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Brief paper

# Observability criteria for impulsive control systems with applications to biomedical engineering processes ☆

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ABSTRACT

One of the fundamental properties of the impulsive systems is analyzed: observability. Algebraic criteria for testing this property are obtained for the nonlinear case, considering continuous and discrete outputs. For this class of systems, observability is explored not only through time derivatives of the output, but also considering few discrete measurements at different time-instants. In this context, it is shown that nonlinear impulsive control systems can be strongly observable or observable over a finite time interval. A new rank condition based on the structure of the impulses is found to characterize observability of linear impulsive systems. It generalizes the celebrated Kalman criterion, for both kind of outputs, discrete and continuous. Finally, these results are tested and illustrated both on academic examples and on two impulsive dynamical models from biomedical engineering science.



## 1. Introduction

Impulsive control systems (ICS) are encountered in various areas as biology, health, robotics and others. For instance, a diabetic type I patient model will be shortly considered herein, for which new specific mathematical tools are needed for analysis, observation and control. Glycemia regulation is performed in real life by appropriate insulin injections and eventually compensatory snacks, to maintain glucose levels within the predefined target range. These inputs can be approximated as impulses whenever insulin bolus is injected, and are adjusted based on discrete glycemia measurements from blood samples taken at various times during the day (Huang, Li, Song, & Guo, 2012). The intake of 'meals' will affect the level of glucose of the patient, and therefore is considered as an impulse disturbance. In this context, ICS seems the appropriate tool to analyze its dynamics.

Another interesting example of ICS is the model of the dynamics of the human immunodeficiency virus (HIV), initially described in Perelson, Kirschner, and Boer (1993). The intake of drugs once or twice a day can be interpreted as an impulse input (Bellman, 1971), with a fixed time interval. Besides, the measurement of its outputs are far from being continuous since blood samples are taken at most every three or six months. In this framework, ICS is a more pragmatic point of view. The accessibility of this ICS was explored in Rivadeneira and Moog (2012).

More generally, impulsive control systems define a class of systems whose state trajectories are piecewise continuous, with discontinuities of the first kind or 'jumps' at some discrete time instants. The dynamics is modeled by algebraic discrete equations or by introducing impulses into the differential equations.

Observability in linear ICS has been investigated in Guan, Qian, and Yu (2002), Medina and Lawrence (2008), Shi and Xie (2012) and Xie and Wang (2005). The definition used therein establishes that observability depends on measurements of the output on a finite-time interval $[0, t_f]$. When a continuous output is considered, the most popular tool remains a Kalman type observability matrix $\mathcal{O}$ (Guan et al., 2002; Xie & Wang, 2005; Zhao & Sun, 2009), but with a very restrictive assumption over the class of considered impulsive systems. Discontinuities in the state of the form $x(\tau_k^+) = A_I x(\tau_k)$ are allowed, where $A_I$ defines a diagonal matrix. A different class of impulsive control systems is considered in Medina and Lawrence (2009), for which the states evolve in continuous form but the output is available for measurement at discrete times only. Suitable criteria based on geometric properties of the invariant observable space and the observability Gramian were worked out for this case.

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In this paper and for the first-time ever, observability is investigated in nonlinear ICS. The dual property of accessibility of nonlinear ICS was characterized in Rivadeneira and Moog (2012) and the basis of the impulsive exact linearization was stated.

The results of the paper are in threefold: (i) The sufficient and necessary conditions are provided to test observability for linear systems with discrete time outputs. This condition has to be viewed as an extended Kalman criterion for observability. Also, the equivalence between the algebraic condition and the observability Gramian is detailed. (ii) The nonlinear case with continuous output and discrete measurements is tackled. Two definitions of observability are introduced with their respective criterion. Strong observability and observability over a finite time interval, are more natural for the latter nonlinear ICS. (iii) Observability is tested on two important models borrowed from biomedical engineering science: HIV and diabetic type I patient models. A brief description of the glycemia dynamical model is given for diabetic patients in the framework of ICS.

## 2. Preliminaries

A plant is an impulsive control system when there is a set of time instants $T = \{\tau_k\}$, $\tau_k \in \mathbb{R}, \tau_k < \tau_{k+1} < \infty$, and a set of inputs $U_k \in \mathbb{R}^n$, $k = 1, 2, \ldots$, such that the state $x \in \mathbb{R}^n$ is discontinuous at each $\tau_k$ according to $x(\tau_k^+) = f_I(x(\tau_k)) + U(k, x)$. Note that the control instants are not necessarily equidistant, the control $U(k, x)$ yields a discontinuity of $x$ at instant $\tau_k$, the function $f_I(x)$ defines discontinuities of the first kind (or 'natural jumps') in the state variable, and the system is left-continuous, *i.e.* $x(\tau_k^-) = x(\tau_k)$.

The class of dynamic systems of interest basically consists of objects defined by a set of impulsive first-order differential equations of the form (Rivadeneira & Moog, 2012; Yang, 2001)

$$\begin{cases} \dot{x}(t) = f(x), & x(t_0) = x(t_0^+) = x_0, \quad t \neq \tau_k, \\ x(\tau_k^+) = f_I(x(\tau_k)) + g(x(\tau_k))u(\tau_k), & t = \tau_k, \; k \in \mathbb{N}, \\ y_c(t) = h_c(x(t)), & \text{or} \\ y_d[k] = h_d(x(\tau_k)), & k \in \mathbb{N} \end{cases} \tag{1}$$

where the state $x \in \mathcal{X} \in \mathbb{R}^n$, the input $u \in \mathbb{R}^m$, $y_c \in \mathcal{Q} \in \mathbb{R}^q$ is a continuous output, $y_d \in \mathbb{R}^q$ is a set of discrete measurements, and the independent variable $t \in \mathbb{R}$ denotes the time. The functions $f(x)$, $f_I(x) \in \mathbb{R}^n$ and $g(x) \in \mathbb{R}^{n\times m}$ are analytical vector fields, and the spaces $\mathcal{X}$ and $\mathcal{Q}$ are analytic manifolds.

Note that the first two equations of system (1) can be written alternatively as (Rivadeneira & Moog, 2012)

$$\dot{x}(t) = f(x(t)) + (f_1(x(t)) + g(x(t))u(t))\delta(t - \tau_k),$$
$$x(t_0) = x_0, \tag{2}$$

where $f_1 = f_I(x) - x$, and $\delta$ is the impulse applied at times $\tau_k$, $k \in \mathbb{N}$. For the special case where $f_I(x) = x$, then (2) reduces to

$$\dot{x}(t) = f(x(t)) + g(x(t))u(t)\delta(t - \tau_k), \tag{3}$$
$$x(t_0) = x_0. \tag{4}$$

Actually, the nonlinear ICS (1) is an autonomous system in the intervals $]\tau_{k-1}, \tau_k[$, $k = \{1, 2, \ldots\}$. For simplicity, assume that $t_0 = 0$, there is no impulse applied to the system in the interval $[0, \tau_1[$, and $u(\tau_i) = u_i$. Let $\Psi(t, 0, x_0)$ be a solution of the autonomous system[1] of the first equation in (1) for $t \in [0, \tau_1[$, *i.e.* $x(t) = \Psi(t, 0, x_0)$, $t \in [0, \tau_1[$. At $t = \tau_1$,

$$x(\tau_1^+) = f_I(x(\tau_1)) + g(x(\tau_1))u_1 \tag{5}$$
$$= f_I\left(\Psi(\tau_1, 0, x_0)\right) + g\left(\Psi(\tau_1, 0, x_0)\right)u_1. \tag{6}$$

[1] The existence and uniqueness of the solution $\Psi(\cdot)$ is assumed. However, this is still an active field of research. See Ref. Lakshmikantham, Bainov, and Simeonov (1989) for an introduction.

Now, for $t \in [\tau_1, \tau_2[$, where the first impulse has been already applied to the system, the state trajectory $x(t)$ is

$$x(t) = \Psi(t, \tau_1, x(\tau_1^+))$$
$$= \Psi\left(t, \tau_1, f_I\left(\Psi(\tau_1, 0, x_0)\right) + g\left(\Psi(t, 0, x_0)\right)u_1\right).$$

In general, the state $x(t)$ in the interval $[\tau_{k-1}, \tau_k[$ follows the recursive equation

$$x(t) = \Psi(t, \tau_{k-1}, x(\tau_{k-1}^+)), \quad t \in [\tau_{k-1}, \tau_k[, \tag{7}$$
$$x(\tau_k^+) = f_I\left(x(\tau_k)\right) + g(x(\tau_k))u_k, \quad t = \tau_k,\; k \in \mathbb{N}$$

where $\tau_0 = 0$, $x(\tau_0^+) = x_0$, and $k - 1$ impulses have been applied to the system. Note that $x(\cdot)$, and $g(\cdot)$ depend on $x_0$ and $u_i$.

If $f(x) = Ax$, $g(x) = B$, and $f_I(x) = A_Ix$, this system is a linear ICS and can be expressed as (Medina & Lawrence, 2008)

$$\begin{cases} \dot{x}(t) = Ax(t), & x(0^+) = x_0, \quad t \neq \tau_k, \\ x(\tau_k^+) = A_Ix(\tau_k) + Bu(\tau_k), & k \in \mathbb{N}, \\ y_c(t) = C_cx(t), & \text{or} \\ y_d(t) = C_dx(t), & \end{cases} \tag{8}$$

where $A$, $B$, $A_I$, and $C_c$ (or $C_d$) have appropriate dimensions.

The state response for this class of systems can be generated explicitly as follows. Let us denote the final time as $t_f = \tau_{k+1}$, the set of time instants as $T = \{\tau_1, \ldots \tau_k\}$ such that $\Delta_i$ is equal to $\Delta_i = \tau_{i+1} - \tau_i$, and verifies that $\Delta_0 = \tau_1$, and $\Delta_k = t_f - \tau_k$. The state transition matrix of (8) is calculated recursively using (7) and results in $\Phi(t_f, 0) = e^{A\Delta_k}A_Ie^{A\Delta_{k-1}}\cdots A_Ie^{A\Delta_1}A_Ie^{A\Delta_0}$.

The state transition matrix is invertible for all $t \in [0, t_f]$ if only if the matrix $A_I$ is invertible, and in this case, $\Phi(0, t_f) = \Phi^{-1}(t_f, 0)$ (see Medina & Lawrence, 2008 for more details). The state response of system (8) on $[0, t]$ with $k$ impulses applied to the system is $x(t) = \Phi(t, 0)x_0 + \sum_{j=1}^{k}\Phi(t, \tau_j)Bu_j$. Note that if $B = 0$ and $A_I = I$, the state transition matrix for LTI systems is recovered, that is, $\Phi(t, t_0) = e^{At}$ and the state response is just $x(t) = e^{At}x_0$. Now, if $B \neq 0$ but $A_I = I$, the state response equation becomes $x(t) = e^{At}\left(x_0 + \sum_{j=1}^{k} e^{-A\tau_j}Bu_j\right)$, which agrees with results in Yang (2001).

## 3. Observability for nonlinear impulsive systems

### 3.1. Strong observability

In standard nonlinear control systems (where the impulses are not involved), this property has been extensively developed, not only considering continuous outputs (Conte, Moog, & Perdon, 2007), but also discrete ones (Califano, Monaco, & Normand-Cyrot, 2003; Moral & Grizzle, 1995). A standard nonlinear control system with continuous output is called strongly observable, if the state can be deduced from the knowledge of the output and its time derivatives. For nonlinear ICS, the same notion will be maintained even if impulses are taken into account in the dynamics.

**Definition 1.** System (1) is said to be strongly observable at point $t = 0$, if there exist an integer $n$, and locally a function $\varphi$ such that $x(0) = \varphi\left(y_c(0), \dot{y}_c(0), \ldots, y_c^{(n-1)}(0)\right)$.

**Theorem 1.** *System (1) is strongly observable at point $t = 0$, if and only if*

$$Rank \begin{bmatrix} dh_c(x(0)) \\ dL_f h_c(x(0)) \\ \vdots \\ dL_f^{n-1} h_c(x(0)) \end{bmatrix} = n. \quad (9)$$

**Proof.** It is straightforward from Theorem 4.10 in Conte et al. (2007, pp. 57–59) and the Inverse Function Theorem.

**Remark 1.** The rank condition of Theorem 1 proves that it is possible to observe the state $x(t)$ at any $t \neq \tau_k$. But at $t = \tau_k$, a necessary and sufficient condition for which system (1) is strongly observable is

$$Rank \left[ \frac{\partial \left( y_{c1}, \ldots, y_{c1}^{(\rho_1-1)}, \ldots, y_{cq}, \ldots, y_{cq}^{(\rho_q-1)} \right)}{\partial x} \right] = n,$$

where $\rho_i$ is the impulsive relative degree $d_i^0$ of the $i$th output $y_{ci}$, and $q$ is the dimension of the output space.

Note that for $t = \tau_k$, it is necessary that the impulsive input does not affect any component of the output of the system. The definition of impulsive relative degree can be found in Rivadeneira and Moog (2012).

If system (1) is a linear ICS then the criterion in Theorem 1 reduces to $Rank[\mathcal{O}] = Rank \begin{bmatrix} C_c^T & A^T C_c^T & \cdots & (A^{n-1})^T C_c^T \end{bmatrix} = n$. Note that the latter works out properly in the first interval of time, $t \in [0, \tau_1[$, where the first impulse is not applied yet. For the linear case, this criterion is a sufficient (but not necessary) condition of observability for linear ICS, in contrast with the standard case without impulses. A hint for necessity is provided by the following example.

**Example 1.** A linear ICS (8) with $y_c(t) = C_c x(t)$ is described by $A = 0, B = 0, C_c = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix}$, and $A_I = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$. This system is not strongly observable as $Rank[\mathcal{O}] = 1$, and Theorem 4.2 in Guan et al. (2002) cannot be used since $A_I$ is not diagonal. To test observability let us compute the output at different times $\tau_k$: $y_c(0) = x_1(0), y_c(\tau_1) = x_1(0), y_c(\tau_2) = x_2(0), y_c(\tau_3) = x_3(0)$. Note that these algebraic equations allow to derive the initial state from the knowledge of the output, over a sufficient number of impulses in $[0, t_f]$, and the system has to be considered observable. Note also that the time derivatives of the output at several time instants $\tau_k$ can be added to the criterion to decide observability.

Strong observability is not developed for the case of discrete output $y_d[k]$ because impulses are taken at the same time that the output is measured. Then, in the interval $0 \leq t < \tau_1$, there is enough information of the output to determine observability.

### *3.2. Observability on some time interval*

A discrete output $y_d[k]$ is considered first.

**Definition 2.** System (1) is said to be observable on some finite time interval $[0, t_f]$, if there exist an integer $l$, at least $l$ impulses, and locally a function $\xi$ such that $x(0) = \xi\ (y_d[0], y_d[1], \ldots, y_d[l])$.

**Theorem 2.** *System (1) is observable on some finite time interval $[0, t_f]$, if and only if*

$$Rank \left[ \frac{\partial\ (y_d[0], \ldots, y_d[l])}{\partial x} \right] = n. \quad (10)$$

**Proof.** Consider system (1) with discrete output $y_d[k] = h_d(x(\tau_k))$. At times $\{\tau_0, \tau_1, \ldots, \tau_l\}$ the output is $y_d[0] = h_d(x(0)), y_d[1] = h_d(x(\tau_1)) = h_d(\Psi(\tau_1, 0, x_0)), \ldots, y_d[l] = h_d(x(\tau_l)) = h_d(\Psi(\tau_l, \tau_{l-1}, \Psi(\tau_{l-1}, \tau_{l-2}, x(\tau_{l-1}^+))))$ where $x(\tau_j^+)$ is function of $x(0)$ according to (7). If the rank condition (10) is fulfilled, then by the Inverse Function Theorem there exists a function $\xi$ such that $x(0) = \xi\ (y_d[0], \ldots, y_d[l])$.

In the special case where $f_I(x) = x$, the rank condition becomes

$$Rank \begin{bmatrix} L_{M_0} h_d(x(0)) \\ L_{M_1} h_d(x(0)) \\ \vdots \\ L_{M_l} h_d(x(0)) \end{bmatrix} = n, \quad (11)$$

where $L_{M_i} h_d(x(0)) = \frac{\partial h_d}{\partial x} M_i$, and $M_i = \frac{\partial \Psi(\tau_i, 0, x_0)}{\partial x_0}$. Note that a lower bound for $l$ is $\lceil n/q \rceil$ since that matrix has $ql$ rows and $Rank[\cdot] = n$.

*The linear case*: A comparable definition of observability over a finite interval along the lines of the observability Gramian can be found in Medina and Lawrence (2009), in which it is adapted to linear ICS with discrete measurements of the output. It reads as: given the impulsive system (8) on $[0, t_f]$, and the time instants $T = \{\tau_k\},\ k = 0, 1, 2, \ldots, l,\ t_0 = 0$, and $t_f \in [\tau_{l-1}, \tau_l[$, the observability Gramian $M_{\mathcal{O}_I}(0, t_f)$ is defined by $M_{\mathcal{O}_I}(0, t_f) = \sum_{j=0}^{l-1} \Phi^T(\tau_j, 0) C_d^T C_d \Phi(\tau_j, 0)$. For the zero input and $x(0) = x_0$, $x_0^T M_{\mathcal{O}_I}(0, t_f) x_0 = \sum_{j=0}^{l-1} \|y_d[j]\|^2$, from which it follows that for an observable system on $[0, t_f]$, the observability Gramian is positive definite for any impulse time set $T$, and any finite interval containing at least $l$ impulse times. Conversely, if there exists an integer $l$ such that the observability Gramian is positive definite for any impulse time set and any finite interval containing at least $l$ impulse times, then the system is observable (Medina & Lawrence, 2009). Note that this criterion meets Definition 2.

The next theorem asserts the equivalence between the observability Gramian criterion and a new algebraic rank condition for the linear case.

**Theorem 3.** *The following statements are equivalent*

1. *System (8) is observable on some finite time interval $[0, t_f]$,*
2. $Rank \begin{bmatrix} C_d \\ C_d e^{A\Delta_0} \\ \vdots \\ C_d e^{A\Delta_{l-1}} \ldots A_I e^{A\Delta_0} \end{bmatrix} = n,$
3. $M_{\mathcal{O}_I}(0, t_f) > 0.$

**Proof.** The input $u(\tau_k) = 0,\ \forall k$ is considered without loss of generality. Proof of (1) $\Rightarrow$ (2) is obtained from Theorem 2, but considering linear ICS and the proof of (1) $\Leftrightarrow$ (3) can be found in Medina and Lawrence (2009), so it will be omitted here.

(2) $\Rightarrow$ (3). System (8) is assumed to be observable. Suppose that there is a vector $w \neq 0$ such that $w^T M_{\mathcal{O}_I}(0, t_f) w$ is singular, *i.e.*

$$w^T M_{\mathcal{O}_I}(0, t_f) w = w^T \left( \sum_{j=0}^{l-1} \Phi^T(\tau_j, 0) C_d^T C_d \Phi(\tau_j, 0) \right) w = 0,$$

which leads to

$$w^T M_{\mathcal{O}_I}(0, t_f) w = \left( \sum_{j=0}^{l-1} w^T \Phi^T(\tau_j, 0) C_d^T C_d \Phi(\tau_j, 0) w \right) = 0,$$

and

$$w^T M_{\mathcal{O}_I}(0, t_f) w = \sum_{j=0}^{l-1} \left\| C_d \Phi(\tau_j, 0) w \right\|^2 = 0.$$

From the last equation,

$$C_d\Phi(\tau_j,0)w=0\quad \forall\tau_j\, j=0,1,2,\ldots. \tag{12}$$

The latter is evaluated at $l$ impulses applied to system (8) at time instant $\tau_j$, and yields

$$\begin{bmatrix} C_d \\ C_d\Phi(\tau_1,0) \\ \vdots \\ C_d\Phi(\tau_{l-1},0)\end{bmatrix} w = \tag{13}$$

$$\begin{bmatrix} C_d \\ C_d e^{A\Delta_0} \\ \vdots \\ C_d e^{A\Delta_{l-1}}\cdots A_I e^{A\Delta_0}\end{bmatrix} w = \mathcal{O}_I w = 0. \tag{14}$$

As system (8) is observable, $rank[\mathcal{O}_{\mathcal{I}}]=n$. In consequence, $w=0$, which stands in contradiction and proofs that $M_{\mathcal{O}_{\mathcal{I}}}(0,t_f)$ is a positive definite matrix. That completes the proof.

**Remark 2.** Assuming that $A_I$ is the identity matrix, the following statements are equivalent

1. System (8) is observable on some finite time interval $[0,t_f]$,
2. $Rank\begin{bmatrix}C_d^T & (C_d e^{A\tau_1})^T & \cdots & (C_d e^{A\tau_{l-1}})^T\end{bmatrix}=n$,
3. $Rank\begin{bmatrix}C_d^T & A^T C_d^T & \cdots & (A^{n-1})^T C_d^T\end{bmatrix}=n$.

**Proof.** The only implication one has to prove is (2) ⇒ (3).

Before going on with statement 3, Lemma 2.3.1, and Lemma 2.3.2 from Yang (2001, pp. 30–31) yield

$$e^{A\lambda}=\sum_{i=0}^{n-1} f_i(\lambda)A^i,\qquad f_0(\lambda)=1,\qquad f_{i+1}(0)=0, \tag{15}$$

where $f_i(\lambda)$ is a scalar function, and such functions $f_i(\lambda)$ are linearly independent in any open interval $]t_1,t_2[$. The observability condition is given by statements 1–2. From the second statement, and applying the last equation, it is obtained

$$\Psi\mathcal{O}_d=\begin{pmatrix} C_d \\ C_d(I+f_1(\tau_1)A+\cdots+f_{n-1}(\tau_1)A^{n-1}) \\ C_d(I+f_1(\tau_2)A+\cdots+f_{n-1}(\tau_2)A^{n-1}) \\ \vdots \\ C_d(I+f_1(\tau_{l-1})A+\cdots+f_{l-1}(\tau_{l-1})A^{n-1})\end{pmatrix}, \tag{16}$$

where

$$\Psi=\begin{pmatrix} I_q & 0 & \cdots & 0 \\ I_q & f_1(\tau_1)I_q & \cdots & f_{n-1}(\tau_1)I_q \\ \vdots & \vdots & \ddots & \vdots \\ I_q & f_1(\tau_{l-1})I_q & \cdots & f_{n-1}(\tau_{l-1})I_q\end{pmatrix}, \tag{17}$$

$Iq$ is the $q\times q$ identity matrix, and

$$\mathcal{O}_d=\begin{pmatrix} C_d \\ C_dA \\ \vdots \\ C_dA^{n-1}\end{pmatrix}. \tag{18}$$

It is necessary to show that the matrix $\Psi$ has full rank that is no less than $n$ for some time moments $\tau_k$, $k=0,\ldots,l$. By using the Kronecker index with

$$Q\triangleq\begin{pmatrix} 1 & 0 & \cdots & 0 \\ 1 & f_1(\tau_1) & \cdots & f_{n-1}(\tau_1) \\ \vdots & \vdots & \ddots & \vdots \\ 1 & f_1(\tau_{l-1}) & \cdots & f_{n-1}(\tau_{l-1})\end{pmatrix}, \tag{19}$$

$\Psi=Q\oplus I_q$ has full rank if and only if $Q$ has full rank. $\Psi$ has $lq$ rows, hence $Rank(\Psi)\geq n$ implies $l\geq n/q$, this provides a lower bound for $l$. Let $l=\lceil n/q\rceil$, then since $q\geq 1$, $l\leq n$ and by Lemma 2.3.2 we can apply Lemma 2.3.3[2] in Yang (2001, pp. 33) so that there exist $\tau_1,\tau_2,\ldots,\tau_l$ such that $Q$ has full rank.

Now, a continuous output is taken into account and assume that the rank condition of Theorem 1 is not fulfilled, *i.e.* the nonlinear ICS is not strongly observable.

**Example 2.** Consider the nonlinear ICS

$$\begin{aligned} &\dot{x}_1(t)=x_2(t)x_4(t),\\ &\dot{x}_2(t)=0,\\ &\dot{x}_3(t)=x_1(t),\\ &\dot{x}_4(t)=0,\\ &x_1(\tau_k^+)=x_1(\tau_k)+x_2(\tau_k)u_1(\tau_k),\\ &y_c(t)=h_c(x(t))=x_3(t). \end{aligned} \tag{20}$$

At time $t=0$, we have

$$y_c(0)=h_c(x(0))=x_3(0), \tag{21}$$

$$\dot{y}_c(0)=L_fh_c(x(0))=x_1(0), \tag{22}$$

$$y_c^{(2)}(0)=L_f^2h_c(x(0))=x_2(0)x_4(0), \tag{23}$$

$$y_c^{(3)}(0)=0. \tag{24}$$

It is not possible to get the state $x(0)$ from $(y_c(0),\dot{y}_c(0),y_c^{(2)}(0),y_c^{(3)}(0))$, and the system is not strongly observable at any time $t$. However, the initial state may be observed through another way. Since the relative degree is 2, the information of the output and its first derivative at time instants $\tau_k^+$ could be added to retrieve the initial states. The second derivative of the output cannot be used because it involves impulses at times $t=\tau_k$. So, the first derivative displays a discontinuity of the form

$$\begin{aligned} &\dot{y}_c(\tau_k^+)=L_fh_c(x(\tau_k))+L_gL_fh_c(x(\tau_k))u(\tau_k),\\ &\dot{y}_c(\tau_k^+)=x_1(\tau_k)+x_2(\tau_k)u_1(\tau_k). \end{aligned} \tag{25}$$

In particular, at time $\tau_1^+$ and by Eq. (7)

$$\begin{aligned} \dot{y}_c(\tau_1^+)&=x_1(\tau_1)+x_2(\tau_1)u_1(\tau_1)\\ &=x_1(0)+x_2(0)x_4(0)\tau_1+x_2(0)u_1(\tau_1). \end{aligned} \tag{26}$$

Now, 4 Eqs. (21)–(26) are obtained which can be solved in the 4 unknowns. The determinant of the Jacobian matrix of the system of algebraic equations in $x(0)$ is computed as

$$Det\left[\begin{pmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & \tau_1 & 0 & u(\tau_1)\end{pmatrix}\right]=u_1(\tau_1)\neq 0. \tag{27}$$

Since the determinant is always different from 0, by the Inverse Function Theorem, there (locally) exists a function $\zeta$ such that $x(0)=\zeta\left(y_c(0),\dot{y}_c(0),y_c^{(2)}(0),y_c(\tau_1^+),\dot{y}_c(\tau_1^+)\right)$, then the state $x_0$ still can be deduced from the output.

**Definition 3.** System (1) is said to be observable on some finite time interval $[0,t_f]$, if there exist a finite $t_f$, integers $s,l,\rho$, at least $l$ impulses, and locally a function $\zeta$, such that $x(0)=\zeta\left(y_c(0),\dot{y}_c(0),\ldots,y_c^{(s)}(0),{y_c}^{(\rho-1)}(\tau_1^+),\ldots,{y_c}^{(\rho-1)}(\tau_l^+)\right)$.

[2] Lemma: if $f_i(\tau_i)$ $i=1,\ldots,p$ are linearly independent functions in every open interval, then for a given $p\leq n$, and $]t_s,t_{s+1}[$ there exist real numbers $\tau_i$ such that $Q$ has rank $p$.

Note that this definition of observability is weaker than Definition 1 since not only the knowledge of the output is required, but also the information about impulses. The input $u(\tau_k)$ does not change this notion because if controls are present, they are part of the impulses which are applied. Note also that the integer $\rho$ could be different for each component of the output, depending on its impulsive relative degree (see Rivadeneira & Moog, 2012).

Let us define $\Phi = (y_c(0), \dots, y_c^{(s)}(0), {y_c}^{(\rho-1)}(\tau_1^+), \dots, {y_c}^{(\rho-1)}(\tau_l^+))$ then

**Theorem 4.** *System (1) is said to be observable on some finite time interval* $[0, t_f]$*, if and only if*

$$Rank\left[\frac{\partial\Phi}{\partial x}\right] = n. \tag{28}$$

**Proof.** Consider the system (1) in the form (2). Assume that Theorem 1 is not fulfilled, there is no impulse applied to the system at $t = 0$, and for simplicity $f_I = x$. The output and its time derivatives at $t = 0$ are

$$\begin{array}{rcl} y_c(0) & = & h_c(x(0)), \\ \vdots & = & \vdots \\ y_c^{(s)}(0) & = & L_f^s h_c(x(0)). \end{array} \tag{29}$$

System (1) is not strongly observable because $Rank\left[\frac{\partial\left(y_c,\dots,y_c^{(s)}\right)}{\partial x}\right] < n$. Let $\rho$ be the impulsive relative degree $d^0$ of the each component of the output. Note that $\rho$ could have different value for each component of the output. Precisely, the expression containing the most information about the system is the time derivative of the output $y_c$ around $\tau_k$ of order $\rho - 1$, that is ${y_c}^{(\rho-1)}(\tau_k^+) = L_f^{\rho-1}h_c(x(\tau_k)) + L_gL_f^{\rho-1}h_c(x(\tau_k))u(\tau_k)$.

At time instants $\{\tau_1, \dots, \tau_l\}$, $\{{y_c}^{(\rho-1)}(\tau_1^+), \dots, {y_c}^{(\rho-1)}(\tau_l^+)\}$ are functions of $x_0$ by means of (7) and ${y_c}^{(\rho-1)}(\tau_k^+)$, more precisely

$$\begin{array}{rcl} {y_c}^{(\rho-1)}(\tau_1^+) & = & L_f^{\rho-1}h_c(x(\tau_1)) + L_gL_f^{\rho-1}h_c(x(\tau_1))u(\tau_1), \\ x(\tau_1) & = & \Psi(\tau_1, 0, x_0), \\ \vdots & = & \vdots \\ {y_c}^{(\rho_i-1)}(\tau_l^+) & = & L_f^{\rho-1}h_c(x(\tau_l)) + L_gL_f^{\rho-1}h_c(x(\tau_l))u(\tau_l), \\ x(\tau_l) & = & \Psi(\tau_l, \tau_{l-1}, x(\tau_{l-1}^+)). \end{array} \tag{30}$$

From (29), (30) and if the rank condition (28) is fulfilled, by the Inverse Function Theorem we can state that there locally exists a function $\zeta$ such that $x(0) = \zeta(y_c(0), \dots, y_c^{(s)}(0), {y_c}^{(\rho-1)^+}(\tau_1^+), \dots, {y_c}^{(\rho-1)}(\tau_l^+))$. Then by Definition 3, the system (1) is observable on some time interval $[0, t_f]$.

In linear ICS with continuous output, this notion of observability on some time interval can be stated as it is suggested by Theorem 4.2 in Guan et al. (2002, pp. 255–256) assuming diagonal matrices or more general as in Shi and Xie (2012). The strong observability can be seen as a sufficient condition of observability in ICS, and that coincides with Theorem 4.2 in Guan et al. (2002).

Note that Definition 3 encompasses the similar ones described in Guan et al. (2002), Shi and Xie (2012) and Zhao and Sun (2009, 2012) for linear ICS with continuous output. The following remark comprises the different criteria for observability over some finite interval for this case:

**Theorem 5.** *System (8) is observable on some finite interval* $[0, t_f]$ *if and only if* $Rank\begin{bmatrix}\mathcal{O}\\ \mathcal{O}_{\mathcal{I}}\end{bmatrix} = n$.

Note that this theorem is equivalent to the criterion developed in Shi and Xie (2012). If a linear ICS is not strongly observable then $Rank[\mathcal{O}] < n$. Therefore the information necessary to complete the rank could be collected from the information of the output at times $\tau_k$, that is the part of the $Rank[\mathcal{O}_{\mathcal{I}}]$.

## 4. Application to biomedical engineering processes

### 4.1. Case-study: impulsive HIV dynamics

Several nonlinear models have been developed to describe the dynamics of HIV-1 virus which take into account the kinetics of HIV infection with different cells populations *e.g.* macrophages, CTL cells, latently infected $CD4^+$ T cells (cluster of differentiation 4, the cells commonly known as helper T cells or T4 cells) as well the inclusion of the lymphoid compartments in their models (Perelson et al., 1993). However, for the control and parameter estimation based on clinical data, the dynamics of the infection can be modeled by relatively simple ordinary differential equations for the interactions of healthy $CD4^+$ cells ($T$), infected $CD4^+$ cells ($y$), free viruses ($z$) (Rivadeneira & Moog, 2012).

In this paper, the used model takes into account the interaction of the intake of drugs and its concentration in blood according to the notions of pharmacokinetics and pharmacodynamics described in Legrand et al. (2003). Consequently, the impulsive model is:

$$\begin{array}{ll} \dot{T}(t) = s - \delta T(t) - \beta T(t)z(t), & \\ \dot{y}(t) = \beta T(t)z(t) - \mu y(t), & t \neq \tau_k, \\ \dot{z}(t) = \left(1 - \dfrac{w(t)}{w(t) + w_{50}}\right)ky(t) - cz(t), & \\ \dot{w}(t) = -Kw(t), & \\ w(\tau_k^+) = w(\tau_k) + u(\tau_k), & k \in \mathbb{N}, \end{array} \tag{31}$$

where healthy $CD4^+$ cells ($T$) are produced from the thymus at a constant rate $s$ and die with a half life time equal to $\frac{1}{\delta}$. The healthy cells are infected by the virus at a rate that is proportional to the product of their population and the amount of free virus particles. The proportionality constant $\beta$ is an indication of the effectiveness of the infection process. The infected $CD4^+$ cells ($y$) result from the infection of healthy $CD4^+$ cells and die at a rate $\mu$. Free virus particles ($z$) are produced from infected $CD4^+$ cells at a rate $k$ and die with a half life time equal to $\frac{1}{c}$. The parameter $w_{50}$ represents the concentration of drug that lowers the viral load by 50%, and the parameter $\eta$ is the efficacy of an anti-HIV treatment (in general a cocktail drugs of RT and P inhibitors).

Only the physical output $h_c = T(t)$ is taken into account and considered to be continuous. Applying Theorem 1, one computes

$$Rank\,[\mathcal{O}] = Rank\left[\begin{pmatrix} dh_c(0) \\ dL_f h_c(0) \\ dL_f^2 h_c(0) \\ dL_f^3 h_c(0) \end{pmatrix}\right] = 4, \tag{32}$$

whose determinant

$$Det\,[\mathcal{O}] = \frac{\beta^3k^2w_{50}^2(T_0)^3(Kw_{50}y_0 - \beta(w_0 + w_{50})z_0)}{(w_0 + w_{50})^4}$$

is different from zero if $Kw_{50}y_0 \neq \beta(w_0 + w_{50})z_0$. As a conclusion, this nonlinear model of HIV is strongly observable.

### 4.2. Case-study: diabetic type I patient model

The following model is considered as a first approximation dynamics for diabetic type I patients based on Bergman's model (Bergman, Phillips, & Cobelli, 1981)

$$\begin{aligned}
&\dot{x}_1(t) = -x_1x_2, \quad x_1(0) = x_{10},\\
&\dot{x}_2(t) = -a_2x_2 + a_3x_3, \quad x_2(0) = x_{20},\\
&\dot{x}_3(t) = -a_4x_3, \quad x_3(0) = x_{30},\\
&x_1(\tau_k^+) = x_1(\tau_k) + J_p(\tau_k), \quad k \in \mathbb{N},\\
&x_3(\tau_k^+) = x_3(\tau_k) + \frac{1}{V}u(\tau_k), \quad k \in \mathbb{N},\\
&y_d[k] = x_1(\tau_k) \quad k \in \mathbb{N},
\end{aligned} \tag{33}$$

where $x_1$ is the deviation of the blood glucose concentration from its basal value (assumed to be $G_b = 1.0$ g/l), $x_2$ is proportional to insulin in a remote compartment, and $x_3$ is the plasma insulin concentration in $U/l$, the control variable $u$ represents a sudden change in the insulin concentration due to an injection of insulin, and $y$ is a discrete measurement of the deviation of the blood glucose concentration. The parameter $a_2$ represents the decrease of glucose under the action of insulin, and $a_4$ is the natural absorption rate of insulin in the body.

An other factor that changes the concentration of the blood glucose is the input of meals. As a first approximation, these inputs can be seen, in a day of the patient, as an impulsive jump of the glucose concentration in the body and after a digestion process as a variation in the blood concentration, and it is represented by $J_p$.

Since this model is a nonlinear ICS with discrete measurements, the observability is checked through Theorem 2. It is considered that the measurements are made before the intake of the meal and are performed by the patient.

The impulsive control system (33),during an interval $\tau_i^+ \le t < \tau_{i+1}$, is an autonomous and invariant system. The solution of this dynamical system is:

$$\begin{aligned}
x_3(t) &= e^{-a_4(t-\tau_i^+)}x_3(\tau_i^+),\\
x_2(t) &= e^{-a_2(t-\tau_i^+)}x_2(\tau_i^+)\\
&\quad + \frac{a_3}{a_2-a_4}(e^{-a_2(t-\tau_i^+)} + e^{-a_4(t-\tau_i^+)})x_3(\tau_i^+),\\
x_1(t) &= e^{\xi(t)}x_1(\tau_i^+),\\
\xi(t) &= \frac{(-1+e^{-a_2(t-\tau_i^+)})}{a_2}x_2(\tau_i^+)\\
&\quad - \frac{a_3}{a_2a_4}x_3(\tau_i^+)\left(1 + \frac{a_4e^{-a_2(t-\tau_i^+)} + a_2e^{-a_4(t-\tau_i^+)}}{(a_2-a_4)}\right).
\end{aligned} \tag{34}$$

The necessary discrete measurements of the output are $y_d[0] = x_1(0)$, $y_d[1] = x_1(\tau_1)$, and $y_d[2] = x_1(\tau_2)$.

From Eq. (34), and taking $0 \le t < \tau_1$,

$$\begin{aligned}
x_1(\tau_1) &= e^{\xi(\tau_1)}x_1(0),\\
\xi(\tau_1) &= \frac{(-1+e^{-a_2\tau_1})}{a_2}x_2(0)\\
&\quad - \frac{a_3}{a_2a_4}x_3(0)\left(1 + \frac{(a_4e^{-a_2\tau_1} + a_2e^{-a_4\tau_1})}{(a_2-a_4)}\right),
\end{aligned} \tag{35}$$

and

$$\begin{aligned}
x_1(\tau_2) &= e^{\xi(\tau_2)}x_1(\tau_1^+),\\
\xi(\tau_2) &= \frac{(-1+e^{-a_2\tau_2})}{a_2}x_2(\tau_1) - \frac{a_3}{a_2a_4}x_3(\tau_1^+)\\
&\quad \times \left(1 + \frac{1}{(a_2-a_4)}(a_4e^{-a_2\tau_2} + a_2e^{-a_4\tau_2})\right)
\end{aligned} \tag{36}$$

$$x_1(\tau_1^+) = x_1(\tau_1) + J_p(\tau_1), \tag{37}$$

$$x_2(\tau_1) = e^{-a_2\tau_1}x_2(0) + \frac{a_3}{a_2-a_4}(e^{-a_2\tau_1}), \tag{38}$$

$$x_3(\tau_1^+) = e^{-a_4\tau_1}x_3(0) + \frac{1}{V}u(\tau_1). \tag{39}$$

Note that the set of measurements $(y_d[0], y_d[1], y_d[2])$ depends on $(x_1(0), x_2(0), x_3(0))$. So, now the rank of the matrix $\left[\frac{\partial(y_d[0], y_d[1], y_d[2])}{\partial x}\right]$ must be calculated. For that, the software Mathematica™ was used and yielded that the rank is 3. As a conclusion, the nonlinear ICS is observable over the interval $[0, \tau_2[$.

## 5. Conclusions and perspectives

The observability of linear impulsive control systems with discrete outputs has been fully characterized. Criteria have been given in terms of suitable rank conditions. Throughout the paper it was shown that these criteria have to be stated in terms of the matrices $(C_d, A, A_I)$. Obviously, the special case without impulses reduces to the well-known Observability Kalman Criterion for linear control systems.

This result generalizes and unifies criteria that can be found in the current literature, using the observability Gramian and algebraic conditions.

Observability is analyzed in nonlinear ICS for the first time, continuous and discrete outputs are considered. Useful criteria were developed to characterize this property, showing that an ICS can be strongly observable or observable over a finite time interval. This last definition is interesting and more natural due to the use of both time-shifts and time derivatives of the output. Thus, it may avoid the computation of input time derivatives for observation of the state. A good understanding of those notions of observability is obviously mandatory for the design of effective observers for ICS.

Future research challenges include the design of effective nonlinear impulsive observers. Parameter identifiability and the effective identification are connected problems which are worth to investigate within a similar approach. Its application to real life parameter identification of the glycemia dynamics *e.g.* is promising for diagnosis for patients. The latter is open for further research as well.

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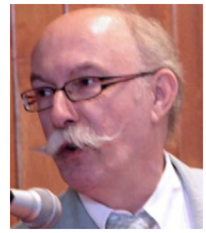

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