ON THE SMALLEST LAPLACE EIGENVALUE FOR NATURALLY REDUCTIVE METRICS ON COMPACT SIMPLE LIE GROUPS

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Abstract. Eldredge, Gordina and Saloff-Coste recently conjectured that, for a given compact connected Lie group G, there is a positive real number C such that $\lambda_1(G,g)\operatorname{diam}(G,g)^2 \leq C$ for all left-invariant metrics g on G. In this short note, we establish the conjecture for the small subclass of naturally reductive left-invariant metrics on a compact simple Lie group.

For an arbitrary compact homogeneous Riemannian manifold (M,g), Peter Li [Li80] proved that

$$\lambda_1(M,g) \geq \frac{\pi^2/4}{\operatorname{diam}(M,g)^2}. \tag{1}$$

Here, $\lambda_1(M,g)$ denotes the smallest positive eigenvalue of the Laplace–Beltrami operator on (M,g) and $\operatorname{diam}(M,g)$ is the diameter of (M,g), that is, the maximum Riemannian distance between two points in M. This lower bound has been recently improved by Judge and Lyons [JL19, Thm. 1.3].

In contrast, there is no uniform upper bound for the term $\lambda_1(M,g)\operatorname{diam}(M,g)^2$ among all compact homogeneous Riemannian manifolds. For instance, the product (M_n,g_n) of n d-dimensional round spheres of constant curvature one satisfies $\lambda_1(M_n,g_n)=d$ and $\operatorname{diam}(M_n,g_n)=\sqrt{n}\pi$ goes to infinity when $n\to\infty$.

Eldredge, Gordina and Saloff-Coste have recently conjectured the existence of a uniform upper bound valid on special classes of homogeneous Riemannian manifolds, namely, the space of left-invariant metrics on a fixed compact connected Lie group.

Conjecture 1. [EGS18, (1.2)] Given G a compact connected Lie group, there exists $C>0$ (depending only on G) such that

$$\lambda_1(M,g) \leq \frac{C}{\operatorname{diam}(M,g)^2} \tag{2}$$

for all left-invariant metrics g on G.

Among many other results, they confirm its validity for SU(2) in [EGS18, Thm. 8.5]. Explicit values of C for SU(2) and SO(3) can be found in [La19, Thm. 1.4].

The main goal of this article is to give a simple and short proof of the validity of the weaker conjecture after restricting to naturally reductive left-invariant metrics on a compact connected simple Lie group G (Theorem 4 below). The reader should not consider this result as a strong evidence of Conjecture 1.

We next define naturally reductive metrics (see for instance [Be, §7.G]). Let (M,g) be a homogeneous Riemannian manifold. We fix a base point $m\in M$ and H a transitive group of isometries of (M,g). Let K be the isotropy subgroup at m, that is, $K=\{a\in H : a\cdot m = m\}$.

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The Lie algebra $\mathfrak{h}$ of H decomposes into a sum $\mathfrak{h} = \mathfrak{k} \oplus \mathfrak{p}$, where $\mathfrak{k}$ denotes the Lie algebra of K and $\mathfrak{p}$ is $\operatorname{Ad}(K)$-invariant. For $X \in \mathfrak{h}$, we write $X = X_\mathfrak{k} + X_\mathfrak{p}$ according to this decomposition. We have the identifications $M \equiv H/K$ and $T_mM \equiv \mathfrak{p}$, and the metric g on M corresponds to an $\operatorname{Ad}(K)$-invariant inner product $\langle \cdot, \cdot \rangle_m$ on $\mathfrak{p}$.

Definition 2. A Riemannian manifold (M, g) is said to be *naturally reductive* if it admits a transitive action by isometries by a Lie group H and an $\operatorname{Ad}(K)$-invariant complement $\mathfrak{p}$ as above such that

$$\langle [Z, X]_\mathfrak{p}, Y\rangle_m + \langle X, [Z, Y]_\mathfrak{p}\rangle_m = 0 \tag{3}$$

for all $X, Y, Z \in \mathfrak{g}$. (Here, $[\cdot, \cdot]$ denotes the bracket of the Lie algebra $\mathfrak{h}$.)

A naturally reductive space can be seen as a generalization of a symmetric space. Among their nice geometric properties, we have that every geodesic is an orbit of an one-parameter group of isometries. Normal homogeneous spaces are also naturally reductive. However, the class of naturally reductive spaces is much broader and contains many other interesting cases.

We now give a simple construction of naturally reductive metrics in the case of interest of this paper, that is, when M is a simple compact connected Lie group.

Remark 3. Let G be a semisimple compact connected Lie group. It is well known that the space of left-invariant metrics on G is in 1-to-1 correspondence with the space of inner products on its Lie algebra $\mathfrak{g}$. Let K be a closed subgroup of G and let $\mathfrak{p}$ denote the orthogonal complement of $\mathfrak{k}$ in $\mathfrak{g}$ with respect to the Killing form $B_\mathfrak{g}$ of $\mathfrak{g}$. (We recall that $B_\mathfrak{g}$ is an $\operatorname{Ad}(G)$-invariant negative definite bilinear form on $\mathfrak{g}$.) Given h a bi-invariant metric on K and α a positive real number, we define the left-invariant metric $g_{h,\alpha}$ on G induced by the inner product on $\mathfrak{g}$ given by

$$g_{\alpha,h}(X, Y) = h(X_\mathfrak{k}, Y_\mathfrak{k}) + \alpha\,(-B_\mathfrak{g})(X_\mathfrak{p}, Y_\mathfrak{p}) \qquad \text{for } X, Y \in \mathfrak{g}. \tag{4}$$

D'Atri and Ziller proved that $g_{\alpha,h}$ is naturally reductive for all α, h as above (see [DZ79, Thm. 1]). (Note that the transitive group H as in Definition 2 is $G \times K$ acting on G as $(a, b) \cdot x = axb^{-1}$ for $a, x \in G$, $b \in K$.) Moreover, they also proved that any naturally reductive metric on G is isometric to one of the above form when G is assumed simple (see [DZ79, Thm. 3]).

We are now in position to prove the main theorem.

Theorem 4. *Let G be a compact connected simple Lie group. There exists $C = C(G) > 0$ such that*

$$\lambda_1(G, g) \leq \frac{C}{\operatorname{diam}(G, g)^2} \tag{5}$$

for all naturally reductive metrics g on G.

Proof. Let G be a compact connected simple Lie group. We pick any naturally reductive metric on G, that is, $g_{\alpha,h}$ as in (4), for some choice of K, α, and h as in Remark 3. Since the term $\lambda_1(M, g) \operatorname{diam}(M, g)^2$ is invariant under positive scaling of g, we can assume without loosing generality that $\alpha = 1$. We abbreviate $g_h = g_{h,1}$. Moreover, we avoid the case when $\mathfrak{p} = 0$ for being trivial since g_h is necessarily a negative multiple of the Killing form $B_\mathfrak{g}$ of $\mathfrak{g}$. Under this new assumption, $\mathfrak{p} \neq 0$, we will not use that h is a bi-invariant metric on K. More precisely, g_h is defined as in (4) with $\alpha = 1$ and h any inner product on $\mathfrak{k}$.

We need to introduce some notions to give an upper bound for the diameter. Recall that a sub-Riemannian manifold is a triple $(M, \mathcal{H}, s)$, where M is a smooth manifold, $\mathcal{H}$ is a subbundle of TM and $s = (s_m)_{m\in M}$ denotes a family of inner product on $\mathcal{H}$ which smoothly vary with the base point (see [Mo] for a general reference). A smooth curve γ on $(M, \mathcal{H}, g)$ is called *horizontal* if $\gamma'(t) \in \mathcal{H}_{\gamma(t)}$ for all t. When $\mathcal{H}$ satisfies the *bracket-generating condition* (i.e. the Lie algebra generated by vector fields in $\mathcal{H}$ spans at every point the tangent space of M), also known as *Hörmander condition*, the Chow–Rashevskii Theorem ensures that $\operatorname{diam}(M, \mathcal{H}, s) < \infty$ when M is compact, that is, any two points in M can be joined by a horizontal curve on $(M, \mathcal{H}, s)$. It follows immediately that, if g is a Riemannian metric on M and we define the sub-Riemannian metric s on $(M, \mathcal{H})$ given by $s_m = g_m|_{\mathcal{H}_m}$ for all $m \in M$, then

$$\operatorname{diam}(M, g) \leq \operatorname{diam}(M, \mathcal{H}, s). \tag{6}$$

We consider the sub-Riemannian manifold $(G, \mathcal{H}, s)$ determined by

$$\mathcal{H} = \bigcup_{a \in G} dL_a(\mathfrak{p}), \qquad s_a\big(dL_a(X), dL_a(Y)\big) = -B_{\mathfrak{g}}(X, Y), \tag{7}$$

for $X, Y \in \mathfrak{p}$ and $a \in G$. Here, $L_a : G \to G$ is given by $L_a(x) = ax$ and $\mathfrak{p}$ is seen as a subspace of $T_eG \equiv \mathfrak{g}$. Sub-Riemannian structures on arbitrary Lie groups of this form are called left invariant. Since s is the restriction of g_h to $\mathcal{H}$, (6) gives

$$\operatorname{diam}(G, g_h) \leq \operatorname{diam}(G, \mathcal{H}, s). \tag{8}$$

We next show that $\operatorname{diam}(G, \mathcal{H}, s) < \infty$. It is sufficient to show that $\mathcal{H}$ satisfies the bracket-generating condition by Chow–Rashevskii Theorem. Since $\mathcal{H}$ is left invariant, this condition is equivalent to show that the Lie subalgebra of $\mathfrak{g}$ generated by $\mathfrak{p}$, say $\mathfrak{a}$, satisfies $\mathfrak{a} = \mathfrak{g}$. We next prove that $\mathfrak{a}$ is an ideal of the simple Lie algebra $\mathfrak{g}$, giving the required assertion. For $X = X_{\mathfrak{k}} + X_{\mathfrak{p}} \in \mathfrak{g}$ and $Y \in \mathfrak{a}$, we have that $[X, Y] = [X_{\mathfrak{k}}, Y] + [X_{\mathfrak{p}}, Y]$, so it remains to show that $[X_{\mathfrak{k}}, Y] \in \mathfrak{a}$ since $[X_{\mathfrak{p}}, Y] \in \mathfrak{a}$ is clear. The subspace $\mathfrak{a}$ is spanned by elements of the form $[Y_1, [Y_2, \cdots, [Y_{n-1}, Y_n] \cdots]]$ with $Y_1, \dots, Y_n \in \mathfrak{p}$. We next show by induction on n that $[X_{\mathfrak{k}}, Y] \in \mathfrak{a}$ for such element Y. The case $n = 1$ is clear since $[X_{\mathfrak{k}}, Y] = [X_{\mathfrak{k}}, Y_1] \in \mathfrak{p}$ because $Y = Y_1 \in \mathfrak{p}$ and $\mathfrak{p}$ is $\operatorname{ad}(\mathfrak{k})$-invariant. Furthermore, the Jacobi identity implies

$$[X_{\mathfrak{k}}, Y] = \big[[X_{\mathfrak{k}}, Y_1], [Y_2, \cdots, [Y_{n-1}, Y_n] \cdots]\big] + \big[Y_1, [X_{\mathfrak{k}}, [Y_2, \cdots, [Y_{n-1}, Y_n] \cdots]]\big], \tag{9}$$

thus the inductive step also follows. This concludes the proof of $\mathfrak{a} = \mathfrak{g}$ and its consequence $\operatorname{diam}(G, \mathcal{H}, s) < \infty$.

We now consider the Riemannian submersion with totally geodesic fibers given by

$$(K, h) \longrightarrow (G, g_h) \xrightarrow{\pi} (G/K, -B_{\mathfrak{g}}|_{\mathfrak{p}}), \tag{10}$$

which is a particular case of the general construction in [BB82, §2.2] and [Be, Thm. 9.80]. The spectral theory of Riemannian submersions with totally geodesic fibers has been intensively studied; see for instance [GLP]. It is well known that if f is an eigenfunction of the Laplace–Beltrami operator Δ_b of the base space $(G/K, -B_{\mathfrak{g}}|_{\mathfrak{p}})$ with corresponding eigenvalue λ, then $f \circ \pi$ is an eigenfunction of the Laplace–Beltrami operator Δ_{g_h} of (G, g_h) with corresponding eigenvalue λ. Consequently, the smallest positive eigenvalue of Δ_b is an upper bound for the smallest positive eigenvalue of Δ_{g_h}, that is

$$\lambda_1(G, g_h) \leq \lambda_1(G/K, -B_{\mathfrak{g}}|_{\mathfrak{p}}). \tag{11}$$

We have shown that $\lambda_1(G, g_h) \operatorname{diam}(G, g_h)^2 \leq \lambda_1(G/K, -B_{\mathfrak{g}}|_{\mathfrak{p}}) \operatorname{diam}(G, \mathcal{H}, s)^2$ for every left-invariant metric h on $\mathfrak{k}$, when $\mathfrak{p} \neq 0$. Note that the right hand side depends on K and $\mathfrak{p}$, but not on h. We conclude the proof by a finiteness argument on the choice of K and $\mathfrak{p}$.

There is a finite collection $\mathcal{K}$ of closed subgroups of G such that, for any naturally reductive left-invariant metric g on G, there are $K \in \mathcal{K}$, $\alpha > 0$, and h a bi-invariant metric on $\mathfrak{k}$ such that (G, g) is isometric to $(G, g_{h,\alpha})$ as in Remark 3 (see [GS10, Cor. 3.7]). Hence, by taking

$$C = \max_{K \in \mathcal{K} \smallsetminus \{G\}} \left\{ \lambda_1(G/K, -B_{\mathfrak{g}}|_{\mathfrak{p}})\ \operatorname{diam}(G, \mathfrak{p}, -B_{\mathfrak{g}}|_{\mathfrak{p}})^2,\ \lambda_1(G, -B_{\mathfrak{g}})\ \operatorname{diam}(G, -B_{\mathfrak{g}})^2 \right\}, \tag{12}$$

we conclude that (13) holds for all naturally reductive left-invariant metrics on G. □

Inside the proof, for a fixed closed subgroup K of G with $\dim K < \dim G$ (i.e. $\mathfrak{p} \neq 0$), it was not used that h is a bi-invariant inner product on $\mathfrak{k}$. Thus, it was also proven the following statement.

Theorem 5. *Let G be a compact connected simple Lie group, let K be a closed subgroup of G of dimension strictly less than $\dim G$, and let $\mathfrak{p}$ denote the orthogonal complement of $\mathfrak{k}$ in $\mathfrak{g}$ with respect to the Killing form $B_\mathfrak{g}$ of $\mathfrak{g}$. There exists $C = C(G,K) > 0$ such that*

$$\lambda_1(G,g) \leq \frac{C}{\operatorname{diam}(G,g)^2} \tag{13}$$

for every left-invariant metric g on G satisfying that $g(\mathfrak{k},\mathfrak{p}) = 0$ and $g|_\mathfrak{p}$ is a negative multiple of $B_\mathfrak{g}|_\mathfrak{g}$.

Theorem 4 applied to the well-known situation $G = \mathrm{SU}(2)$ returns that (2) is valid for a codimension one subspace of the space of left-invariant metrics up to isometry, as shown in the next example. For higher-dimensional compact connected simple Lie groups, the analogous codimension increases considerably.

Example 6. We consider $G = \mathrm{SU}(2)$, which is diffeomorphic to the 3-sphere S^3. The elements

$$X_1 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \qquad X_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad X_3 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \tag{14}$$

form a basis of the Lie algebra $\mathfrak{su}(2)$. For positive real numbers a_1, a_2, a_3, let $g_{(a_1,a_2,a_3)}$ denote the left-invariant metric on $\mathrm{SU}(2)$ induced by the inner product on $\mathfrak{su}(2)$ given by $g_{(a_1,a_2,a_3)}(X_i, X_j) = a_i^2\, \delta_{i,j}$. Note that $g_{(1,1,1)}$ is a negative multiple of $B_{\mathfrak{su}(2)}$. Although the dimension of the space of inner products on $\mathfrak{su}(2)$ is 6, Milnor [Mi76] proved that every left-invariant metric on $\mathrm{SU}(2)$ is isometric to $g_{(a_1,a_2,a_3)}$ for some $a_1, a_2, a_3 > 0$. (Permutations of (a_1, a_2, a_3) do note change the isometry class of $g_{(a_1,a_2,a_3)}$; see e.g. [EGS18, Lem. 2.8].)

The only proper closed connected subgroup of G up to conjugation is the torus

$$T := \left\{ \begin{pmatrix} e^{i\theta} & \\ & e^{-i\theta} \end{pmatrix} : \theta \in \mathbb{R} \right\}. \tag{15}$$

Up to homotheties, there is only one bi-invariant metric on T. Since the Lie algebra of T is $\mathfrak{t} := \mathrm{Span}_\mathbb{R}\{X_1\}$, D'Atri and Ziller's Theorem mentioned in Remark 3 ([DZ79, Thm. 3]) ensures that every naturally reductive left-invariant metric on $\mathrm{SU}(2)$ is isometric to

$$g_{\alpha,\beta} := \beta\, g_{(1,1,1)}|_\mathfrak{t} \oplus \alpha\, g_{(1,1,1)}|_\mathfrak{p} = g_{(\sqrt{\beta},\sqrt{\alpha},\sqrt{\alpha})} \tag{16}$$

for some $\alpha, \beta > 0$, where $\mathfrak{p} = \mathrm{Span}_\mathbb{R}\{X_2, X_3\}$. These metrics are precisely the 3-dimensional Berger spheres.

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