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The k-limited packing and k-tuple domination problems in strongly chordal, P_4-tidy and split graphs [1]

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Abstract

The notion of k-limited packing in a graph is a generalization of 2-packing. For a given non negative integer k, a subset B of vertices is a k-limited packing if there are at most k elements of B in the closed neighborhood of every vertex. On the other side, a k-tuple domination set in a graph is a subset of vertices D such that every vertex has at least k elements of D in its closed neighborhood. In this work we first reveal a strong relationship between these notions, and obtain from a result due to Liao and Chang (2002), the polynomiality of the k-limited packing problem for strongly chordal graphs.

We also prove that, in coincidence with the case of domination, the k-limited packing problem is NP-complete for split graphs. Finally, we prove that both problems are polynomial for the non-perfect class of P_4-tidy graphs, including the perfect classes of P_4-sparse graphs and cographs.

Keywords: k-tuple dominating set, k-limited packing, P_4-tidy graphs, split graphs, computational complexity

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1 Introduction

The concepts of packing and domination in graph theory are good models for many location problems in operation research. As was introduced in [3], in the packing case we consider the location of necessary but obnoxious facilities in specified placements, in such a way that no more than a fixed number k of these facilities should be placed inside the "neighborhood" of each placement. We are interested in placing the maximum number of facilities in a given scenario. In contrast, in the dominating case at least a fixed number k of facilities are required inside the "neighborhood" of each placement. In this case, we wish to place the minimum number of facilities that satisfy the requirements.

For both problems it is usual to model the scenario by a graph G, where the possible locations for the facilities correspond to the subsets of its vertex set.

In [1] we have presented a linear-time algorithm for the k-limited packing problem in trees, for any non negative integer k. The proof was based on a generalized version of a k-limited packing presented in detailed in the next section.

On the other side, Liao and Chang ([9] and [10]) also provided a linear-time algorithm for the k-tuple dominating problem for strongly chordal graphs, which in particular includes trees. The algorithm is also based on a generalized notion of a k-tuple dominating set. Besides, they showed that the problem is NP-complete for split graphs and left open the complexity of this problem for other subclasses of perfect graphs.

In this paper we show that the generalized versions of the k-limited packing and k-tuple dominating problems mentioned above are equivalent. From this equivalence and the results by Liao and Chang in [10], we obtain for the generalized version of the k-limited packing problem, the polynomiality for strongly chordal graphs and the NP-completeness for split graphs. Besides, we prove that the problem remains NP-complete for split graphs when restricted to the particular instances corresponding to the k-limited packing problem.

Finally, we analyze the behavior of the the k-limited packing and k-tuple domination numbers for the union and join of two given graphs. Our results allow us to prove the polynomiality of the k-limited packing and k-tuple domination problems for P_4-tidy graphs. Since P_4-tidy graphs generalize P_4-sparse graphs, our results provide another class of perfect graphs —P_4-sparse— for

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which the k-dominating problem is polynomial, as asked by Liao and Chang [10].

2 Basic definitions and notation

Graphs in this work are simple and for a graph G, $V(G)$ and $E(G)$ denotes respectively, its vertex and edge sets. A graph G is *trivial* if it has at most one vertex.

For $v \in V(G)$, $N[v]$ denotes the *closed neighborhood* of v in G. The *degree of* v in G, denoted by $\delta(v)$, is $|N[v]| - 1$.

Given two graphs G_1 and G_2, with $V(G_1) \cap V(G_2) = \emptyset$, the *(disjoint) union* of G_1 and G_2, denoted by $G_1 \cup G_2$, is the graph with $V(G) = V(G_1) \cup V(G_2)$ and $E(G) = E(G_1) \cup E(G_2)$. The *join* of G_1 and G_2, denoted by $G_1 \vee G_2$, is the graph with $V(G) = V(G_1) \cup V(G_2)$ and $E(G) = E(G_1) \cup E(G_2) \cup \{ij : i \in V(G_1), j \in V(G_2)\}$.

Given a graph G and a non negative integer k, $B \subseteq V(G)$ is a k-limited packing in G if $|N[v] \cap B| \leq k$, for every $v \in V(G)$. On the other side, $D \subseteq V(G)$ is a k-tuple dominating set in G if $|N[v] \cap D| \geq k$, for every $v \in V(G)$.

Both concepts have been generalized in [1] and [9], respectively. The notation for the domination case has been slightly modified in order to make it consistent with the notation for the packing case.

Definition 2.1 Let G be a graph.

- Given $\mathbf{c} = (c_v) \in \mathbb{Z}_+^{V(G)}$ and $\mathcal{A} \subseteq V(G)$, $B \subseteq V(G)$ is a $(\mathbf{c}, \mathcal{A})$-*limited packing* in G if $B \subseteq \mathcal{A}$ and $|B \cap N[v]| \leq c_v$, for every $v \in V(G)$.
- Given $\mathbf{r} = (r_v) \in \mathbb{Z}_+^{V(G)}$ and $\mathcal{R} \subseteq V(G)$, $D \subseteq V(G)$ is a $(\mathbf{r}, \mathcal{R})$-*tuple dominating set* in G if $\mathcal{R} \subseteq D$ and $|D \cap N[v]| \geq r_v$, for every $v \in V(G)$.

Given a graph G, a vector $\mathbf{c} \in \mathbb{Z}_+^{V(G)}$, $\mathcal{A} \subseteq V(G)$ and $b \in \mathbb{Z}_+$, the $(\mathbf{c}, \mathcal{A})$-*limited packing problem* is to decide if there exists a $(\mathbf{c}, \mathcal{A})$-limited packing in G of size at least b. The $(\mathbf{c}, \mathcal{A})$-*limited packing number*, $L_{\mathbf{c},\mathcal{A}}(G)$, is defined as the maximum cardinality of a $(\mathbf{c}, \mathcal{A})$-limited packing in G. Clearly, if $c_v \geq \delta(v)$ or $c_v \geq |\mathcal{A}|$ for every $v \in V(G)$, we have $L_{\mathbf{c},\mathcal{A}}(G) = |\mathcal{A}|$.

When $\mathcal{A} = V(G)$ and $c_v = k$ for every $v \in V(G)$, $(\mathbf{c}, \mathcal{A})$-limited packings are k-limited packings and $L_{\mathbf{c},\mathcal{A}}(G)$ is denoted by $L_k(G)$. Notice that $L_0(G) = 0$ for every graph G.

On the other side, given a graph G, a vector $\mathbf{r} \in \mathbb{Z}_+^{V(G)}$, $\mathcal{R} \subseteq V(G)$ and $d \in \mathbb{Z}_+$, the $(\mathbf{r}, \mathcal{R})$-*tuple domination problem* is to decide if there exists a

$(\mathbf{r},\mathcal{R})$-tuple dominating set in G of size at most d. The $(\mathbf{r},\mathcal{R})$-*tuple dominating number*, $\gamma_{\mathbf{r},\mathcal{R}}(G)$, is defined as the minimum cardinality of an $(\mathbf{r},\mathcal{R})$-tuple dominating set in G. It is clear that, if $r_v \geq \delta(v) + 2$ for some $v \in V(G)$, there does not exist an $(\mathbf{r},\mathcal{R})$-tuple dominating set in G; in this case we define $\gamma_{\mathbf{r},\mathcal{R}}(G) := +\infty$.

When $\mathcal{R} = \emptyset$ and $r_v = k$ for every $v \in V(G)$, $(\mathbf{r},\mathcal{R})$-tuple dominating sets are k-tuple dominating sets and $\gamma_{\mathbf{r},\mathcal{R}}(G)$ is denoted by $\gamma_k(G)$. Notice that $\gamma_0(G) = 0$ for every graph G.

3 The k-limited packing number for strongly chordal graphs and for split graphs

Let us begin this section by remarking the following, which implies that the $(\mathbf{c},\mathcal{A})$-limited packing and $(\mathbf{r},\mathcal{R})$-tuple domination problems are equivalent:

Remark 3.1 Let G be a graph, $\mathcal{A} \subseteq V(G)$ and $\mathbf{c} \in \mathbb{Z}_+^{V(G)}$. Then, $B \subseteq \mathcal{A}$ is a $(\mathbf{c},\mathcal{A})$-limited packing in G if and only if $D := V(G) - B$ is a $(\mathbf{r}, V(G) \setminus \mathcal{A})$-tuple dominating set in G, where $r_v = \max\{0, \delta(v) + 1 - c_v\}$.

Clearly, from this remark, any algorithm that solves one of the problems also solves the other. Since the $(\mathbf{r},\mathcal{R})$-tuple domination problem is linear for strongly chordal graphs [10], we obtain:

Corollary 3.2 *The $(\mathbf{c},\mathcal{A})$-limited packing problem can be solved in linear time for strongly chordal graphs.*

Let us observe that the polynomiality of both problems on strongly chordal graphs can be also derived from the total balancedness of their incidence matrices of the closed neighborhoods (see, for example, [2] and [5]). The step forward provided by the above corollary is the linearity in the time resolution of the corresponding algorithms.

A graph is split if its vertex set can be partitioned into a clique Q and an stable set S. In [10] it is proved that the k-tuple domination problem is NP-complete for split graphs, via a reduction of the vertex cover problem. Given a graph $G = (V,E)$, they construct a split graph $G' = (V',E')$ with $V' = V \cup S \cup E$, where $S = \{s_1 \cdots, s_{k-1}\}$ and $E' = \{uv : u \neq v,\ u,v \in V \cup E\} \cup \{ve : v \in V,\ e \in E,\ v \text{ endpoint of } e\} \cup \{s_i e : s_i \in S,\ e \in E\}$. It is not difficult to see that G has a vertex cover of size α if and only if G' has a k-tuple dominating set of size $\alpha + k - 1$.

As a consequence, the $(\mathbf{r},\mathcal{R})$-tuple domination problem and the $(\mathbf{c},\mathcal{A})$-limited packing problem are NP-complete.

To end this section we show that the $(\mathbf{c},\mathcal{A})$-limited packing problem remains NP-complete on the particular instances corresponding to a fixed k. We have:

Theorem 3.3 *The k-limited packing problem is NP-complete for split graphs.*

The proof is based on a reduction of the stable set problem. Given a graph G with $V(G) = \{v_j : \ j = 1 \ldots n\}$ and $E(G) = \{e_l : \ l = 1, \cdots, m\}$, we construct the split graph G' with $V(G') = Q \cup S$ with $Q = E(G)$ and S is the union of k "copies" of $V(G)$, that is $S = \bigcup_{i=1}^{k} \{v_j^i : 1 \leq j \leq n\}$. For each vertex $e = v_p v_q \in Q$, $N[e] \cap S = \bigcup_{i=1}^{k} \{v_p^i, v_q^i\}$. We can prove that there exists a stable set in G of size α if and only if there exists a k-limited packing in G' of size $k\alpha$. te

In the next section we study the k-limited packing and the k-tuple domination numbers on families of graphs where both parameters can be polynomially obtained.

4 The k-limited packing and the k-tuple domination numbers of P_4-tidy graphs

P_4-tidy graphs were introduced by Rusu (see [4]), generalizing cographs and P_4-sparse graphs [6]. Let us take as definition of P_4-tidy graphs a characterization given in [4]. Before presenting it, let us recall the following definitions. A *spider* is a graph whose vertex set can be partitioned into S, C and R, where $S = \{s_1 \cdots, s_r\}$ is a stable set, $C = \{c_1, \cdots, c_r\}$ is a clique, $r \geq 2$; and the *head* R is allowed to be empty. Moreover, all vertices in R are adjacent to all vertices in C and non-adjacent to all vertices in S. In (a *thin spider*) s_i is adjacent to c_j if and only if $i = j$ and in (a *thick spider*), s_i is adjacent to c_j if and only if $i \neq j$. It is straightforward that the complement of a thin spider is a thick spider, and vice-versa. The triple (S, C, R) is called the *spider partition* and can be found in linear time [7].

One one hand, given a graph G which is not a spider, it is P_4-tidy if and only if, in case G and its complementary graph $\overline{G}$ are connected, then G is a trivial graph, P_5, $\overline{P}_5$, or C_5. On the other hand, given a spider G with partition (S, C, R), G is P_4-tidy if and only if the subgraph induced by R is P_4-tidy.

It is not difficult to prove that the family of P_4-tidy graphs is hereditary and self-complementary (for details see [4], [6] and [8]).

Therefore, given a non trivial P_4-tidy graph G distinct from P_5, $\overline{P}_5$, C_5 and a spider, G is the union or the join of two P_4-tidy graphs strictly "smaller" that G.

This fact leads us to study the packing and domination parameters under the graph union and join operations. We obtain the following result:

Proposition 4.1 *Let G_1 and G_2 two graphs and k a non negative integer number. Then,*

(i) $L_k(G_1 \cup G_2) = L_k(G_1) + L_k(G_2)$ *and* $\gamma_k(G_1 \cup G_2) = \gamma_k(G_1) + \gamma_k(G_2)$.

(ii) $L_k(G_1 \vee G_2) = \max\{s + r : s, r \leq k,\ s, r \in \mathbf{Z}_+,\ s \leq L_{k-r}(G_1),\ r \leq L_{k-s}(G_2)\}$.

(iii) $\gamma_k(G_1 \vee G_2) = \min\{s + r : s, r \leq k,\ s, r \in \mathbf{Z}_+,\ \gamma_{k-r}(G_1) \leq s \leq |V(G_1)|,\ \gamma_{k-s}(G_2) \leq r \leq |V(G_2)|\}$.

The result for the union is straightforward. For the join, the proof is based on the following fact: if $B \subset V(G_1) \cup V(G_2)$ with $|B \cap V(G_1)| = s$ and $|B \cap V(G_2)| = r$, then B is a k-limited packing in $G_1 \vee G_2$ if and only if $B \cap V(G_1)$ is a $(k-r)$-limited packing in G_1 and $B \cap V(G_2)$ is a $(k-s)$-limited packing in G_2. The same remark is valid for the behavior of $\gamma_k(G_1 \vee G_2)$ on the join.

Let us recall that, given k, if we are able to calculate in polynomial time $L_s(G)$, for $s \leq k$ for P_5, $\overline{P}_5$, C_5 and spiders, we can calculate in polynomial time $L_k(G)$ for every P_4-tidy graph. The same can be said for the k-tuple dominating number. Let us recall that $L_0(G) = \gamma_0(G) = 0$ for every G. We list some easily verified facts:

(i) $L_1(C_5) = 1$, $L_2(C_5) = 3$ and $L_k(C_5) = 5$ for every $k \geq 3$.
$\gamma_1(C_5) = 2$, $\gamma_2(C_5) = 4$, $\gamma_3(C_5) = 5$ and $\gamma_k(C_5) = +\infty$ for every $k \geq 4$.

(ii) $L_1(P_5) = 2$, $L_2(P_5) = 4$ and $L_k(P_5) = 5$ for every $k \geq 3$.
$\gamma_1(P_5) = 2$, $\gamma_2(P_5) = 4$ and $\gamma_k(P_5) = +\infty$ for every $k \geq 3$.

(iii) $L_1(\overline{P}_5) = 1$; $L_2(\overline{P}_5) = 3$, $L_3(\overline{P}_5) = 4$ and $L_k(\overline{P}_5) = 5$ for every $k \geq 4$.
$\gamma_1(\overline{P}_5) = 2$; $\gamma_2(\overline{P}_5) = 3$, $\gamma_3(\overline{P}_5) = 5$ and $\gamma_k(\overline{P}_5) = +\infty$ for every $k \geq 4$.

It remains to obtain the parameters for spiders. We obtain that:

Proposition 4.2 *Let G a spider with spider partition (S, C, R). If $k \geq 2|S| + |R|$, then $L_k(G) = 2|S| + |R|$. Moreover, if $1 \leq k \leq 2|S| + |R| - 1$, we have:*

(i) *If G is thin, then* $L_k(G) = |S| + \min\{k - 1, |S| + |R|\}$.

(ii) *If G is thick, then $L_k(G) = k + 1$ if $k \geq |S| - 1$, and $L_k(G) = k$, if*

$1 \leq k \leq |S| - 2$.

Proposition 4.3 *Let k be a non negative integer and G a spider with spider partition (S, C, R).*

(i) *If G is thin, then $\gamma_1(G) = |S|$, $\gamma_2(G) = 2|S|$ and $\gamma_k(G) = +\infty$ for every $k \geq 3$.*

(ii) *If G is thick, then $\gamma_k(G) = k + 1$ for $1 \leq k \leq |S|$, $\gamma_{|S|}(G) = 2|S|$ and $\gamma_k(G) = +\infty$ for every $k \geq |S| + 1$.*

The proofs for thin spiders are almost straightforward. For thick spiders, the key is to focus on the neighborhood of vertices of the stable set S.

As a corollary of the above results we obtain:

Theorem 4.4 *The k-limited packing and the k-tuple domination numbers can be calculated in polynomial time for P_4-tidy graphs, for any k.*

5 Final remarks

In relation with the question raised by Liao and Chang in [10] we remark that, from theorem 4.4, the k-tuple domination number —and also the k-limited packing number— may be calculated in polynomial time for the class of P_4-sparse graphs, for any k, providing in this way another class of perfect graphs where the k-tuple domination problem becomes polynomial-time solvable.

Finally, our guiding intuition is that the results concerning the union and join operations presented in this work (proposition 4.1) may be extended to the generalized versions of both problems treated in this work, and therefore, following remark 3.1, one of the related parameters may be obtained from the other.

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