Acta Mathematica Sinica, English Series
Sep., 201x, Vol. x, No. x, pp. 1–12
Published online: August 15, 201x
DOI: 0000000000000000
Http://www.ActaMath.com

Weighted inequalities for fractional type operators with some homogeneous kernels

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Abstract In this paper we study integral operators of the form

$$T_\alpha f(x) = \int_{\mathbb{R}^n} |x - A_1 y|^{-\alpha_1} \dots |x - A_m y|^{-\alpha_m} f(y) dy,$$

where A_i are certain invertible matrices, $\alpha_i > 0$, $1 \leq i \leq m$, $\alpha_1 + \dots + \alpha_m = n - \alpha$, $0 \leq \alpha < n$. For $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$ we obtain the $L^p(\mathbb{R}^n, w^p) - L^q(\mathbb{R}^n, w^q)$ boundedness for weights w in $A(p, q)$ satisfying that there exists $c > 0$ such that $w(A_i x) \leq c w(x)$, a.e.$x \in \mathbb{R}^n$, $1 \leq i \leq m$. Moreover we obtain the appropriate weighted BMO and weak type estimates for certain weights satisfying the above inequality. We also give a Coifman Type estimate for these operators.

Keywords Fractional operators, Calderón-Zygmund operators, BMO , Muckenhoupt weights.

MR(2000) Subject Classification 42B20, 42B25

1 Introduction

In this paper we will study integral operators of the form

$$T_\alpha f(x) = \int_{\mathbb{R}^n} |x - A_1 y|^{-\alpha_1} \dots |x - A_m y|^{-\alpha_m} f(y) dy, \tag{1.1}$$

for certain invertible matrices A_i , $\alpha_i > 0$, $1 \leq i \leq m$, $\alpha_1 + \dots + \alpha_m = n - \alpha$, $0 \leq \alpha < n$. We observe if $f \in L^\infty_c(\mathbb{R}^n, dx)$ then $T_\alpha f(x) < \infty$ a.e. $x \in \mathbb{R}^n$.

In [1] Ricci and Sjögren obtained the $L^p(\mathbb{R}, dx)$ boundedness, $p > 1$, for a family of maximal operators on the three dimensional Heisenberg group. Some of these operators arise in the study of the boundary behavior of Poisson integrals on the symmetric space $SL(\mathbb{R}^3)/SO(3)$. To get the principal result, they studied the boundedness, on $L^2(\mathbb{R}, dx)$ of the integral operator

$$Tf(x) = \int |x - y|^{-\alpha} |x + y|^{\alpha - 1} f(y) dy,$$

$0 < \alpha < 1$.

Received x x, 201x, accepted x x, 201x
The authors are partially supported by CONICET and SECYT-UNC

In [2] Godoy and Urciuolo study integral operators of the form

$$Tf(x) = \int_{\mathbb{R}^n} |x-y|^{-\alpha} |x+y|^{-n+\alpha} f(y)dy,$$

$0 < \alpha < n$. They obtain the $L^p(\mathbb{R}^n, dx)$ boundedness and the weak type $(1,1)$ of them.

We recall that a weight w is a locally integrable and non negative function. The Muckenhoupt class A_p, $1 < p < \infty$ is defined as the class of weights w such that

$$\sup_Q \left[\left(\frac{1}{|Q|} \int_Q w \right) \left(\frac{1}{|Q|} \int_Q w^{-\frac{1}{p-1}} \right)^{p-1} \right] < \infty.$$

For $p = 1$, A_1 is the class of weights w satisfying that there exists $c > 0$ such that $Mw(x) \le cw(x)$ a.e. $x \in \mathbb{R}^n$, where M is the Hardy-Littlewood maximal function. Also $A_\infty = \cup_{1 \le p < \infty} A_p$. In [3] we considered integral operators of the form (1.1) for $\alpha = 0$ and $A_i = a_i I$, $i = 1, ..., m$. We obtain the $L^p(\mathbb{R}^n, w)$ boundedness of them, and a weighted $(1,1)$ inequality, for weights w in A_p, $p \ge 1$, satisfying that there exists $c > 0$ such that $w(a_i x) \le cw(x)$, a.e. $x \in \mathbb{R}^n$, $1 \le i \le m$. Moreover we prove that $\|Tf\|_{BMO} \le c \|f\|_\infty$ for a wide family of functions $f \in L^\infty(\mathbb{R}^n, dx)$, where $BMO = BMO(\mathbb{R}^n)$ is the classical space of function with bounded mean oscillation defined by John and Nirenberg in [4].

In [5] Rocha and Urciuolo consider the operator T_α in the case that the matrices $A_1, \ldots, A_m$ satisfy the following hypothesis

(H) *A_i is invertible and $A_i - A_j$ is invertible for $i \ne j$, $1 \le i, j \le m$.*

They obtain that T_α is bounded from $H^p(\mathbb{R}^n, dx)$ into $L^q(\mathbb{R}^n, dx)$, for $0 < p < \frac{n}{\alpha}$ and $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$.

For $0 \le \alpha < n$ we take the fractional maximal function as

$$M_\alpha f(x) = \sup_Q \frac{1}{|Q|^{1-\frac{\alpha}{n}}} \int_Q |f(x)|\, dx,$$

where the supremum is taken along all the cubes Q such that x belongs to Q. We observe that $M = M_0$. It is well known (see [6]) that M_α is bounded on $L^p(\mathbb{R}^n, w^p)$ into $L^q(\mathbb{R}^n, w^q)$, for $1 < p < \frac{n}{\alpha}$ and $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$, if and only if

$$\sup_Q \left[\left(\frac{1}{|Q|} \int_Q w^q \right)^{\frac{1}{q}} \left(\frac{1}{|Q|} \int_Q w^{-p'} \right)^{\frac{1}{p'}} \right] < \infty. \tag{1.2}$$

The class of functions that satisfy (1.2) is called $A(p,q)$.

For $p = 1$, the class $A(1,q)$ should be interpreted as the class of weights w satisfying

$$\sup_Q \left[\left(\frac{1}{|Q|} \int_Q w^q \right)^{\frac{1}{q}} \left(\left\| w^{-1} \chi_Q \right\|_\infty \right) \right] < \infty, \tag{1.3}$$

also for $p > 1$, $A(p, \infty)$ is the class of weights w satisfying

$$\sup_Q \left[\left(\|w \chi_Q\|_\infty \right) \left(\frac{1}{|Q|} \int_Q w^{-p'} \right)^{\frac{1}{p'}} \right] < \infty.$$

We note that the statement $w \in A(\infty,\infty)$ is equivalent to $w^{-1} \in A_1$. We recall that $f \in L^1_{loc}(\mathbb{R}^n, dx)$ belongs to BMO if there exist $c > 0$ such that

$$\frac{1}{|Q|}\int \left| f(x) - \frac{1}{|Q|}\int_Q |f| \right| dx \le c$$

for all cube $Q \subset \mathbb{R}^n$. The smallest bound c for which the above inequality holds is called $\|f\|_*$.

There is also a weighted version of BMO, this is $BMO(w)$ that is described by the semi norm

$$|||f|||_w = \sup_Q \|w\chi_Q\|_\infty \left(\frac{1}{|Q|}\int_Q \left| f(x) - \frac{1}{|Q|}\int_Q f \right| dx \right). \tag{1.4}$$

In [6] Muckenhoupt and Wheeden study the classical fractional integral operator I_α. They obtain the following endpoint results, if $w \in A\left(\frac{n}{\alpha},\infty\right)$ then

$$|||I_\alpha f|||_w \le c\left(\int (|f|\, w)^{\frac{n}{\alpha}}\right)^{\frac{\alpha}{n}}, \tag{1.5}$$

also if $w \in A(1, \frac{n}{n-\alpha})$ they obtain the weighted weak type $(1, \frac{n}{n-\alpha})$ estimate.

In this paper we study the operator T_α defined as in (1.1) for matrices A_i satisfying the hypothesis (H). Throughout this paper we will consider weights w such that there exists $c > 0$ with

$$w(A_i x) \le c w(x), \tag{1.6}$$

a.e. $x \in \mathbb{R}^n$, $1 \le i \le m$.

In §2 we obtain a Coifman type estimate for this operator, namely we find which is the maximal operator that controls T_α in weighted $p-$norms, for any $w \in A_\infty$ satisfying (1.6). A fundamental tool to prove this result is the inequality (2.1). As a consequence of this theorem we get the $L^p(\mathbb{R}^n, w^p)$ - $L^q(\mathbb{R}^n, w^q)$ boundedness for $1 < p < \frac{n}{\alpha}$ and $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$, for w in $A(p,q)$ satisfying (1.6).

In §3 we prove an inequality analogous to (1.5) for the operator T_α and weights w in $A\left(\frac{n}{\alpha},\infty\right)$ satisfying (1.6). We also prove a weighted weak type $(1, \frac{n}{n-\alpha})$ estimate for T_α and weights w in $A(1, \frac{n}{n-\alpha})$ satisfying (1.6).

Throughout this paper c and C will denote positive constants, not the same at each occurrence.

2 Main Results

The following result is a Coifman type estimate for the operator T_α.

Theorem 2.1 *Let $0 \le \alpha < n$ and $\alpha_1, \dots, \alpha_m > 0$ such that $\alpha_1 + ... + \alpha_m = n - \alpha$. Let T_α be defined as in (1.1) where $A_1, \dots, A_m$ satisfy the hypothesis (H). If $0 < p < \infty$ and $w \in A_\infty$ satisfies (1.6) then there exists $C > 0$ such that*

$$\int_{\mathbb{R}^n} |T_\alpha f(x)|^p w(x)\, dx \le C \int_{\mathbb{R}^n} |M_\alpha f(x)|^p w(x)\, dx, \qquad f \in L^\infty_c(\mathbb{R}^n, dx)$$

always holds if the left hand side is finite.

Proof Let $f \in L^\infty_c(\mathbb{R}^n, dx)$, $f \geq 0$ and $0 < \delta < 1$. We prove now that there exists $C > 0$ such that

$$M^\sharp_\delta(T_\alpha f)(x) \leq C \sum_{i=1}^m M_\alpha f(A_i^{-1}x), \tag{2.1}$$

where $M^\sharp_\delta f = (M^\sharp |f|^\delta)^{1/\delta}$ with

$$M^\sharp f(x) = \sup_{B \ni x} \inf_{a \in \mathbb{R}} \frac{1}{|B|} \int_B |f(y) - a|\, dy.$$

In [5] Rocha and Urciuolo prove that T_α is a bounded operator from $L^s(\mathbb{R}^n, dx)$ into $L^q(\mathbb{R}^n, dx)$, for $1 < s < \frac{n}{\alpha}$, and $\frac{1}{q} = \frac{1}{s} - \frac{\alpha}{n}$, so $T_\alpha(f) \in L^1_{loc}(\mathbb{R}^n, dx)$ and $M^\sharp_\delta(T_\alpha f)(x)$ is well defined for all $x \in \mathbb{R}^n$. Let $x \in \mathbb{R}^n$ and let $B = B(x_B, r)$ be a ball that contains x, centered at x_B with radius r, and $T_\alpha f(x_B) < \infty$. We write $\widetilde{B} = B(x_B, 4r)$, and for $1 \leq i \leq m$ we also set $\widetilde{B}_i = A_i^{-1}\widetilde{B}$. Let $f_1 = f\chi_{\bigcup_{1\leq i \leq m} \widetilde{B_i}}$ and let $f_2 = f - f_1$.

We choose $a = T_\alpha f_2(x_B)$. We consider first the case $0 < \alpha < n$. By Jensen's inequality and from the inequality

$$|t^\delta - s^\delta|^{1/\delta} \leq |t - s|,$$

which holds for any positive t, s,

$$\begin{aligned}
\left(\frac{1}{|B|}\int_B |(T_\alpha f)^\delta(y) - a^\delta| dy\right)^{1/\delta} &\leq \left(\frac{1}{|B|}\int_B |T_\alpha f(y) - a| dy\right) \\
&\leq \left(\frac{1}{|B|}\int_B T_\alpha f_1(y) dy\right) + \left(\frac{1}{|B|}\int_B |T_\alpha f_2(y) - a| dy\right) \\
&= I + II.
\end{aligned}$$

$$\begin{aligned}
I &= \frac{1}{|B|}\int_B T_\alpha f_1(y) dy \\
&\leq \frac{1}{|B|}\int_B \sum_{i=1}^m \int_{\widetilde{B}_i} |y - A_1 z|^{-\alpha_1} \ldots |y - A_m z|^{-\alpha_m} f(z) dz\, dy \\
&\leq \sum_{i=1}^m \frac{1}{|B|}\int_{\widetilde{B}_i} f(z) \int_B |y - A_1 z|^{-\alpha_1} \ldots |y - A_m z|^{-\alpha_m} dy\, dz.
\end{aligned}$$

If $z \in \widetilde{B}_i$

$$\begin{aligned}
&\int_B |y - A_1 z|^{-\alpha_1} \ldots |y - A_m z|^{-\alpha_m} dy \\
&\leq \sum_{j=1}^m \int_{\{y \in B : |y - A_j z| \leq |y - A_l z|,\ 1 \leq l \leq m\}} |y - A_1 z|^{-\alpha_1} \ldots |y - A_m z|^{-\alpha_m} dy \\
&\leq \sum_{j=1}^m \int_{\{y \in B : |y - A_j z| \leq |y - A_l z|,\ 1 \leq l \leq m\}} |y - A_j z|^{\alpha - n} dy \qquad (2.2) \\
&\leq \sum_{j=1}^m \int_{B(x_B, 6r)} |y - A_j z|^{\alpha - n} dy \\
&\leq C r^\alpha,
\end{aligned}$$

the last inequality follows since we take $y \in B$ such that $|y-A_jz| \leq |y-A_lz|$, for all $1 \leq l \leq m$, so in particular

$$|A_jz - x_B| \leq |A_jz - y| + |y - x_B| \leq |A_iz - y| + |y - x_B| \leq |A_iz - x_B| + 2|y - x_B| \leq 6r,$$

and so $A_jz \in B(x_B, 6r)$. Then

$$I \leq C \sum_{i=1}^{m} \frac{1}{|\widetilde{B}_i|^{1-\frac{\alpha}{n}}} \int_{\widetilde{B}_i} f(z)\, dz \leq C \sum_{i=1}^{m} M_\alpha f(A_i^{-1}x).$$

On the other hand

$$\begin{aligned} II &= \frac{1}{|B|} \int_B |T_\alpha f_2\,(y) - T_\alpha f_2\,(x_B)|\, dy \\ &\leq \frac{1}{|B|} \int_B \int_{\left(\bigcup\limits_{1\leq k\leq m} \widetilde{B}_k\right)^c} |K(y,z) - K(x_B,z)| f(z) dz dy \\ &\leq \sum_{j=1}^{m} \frac{1}{|B|} \int_B \int_{Z_j} |K(y,z) - K(x_B,z)| f(z) dz dy, \end{aligned}$$

where

$$K(x,y) = |x - A_1y|^{-\alpha_1} \ldots |x - A_my|^{-\alpha_m} \tag{2.3}$$

and

$$Z_j = \left(\bigcup_{1\leq k\leq m} \widetilde{B}_k \right)^c \bigcap \{z : |x_B - A_jz| \leq |x_B - A_iz|, \text{for } 1 \leq i \leq m\}. \tag{2.4}$$

We estimate now $|K(y,z) - K(x_B,z)|$ for $y \in B$ and $z \in Z_j$. By the mean value theorem we obtain

$$|K(y,z) - K(x_B,z)| \leq |x_B - y| \sum_{i=1}^{m} \frac{\alpha_i}{|\xi - A_iz|^{\alpha_i+1} \prod\limits_{l\neq i} |\xi - A_lz|^{\alpha_l}},$$

for some ξ between x_B and y. But

$$|\xi - A_iz| \geq |x_B - A_iz| - |\xi - x_B| \geq \frac{|x_B - A_iz|}{2},$$

for $i = 1, \ldots, m$, thus

$$|K(y,z) - K(x_B,z)| \leq c \frac{|x_B - y|}{|x_B - A_jz|^{n-\alpha+1}}. \tag{2.5}$$

For $1 \le j \le m$, we denote $m_j = \min\{|A_j y| : |y| = 1\}$ and $n_j = \min\{|A_j^{-1} y| : |y| = 1\}$.

$$\begin{aligned}
&\frac{1}{|B|}\int_B \int_{Z_j} |K(y,z) - K(x_B,z)|\, f(z)dzdy \\
&\le \frac{c}{|B|}\int_B \int_{Z_j} \frac{|x_B - y|}{|x_B - A_j z|^{n-\alpha+1}} f(z)dzdy \\
&\le \frac{c}{|B|}\int_B \int_{|A_j^{-1}x_B - z| \ge 4n_j r} \frac{|x_B - y|}{|x_B - A_j z|^{n-\alpha+1}} f(z)dzdy \\
&\le \frac{c}{(m_j)^{n-\alpha+1}\,|B|}\int_B \sum_{k=2}^{\infty} \int_{2^k n_j r \le |A_j^{-1}x_B - z| < 2^{k+1} n_j r} \frac{|x_B - y|}{\left|A_j^{-1}x_B - z\right|^{n-\alpha+1}} f(z)dzdy \\
&\le \frac{c}{(m_j)^{n-\alpha+1}} \sum_{k=2}^{\infty} \frac{r}{(2^k n_j r)^{n-\alpha+1}} \int_{\left|A_j^{-1}x_B - z\right| < 2^{k+1} n_j r} f(z)dz \\
&\le \frac{c}{(m_j)^{n-\alpha+1} n_j} \sum_{k=2}^{\infty} \frac{1}{2^k} M_\alpha f(A_j^{-1}x) \\
&\le C_j M_\alpha f(A_j^{-1}x).
\end{aligned}$$

Thus

$$II \le C \sum_{j=1}^{m} M_\alpha f(A_j^{-1}x),$$

and so (2.1) follows in the case $\alpha > 0$. To prove (2.1) for $\alpha = 0$, we estimate

$$\begin{aligned}
\left(\frac{1}{|B|}\int_B |(T_0 f)^\delta(y) - a^\delta| dy\right)^{1/\delta} &\le \left(\frac{C}{|B|}\int_B (T_0 f_1)^\delta(y) dy\right)^{1/\delta} + \left(\frac{C}{|B|}\int_B |(T_0 f_2)^\delta(y) - a^\delta| dy\right)^{1/\delta} \\
&= I + II.
\end{aligned}$$

To estimate I we observe that T_0 is of weak type $(1,1)$ with respect to the Lebesgue measure. To prove this result we perform the classical Calderón-Zygmund decomposition $f = g + b$. Since T_0 is bounded on $L^2(\mathbb{R}^n, dx)$ (see [5]) we obtain that

$$|\{x : |T_0 g(x)| > \lambda\}| \le \frac{C}{\lambda}||f||_1.$$

On the other hand, as above, it is easy to check that the kernel K satisfies the "Hörmander type" inequality

$$\int_{(\widetilde{B}_k)^c} |K(y,z) - K(x_B,z)| dz \le C,$$

where $y \in B(x_B, r)$, $\widetilde{B}_k = A_k^{-1}\widetilde{B}$ and $\widetilde{B} = B(x_B, 4r)$. As usual we obtain that

$$|\{x : |T_0 b(x)| > \lambda\}| \le \frac{C}{\lambda}||f||_1.$$

We use now Kolmogorov's inequality (see exercise 2.1.5. p. 91 in [7]) to get

$$\begin{aligned}
I &\le \frac{C}{|B|}\int_{\mathbb{R}^n} f_1(y) dy \le \sum_{j=1}^{m} \frac{C}{|B|}\int_{\widetilde{B}_j} f(y) dy \\
&\le C \sum_{j=1}^{m} M f(A_j^{-1}x).
\end{aligned}$$

To estimate II, we first use Jensen's inequality and then we proceed just as in the case $0 < \alpha < n$ to get

$$II \leq C \sum_{j=1}^{m} Mf(A_j^{-1}x),$$

and so (2.1) follows in this case.

Let $w \in A_\infty$, then there exists $r > 1$ such that $w \in A_r$. For $0 < p < \infty$ we take $0 < \delta < 1$, such that $1 < r < p/\delta$, thus $w \in A_{p/\delta}$. If $||T_\alpha f||_{p,w} < \infty$ then also $||(T_\alpha f)^\delta||_{\frac{p}{\delta},w} < \infty$. Under these conditions we can apply Theorem 2.20 in [8], p. 410, and from (2.1) we get

$$\begin{aligned}
\int_{\mathbb{R}^n} |T_\alpha f(x)|^p w(x)\, dx &\leq \int_{\mathbb{R}^n} (M(T_\alpha f)^\delta(x))^{p/\delta} w(x)\, dx \\
&\leq \int_{\mathbb{R}^n} (M_\delta^\sharp(T_\alpha f)(x))^p w(x)\, dx \\
&\leq C \int_{\mathbb{R}^n} \left(\sum_{i=1}^{m} M_\alpha f(A_i^{-1}x) \right)^p w(x)\, dx \\
&\leq C \sum_{i=1}^{m} \int_{\mathbb{R}^n} (M_\alpha f)^p(x) w(A_i x)\, dx \\
&\leq C \int_{\mathbb{R}^n} (M_\alpha f(x))^p w(x)\, dx,
\end{aligned}$$

where the last inequality follows since w satisfies (1.6). □

Lemma 2.2 *Let $0 \leq \alpha < n$ and $\alpha_1, \ldots, \alpha_m > 0$ such that $\alpha_1 + \ldots + \alpha_m = n - \alpha$. Let T_α be defined as in (1.1) where $A_1, \ldots, A_m$ satisfy the hypothesis (H). If $1 < p < \frac{n}{\alpha}$, $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$, $w \in A(p,q)$, and $f \in L_c^\infty(\mathbb{R}^n, dx)$ then $T_\alpha(f) \in L^q(\mathbb{R}^n, w^q)$.*

Proof Let $\mathcal{M}_j = \max\{|A_j y| : |y| = 1\}$, and let $\mathcal{M} = \max\{\mathcal{M}_j : 1 \leq j \leq m\}$. If supp$f \subset B(0,R)$ and $|x| > 2\mathcal{M}R$, then $|K(x,y)| \leq \frac{C}{|x|^{n-\alpha}}$ and so

$$\begin{aligned}
\int_{|x|>2\mathcal{M}R} |T_\alpha f|^q w^q dx &\leq C_R \int_{|x|>2\mathcal{M}R} \frac{w^q(x)}{|x|^{(n-\alpha)q}}\, dx \\
&\leq C_R \sum_{k=1}^{\infty} \int_{2^k\mathcal{M}R \leq |x| < 2^{k+1}\mathcal{M}R} \frac{w^q(x)}{|x|^{(n-\alpha)q}}\, dx \\
&\leq C_R \sum_{k=1}^{\infty} 2^{-k(n-\alpha)q}\, w^q(B(0, 2^{k+1}\mathcal{M}R))\, dx,
\end{aligned}$$

where $w(B) = \int_B w(x)dx$. Since $w^q \in A_r$ with $r = 1 + q/p'$, there exists $\tilde{r} < r$ such that $w^q \in A_{\tilde{r}}$, thus $w^q(B(0, 2^{k+1}\mathcal{M}R)) \leq C(R, w, n)2^{kn\tilde{r}}$ (see Lemma 2.2 in [8], p.396). We observe that $q(n-\alpha) = nr > n\tilde{r}$ and so the last sum is finite. Now by Hölder's inequality, for any $\epsilon > 0$,

$$\int_{|x|<2\mathcal{M}R} |T_\alpha f|^q w^q dx \leq \left(\int_{|x|<2\mathcal{M}R} |T_\alpha f|^{q\frac{1+\epsilon}{\epsilon}} \right)^{\frac{\epsilon}{1+\epsilon}} \left(\int_{|x|<2\mathcal{M}R} w^{q(1+\epsilon)} \right)^{\frac{1}{1+\epsilon}},$$

by reverse Hölder's inequality we can choose $\epsilon > 0$ such the second integral is finite. The first one is finite since $T_\alpha f \in L^{q\frac{1+\epsilon}{\epsilon}}(\mathbb{R}^n, dx)$ (see [5]). □

From this Lemma and Theorem 2.1 we get the following

Theorem 2.3 *Let $0 \le \alpha < n$ and $\alpha_1, \dots, \alpha_m > 0$ such that $\alpha_1 + \dots + \alpha_m = n - \alpha$. Let T_α be defined as in (1.1) where $A_1, \dots, A_m$ satisfy the hypothesis (H). If $1 < p < \frac{n}{\alpha}$, $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$, and $w \in A(p,q)$ satisfies (1.6) then there exits $C > 0$ such that*

$$\left(\int_{\mathbb{R}^n} |T_\alpha f(x)|^q w^q(x)\, dx \right)^{\frac{1}{q}} \le C \left(\int_{\mathbb{R}^n} |f(x)|^p w^p(x)\, dx \right)^{\frac{1}{p}}, \quad f \in L_c^\infty(\mathbb{R}^n, dx). \tag{2.6}$$

Proof Since $w \in A(p,q)$ for $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$ then $w^q \in A_{1+q/p'} \subset A_\infty$.Without loss of generality we take $f \in L_c^\infty(\mathbb{R}^n, dx)$. By Lemma 2.2 we have that $T_\alpha f \in L^q(\mathbb{R}^n, w^q)$. Moreover we recall that $w \in A(p,q)$ implies that M_α is bounded from $L^p(\mathbb{R}^n, w^p)$ into $L^q(\mathbb{R}^n, w^q)$, so we apply Theorem 2.1 to obtain

$$\left(\int |T_\alpha f|^q w^q dx \right)^{\frac{1}{q}} \le C \left(\int (M_\alpha f)^q w^q dx \right)^{\frac{1}{q}} \le C \left(\int |f|^p w^p dx \right)^{\frac{1}{p}}. \qquad \square$$

Remark 2.4 The inequality in (2.6) still holds for $f \in L^p(\mathbb{R}^n, w^p)$. Indeed if $f \ge 0$ we define $f_N(x) = f \chi_{\{x: f(x) \le N\}} \chi_{\{x:|x| \le N\}}$, then (2.6) can be applied to f_N. Taking the limit as $N \to \infty$ and using the monotone convergence theorem,(2.6) follows for general f.

3 Endpoint results

In this paragraph we obtain an estimation of the type (1.5) for the operator T_α and for certain weights in the class $A(\frac{n}{\alpha}, \infty)$. We also prove that T_α satisfies a weighted weak type $(1, \frac{n}{n-\alpha})$ estimate for certain weights in $A(1, \frac{n}{n-\alpha})$.

In the following theorem, if $\alpha = 0$ we understand that $\left(\int (|f|\, w)^{\frac{n}{\alpha}} \right)^{\frac{\alpha}{n}} = ||fw||_\infty$.

Theorem 3.1 *Let $0 \le \alpha < n$ and $\alpha_1, \dots, \alpha_m > 0$ such that $\alpha_1 + \dots + \alpha_m = n - \alpha$. Let T_α be defined as in (1.1) where $A_1, \dots, A_m$ satisfy the hypothesis (H). If $w \in A(n/\alpha, \infty)$ and satisfies (1.6), then there exits $C > 0$ such that*

$$\| |T_\alpha f| \|_w \le C \left(\int (|f|\, w)^{\frac{n}{\alpha}} \right)^{\frac{\alpha}{n}}, \quad f \in L_c^\infty(\mathbb{R}^n, dx).$$

Proof We recall that

$$\| |T_\alpha f| \|_w = \sup_Q \|w \chi_Q\|_\infty \left(\frac{1}{|Q|} \int_Q \left| T_\alpha f(y) - \frac{1}{|Q|} \int_Q T_\alpha f \right| dy \right),$$

without loss of generality we can replace the cubes Q by balls B in (1.3) and in (1.4). Let $B = B(x_B, r)$ be a ball centered at x_B with radius r. We write $\widetilde{B} = B(x_B, 4r)$, and for $1 \le i \le m$ we also set $\widetilde{B}_i = A_i^{-1} \widetilde{B}$. Let $f_1 = f \chi_{\bigcup_{1 \le i \le m} \widetilde{B_i}}$ and let $f_2 = f - f_1$.

$$\begin{aligned} &\|w \chi_B\|_\infty \left(\frac{1}{|B|} \int_B |T_\alpha f(y) - \frac{1}{|B|} \int_B T_\alpha f| dy \right) \\ &\le \|w \chi_B\|_\infty \left(\frac{1}{|B|} \int_B |T_\alpha f_1(y) - \frac{1}{|B|} \int_B T_\alpha f_1| dy \right) \\ &+ \|w \chi_B\|_\infty \left(\frac{1}{|B|} \int_B |T_\alpha f_2(y) - \frac{1}{|B|} \int_B T_\alpha f_2| dy \right) = I + II. \end{aligned} \tag{3.1}$$

If $0<\alpha<n$

$$
\begin{aligned}
I &\leq \frac{2\,\|w\chi_B\|_\infty}{|B|}\int_B |T_\alpha f_1|(y)dy \\
&\leq \frac{2\,\|w\chi_B\|_\infty}{|B|}\int_B \sum_{i=1}^m \int_{\widetilde{B}_i} |y-A_1z|^{-\alpha_1}\ldots|y-A_mz|^{-\alpha_m}\,|f(z)|dz\,dy \\
&\leq \sum_{i=1}^m \frac{2\,\|w\chi_B\|_\infty}{|B|}\int_{\widetilde{B}_i}|f(z)|\int_B |y-A_1z|^{-\alpha_1}\ldots|y-A_mz|^{-\alpha_m}\,dy\,dz.
\end{aligned}
$$

If $z\in\widetilde{B}_i$, as in (2.2) we have

$$\int_B |y-A_1z|^{-\alpha_1}\ldots|y-A_mz|^{-\alpha_m}\,dy\leq Cr^\alpha,$$

Then

$$I\leq C\frac{\|w\chi_B\|_\infty}{|B|^{1-\frac{\alpha}{n}}}\sum_{i=1}^m\int_{\widetilde{B}}|f(A_i^{-1}z)|w(z)w^{-1}(z)\;dz.$$

Now we use Hölder's inequality to obtain that

$$
\begin{aligned}
&\int_{\widetilde{B}}|f(A_i^{-1}z)|w(z)w^{-1}(z)\;dz \\
&\leq\left(\int_{\widetilde{B}}(|f(A_i^{-1}z)|w(z))^{\frac{n}{\alpha}}\;dz\right)^{\frac{\alpha}{n}}\left(\int_{\widetilde{B}}(w(z))^{\frac{-n}{n-\alpha}}dz\right)^{\frac{n-\alpha}{n}}.
\end{aligned}
$$

From this inequality and the hypothesis about w, we get that

$$
\begin{aligned}
I &\leq C\sum_{i=1}^m\left(\int_{\widetilde{B}}(|f(A_i^{-1}z)|w(z))^{n/\alpha}\;dz\right)^{\alpha/n}\leq C\sum_{i=1}^m\left(\int_{\mathbb{R}^n}(|f(z)|w(A_iz))^{n/\alpha}\;dz\right)^{\alpha/n} \\
&\leq C\left(\int_{\mathbb{R}^n}(|f(z)|w(z))^{n/\alpha}\;dz\right)^{\alpha/n}.
\end{aligned}
$$

On the other hand

$$
\begin{aligned}
II &= \|w\chi_B\|_\infty\left(\frac{1}{|B|}\int_B\left|T_\alpha f_2(y)-\frac{1}{|B|}\int_B T_\alpha f_2(t)dt\right|dy\right) \\
&\leq \frac{\|w\chi_B\|_\infty}{|B|}\int_B\frac{1}{|B|}\int_B\int_{\left(\bigcup\limits_{1\leq k\leq m}\widetilde{B}_k\right)^c}|K(y,z)-K(t,z)||f(z)|dzdtdy \\
&\leq \|w\chi_B\|_\infty\sum_{j=1}^m\frac{1}{|B|}\int_B\frac{1}{|B|}\int_B\int_{Z_j}|K(y,z)-K(t,z)||f(z)|dzdtdy,
\end{aligned}
$$

where $K(x,y)$ and Z_j are defined in (2.3) and (2.4) respectively.

Now $|K(y,z)-K(t,z)|\leq|K(y,z)-K(x_B,z)|+|K(x_B,z)-K(t,z)|$ and proceeding as in the proof of (2.5) we get for $z\in Z_j$ and $y,t\in B$

$$
\begin{aligned}
|K(y,z)-K(t,z)| &\leq C\left(\frac{|x_B-y|}{|x_B-A_jz|^{n-\alpha+1}}+\frac{|x_B-t|}{|x_B-A_jz|^{n-\alpha+1}}\right) \\
&\leq \frac{Cr}{|x_B-A_jz|^{n-\alpha+1}}\,.
\end{aligned}
$$

Then

$$\begin{aligned} II &\leq C\,\|w\chi_B\|_\infty\; r\sum_{j=1}^{m}\int_{Z_j}\frac{|f(z)|}{|x_B-A_jz|^{n-\alpha+1}}dz \\ &\leq C\,\|w\chi_B\|_\infty\; r\sum_{j=1}^{m}\int_{|x_B-A_jz|\geq 4r}\frac{|f(z)|}{|x_B-A_jz|^{n-\alpha+1}}dz \\ &\leq C\,\|w\chi_B\|_\infty\; r\sum_{j=1}^{m}\int_{|x_B-x|\geq 4\,r}\frac{|f(A_j^{-1}x)|}{|x_B-x|^{n-\alpha+1}}w(x)w^{-1}(x)dx. \end{aligned} \tag{3.2}$$

Again from Hölder's inequality we obtain

$$\begin{aligned} II &\leq C\,\|w\chi_B\|_\infty\; r\left(\int_{|x_B-x|\geq 4\,r}\left(\frac{w^{-1}(x)}{|x_B-x|^{n-\alpha+1}}\right)^{\frac{n}{n-\alpha}}dx\right)^{\frac{n-\alpha}{n}} \\ &\times\sum_{j=1}^{m}\left(\int_{\mathbb{R}^n}(|f(A_j^{-1}x)|w(x))^{\frac{n}{\alpha}}dx\right)^{\frac{\alpha}{n}}. \end{aligned}$$

Now,

$$\begin{aligned} &\int_{|x_B-x|\geq 4\,r}\left(\frac{w^{-1}(x)}{|x_B-x|^{n-\alpha+1}}\right)^{\frac{n}{n-\alpha}}dx \\ &\qquad =\sum_{k=2}^{\infty}\int_{2^kr\leq|x_B-x|<2^{k+1}r}\left(\frac{w^{-1}(x)}{|x_B-x|^{n-\alpha+1}}\right)^{\frac{n}{n-\alpha}}dx \\ &\qquad \leq\sum_{k=2}^{\infty}\frac{1}{(2^kr)^{(n-\alpha+1)\frac{n}{n-\alpha}}}\int_{|x_B-x|<2^{k+1}r}(w^{-1}(x))^{\frac{n}{n-\alpha}}dx, \end{aligned} \tag{3.3}$$

then, since $w\in A(n/\alpha,\infty)$,

$$\|w\chi_B\|_\infty\; r\left(\int_{|x_B-x|\geq 4\,r}\left(\frac{w^{-1}(x)}{|x_B-x|^{n-\alpha+1}}\right)^{\frac{n}{n-\alpha}}dx\right)^{\frac{n-\alpha}{n}}\leq C,$$

so

$$II\leq C\sum_{j=1}^{m}\left(\int_{\mathbb{R}^n}(|f(A_j^{-1}x)|w(x))^{\frac{n}{\alpha}}dx\right)^{\frac{\alpha}{n}}\leq C\left(\int_{\mathbb{R}^n}(|f(x)|w(x))^{\frac{n}{\alpha}}dx\right)^{\frac{\alpha}{n}},$$

where the last inequality follows from (1.6).

If $\alpha=0$, since $w\in A(\infty,\infty)$ then there exists $r>1$ such that $w^{-r}\in A_1$. From Hölder's inequality, the $L^r(\mathbb{R}^n,dx)$ boundedness of T_0 and since w satisfies (1.6) we get

$$\begin{aligned} I &\leq\frac{2\,\|w\chi_B\|_\infty}{|B|}\int_B|T_0f_1(y)|dy\leq 2\,\|w\chi_B\|_\infty\left(\frac{1}{|B|}\int_B|T_0f_1(y)|^rdy\right)^{\frac{1}{r}} \\ &\leq C\|w\chi_B\|_\infty\left(\frac{1}{|B|}\int_{\mathbb{R}^n}|f_1(y)|^rdy\right)^{\frac{1}{r}}\leq C\|w\chi_B\|_\infty\sum_{j=1}^{m}\left(\frac{1}{|B|}\int_{\widetilde{B}_j}|f(y)|^rdy\right)^{\frac{1}{r}} \\ &\leq C\|w\chi_B\|_\infty\sum_{j=1}^{m}\left(\frac{1}{|B|}\int_B|f(A_j^{-1}y)|^rw^r(y)w^{-r}(y)dy\right)^{\frac{1}{r}} \\ &\leq C||fw||_\infty\|w\chi_B\|_\infty\left(\frac{1}{|B|}\int_Bw^{-r}(y)dy\right)^{\frac{1}{r}}\leq C||fw||_\infty. \end{aligned}$$

We observe that (3.2) still holds for $\alpha = 0$, and since w satisfies (1.6) we get that

$$II \leq C \, \|w\chi_B\|_\infty \, ||fw||_\infty \, r \sum_{j=1}^{m} \int_{|x_B - x| \geq 4\, r} \frac{w^{-1}(x)}{|x_B - x|^{n+1}} dx \leq C||fw||_\infty,$$

where the last inequality follows from (3.3) and from the condition $w \in A(\infty, \infty)$. Taking sup over all balls B in (3.1) we get the Theorem. □

Theorem 3.2 *Let $0 \leq \alpha < n$ and $\alpha_1, \ldots, \alpha_m > 0$ such that $\alpha_1 + ... + \alpha_m = n - \alpha$. Let T_α be defined as in (1.1) where $A_1, \ldots, A_m$ satisfy the hypothesis (H). If $w \in A(1, \frac{n}{n-\alpha})$ and satisfies (1.6) then there exists $C > 0$ such that*

$$\sup_{\lambda > 0} \lambda (w^{\frac{n}{n-\alpha}} \{x : |T_\alpha f(x)| > \lambda\})^{\frac{n-\alpha}{n}} \leq C \int |f(x)| w(x) dx, \quad f \in L^1(\mathbb{R}^n, w).$$

Proof Without loss of generality we suppose $f \in L^\infty_c(\mathbb{R}^n, dx)$. Given $w \in A_\infty$ there exists $\delta > 0$ and $C > 0$ such that

$$w\{x : Mf(x) > 2\lambda, M^\sharp f(x) \leq \gamma\lambda\} \leq C\gamma^\delta w\{x : Mf(x) > \lambda\},$$

for any $\gamma > 0$ (see [9] p.146).

For $q \geq 1$,

$$\begin{aligned}
\sup_{0<\lambda<N} \lambda^q w\{x : Mf(x) > 2\lambda\} &\leq \sup_{0<\lambda<N} \lambda^q w\{x : Mf(x) > 2\lambda, M^\sharp f(x) \leq \gamma\lambda\} \\
&\quad + \sup_{0<\lambda<N} \lambda^q w\{x : M^\sharp f(x) > \gamma\lambda\} \\
&\leq \sup_{0<\lambda<N} C\lambda^q \gamma^\delta w\{x : Mf(x) > \lambda\} + \sup_{0<\lambda<N} \lambda^q w\{x : M^\sharp f(x) > \gamma\lambda\} \\
&\leq \sup_{0<\lambda<2N} C\lambda^q \gamma^\delta w\{x : Mf(x) > \lambda\} + \sup_{0<\lambda<N} \lambda^q w\{x : M^\sharp f(x) > \gamma\lambda\}.
\end{aligned}$$

The left side of the inequality can be written as

$$\sup_{0<\lambda<2N} 2^{-q} \lambda^q w\{x : Mf(x) > \lambda\},$$

so we choose γ such that $C\gamma^\delta < 2^{-q-1}$ to obtain

$$\sup_{\lambda>0} \lambda^q w\{x : Mf(x) > \lambda\} \leq C \sup_{\lambda>0} \lambda^q w\{x : M^\sharp f(x) > \gamma\lambda\}.$$

We observe that (2.1) still holds for $\delta = 1$ if $0 < \alpha < n$, also $w \in A(1, \frac{n}{n-\alpha})$ implies $w^{\frac{n}{n-\alpha}} \in A_\infty$. So for $0 < \alpha < n$, $q = \frac{n}{n-\alpha}$, we obtain

$$\begin{aligned}
\sup_{\lambda>0} \lambda (w^{\frac{n}{n-\alpha}} \{x : |T_\alpha f|(x) > \lambda\})^{\frac{n-\alpha}{n}} &\leq C \sup_{\lambda>0} \lambda (w^{\frac{n}{n-\alpha}} \{x : MT_\alpha f(x) > \lambda\})^{\frac{n-\alpha}{n}} \\
&\leq C \sup_{\lambda>0} \lambda (w^{\frac{n}{n-\alpha}} \{x : M^\sharp T_\alpha f(x) > \gamma\lambda\})^{\frac{n-\alpha}{n}} \\
&\leq C \sup_{\lambda>0} \lambda (w^{\frac{n}{n-\alpha}} \{x : \sum_{i=1}^{m} M_\alpha f(A_i^{-1}x) > C\gamma\lambda\})^{\frac{n-\alpha}{n}}.
\end{aligned}$$

Since w satisfies (1.6), it is easy to check that

$$w^{\frac{n}{n-\alpha}} \{x : M_\alpha f(A_i^{-1}x) > \lambda\} \leq C_i w^{\frac{n}{n-\alpha}} \{x : M_\alpha f(x) > \lambda\},$$

so
$$\sup_{\lambda>0} \lambda (w^{\frac{n}{n-\alpha}}\{x : |T_\alpha f|(x) > \lambda\})^{\frac{n-\alpha}{n}} \leq C \sup_{\lambda>0} \lambda (w^{\frac{n}{n-\alpha}}\{x : M_\alpha f(x) > \lambda\})^{\frac{n-\alpha}{n}}$$
$$\leq C \int |f(x)| w(x) dx.$$
where the last inequality follows since $w \in A(1, \frac{n}{n-\alpha})$.

The proof for $\alpha = 0$ is analogous to the proof of Theorem 1 b) in [3].

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