infty$, let v be a weight such that*

$$\|v^{-1/p}(y)\,(\chi_{(0,1)}(y) + e^{-c_0 y}\, y^{\alpha/2+1}\chi_{[1,\infty)}(y))\|_{L^{p'}((0,\infty),dy)} = C_v < \infty\,, \tag{9}$$

where c_0 is a constant less than $1/4$. (The constant c_0 can be choosed equal to $1/16$.) Then there exists a weight u such that $\mathcal{R}_+^\alpha f$ maps $L^p((0,\infty), v(x)dx)$ into $L^p((0,\infty), u(x)dx)$ for $p > 1$, and $L^1((0,\infty), v(x)dx)$ into weak-$L^1((0,\infty), u(x)dx)$ for $p = 1$.

The proof of Theorem 1.1 is contained in Section 2. In Section 3, the proof of Theorem 1.2 and its corollary is given. The proof of Theorem 1.4 is obtained in Section 4 .

2. Local maximal Hardy-Littlewood operators. We shall need the following Whitney type covering Lemma (see [18, p. 167] for details).

LEMMA 2.1. *Given a positive constant c, there exists a collection $\mathcal{W}_c$ of cubes, say $\mathcal{W}_c = \{Q_1, Q_2, \dots\}$, such that*

$$\bigcup_k Q_k = \Omega = \boldsymbol{R}^n \setminus F\,, \tag{10}$$

the interiors Q_k° are pairwise disjoint, and

$$c\,\mathrm{diam}(Q_k) \le \mathrm{dist}(Q_k, F) \le 2(1+c)\,\mathrm{diam}(Q_k)\,. \tag{12}$$

DEFINITION 2.2. For a given $p \ge 1$, we say that a non-negative weight w belongs to $L^p_{\mathrm{loc},\mathcal{F}_\gamma}$ if $\int_Q w^p \le C_{w,Q} < \infty$ for any cube $Q \in \mathcal{F}_\gamma$. Also, we understand that a non-negative weight w is in $L^p_{\mathrm{loc}}(\Omega)$ if $\int_E w^p < \infty$ for any bounded set E such that $\bar{E} \subset \Omega$.

REMARK 2.3. We note that a weight w is in $L^p_{\mathrm{loc}}(\Omega)$ if and only if $w \in L^p_{\mathrm{loc},\mathcal{F}_\gamma}$ for any $0 < \gamma < 1$. In fact, the inclusion $L^p_{\mathrm{loc}}(\Omega) \subset L^p_{\mathrm{loc},\mathcal{F}_\gamma}$ is clear. On the other hand, for a given bounded set E such that $\bar{E} \subset \Omega$, consider a Whitney type covering for Ω as in Lemma 2.1 above for $c = 2/\gamma$. Then there exists a positive integer N verifying

$$\bar{E} \subset \bigcup_{k=1}^N (2Q(x_k, r_k))^\circ = \bigcup_{k=1}^N (2Q_k)^\circ\,.$$

Here we write the cube $Q(x, 2r)$ by $2Q$ when $Q = Q(x, r)$. By (12) we have that

$$2r_k \le \frac{\gamma}{2}\,\mathrm{dist}(Q_k, F) \le \frac{\gamma}{2} 3r_k + \frac{\gamma}{2}\,\mathrm{dist}(2Q_k, F)\,. \tag{13}$$

Thus,

$$2r_k < \frac{\gamma}{2 - 3\gamma/2}\,\mathrm{dist}(2Q_k, F) < \gamma\,\mathrm{dist}(2Q_k, F)\,. \tag{14}$$

Hence $\tilde{Q}_k = 2Q_k \in \mathcal{F}_\gamma$, and consequently

$$\int_{\bar{E}} w^p \le \sum_{k=1}^N \int_{\tilde{Q}_k} w(x)^p\,dx < \infty\,, \tag{15}$$

which implies that $w \in L^p_{\mathrm{loc}}(\Omega)$.

The local maximal Hardy-Littlewood operator M^κ_{loc} is defined by

$$M^\kappa_{\mathrm{loc}} f(x) = \sup_{0<a<x<b<\kappa a} \frac{1}{b-a} \int_a^b |f(y)|\,dy\,, \quad \kappa = 16\,. \tag{16}$$

As a consequence of Theorem 1.1, for $\Omega = (0, \infty)$ we have the following result.

THEOREM 2.4. *For a given real number p, $1 \le p < \infty$, and $\kappa > 1$, the following statements are equivalent:*

(i) *The weight v satisfies*

$$v^{-1/p} \in L^{p'}_{\mathrm{loc}}(0,\infty) \quad \textit{for } p > 1\,, \quad \textit{and} \quad v^{-1} \in L^\infty_{\mathrm{loc}}(0,\infty) \textit{ for } p = 1\,. \tag{17}$$

(ii) *There exists a weight u such that $M^{\kappa}_{\mathrm{loc}} f(x)$ is bounded from $L^p((0,\infty), v(x)dx)$ to weak-$L^p((0,\infty), u(x)dx)$.*

(iii) *There exists a weight u such that $M^{\kappa}_{\mathrm{loc}} f(x)$ is bounded from $L^p((0,\infty), v(x)dx)$ to $L^p((0,\infty), u(x)dx)$ for $p > 1$ and from $L^1((0,\infty), v(x)dx)$ to weak-$L^1((0,\infty), u(x)dx)$ for $p = 1$.*

Before proving Theorems 1.1 and 2.4, we consider the following version of the Theorem of Rubio de Francia (see, for example, [6, p. 554]).

THEOREM 2.5. *Let (X, μ) a measurable space, $\boldsymbol{B}$ a Banach space and T a sublinear operator $T : \boldsymbol{B} \to L^q(X)$ for some $q < p$. Then the inequality*

$$\left\| \left(\sum_j |Tf_j|^p \right)^{1/p} \right\|_{L^q(X)} \le C \left(\sum_j \|f_j\|^p_{\boldsymbol{B}} \right)^{1/p} \tag{18}$$

holds if and only if there exists a non-negative weight u such that

$$\|Tf\|_{L^p(u)} \le C \|f\|_{\boldsymbol{B}} . \tag{19}$$

This theorem suggests the following vector-valued inequality.

PROPOSITION 2.6. *Let p be a real number with $1 < p < \infty$, and let $\{Q_k = Q(x_k, r_k)\}$ be a sequence of cubes satisfying the condition of Lemma 2.1 for $c = 1/\gamma$. If $v^{-1/p} \in L^{p'}_{\mathrm{loc}}(\Omega)$ and $0 < q < 1 < p$, then there exists a positive constant $C_{k,v}$ such that the following inequality holds.*

$$\left\| \left(\sum_j |\mathcal{M}_{\mathcal{F}_\gamma} f_j|^p \right)^{1/p} \right\|_{L^q(Q_k)} \le C_{k,v} \left(\sum_j \|f_j\|^p_{L^p(\Omega,v)} \right)^{1/p} . \tag{20}$$

PROOF. By replacing $\gamma/2$ by γ in (13), we clearly have $2r_k < 3\gamma r_k + \gamma \operatorname{dist}(2Q_k, F)$. Since $\gamma < 1/4$, we get that $\overline{2Q_k} \subset \Omega$. Now we can write

$$f_j = f_j \chi_{2Q_k} + f_j \chi_{(2Q_k)^c} =: f'_j + f''_j .$$

For each k, we first estimate $\mathcal{M}_{\mathcal{F}_\gamma} f'_j$ on the cube Q_k. Since $\mathcal{M}_{\mathcal{F}_\gamma}$ is of weak type (1, 1), by invoking the Theorem of Kolmogorov for $q < 1$ and applying the condition $v^{-1/p} \in L^{p'}_{\mathrm{loc}}(\Omega)$, we have

$$\begin{aligned}
\left\| \left(\sum_j |\mathcal{M}_{\mathcal{F}_\gamma} f'_j|^p \right)^{1/p} \right\|_{L^q(Q_k)} &\le C|Q_k|^{1/q-1} \left\| \left(\sum_j |\mathcal{M}_{\mathcal{F}_\gamma} f'_j|^p \right)^{1/p} \right\|_{\text{weak-}L^1(Q_k, dx)} \\
&\le C|Q_k|^{1/q-1} \int_{2Q_k} \left(\sum_j |f_j(x)|^p \right)^{1/p} v^{1/p}(x) v^{-1/p}(x) dx \\
&\le C|Q_k|^{1/q-1} \left(\sum_j \|f_j\|^p_{L^p(\Omega,v)} \right)^{1/p} \left(\int_{2Q_k} v^{-p'/p}(x) dx \right)^{1/p'} .
\end{aligned}$$

Therefore,

$$\left\| \left(\sum_j |\mathcal{M}_{\mathcal{F}_\gamma} f_j'|^p \right)^{1/p} \right\|_{L^q(Q_k)} \le C_{k,v} |Q_k|^{1/q-1} \left(\sum_j \|f_j\|^p_{L^p(\Omega,v)} \right)^{1/p} . \tag{21}$$

Now we estimate the term dealing with f_j''. Suppose that $x \in Q_k$. We have to analyze the expression

$$\mathcal{M}_{\mathcal{F}_\gamma} f''(x) = \sup_{Q \in \mathcal{F}_\gamma, Q \ni x} \frac{1}{|Q|} \int_{Q \cap (2Q_k)^c} |f(y)| dy .$$

Let $Q = Q(x_0, r)$ be a cube in $\mathcal{F}_\gamma$. Consider a point $y \in Q \cap (2Q_k)^c$. Since $x \in Q_k$, we have

$$2r_k \le |y - x_k| \le |y - x| + |x - x_k| \le 2r + r_k . \tag{22}$$

Hence, we must have $r_k \le 2r$, and hence $|2Q| \ge |Q_k|$. From that, we obtain

$$\mathcal{M}_{\mathcal{F}_\gamma} f''(x) \le \frac{2^n}{|Q_k|} \sup_{Q \in \mathcal{F}_\gamma, Q \ni x} \int_{Q \cap (2Q_k)^c} |f(y)| dy . \tag{23}$$

Now, we look for a ball $B_k = B(x_k, R_k)$ such that its center x_k is the same as that of Q_k, $Q \subset B_k$ and $\overline{B_k} \subset \Omega$. In view of this, for $y \in Q$, we get from (22) that

$$\begin{aligned} |y - x_k| &\le 2r + r_k \\ &\le 2\gamma d(x, \partial F) + r_k \\ &\le 2\gamma |x - x_k| + 2\gamma d(x_k, \partial F) + r_k \\ &\le (1 + 2\gamma) r_k + 2\gamma d(x_k, \partial F) \\ &=: R_k . \end{aligned}$$

We claim that this value R_k fulfills our goals. Indeed, the calculations above give us $Q \subset B_k$. Also, since $\gamma < 1/4$, by using (12), we get

$$R_k < \left(\frac{1 + 2\gamma}{2} + 2 \right) \gamma \; d(x_k, \partial F) < \frac{11}{16} d(x_k, \partial F) .$$

Then, for $x \in Q_k$, it follows that

$$\bigcup \{Q;\, Q \in \mathcal{F}_\gamma, x \in Q\} \subset \overline{B_k} \subset \Omega .$$

Now, from (23) and the fact that $v^{-1/p} \in L^{p'}_{\mathrm{loc}}(\Omega)$, we can write

$$\begin{aligned} \mathcal{M}_{\mathcal{F}_\gamma} f_j''(x) &\le \frac{2^n}{|Q_k|} \int_{B_k} |f_j| v^{1/p} v^{-1/p} \\ &\le \frac{2^n}{|Q_k|} \left(\int_{B_k} v^{-p'/p}(y) dy \right)^{1/p'} \|f_j\|_{L^p(\Omega,v)} \\ &=: C_{k,v} |Q_k|^{-1} \|f_j\|_{L^p(\Omega,v)} . \end{aligned} \tag{24}$$

Then, it follows

$$\Big(\int_{Q_k}\Big(\sum_j \big|\mathcal{M}_{\mathcal{F}_\gamma} f_j''(x)\big|^p\Big)^{q/p}\Big)^{1/q} \le C_{k,v}|Q_k|^{1/q-1}\Big(\sum_j \|f_j\|^p_{L^p(\Omega,v)}\Big)^{1/p}. \tag{25}$$

Putting (21) and (25) together, we can finally obtain the inequality (20), and the proof is finished. □

REMARK 2.7. We note that Proposition 2.6 holds for the local maximal Hardy-Littlewood operator defined by (16) for any $\kappa > 1$. In fact, we take an interval $Q_k = Q(x_k, r_k)$ verifying Lemma 2.1 in such a way that $\overline{2Q_k} \subset \Omega$ (it is enough to take $c > 3/2$). Then (21) holds for $M^\kappa_{\mathrm{loc}} f_j'(x)$. In order to estimate the expression

$$M^\kappa_{\mathrm{loc}} f_j''(x) = \sup_{0<a<x<b<\kappa a} \frac{1}{b-a}\int_{(a,b)\cap(2Q_k)^c} |f(y)|dy\,,$$

by (22), we clearly have

$$M^\kappa_{\mathrm{loc}} f_j''(x) \le \frac{2^n}{|Q_k|}\sup_{0<a<x<b<\kappa a}\int_{(a,b)\cap(2Q_k)^c} |f(y)|dy\,. \tag{26}$$

But, for $x \in Q_k$,

$$\bigcup\{(a,b);\, a<x<b<\kappa a\} \subset (x/\kappa, \kappa x) \subset \left[\frac{x_k - r_k}{\kappa}, \kappa(x_k + r_k)\right] \subset (0,\infty)\,.$$

Therefore, (25) follows for $M^\kappa_{\mathrm{loc}} f_j''(x)$.

PROOF OF THEOREM 1.1. First, let us suppose that $v^{-1/p} \in L^{p'}_{\mathrm{loc}}(\Omega)$. By applying Proposition 2.6 above for $0 < q < 1 < p$, we have that the operator $\mathcal{M}_{\mathcal{F}_\gamma}$ satisfies the condition of Theorem 2.5 in each set Q_k. Hence a family of weights u_k supported in Q_k satisfying (19) in Theorem 2.5 can be found. Thus the weight defined by

$$u(x) := \sum_k 2^{-kp} c_k^{-p} u_k(x)\chi_{Q_k}(x)$$

satisfies (i) as we wanted.

For the case $p = 1$, we use the weak-$(1,1)$ continuity of the Hardy-Littlewood maximal operator, and then

$$\begin{aligned}
|\{x \in Q_k;\, \mathcal{M}_{\mathcal{F}_\gamma} f'(x) > \lambda\}| &\le \frac{C}{\lambda}\|f'\|_{L^1} = \frac{C}{\lambda}\int_{2Q_k}|f(x)|dx \\
&\le \frac{C}{\lambda}\Big(\int_{2Q_k}|f(x)|v(x)dx\Big)\Big\|v^{-1}(\cdot)\chi_{2Q_k}(\cdot)\Big\|_{L^\infty(\boldsymbol{R}^n,dx)} \\
&\le \frac{C_{k,v}}{\lambda}\int |f(x)|v(x)dx\,.
\end{aligned} \tag{27}$$

Pasting (24) and (27) together, we get

$$|\{x \in Q_k;\, \mathcal{M}_{\mathcal{F}_\gamma} f(x) > \lambda\}| \le \frac{C_{k,v}}{\lambda}\int |f(x)|v(x)dx\,. \tag{28}$$

Hence the weight $u(x) = \sum_k 1/2^k C_{k,v} \chi_{Q_k}(x)$ gives (ii)

Now, suppose that (ii) holds. Let $Q(x_0, R)$ a fixed cube such that $4Q(x_0, R) \in \mathcal{F}_\gamma$. Since then $Q(x_0, R) \subset Q(x, 2R)$ for $x \in Q(x_0, R)$ and $Q(x, 2R) \in \mathcal{F}_\gamma$, for any nonnegative f, we have

$$\mathcal{M}_{\mathcal{F}_\gamma} f(x) \geq \frac{1}{R^n} \int_{Q(x,2R)} f(y)dy > \frac{1}{R^n} \int_{Q(x_0,R)} f(y)dy,\ x \in Q(x_0, R)\,.$$

Therefore, by (ii) and the Kolmogorov inequality, we have for $s < p$,

$$\begin{aligned}\left(\frac{1}{R^n} \int_{Q(x_0,R)} f(y)dy\right) \big(u(Q(x_0, R))\big)^{1/s} &\\ \leq C\left(\int_{Q(x_0,R)} \mathcal{M}_{\mathcal{F}_\gamma} f(x)^s u(x)dx\right)^{1/s} &\\ \leq Cu(Q(x_0, R))^{1/s-1/p}\left(\int f^p(y)v(y)dy\right)^{1/p}.&\end{aligned}$$

Given an arbitrary positive function $g \in L^p(Q(x_0, R))$, we choose $f = gv^{-1/p}$, then we have

$$\left(\frac{1}{R^n} \int_{Q(x_0,R)} g(y)v^{-1/p}(y)dy\right) \leq Cu(Q(x_0, R))^{-1/p}\left(\int g^p(y)dy\right)^{1/p}.$$

By duality we conclude that $v^{-1/p}$ belongs to $L^{p'}(Q(x_0, R), dx)$. Since any cube $Q \in \mathcal{F}_\gamma$ can be covered by a finite family of cubes Q_i with the property that $4Q_i \in \mathcal{F}_\gamma$, the statement (iii) follows. □

PROOF OF THEOREM 2.4. We first note that the case $\kappa \leq 3/2$ is contained in Theorem 1.1. For any $\kappa > 1$, assume first that (iii) holds. Then, the statement (i) follows as in Theorem 1.1, in view of Remark 2.7. To prove the implication (ii) $\Rightarrow$ (iii), we proceed as in the proof of Theorem 1.1, but taking this time an interval $Q \subset \Omega$ such that $4Q \in \mathcal{F}_\kappa$, where

$$\mathcal{F}_\kappa = \{Q = (a, b) \subset \Omega;\, a < b < \kappa a\}\,.$$

□

3. The heat diffusion Laguerre operators. For the heat diffusion semigroup T_t, we recall the well-known expression

$$T_t f(x) = \int K_\alpha(t, x, y) f(y)\, dy\,,$$

where

$$K_\alpha(t, x, y) = W_\alpha\left(\frac{1 - e^{t/2}}{1 + e^{t/2}}, x, y\right)$$

with

$$W_\alpha(s, x, y) = \frac{1 - s^2}{4s} e^{-(s+1/s)(x^{1/2}-y^{1/2})^2/4} e^{-(s+1/s)(xy)^{1/2}/2} I_\alpha\left(\frac{1 - s^2}{2s}(xy)^{1/2}\right) \tag{29}$$

after the change of parameters $s = (1 - e^{-t/2})/(1 + e^{-t/2})$. Here, I_α is the modified Bessel function of order α, any $\alpha > -1$. In this way, we have

$$W_\alpha^* f(x) = \sup_{0<s<1} \int W_\alpha(s, x, y) f(y)\, dy\,.$$

The proof of Theorem 1.2 is based on the pointwise estimate (see [2])

$$W_\alpha^* f(x) \le (H_0^{\alpha/2} + M_{\mathrm{loc}}^{\kappa} + T^{\alpha/2}) f(x)\,, \tag{30}$$

where $H_0^{\alpha/2}$ is a Hardy operator modified at the origin

$$H_0^{\alpha/2} f(x) := x^{-\alpha/2-1} \int_0^x f(y) y^{\alpha/2}\, dy, \quad \alpha/2 > -1\,, \tag{31}$$

and $T^{\alpha/2}$ is the maximal operator

$$T^{\alpha/2} f(x) = \sup_{0<s<1} |T_s^{\alpha/2} f(x)|\,,$$

with $T_s^{\alpha/2}$ the operator given by

$$\begin{aligned} T_s^{\alpha/2} f(x) &= x^{\alpha/2} \int_x^\infty \left[\left(\frac{y}{s}\right)^{\alpha+1} e^{-c_0 y/s} \right] y^{-\alpha/2-1} f(y)\, dy \\ &=: x^{\alpha/2} \int_x^\infty \phi(s, y)\, y^{-\alpha/2-1} f(y)\, dy\,. \end{aligned} \tag{32}$$

Here c_0 is equal to $1/16$. Besides, we have implicitly defined

$$\phi(s, y) = \left(\frac{y}{s}\right)^{\alpha+1} e^{-c_0 y/s}\,. \tag{33}$$

REMARK 3.1. We note that the constant c_0 is not optimal. This depends on the local region considered. It is not hard to check that c_0 tends to $1/4$ when κ tends to ∞. The constant $c_0 = 1/16$ is obtained by considering in [2, pp. 343] the local region defined by

$$\frac{x}{4} < y < 4x\,, \tag{34}$$

where $\kappa = 16$. In this way, for the global case at infinity $4x < y < \infty$, we have

$$e^{-|x^{1/2} - y^{1/2}|^2/4s} \le e^{-y/16s} \equiv e^{-c_0 y/s}\,, \tag{35}$$

whose right-hand side appeared in (32).

In view of (30), we devote the rest of this section to solve the *weighted problem* for the operators $H_0^\alpha f$ and $T^{\alpha/2} f$. Since the operators $T^{\alpha/2}$, $\alpha \ge 0$, are pointwise bounded by Hardy type modified operators at the infinity, they are of weak type $(1, 1)$ (see [1]). Therefore, we can proceed in a similar way as in Proposition 2.6, obtaining the following Proposition.

PROPOSITION 3.2. *Let $1 \le p < \infty$ and $\alpha \ge 0$. Assume that the weight v satisfies*

$$\| v^{-1/p}(y)\, (\chi_{(0,1)}(y) + e^{-c_0 y} y^{\alpha/2} \chi_{[1,\infty)}(y)) \|_{L^{p'}((0,\infty), dy)} = C_v < \infty\,. \tag{36}$$

Then, for $0 < q < 1 < p$,

$$\left\| \left(\sum_j |T^{\alpha/2} f_j|^p(\cdot) \right)^{1/p} \right\|_{L^q((0,\infty)dx)} \leq C \left(\sum_j \|f_j\|^p_{L^p((0,\infty),v(x)dx)} \right)^{1/p}.$$

Moreover, there exists a weight u *such that* $T^{\alpha/2} f(x)$ *is bounded from* $L^p((0,\infty), v(x)dx)$ *to* $L^p((0,\infty), u(x)dx)$ *for* $p > 1$, *and from* $L^1((0,\infty), v(x)dx)$ *to weak-*$L^1((0,\infty), u(x)dx)$ *for* $p = 1$.

PROOF. Before starting, we set the constant

$$A := \frac{\alpha + 1}{c_0}.$$

Let $\{Q_k\}_{k\geq 1}$ be the sequence of intervals $Q_k = (A^k, A^{k+1}]$ for $k \geq 1$, and denote $Q_0 = (0, A]$. Also, let us write

$$f_j = f_j \chi_{(0,A^{k+1}]} + f_j \chi_{(A^{k+1},\infty)} =: f_j' + f_j''.$$

Proceeding as in Proposition 2.6, we can see that

$$\begin{aligned} &\left\| \left(\sum_j |T^{\alpha/2} f_j'|^p \right)^{1/p} \right\|_{L^q(Q_k)} \\ &\leq C|Q_k|^{1/q-1} \left(\sum_j \|f_j\|^p_{L^p((0,\infty),v)} \right)^{1/p} \left(\int_{(0,A^{k+1}]} v^{-p'/p}(x)dx \right)^{1/p'} \qquad (37) \\ &\leq C'_{k,v} |Q_k|^{1/q} \left(\sum_j \|f_j\|^p_{L^p((0,\infty),v)} \right)^{1/p}, \end{aligned}$$

where

$$C'_{k,v} := |Q_k|^{-1} \left(\int_{(0,A^{k+1}]} v^{-p'/p}(x)dx \right)^{1/p'}.$$

Now, we analyze $T^{\alpha/2} f_j''(x)$ for $x \in Q_k$. The function $\phi(s, y)$ (see equation (33)) is increasing in the variable s whenever $y \geq A$; therefore, it attains its maximum at $s = 1$. Finally, with the aid of Hölder inequality, we have

$$\begin{aligned} T^{\alpha/2} f_j''(x) &\leq \sup_{0<s<1} A^{(k+1)\alpha/2} \int_{A^{k+1}}^{\infty} \phi(s, y) y^{-\alpha/2-1} f_j(y)dy \qquad (38) \\ &\leq C''_{k,v} \|f\|_{L^p((0,\infty),v(x)dx)}, \end{aligned}$$

where

$$C''_{k,v} \doteq \left(A^{p'(k+1)\alpha/2} \int_1^{\infty} e^{-p'c_0 y} y^{p'\alpha/2} v^{-p'/p}(y)\, dy \right)^{1/p'} < \infty. \qquad (39)$$

By denoting $C_{k,v} = C'_{k,v} + C''_{k,v}$, we clearly have

$$C_{k,v} \leq C_k \left(\int_0^{\infty} v^{-p'/p}(y) \left(\chi_{(0,1)} + e^{-p'c_0 y} y^{p\alpha/2} \chi_{[1,\infty)}(y) \right) dy \right)^{1/p'}.$$

Putting (37) and (38) together, we see that the operator satisfies the condition of Theorem 2.5 on each set Q_k. Therefore, as before, a family of weights u_k supported in Q_k satisfying the statement (19) can be found. Finally, we define

$$u(x) = \sum_{k=0}^{\infty} 2^{-kp} c_k^{-p} u_k(x) \chi_{Q_k}(x)\,,$$

which complete the proof when $1 < p$. For the case $p = 1$, using the weak-$(1, 1)$ continuity of the operator $T^{\alpha/2} f(x)$ and proceeding as the corresponding bound of $\mathcal{M}_{\mathcal{F}_\gamma}$, we have

$$\begin{aligned} |\{x \in Q_k;\, T^{\alpha/2} f'(x) > \lambda\}| &\le \frac{C}{\lambda} \|f'\|_{L^1} = \frac{C}{\lambda} \int_{2Q_k} |f(x)| dx \\ &\le \frac{C}{\lambda} \left(\int_{2Q_k} |f(x)| v(x)\, dx \right) \left\| v^{-1}(\cdot) \chi_{2Q_k}(\cdot) \right\|_{L^\infty(\boldsymbol{R}^n, dx)} \qquad (40) \\ &\le \frac{C'_{k,v}}{\lambda} \int |f(x)| v(x) dx\,. \end{aligned}$$

From(40) and (38), by taking

$$C''_{k,v} \doteq A^{(k+1)\alpha/2} \| e^{-c_0 y} y^{\alpha/2} v^{-1/p}(y) \chi_{[1,\infty)}(y) \|_{L^\infty((0,\infty), dy)} \quad \text{and } C_{k,v} = C'_{k,v} + C''_{k,v}\,,$$

we get

$$|\{x \in Q_k;\, T^{\alpha/2} f(x) > \lambda\}| \le \frac{C_{k,v}}{\lambda} \int |f(x)|\, v(x) dx\,. \qquad (41)$$

Hence the weight $u(x) = \sum_k 2^{-kp} C_{k,v}^{-1} \chi_{Q_k}(x)$ gives the conclusion.

□

Now, pay attention to the Hardy operators. Since $H_0^{\alpha/2}$ has weak type $(1, 1)$ whenever $\alpha \ge 0$ (see [1]), we can apply similar procedures as for $T^{\alpha/2}$. Again, for $Q_k = (A^k, A^{k+1}]$ we take $f_j =: f'_j + f''_j$, with f'_j supported in $(0, A^{k+1}]$ and f''_j supported in (A^{k+1}, ∞). However, observe that $H_0^{\alpha/2} f''_j(x) = 0$ because $f''_j(x)$ is zero in the domain of integration in (31). For that reason, it is enough to look only through the term $H_0^{\alpha/2} f'_j(x)$. But, since $H_0^{\alpha/2}$ has weak type $(1, 1)$, we can follow the same lines as before, obtaining the next result.

PROPOSITION 3.3. *Let $1 \le p < \infty$ and $\alpha \ge 0$. Let v be a weight satisfying*

$$v^{-1/p} \in L^{p'}_{\mathrm{loc}}(0, \infty)\,. \qquad (42)$$

If $0 < q < 1 < p$, then

$$\left\| \left(\sum_j |H_0^{\alpha/2} f_j|^p(\cdot) \right)^{1/p} \right\|_{L^q((0,\infty), dx)} \le C \left(\sum_j \|f_j\|^p_{L^p((0,\infty), v)} \right)^{1/p}.$$

Moreover, there exists a weight u such that $H_0^{\alpha/2} f(x)$ is bounded from $L^p((0, \infty), v(x)dx)$ to $L^p((0, \infty), u(x)dx)$ for $p > 1$, and from $L^1((0, \infty), v(x)dx)$ to weak-$L^1((0, \infty), u(x)dx)$ for $p = 1$.

PROOF OF THEOREM 1.2. By taking $\alpha \geq 0$, applying (30), Theorem 2.4 and Propositions 3.2 and 3.3, we prove Theorem 1.2, as desired. □

PROOF OF COROLLARY 1.3. Since the family $\{e^{-tL_\alpha}\}$ is a symmetric diffusion semigroup for $\alpha \geq 0$, we deduce (see [17])

$$T_t f(x) \to f(x), \quad \text{as } t \to 0,$$

for all $f \in L^2(0,\infty)$, in particular for any f in the set $\mathcal{C}_0$ consisting of all continuous functions on $(0,\infty)$ with compact support, which is dense in $L^p(v)$ for all $1 \leq p < \infty$. Finally, by using Theorem 1.2, we have that $T_t f(x) \to f(x)$ as $t \to 0$, almost every $x \in (0,\infty)$, for all $f \in L^p((0,\infty), v)$ with $1 \leq p < \infty$. □

4. Riesz-Laguerre operators. In this Section we study the *weighted problem* for the Riesz-Laguerre operators $\mathcal{R}^\alpha$, $\alpha \geq 0$. Their properties are studied in [7] and [9]. For the self-adjoint differential operator L_α, we shall consider the Riesz-potentials (see [20])

$$L^{-\sigma} f(x) = \frac{1}{\Gamma(\sigma)} \int_0^\infty T_t f(x) t^{\sigma-1} dt, \ \sigma > 0,$$

where $T_t = e^{-tL_\alpha}$ is the heat diffusion semigroup. In order to define the corresponding Riesz transforms, the following first order derivatives can be considered (see [9]).

$$\delta^\alpha = \sqrt{x}\frac{d}{dx} + \frac{1}{2}\left(\sqrt{x} - \frac{\alpha}{\sqrt{x}}\right) \quad \text{and} \quad \partial^{\alpha+1} = -\sqrt{x}\frac{d}{dx} + \frac{1}{2}\left(\sqrt{x} - \frac{\alpha+1}{\sqrt{x}}\right),$$

which satisfies

$$L_\alpha - \left(\frac{\alpha+1}{2}\right) = (\delta_y^\alpha)^* \delta_y^\alpha = \partial_y^{\alpha+1} \delta_y^\alpha, \quad L_\beta - \left(\frac{\beta-1}{2}\right) = (\partial_y^\beta)^* \partial_y^\beta = \delta_y^{\beta-1} \partial_y^\beta. \tag{43}$$

Hence the Riesz transforms for the Laguerre function expansions are defined by

$$\mathcal{R}_+^\alpha = \delta_y^\alpha (L_\alpha)^{-1/2}, \quad \alpha > -1, \quad \text{and} \quad \mathcal{R}_-^\beta = \partial_y^\beta (L_\beta)^{-1/2}, \quad \beta > 0. \tag{44}$$

For $f, g \in C_c^\infty(0,\infty)$ with disjoint supports, the following equality holds

$$\langle \mathcal{R}_{(+,-)}^\alpha f, g\rangle = \int_{(0,\infty)} \int_{(0,\infty)} \mathcal{R}_{(+,-)}^\alpha(x,z) f(x) g(z) dx dz. \tag{45}$$

We shall only consider the weighted problem for the Riesz-Laguerre transform R_+^α. To get the kernel of R_+^α, we write

$$\begin{aligned}
\mathcal{R}_+^\alpha(x,z) &= C\left(\sqrt{x}\frac{d}{dx} + \frac{1}{2}\left(\sqrt{x} - \frac{\alpha}{\sqrt{x}}\right)\right)(L^\alpha)^{-1/2}(x,z) \\
&= C\int_0^1 \left(\frac{1}{2}\frac{1-s^2}{2s} z^{1/2} W_{\alpha+1}(s,x,z) - \frac{1}{2}\frac{(1-s)^2}{2s} x^{1/2} W_\alpha(s,x,z)\right) \\
&\qquad \times \left(\log\frac{1+s}{1-s}\right)^{-1/2} \frac{ds}{1-s^2} \\
&= C\int_0^1 R_\alpha(x,z,s)\left(\log\frac{1+s}{1-s}\right)^{-1/2} \frac{ds}{1-s^2},
\end{aligned} \tag{46}$$

where $s = (1 - e^{-t/2})/(1 + e^{-t/2})$ and $W_\alpha(s, x, z)$ is the heat-diffusion kernel given by (29).

PROOF OF THEOREM 1.4. We will use the decomposition $\mathcal{R}_+^\alpha =: \mathcal{R}_{+,\mathrm{loc}}^\alpha + \mathcal{R}_{+,\mathrm{glob}}^\alpha$, where

$$\mathcal{R}_{+,\mathrm{glob}}^\alpha f(x) := \int_0^\infty \mathcal{R}_+^\alpha(x, z)\chi_{\{z\le x/4\}\cup\{z\ge 4x\}} f(z)dz\,.$$

It is shown in [7] that $\mathcal{R}_{+,\mathrm{loc}}^\alpha$ is a local Calderón-Zygmund operator for $\alpha > -1$. Consequently, as is proved in [14], $\mathcal{R}_{+,\mathrm{loc}}^\alpha$ is strong type (p, p) for any $1 < p < \infty$ and weak type $(1, 1)$. Now, for a function f in $L^p((0,\infty), v(x)dx)$, $p > 1$, we can write $f =: f_k' + f_k''$, by setting $f_k' = f\chi_{[0,2^{k+3})}$. First, in the case that $f \in C_c^\infty(0,\infty)$ and $x \in Q_k = (2^k, 2^{k+1}]$, we have

$$\mathcal{R}_{+,\mathrm{loc}}^\alpha f_k''(x) = \int_{x/4}^{4x} \mathcal{R}^\alpha(x, z) f_k''(z)dz = 0\,. \tag{47}$$

For $\mathcal{R}_{+,\mathrm{loc}}^\alpha f_k'$ we can proceed in the same way as in Proposition 3.2. Applying the Theorem of Rubio de Francia and the weak type (1,1) of $\mathcal{R}_{+,\mathrm{loc}}^\alpha$, we can conclude that

$$\|\mathcal{R}_{+,\mathrm{loc}}^\alpha f\|_{L^p((0,\infty),u(x)dx)} \le C\|f\|_{L^p((0,\infty),v(x)dx)}$$

for all weight v satisfying $v^{-1/p} \in L_{\mathrm{loc}}^{p'}(0,\infty)$. Now we turn to $\mathcal{R}_{+,\mathrm{glob}}^\alpha$. By the estimate obtained in [7, pp. 16–17], for $f \ge 0$, we have

$$\mathcal{R}_{+,\mathrm{glob}}^\alpha f(x) \le Cx^{-1/2}\tilde{W}_{\alpha+1,\mathrm{glob}}^*(z^{1/2}f)(x) + C\tilde{W}_{\alpha,\mathrm{glob}}^* f(x) =: J_1 + J_2\,. \tag{48}$$

Here, $\tilde{W}_{\alpha,\mathrm{glob}}^*$ is defined by

$$\tilde{W}_{\alpha,\mathrm{glob}}^* f(x) := \sup_{0<s\le 1}\int \tilde{W}_\alpha(s, x, z) f(z)dz\,,$$

where $\widetilde{W}_\alpha^*$ is a minor modification of W_α^*, the maximal operator. More precisely,

$$\widetilde{W}_\alpha^* f(x) = \sup_{0<s\le 1}\int \widetilde{W}_\alpha(x, z, s) f(z)dy\,,$$

where $\widetilde{W}_\alpha = \widetilde{M} I_\alpha$ with

$$\widetilde{M} = \frac{1}{2}\frac{1-s^2}{2s} e^{-(s+1/s)|x^{1/2}-z^{1/2}|^2/\mathbf{8}} e^{-(s+1/s)(xz)^{1/2}/2}\,.$$

Therefore, the only difference with W_α is the $1/8$ in the first exponential instead of $1/4$. We refer to [2] in order to check that such a change does not affect the estimate obtained there

$$J_2 := \sup_{0<s<1}\int \widetilde{W}_\alpha(s, x, z)\chi_G(z) f(z)\,dz \le C\Big(H_0^{\alpha/2} f(x) + \sup_{0<s<1} |T_s^{\alpha/2} f(x)|\Big)$$

for all $0 < s < 1$, where $\chi_G(z) = \chi_{\{z\le x/4\}\cup\{z\ge 4x\}}(z)$ and $\sup_{0<s<1} |T_s^{\alpha/2} f(x)|$ is like in (32) with c_0 replaced by $c_0/2$. For J_1, we clearly have

$$J_1 = Cx^{-1/2}\sup_{0<s\le 1}\int \widetilde{W}_{\alpha+1}(s, x, z)\chi_G(z) z^{1/2} f(z)\,dz \le H_0^{(\alpha+2)/2} f(x) + \tilde{T}^{\alpha/2} f(x)$$

where

$$\tilde{T}^{\alpha/2} f(x) := \sup_{0<s\leq 1} x^{\alpha/2} \int_{4x}^{\infty} z^{-\alpha/2-1} \tilde{\phi}(s,z) f(z)\, dz$$

with

$$\tilde{\phi}(s,z) = \left(\frac{z}{s}\right)^{\alpha+2} e^{-(c_0/2)\, z/s}. \tag{49}$$

The function $\tilde{\phi}(s, y)$ satisfies the properties (3.1), (3.2) and (3.3) listed in [2, p. 336]. So the operator $\tilde{T}^{\alpha/2}$ is of weak type (1, 1) and strong type (p, p) for any $1 < p < \infty$. Summing up, the inequality

$$\mathcal{R}^{\alpha}_{+,\mathrm{glob}} f(x) \leq C \left(H_0^{(\alpha+1)/2} + \tilde{T}^{\alpha/2} + H_0^{\alpha/2} + T^{\alpha/2}\right) f(x)$$

holds for $f \geq 0$ and for all $x > 0$.

Now, by applying Proposition 3.2, but this time taking into account the particular expression of the kernel $\tilde{\phi}(s, y)$, and Proposition 3.3, we get the conclusion of Theorem 1.4. □

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