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Integral Equations and Operator Theory

Spectral Shorted Operators

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To the memory of Gert K. Pedersen

Abstract. If $\mathcal{H}$ is a Hilbert space, $\mathcal{S}$ is a closed subspace of $\mathcal{H}$, and A is a positive bounded linear operator on $\mathcal{H}$, the spectral shorted operator $\rho(\mathcal{S}, A)$ is defined as the infimum of the sequence $\Sigma(\mathcal{S}, A^n)^{1/n}$, where $\Sigma(\mathcal{S}, B)$ denotes the shorted operator of B to $\mathcal{S}$. We characterize the left spectral resolution of $\rho(\mathcal{S}, A)$ and show several properties of this operator, particularly in the case that $\dim \mathcal{S} = 1$. We use these results to generalize the concept of Kolmogorov complexity for the infinite dimensional case and for non invertible operators.

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1. Introduction

Let $\mathcal{H}$ be a separable Hilbert space and $L(\mathcal{H})$ the algebra of bounded operators on $\mathcal{H}$. Given a positive (i.e. semidefinite non negative) operator $A \in L(\mathcal{H})$ and a closed subspace $\mathcal{S}$ of $\mathcal{H}$, the shorted operator $\Sigma(\mathcal{S}, A)$ was defined by Krein [8] and Anderson-Trapp [2] by

$$\Sigma(\mathcal{S}, A) = \max\{X \in L(\mathcal{H})^+ : X \le A \quad \text{and} \quad R(X) \subseteq \mathcal{S}\},$$

where the maximum is taken for the natural order relation in $L(\mathcal{H})^+$, the set of positive operators in $L(\mathcal{H})$ (see [8], [16], [15], [1], [2], [14] [9]).

In a previous paper [3], the authors have defined, under the assumption that $\dim \mathcal{H} < \infty$, the *spectral shorted matrix*:

$$\rho(\mathcal{S}, A) = \lim_{m\to\infty} \Sigma(\mathcal{S}, A^m)^{1/m} = \inf_{m\to\infty} \Sigma(\mathcal{S}, A^m)^{1/m}. \tag{1.1}$$

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This paper, which is a continuation of [3], is devoted to study the natural generalization of ρ to the infinite dimensional setting. If $\dim \mathcal{H} = \infty$, $A \in L(\mathcal{H})^+$ and the subspace $\mathcal{S}$ is closed, the operator $\rho(\mathcal{S}, A)$ is also defined by equation (1.1). We call this operator the *spectral shorted operator* associated to $\mathcal{S}$ and A.

Many properties of the spectral shorted matrices proved in [3] also hold for spectral shorted operators, but some of them must be formulated in terms of the spectral measure of A instead of eigenvalues and eigenspaces, as in [3].

As in the matrix case, the properties of ρ are strongly related with the so called *spectral order* of positive operators. Recall the definition of the spectral order $\preccurlyeq$ in $L(\mathcal{H})^+$: given $A, B \in L(\mathcal{H})^+$, we write $A \preccurlyeq B$ if $A^m \leq B^m$ for all $m \geq 1$. The spectral order was extensively studied by M. P. Olson in [11], where the following characterization is proved: given $A, B \in L(\mathcal{H})^+$, then $A \preccurlyeq B$ if and only if $f(A) \leq f(B)$ for every non-decreasing map $f : [0, +\infty) \to \mathbb{R}$.

Section 2 contains preliminaries and a brief account of the main properties of the shorting operation, spectral order and spectral resolutions. In section 3 we collect those properties of ρ which can be generalized to the infinite dimensional setting in a, more or less, direct way. The most subtle tool is the use of continuity of the map $t \mapsto t^r$ (for $0 \leq r \leq 1$) with respect to the strong operator topology on $L(\mathcal{H})^+$. It is used, for instance, for proving that for every $t > 0$,

$$\rho\left(\mathcal{S}, A^t\right) = \rho\left(\mathcal{S}, A\right)^t. \tag{1.2}$$

This relevant property, which is not shared by the usual shorting operation, is one of main reasons to study $\rho(\mathcal{S}, A)$.

The spectral order provides the following link with Krein and Anderson-Trapp definition of the shorted operator: $\rho(\mathcal{S}, A)$ is the biggest (in both orders $\leq$ and $\preccurlyeq$) element D of $L(\mathcal{H})^+$ such that $D \preccurlyeq A$ and $R(D) \subseteq \mathcal{S}$ (see Theorem 3.5). This shows the monoticity of $\rho(\mathcal{S}, \cdot)$ with respect to the preorder $\preccurlyeq$ and allows us to get some results about limits of spectral shorted operators. In this section, we also get a complete characterization of $\rho(\mathcal{S}, A)$ in terms of the (left) spectral resolution of A: for every $0 < \lambda \in \mathbb{R}$,

$$\aleph_{[\lambda,\infty)}(\rho(\mathcal{S}, A)) = \aleph_{[\lambda,\infty)}(A) \wedge P_{\mathcal{S}}.$$

This results allows us to get simple proofs in our context of several properties of spectral shorted matrices. For example, given $A \in L(\mathcal{H})^+$ and two closed subspaces $\mathcal{S}$ and $\mathcal{T}$ of $\mathcal{H}$,

1. $\rho(\mathcal{S} \cap \mathcal{T}, A) = \rho(\mathcal{T}, \rho(\mathcal{S}, A))$.
2. $\sigma(\rho(\mathcal{S}, A)) \subseteq \sigma(A)$.
3. $f(\rho(\mathcal{S}, A)) = \rho(\mathcal{S}, f(A))$, for every non-decreasing *right continuous* positive function f defined on $[0, +\infty)$.
4. $\lambda_{min}(A) P_{\mathcal{S}} \leq \rho(\mathcal{S}, A)$, where $\lambda_{min}(C) = \min \sigma(C)$, for $C \in L(\mathcal{H})^+$.
5. If $\rho(\mathcal{S}, A)$ is considered as acting in $\mathcal{S}$, then

$$\lambda_{min}(\rho(\mathcal{S}, A)) = \min\{\mu \in \sigma(A) : \ P_{\mathcal{S}}\ \aleph_{[\mu,\mu+\varepsilon)}(A) \neq 0 \ \ \forall\, \varepsilon > 0\}.$$

The case $\dim \mathcal{S} = 1$ is extensively studied in section 5. If $\mathcal{S}$ is the subspace generalted by the unit vector ξ, we denote by $\rho\,(A, \xi)$ the unique positive number such that $\rho\,(\mathcal{S}, A) = \rho\,(A, \xi)\, P_{\mathcal{S}}$. The following list contains the main results of this section:

1. If $A \in L(\mathcal{H})^+$ and $\xi \in \mathcal{H}$ is an unit vector, then
$$\rho\,(A, \xi) = \min \sigma\,(\rho\,(\mathcal{S}, A)) = \min \Big\{ \mu \in \sigma\,(A) : \ \aleph_{[\mu, \mu+\varepsilon)}(A)\xi \neq 0 \ \ \forall\, \varepsilon > 0 \Big\}.$$
2. $\rho\,(A, \xi) = \max\{\lambda \in \sigma\,(A) : \xi \in R(\aleph_{[\lambda,\infty)}(A))\}$.
3. If A is invertible, then $\rho\,(A, \xi) = \lim\limits_{m\to\infty} \|A^{-m}\xi\|^{-1/m} = \inf\limits_{m\in\mathbb{N}} \|A^{-m}\xi\|^{-1/m}$.
4. If $R(A)$ is closed and $\xi \in R(A)$, then, $\rho\,(A, \xi) = \lim_{m\to\infty} \|(A^\dagger)^m\xi\|^{-1/m}$, where $A^\dagger$ is the Moore-Penrose pseudo-inverse of A. If $\xi \notin R(A)$, then $\rho\,(A, \xi) = 0$.
5. If $\sigma_{sh}\,(A) = \big\{\rho\,(A, \xi) : \|\xi\| = 1\big\}$, then
$$\sigma_{sh}\,(A) = \sigma_+\,(A) \cup \sigma_{pt}(A) = \big\{\lambda \in \sigma\,(A) : \forall\ \varepsilon > 0\ ,\ \aleph_{[\lambda, \lambda+\varepsilon)}(A) \neq 0\big\},$$
where $\sigma_{pt}(A)$ denotes the point spectrum of A, i.e the set of eigenvalues of A and $\sigma_+\,(A)$ is the set of points in $\sigma\,(A)$ which are limit point of $\sigma\,(A) \setminus \{\lambda\}$ from the right. This shows that $\sigma_{sh}\,(A)$ is dense in $\sigma\,(A)$, but $\sigma_{sh}\,(A) \neq \sigma\,(A)$ in general.
6. $\rho\,(A, \xi) \neq 0$ if and only if $\xi \in R_0(A) : \bigcup_{\lambda>0} R(\aleph_{[\lambda,\infty)}(A)) \subseteq R(A)$.

In [5], J. I. Fujii and M. Fujii defined the Kolmogorov complexity
$$K\,(A, \xi) = \lim_{n\to\infty} \frac{\log(\langle A^n\xi, \xi\rangle)}{n} = \log \lim_{n\to\infty} \langle A^n\xi, \xi\rangle^{1/n}\,. \tag{1.3}$$
for an invertible positive matrix A and a unit vector ξ and proved several properties of K. In [3] it was proved that, if $\mathcal{S}$ is the subspace generated by ξ, then
$$K\,(A, \xi) = \log \rho\,\big(A^{-1}, \xi\big)^{-1}\,.$$
For $\dim \mathcal{H}\infty$ and $A \in L(\mathcal{H})^+$ not necessarily invertible, we define a generalized version of the Kolmogorov complexity as follows: given $\xi \in \mathcal{H}$ and $A \in L(\mathcal{H})^+$, we denote by
$$k\,(A, \xi) = \lim_{n\to\infty} \langle A^n\xi, \xi\rangle^{1/n}\,,$$
so that, $k\,(A, \xi) = \exp K\,(A, \xi)$ if $K\,(A, \xi)$ is defined as in equation (1.3). Our definition is without logarithms in order to avoid the value $-\infty$.

If $\xi \in \mathcal{H}$ and $A \in L(\mathcal{H})^+$, we prove:

1. If $\|\xi\| = 1$, then the sequence $\langle A^n\xi, \xi\rangle^{1/n}$ is increasing. So that, for every $\xi \in \mathcal{H}$, there exists $\lim\limits_{n\to\infty} \langle A^n\xi, \xi\rangle^{1/n}$.
2. $k\,(A, \xi) = k\,(A, a\xi)$ for every $0 \neq a \in \mathbb{C}$.
3. $k\,(A, \xi) = k\,\big(A, \aleph_{[\lambda,\infty)}(A)\xi\big)$ for every $\lambda > 0$ such that $\aleph_{[\lambda,\infty)}(A)\xi \neq 0$.
4. $k\,(A, \xi) \neq 0$ (i.e. $K\,(A, \xi) \neq -\infty$) if and only if $P_{\overline{R(A)}}\,\xi \in R_0(A) \setminus \{0\}$.

5. If $\xi \neq 0$, then $k(A,\xi) \in \sigma(A)$. Moreover,

$$\left\{k(A,\xi) : \xi \neq 0\right\}\left\{\lambda \in \sigma(A) : \aleph_{(\lambda+\varepsilon,\lambda]}(A) \neq 0 \ , \ \forall\, \varepsilon > 0 \ \right\},$$

which is a dense subset of $\sigma(A)$.

6.

$$\begin{aligned} k(A,\xi) &= \min\left\{\lambda \in \sigma(A) : \xi \in R(\aleph_{(-\infty,\lambda]}(A))\right\} \\ &= \max\left\{\mu \in \sigma(A) : \ \aleph_{(\mu-\varepsilon,\mu]}(A)\xi \neq 0 \ \ \forall\, \varepsilon > 0\right\} \\ &= \sup\left\{\mu \in \sigma(A) : \ \aleph_{[\mu,\infty)}(A)\xi \neq 0\right\} . \end{aligned}$$

7. If $R(A)$ is closed, then
   (a) If $\xi \in R(A)$ then $k(A,\xi) = \rho\left(A^\dagger,\xi\right)^{-1}$.
   (b) If $\xi \notin R(A)$, but $P\xi \neq 0$, where $P = P_{R(A)}$, then

$$k(A,\xi) = \rho\left(A^\dagger, \frac{P\xi}{\|P\xi\|}\right)^{-1} .$$

2. Preliminaries

For an operator $A \in L(\mathcal{H})$, we denote by $R(A)$ the range of A, $N(A)$ the null-space of A, $\sigma(A)$ the spectrum of A, A^* the adjoint of A, $\rho(A)$ the spectral radius of A, and $\|A\|$ the operator norm of A. $L(\mathcal{H})_{sa}$ is the space of selfadjoint operators in $L(\mathcal{H})$ and $L(\mathcal{H})^+$ is the subset of $L(\mathcal{H})_{sa}$ of positive (i.e. semidefinite non-negative) operators. If $A \in L(\mathcal{H})_{sa}$, $\lambda_{min}(A) = \min \sigma(A) = \inf_{\|\xi\|=1}\langle A\xi,\xi\rangle$.

Given a closed subspace $\mathcal{S}$ of $\mathcal{H}$, $P_{\mathcal{S}}$ is the orthogonal (i.e. selfadjoint) projection onto $\mathcal{S}$. If P and Q are orthogonal projections, $P \wedge Q$ denotes the orthogonal projection onto $R(P) \cap R(Q)$. If $B \in L(\mathcal{H})$ satisfies $P_{\mathcal{S}}BP_{\mathcal{S}} = B$, we sometimes consider the compression of B to $\mathcal{S}$ (i.e. the restriction of B to $\mathcal{S}$ as a linear transformation form $\mathcal{S}$ to $\mathcal{S}$), and we say that we consider B as *acting* on $\mathcal{S}$. Several times this is done in order to consider $\sigma(B)$ just in terms of the action of B on $\mathcal{S}$. For example, if $B \geq \lambda P_{\mathcal{S}}$ for some $\lambda > 0$, then we can deduce that $0 \notin \sigma(B)$, if we consider B as acting on $\mathcal{S}$.

We use in this note several standard results of spectral theory, functional calculus and weak convergences of operators in $L(\mathcal{H})_{sa}$. About these matters, we refer the reader to the books of Pedersen [13] or Kadison and Ringrose [7]. If $A \in L(\mathcal{H})_{sa}$ we denote by E_A the spectral measure associated to A, defined by $E_A(\Delta) = \aleph_\Delta(A)$, for any Borel set $\Delta \subseteq \mathbb{R}$. By SOT convergence or topology we mean *strong operator topology* of $L(\mathcal{H})_{sa}$. In the following subsections, we state several known results which we shall need in the sequel and which we could not find explicitly mentioned in the literature.

Shorted operators

Definition 2.1 (Krein [8]**, Anderson and Trapp** [1]**,** [2]**).** Given $A \in L(\mathcal{H})^+$ and a closed subspace $\mathcal{S}$ of $\mathcal{H}$, the *shorted operator* of A to $\mathcal{S}$ is defined by

$$\Sigma(\mathcal{S}, A) = \max\{X \in L(\mathcal{H})^+ : X \leq A \quad \text{and} \quad R(X) \subseteq \mathcal{S}\},$$

where the maximum is taken for the natural order relation in $L(\mathcal{H})^+$.

Among many results proved by M.G. Krein [8], Anderson and Trapp [2], and E. L. Pekarev [14], we collect those which are relevant in this paper in the following theorem.

Theorem 2.2. *Let $\mathcal{S}$ and $\mathcal{T}$ be subspaces of $\mathcal{H}$ and let $A, B \in L(\mathcal{H})^+$. Then:*

1. *If $\mathcal{S} \subseteq \mathcal{T}$, then, $\Sigma(\mathcal{S}, A) \leq \Sigma(\mathcal{T}, A)$.*
2. *$\Sigma(\mathcal{S} \cap \mathcal{T}, A) = \Sigma(\mathcal{S}, \Sigma(\mathcal{T}, A))$.*
3. *If $A \leq B$, then, $\Sigma(\mathcal{S}, A) \leq \Sigma(\mathcal{S}, B)$.*
4. *Let $\mathcal{M} = A^{-1/2}(\mathcal{S})$. Then $\Sigma(\mathcal{S}, A) = A^{1/2} P_{\mathcal{M}} A^{1/2}$.*

There are also some results about the continuity of the shorting operation (see [2], Corollary 3).

Proposition 2.3. *Let A_n ($n \in \mathbb{N}$) be a sequence of positive operators such that $A_n \underset{n\to\infty}{\overset{\text{SOT}}{\searrow}} A$. Then, for every closed subspace $\mathcal{S}$ it holds $\Sigma(\mathcal{S}, A_n) \underset{n\to\infty}{\overset{\text{SOT}}{\searrow}} \Sigma(\mathcal{S}, A)$.*

Proposition 2.4. *Let $\mathcal{S}_n$ ($n \in \mathbb{N}$) and $\mathcal{S}$ be closed subspaces such that $P_{\mathcal{S}_n} \underset{n\to\infty}{\overset{\text{SOT}}{\searrow}} P_{\mathcal{S}}$. Then, for every $A \in L(\mathcal{H})^+$, it holds that $\Sigma(\mathcal{S}_n, A) \underset{n\to\infty}{\overset{\text{SOT}}{\searrow}} \Sigma(\mathcal{S}, A)$.*

Proof. Since $\{\Sigma(\mathcal{S}_n, A)\}$ is a non-increasing sequence, it has a strong limit, say L. As $\Sigma(\mathcal{S}_n, A) \leq A$ for all $n \in \mathbb{N}$, then $L \leq A$. On the other hand, $L \leq \Sigma(\mathcal{S}_n, A)$ implies

$$R(L^{1/2}) \subseteq R\left(\Sigma(\mathcal{S}_n, A)^{1/2}\right) \subseteq \mathcal{S}_n \qquad \forall\, n \in \mathbb{N}.$$

Therefore $R(L) \subset \bigcap_{n=1}^{\infty} \mathcal{S}_n = \mathcal{S}$. Finally, if $0 \leq X \leq A$ and $R(X) \subset \mathcal{S}$, then $R(X) \subseteq \mathcal{S}_n$, so that $X \leq \Sigma(\mathcal{S}_n, A)$, for all $n \in \mathbb{N}$. Therefore $X \leq L$. □

Spectral order

The spectral order was considered by Olson (see [11]) with the purpose of reporting an order relation with respect to which the real vector space of selfadjoint operators form a conditionally complete lattice. Throughout this note we shall only use the spectral order for positive operators, and this is the reason why we take the following statement as definition of the spectral order.

Definition 2.5. Let $A, B \in L(\mathcal{H})^+$. We write $A \preccurlyeq B$ if for every $m \in \mathbb{N}$ it holds that $A^m \leq B^m$. The relation $\preccurlyeq$ defined on $L(\mathcal{H})^+$ is a partial order called the *spectral order*.

Examples. Consider $A, B \in L(\mathcal{H})^+$. Then

1. If $AB = BA$ and $A \le B$, then, $A \preccurlyeq B$.
2. If $\dim \mathcal{H} = n < \infty$, then $A \preccurlyeq B$ if and only if there is a positive integer $k \le n$ and an sequence of positive matrices $\{D_i\}_{0 \le i \le k}$ such that, $D_0 = A$, $D_k = B$, $D_i \le D_{i+1}$ and $D_i D_{i+1} = D_{i+1} D_i$ $(i = 0, \cdots, k-1)$ (see [3]).

The next results was proved by Olson in [11].

Theorem 2.6. *Let $A, B \in L(\mathcal{H})^+$. The following statements are mutually equivalent.*

(1) $A \preccurlyeq B$,
(2) $\aleph_{[\lambda,\infty)}(A) \le \aleph_{[\lambda,\infty)}(B)$ *($0 \le \lambda < \infty$),*
(3) $f(A) \le f(B)$ *for every non-decreasing continuous function f on $[0, \infty)$.*

The following result about functions which are continuous relative to the SOT topology of $L(\mathcal{H})^+$ or $L(\mathcal{H})_{sa}$ is a key tool for the extension of the results about spectral shorted operators from matrices to operators in Hilbert spaces. A proof can be found, for example, in Pedersen's book [12], proposition 2.3.2.

Lemma 2.7. *Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function such that $f(0) = 0$ and $|f(t)| \le \alpha|t| + \beta$ for some positive numbers α and β. Then, if $\{A_\alpha\}_{\alpha \in \Lambda}$ is a net in $L(\mathcal{H})_{sa}$ such that $A_\alpha \xrightarrow{SOT} A \in L(\mathcal{H})_{sa}$, it holds that $f(A_\alpha) \xrightarrow{SOT} f(A)$, i.e. $f : L(\mathcal{H})_{sa} \to L(\mathcal{H})_{sa}$ is continuous for the SOT topology. In particular $f(t) = t^r$ for $0 \le r \le 1$ is SOT-continuous in $L(\mathcal{H})^+$.*

We shall use the next corollary of the lemma.

Proposition 2.8. *Let $\{A_n\}$ be a sequence in $L(\mathcal{H})^+$ such that $A_{n+1} \preccurlyeq A_n$, $n \in \mathbb{N}$ and $A_n \underset{n\to\infty}{\overset{SOT}{\searrow}} A \in L(\mathcal{H})^+$. Then, for every $k \in \mathbb{N}$, $A_n^k \underset{n\to\infty}{\overset{SOT}{\searrow}} A^k$. In particular, $A \preccurlyeq A_n$, $n \in \mathbb{N}$.*

Proof. Fix $k \in \mathbb{N}$. Since the sequence $\{A_n\}$ is non increasing with respect to the spectral order, there exists $B \in L(\mathcal{H})^+$ such that $A_n^k \underset{n\to\infty}{\overset{SOT}{\searrow}} B$. By Lemma 2.7, applied to the map $f(t) = t^{1/k}$, we can deduce that $A_n \underset{n\to\infty}{\overset{SOT}{\searrow}} B^{1/k} = A$. So that, $B = A^k$. □

Spectral resolutions

Given $f : \mathbb{R} \to L(\mathcal{H})$, we say that f is a right (resp. left) *spectral resolution* if

1. There exist $m, M \in \mathbb{R}$ such that $f(\lambda) = 0$ for $\lambda < m$ and $f(\lambda) = I$ for $\lambda > M$ (resp. $f(\lambda) = I$ for $\lambda < m$ and $f(\lambda) = 0$ for $\lambda > M$).
2. $f(\lambda)$ is a selfadjoint projection, for every $\lambda \in \mathbb{R}$.
3. If $\lambda \le \mu$ then $f(\lambda) \le f(\mu)$ (resp. $f(\lambda) \ge f(\mu)$) as operators.
4. f is continuous on the right (resp. f is continuous on the left).

Under these hypothesis, by standard results of spectral theory, there exists a unique $A \in L(\mathcal{H})_{sa}$ such that f is its spectral resolution, i.e.

$$f(\lambda) = E_A(\ (-\infty, \lambda]) = \aleph_{(-\infty,\lambda]}\ (A) \tag{2.1}$$

(resp. $f(\lambda) = E_A(\ [\lambda, \infty)\ ) = \aleph_{[\lambda,\infty)}(A)$). Conversely, if $A \in L(\mathcal{H})_{sa}$, then the map f defined by equation (2.1) is a right (resp. left) spectral resolution (see [7],[13]).

The relation between right and left spectral resolutions is given by the following identity: if $A \in L(\mathcal{H})_{sa}$, then $E_A(\ [-\lambda, \infty)\ ) = E_{-A}(\ (-\infty, \lambda])$. On the other hand, if f is a left spectral resolution, then $g(\lambda) = f(-\lambda)$ is a right spectral resolution. Then, if A is the operator associated to g, then $-A$ is the operator associated to f.

3. The spectral shorted operator

In this section we define the spectral shorted operator in the infinite dimensional setting, and we prove its basic properties. All results and proofs of this section are very similar as those which appear in [3] for the finite dimensional case; the main difference is that here we must use SOT-convergence instead of convergence in norm. Thus, in the proof of Proposition 3.4, we need to apply Lemma 2.7 about SOT-continuity of the map $A \mapsto A^r$ for $0 \le r \le 1$. Also Proposition 3.7 is a properly infinite dimensional result.

Throughout this section $A \in L(\mathcal{H})^+$ and $\mathcal{S}$ is a closed subspace of $\mathcal{H}$.

Proposition 3.1. *The map* $t \mapsto \Sigma(\mathcal{S}, A^t)^{1/t},\ t \in [1, \infty)$ *is non-increasing.*

Proof. Fix $t \ge 1$. Then $\Sigma(\mathcal{S}, A^t) \le A^t$. Since $0 \le 1/t \le 1$, by Löwner theorem we can deduce that $\Sigma(\mathcal{S}, A^t)^{1/t} \le A$. On the other hand $R(\Sigma(\mathcal{S}, A^t)^{1/t}) \subseteq \mathcal{S}$. So, by the definition of shorted operator, $\Sigma(\mathcal{S}, A^t)^{1/t} \le \Sigma(\mathcal{S}, A)$. Now, given $1 \le r \le s$, take $t = s/r \ge 1$. By the previous remarks, applied to A^r and t, we have that

$$\Sigma\left(\mathcal{S}, A^r\right) \ge \Sigma\left(\mathcal{S}, A^{rt}\right)^{1/t} = \Sigma\left(\mathcal{S}, A^s\right)^{r/s}.$$

Since $1/r \le 1$, again by Löwner theorem we get $\Sigma\left(\mathcal{S}, A^r\right)^{1/r} \ge \Sigma\left(\mathcal{S}, A^s\right)^{1/s}$. □

Definition 3.2. If $A \in L(\mathcal{H})^+$ and $\mathcal{S}$ is a closed subspace of $\mathcal{H}$, the *spectral shorted operator* of A to $\mathcal{S}$ is defined by

$$\rho(\mathcal{S}, A) = \inf_{t\ge 1} \Sigma(\mathcal{S}, A^t)^{1/t} = \lim_{t\to+\infty} \Sigma(\mathcal{S}, A^t)^{1/t},$$

where the limit is taken in the strong operator topology (SOT).

Remark 3.3. Let $A \in L(\mathcal{H})^+$ and let $\mathcal{S}$ and $\mathcal{T}$ be closed subspaces.

1. If $A = P_{\mathcal{T}}$, then $\rho(\mathcal{S}, A) = \Sigma(\mathcal{S}, A^t)^{1/t} = P_{\mathcal{S}\cap\mathcal{T}}$, for every $t \in [1, \infty)$.
2. If $AP = PA$, then $\rho(\mathcal{S}, A) = \Sigma(\mathcal{S}, A^t)^{1/t} = PA$, for every $t \in [1, \infty)$.
3. $\rho\left(\mathcal{S}, cA\right) = c\,\rho\left(\mathcal{S}, A\right)$ for every $c \in [0, +\infty)$.
4. If $\mathcal{S} \subseteq \mathcal{T}$, then, $\rho\left(\mathcal{S}, A\right) \le \rho\left(\mathcal{T}, A\right)$, since $\Sigma\left(\mathcal{S}, A^t\right)^{1/t} \le \Sigma\left(\mathcal{T}, A^t\right)^{1/t}$ for every $t \ge 1$.

The next result shows one of the main advantages of $\rho(\mathcal{S}, A)$ over $\Sigma(\mathcal{S}, A)$.

Proposition 3.4. *For every $t \in (0, \infty)$ it holds that*

$$\rho(\mathcal{S}, A)^t = \rho\left(\mathcal{S}, A^t\right)$$

In particular, for every $t \in (0, \infty)$

$$\rho(\mathcal{S}, A)^t \leq A^t$$

Proof. Firstly, we prove the statement for $t \geq 1$. By Lemma 2.7, the map $x \to x^r$ is continuous in the strong operator topology when $0 \leq r \leq 1$. So, given $t \in (1, \infty)$, since $st \to \infty$ as $s \to \infty$, we have that

$$\rho\left(\mathcal{S}, A^t\right)^{1/t} = \left(\lim_{s\to\infty} \Sigma\left(\mathcal{S}, (A^t)^s\right)^{1/s}\right)^{1/t} = \lim_{s\to\infty} \Sigma\left(\mathcal{S}, A^{st}\right)^{1/st} = \rho(\mathcal{S}, A),$$

where the limits are taken in the strong operator topology. This proves, for $t \geq 1$, that

$$\rho\left(\mathcal{S}, A^t\right) = \rho(\mathcal{S}, A)^t. \tag{3.1}$$

Now, if $t \in (0, 1)$,

$$\rho\left(\mathcal{S}, A^t\right) = \left(\rho\left(\mathcal{S}, A^t\right)^{1/t}\right)^t \rho\left(\mathcal{S}, (A^t)^{1/t}\right)^t = \rho(\mathcal{S}, A)^t,$$

where in the second equality, we have used equation (3.1) for $\dfrac{1}{t} \geq 1$. □

Recall that given two positive operators A and B we say that

$$A \preccurlyeq B \qquad \text{if} \qquad A^n \leq B^n \quad \forall n \geq 1$$

With respect to this order, the spectral shorted operator has a characterization similar to Krein-Anderson-Trapp's definition of shorted operator.

Theorem 3.5. *If*

$$\mathcal{M}_\rho(\mathcal{S}, A) = \{D \in L(\mathcal{H})^+ : \ D \preccurlyeq A, \ R(D) \subseteq \mathcal{S}\},$$

then $\rho(\mathcal{S}, A) = \max \mathcal{M}_\rho(\mathcal{S}, A)$, where the "maximum" is taken for any of the orders $\leq$ and $\preccurlyeq$.

Proof. Firstly, note that $\rho(\mathcal{S}, A) \in \mathcal{M}_\rho(\mathcal{S}, A)$. In fact, $\rho(\mathcal{S}, A)^m \leq A^m$ for every $m \in \mathbb{N}$ by Proposition 3.4, and $R(\rho(\mathcal{S}, A)) \subseteq \mathcal{S}$ by definition.

Suppose that $D \in \mathcal{M}_\rho(\mathcal{S}, A)$. Fix $m \in \mathbb{N}$. As $D^m \leq A^m$, it holds that $\Sigma(\mathcal{S}, D^m)^{1/m} \leq \Sigma(\mathcal{S}, A^m)^{1/m}$. Since $\Sigma(\mathcal{S}, D^m)^{1/m} = D$, taking $m \to \infty$ we have $D \leq \rho(\mathcal{S}, A)$. This shows that $\rho(\mathcal{S}, A) = \max \mathcal{M}_\rho(\mathcal{S}, A)$ for the usual order.

Note also that, if $D \in \mathcal{M}_\rho(\mathcal{S}, A)$, then for every $k \in \mathbb{N}$, $D^k \preccurlyeq A^k$ and $D^k \in \mathcal{M}_\rho(\mathcal{S}, A^k)$. By the previous case, applied to A^k, one gets

$$D^k \leq \rho\left(\mathcal{S}, A^k\right) = \rho(\mathcal{S}, A)^k \ , \quad k \in \mathbb{N}.$$

Hence $D \preccurlyeq \rho(\mathcal{S}, A)$. □

Corollary 3.6. *Let A and B be positive operators such that $A \preccurlyeq B$ and $\mathcal{S}$ and $\mathcal{T}$ be closed subspaces such that $\mathcal{S} \subseteq \mathcal{T}$. Then $\rho(\mathcal{S}, A) \preccurlyeq \rho(\mathcal{T}, B)$.*

Proof. It suffices to note that $\mathcal{M}_\sigma(\mathcal{S}, A) \subseteq \mathcal{M}_\sigma(\mathcal{T}, B)$. □

Another application of Theorem 3.5 is the following result about the convergence of sequences of spectral shorted operators.

Proposition 3.7. *Let $\{A_n\}$ be a sequence in $L(\mathcal{H})^+$ such that $A_{n+1} \preccurlyeq A_n$, $n \in \mathbb{N}$ and $A_n \xrightarrow[n\to\infty]{SOT} A$, and let $\{\mathcal{S}_n\}$ be a sequence of subspaces such that $\mathcal{S}_{n+1} \subseteq \mathcal{S}_n$. Then*

$$\rho(\mathcal{S}_n, A_n) \underset{n\to\infty}{\overset{SOT}{\searrow}} \rho(\mathcal{S}, A),$$

where $\mathcal{S} = \bigcap_{n=1}^{\infty} \mathcal{S}_n$.

Proof. By Corollary 3.6, for every $n \in \mathbb{N}$, $\rho(\mathcal{S}_{n+1}, A_{n+1}) \leq \rho(\mathcal{S}_n, A_n)$. Then there is a positive operator L such that $\rho(\mathcal{S}_n, A_n) \xrightarrow[n\to\infty]{\text{SOT}} L$. On one hand, by Proposition 2.8, $A \preccurlyeq A_n$, $n \in \mathbb{N}$. As, in addition, $\mathcal{S} \subseteq \mathcal{S}_n$, we have that $\rho(\mathcal{S}, A) \leq \rho(\mathcal{S}_n, A_n)$, $n \in \mathbb{N}$. This shows that $\rho(\mathcal{S}, A) \leq L$. On the other hand, for every $n > m$ and $k \geq 1$, by Corollary 3.6 and the definition of spectral shorted operators,

$$L \leq \rho(\mathcal{S}_n, A_n) \leq \rho(\mathcal{S}_m, A_n) \leq \Sigma\left(\mathcal{S}_m, A_n^k\right)^{1/k}. \tag{3.2}$$

Now fix $k \geq 1$. By Proposition 2.8, $A_n^k \underset{n\to\infty}{\overset{\text{SOT}}{\searrow}} A^k$. Therefore, by Lemma 2.7,

$$\Sigma\left(\mathcal{S}_m, A_n^k\right)^{1/k} \underset{n\to\infty}{\overset{\text{SOT}}{\searrow}} \Sigma\left(\mathcal{S}_m, A^k\right)^{1/k}. \tag{3.3}$$

In a similar way, using Proposition 2.4, we have that

$$\Sigma\left(\mathcal{S}_n, A^k\right)^{1/k} \underset{n\to\infty}{\overset{\text{SOT}}{\searrow}} \Sigma\left(\mathcal{S}, A^k\right)^{1/k}. \tag{3.4}$$

Hence, joining equations (3.2) (3.3) and (3.4), we obtain $L \leq \Sigma\left(\mathcal{S}, A^k\right)^{1/k}$. Finally, since the last inequality is true for every k, by taking limit we have that $L \leq \rho(\mathcal{S}, A)$. □

As the following example shows, the last Proposition does not hold, in general, if the sequence of subspaces fails to be non-increasing.

Example. Let $\mathcal{H}$ be a separable Hilbert space, A a positive operator which is not onto and $\mathcal{L}$ be a proper dense subspace of $\mathcal{H}$ such that $R(A^{1/2}) \cap \mathcal{L} = \{0\}$. Take an orthonormal basis $\{e_n\}$ of $\mathcal{H}$ contained in $\mathcal{L}$, and let $\mathcal{S}_n$ be the span of $\{e_1, \ldots, e_n\}$. Then, $P_{\mathcal{S}_n} \underset{n\to\infty}{\overset{\text{SOT}}{\nearrow}} I$, but, $\rho(\mathcal{S}_n, A) = \Sigma(\mathcal{S}_n, A) = 0$ for all $n \in \mathbb{N}$, because, as it was proved in [2], $R(\Sigma(\mathcal{S}_n, A)^{1/2}) = R(A^{1/2}) \cap \mathcal{S}_n\{0\}$.

4. Main properties of $\rho\,(\mathcal{S}, A)$.

Throughout this section $A \in L(\mathcal{H})^+$ and let $\mathcal{S}$ is a closed subspace of $\mathcal{H}$. It is proven in [3] that, if $\dim \mathcal{H} < \infty$ and $0 < \lambda \in \mathbb{R}$, then

$$\bigoplus_{\mu \geq \lambda} \ker(\rho(\mathcal{S}, A) - \mu I) = \mathcal{S} \cap \bigoplus_{\mu \geq \lambda} \ker(A - \mu I).$$

This can be reformulated, in terms of spectral measures, as

$$\aleph_{[\lambda,\infty)}(\rho\,(\mathcal{S}, A))\aleph_{[\lambda,\infty)}(A) \wedge P_{\mathcal{S}}.$$

This formula, which allows to compute the spectrum and the eigenvectors of $\rho\,(\mathcal{S}, A)$, gives the complete characterization of $\rho\,(\mathcal{S}, A)$ in the matrix case.

In the infinite dimensional case, the result can be proved following the same methods (with considerable more effort). Instead of following this way, it seems more convenient to construct an operator by means of the left spectral resolution given by

$$f(\lambda) = \begin{cases} \aleph_{[\lambda,\infty)}(A) \wedge P_{\mathcal{S}} & \lambda > 0 \\ I & \lambda \leq 0 \end{cases} \tag{4.1}$$

and then to show that its associated operator agrees with $\rho\,(\mathcal{S}, A)$. This can be done by using the characterization of $\rho\,(\mathcal{S}, A)$ given in Theorem 3.5. Note that the verification of the fact that f is, indeed, a left spectral resolution is apparent from the fact that $\lambda \mapsto \aleph_{[\lambda,\infty)}(A)$ is the left spectral resolution of A.

Theorem 4.1. *Let $A \in L(\mathcal{H})^+$ and let $\mathcal{S}$ be a closed subspace of $\mathcal{H}$. Then $\rho\,(\mathcal{S}, A)$ is the operator defined by the left spectral resolution f defined in equation* (4.1). *In other words, for $0 < \lambda \in \mathbb{R}$,*

$$\aleph_{[\lambda,\infty)}(\rho\,(\mathcal{S}, A)) = \aleph_{[\lambda,\infty)}(A) \wedge P_{\mathcal{S}}.$$

Proof. Let B be the operator defined by the spectral resolution f. By Theorem 2.6, it is clear that $B \preccurlyeq A$ and every $D \in \mathcal{M}_\rho(\mathcal{S}, A)$ satisfies $D \preccurlyeq B$. Indeed, suppose that $0 \leq D \preccurlyeq A$ and $R(D) \subseteq \mathcal{S}$. Then, for $\lambda > 0$, $\aleph_{[\lambda,\infty)}(D) \leq \aleph_{[\lambda,\infty)}(A)$ and

$$\aleph_{[\lambda,\infty)}(D) \leq \aleph_{(0,\infty)}(D) \leq P_{\overline{R(D)}} \leq P_{\mathcal{S}}.$$

Therefore $\aleph_{[\lambda,\infty)}(D) \leq \aleph_{[\lambda,\infty)}(A) \wedge P_{\mathcal{S}} \aleph_{[\lambda,\infty)}(B)$. Since $\aleph_{[\lambda,\infty)}(D) = I = \aleph_{[\lambda,\infty)}(B)$ for $\lambda \leq 0$, we get that $D \preccurlyeq B$ by Theorem 2.6. Finally, since

$$\aleph_{[\lambda,\infty)}(\|A\| P_{\mathcal{S}}) = \begin{cases} 0 & \|A\| < \lambda \\ P_{\mathcal{S}} & 0 < \lambda \leq \|A\| \\ I & \lambda \leq 0 \end{cases},$$

we deduce that $B \preccurlyeq \|A\|\; P_{\mathcal{S}}$ and, in particular, $R(B) \subseteq \mathcal{S}$. Then, by Theorem 3.5,

$$B = \max \mathcal{M}_\rho(\mathcal{S}, A) = \rho\,(\mathcal{S}, A)\,. \qquad \square$$

Corollary 4.2. *Let $\mathcal{S}$ and $\mathcal{T}$ be closed subspaces of $\mathcal{H}$. Then*

$$\rho\,(\mathcal{S} \cap \mathcal{T}, A) = \rho\,(\mathcal{T}, \rho\,(\mathcal{S}, A))\,.$$

Proof. It suffices to note that both operators have, as left spectral resolution, the map

$$f(\lambda) = \begin{cases} \aleph_{[\lambda,\infty)}(A) \wedge P_{\mathcal{S}} \wedge P_{\mathcal{T}} & \lambda > 0 \\ I & \lambda \leq 0 \end{cases} . \qquad \square$$

Remark 4.3. Let $A \in L(\mathcal{H})^+$ and let $\mathcal{S}$ and $\mathcal{T}$ be closed subspaces of $\mathcal{H}$. Then

$$\rho\left(\mathcal{S} \cap \mathcal{T}, A\right) \leq \rho\left(\mathcal{T}, \Sigma\left(\mathcal{S}, A\right)\right).$$

Indeed, it can be deduced from inequalities

$$\Sigma\left(\mathcal{S} \cap \mathcal{T}, A^{2^m}\right) \leq \Sigma\left(\mathcal{T}, \Sigma\left(\mathcal{S}, A^{2^m}\right)\right) \leq \Sigma\left(\mathcal{T}, \Sigma\left(\mathcal{S}, A\right)^{2^m}\right) \qquad \forall m \in \mathbb{N}.$$

Note that the mentioned statement can not be deduced from Corollary 4.2.

Proposition 4.4. *Let* $\mu = \min \sigma\left(A\right)$*, then*

$$\mu P \leq \rho(\mathcal{S}, A).$$

In particular, if A *is invertible then* $\rho(\mathcal{S}, A)$ *is invertible if it is considered as acting on* $\mathcal{S}$.

Proof. Note that $\mu^m = \min \sigma\left(A^m\right)$ for all $m \in \mathbb{N}$. Then $\mu^m P_{\mathcal{S}} \leq \mu^m I \leq A^m$ for all $m \in \mathbb{N}$. So that, $\mu P_{\mathcal{S}} \preccurlyeq A$ and the result follows by Theorem 3.5. $\square$

Remark 4.5. Given an operator $A \in L(\mathcal{H})^+$, then $r \notin \sigma\left(A\right)$ if and only if there exists $\varepsilon > 0$ such that $\aleph_{[r-\varepsilon,\, +\infty)}(A)\aleph_{[r+\varepsilon,\, +\infty)}(A)$.

Proposition 4.6. *If* $\rho\left(\mathcal{S}, A\right)$ *is considered as acting on* $\mathcal{S}$*, then*

$$\sigma\left(\rho\left(\mathcal{S}, A\right)\right) \subseteq \sigma\left(A\right).$$

Proof. By Proposition 4.4, if $0 \notin \sigma\left(A\right)$ then $0 \notin \sigma\left(\rho\left(\mathcal{S}, A\right)\right)$. On the other hand, if $r > 0$ and $r \notin \sigma\left(A\right)$, then, by Remark 4.5, there exists $\varepsilon > 0$ such that $\aleph_{[r-\varepsilon,\, +\infty)}(A)\aleph_{[r+\varepsilon,\, +\infty)}(A)$. Hence,

$$\begin{aligned} \aleph_{[r-\varepsilon,\, +\infty)}(\rho\left(\mathcal{S}, A\right)) \quad &= P_{\mathcal{S}} \wedge \aleph_{[r-\varepsilon,\, +\infty)}(A) \\ &= P_{\mathcal{S}} \wedge \aleph_{[r+\varepsilon,\, +\infty)}(A) \\ &\aleph_{[r+\varepsilon,\, +\infty)}(\rho\left(\mathcal{S}, A\right)). \end{aligned}$$

Thus, $r \notin \sigma\left(\rho\left(\mathcal{S}, A\right)\right)$. $\square$

Proposition 4.7. *Let* $f : [0, +\infty) \to [0, +\infty)$ *be a non-decreasing right continuous function. Then*

$$f(\rho\left(\mathcal{S}, A\right)) = \rho\left(\mathcal{S}, f(A)\right) \tag{4.2}$$

Proof. Given $\lambda \geq 0$, since f is non-decreasing and right continuous there exist $\eta \geq 0$ such that $\{\mu : f(\mu) \geq \lambda\} = [\eta, +\infty)$ and, for every $C \in L(\mathcal{H})^+$, $\aleph_{[\lambda,\infty)}(f(C))\aleph_{[\eta,\infty)}(C)$.

If $\eta = 0$, then $\aleph_{[\lambda,\infty)}(f(\rho\left(\mathcal{S}, A\right)))\aleph_{[\lambda,\infty)}(\rho\left(\mathcal{S}, f(A)\right)) = I$. If $\eta > 0$, then

$$\begin{aligned} \aleph_{[\lambda,\infty)}(f(\rho\left(\mathcal{S}, A\right))) &= \aleph_{[\eta,\infty)}(\rho\left(\mathcal{S}, A\right)) = \aleph_{[\eta,\infty)}(A) \wedge P_{\mathcal{S}} \\ &= \aleph_{[\lambda,\infty)}(f(A)) \wedge P_{\mathcal{S}} = \aleph_{[\lambda,\infty)}(\rho\left(\mathcal{S}, f(A)\right)), \end{aligned}$$

which shows that $f(\rho(\mathcal{S},A))$ and $\rho(\mathcal{S},f(A))$ have the same (left) spectral resolution. Hence $f(\rho(\mathcal{S},A)) = \rho(\mathcal{S},f(A))$ □

In the remains of the section we compute the minimum of $\sigma(\rho(\mathcal{S},A))$ and show, in two examples, how to calculate the whole spectrum of $\rho(\mathcal{S},A)$.

Proposition 4.8. *If $\rho(\mathcal{S},A)$ is considered as acting on $\mathcal{S}$, then*

$$\min \sigma(\rho(\mathcal{S},A)) = \max\{\lambda \geq 0 : A^m \geq \lambda^m P_{\mathcal{S}},\ \forall\ m \in \mathbb{N}\}. \tag{4.3}$$

Proof. Note that $A^m \geq \lambda^m P_{\mathcal{S}}$, $m \in \mathbb{N}$, if and only if $\lambda P_{\mathcal{S}} \preccurlyeq A$. On the other hand, since $P_{\mathcal{S}}$ and $\rho(\mathcal{S},A)$ commute, $\lambda P_{\mathcal{S}} \leq \rho(\mathcal{S},A)$ if and only if $\lambda P_{\mathcal{S}} \preccurlyeq \rho(\mathcal{S},A)$ if and only if $\lambda P_{\mathcal{S}} \in \mathcal{M}_\rho(\mathcal{S},A)$ if and only if $\lambda P_{\mathcal{S}} \preccurlyeq A$. □

Theorem 4.9. *If $\rho(\mathcal{S},A)$ is considered as acting on $\mathcal{S}$, then*

$$\begin{aligned} \min \sigma(\rho(\mathcal{S},A)) &= \max\{\lambda \geq 0 : P_{\mathcal{S}} \leq \aleph_{[\lambda,\infty)}(A)\} \\ &= \min\{\mu \in \sigma(A) :\ R(\aleph_{[\mu,\mu+\varepsilon)}(A)) \not\subseteq \mathcal{S}^\perp\ \ \forall\, \varepsilon > 0\} \\ &= \min\{\mu \in \sigma(A) :\ P_{\mathcal{S}}\ \aleph_{[\mu,\mu+\varepsilon)}(A) \neq 0\ \ \forall\, \varepsilon > 0\}. \end{aligned} \tag{4.4}$$

Proof. For any $B \in L(\mathcal{S})^+$, $\min \sigma(B) = \max\{\lambda \geq 0 : \aleph_{[\lambda,\infty)}(B) = I_{\mathcal{S}}\}$. Applying this identity to our problem, we get $\lambda_0 = \min \sigma(\rho(\mathcal{S},A)) = \max\{\lambda \geq 0 : P_{\mathcal{S}} \leq \aleph_{[\lambda,\infty)}(A)\}$. Then $P_{\mathcal{S}} \leq \aleph_{[\lambda_0,\infty)}(A)$ and $P_{\mathcal{S}} \not\leq \aleph_{[\lambda_0+\varepsilon,\infty)}(A)$ for every $\varepsilon > 0$. Then $\lambda_0 \in \{\mu \in \sigma(A) :\ P_{\mathcal{S}}\ \aleph_{[\mu,\mu+\varepsilon)}(A) \neq 0\ \ \forall\, \varepsilon > 0\}$, because if $P_{\mathcal{S}}\ \aleph_{[\lambda_0,\lambda_0+\varepsilon)}(A) = 0$, then

$$P_{\mathcal{S}}\ \aleph_{[\lambda_0+\varepsilon,\infty)}(A) = P_{\mathcal{S}}\ \Big(\aleph_{[\lambda_0,\infty)}(A) - \aleph_{[\lambda_0,\lambda_0+\varepsilon)}(A)\Big) P_{\mathcal{S}}\ \aleph_{[\lambda_0,\infty)}(A) = P_{\mathcal{S}},$$

i.e. $P_{\mathcal{S}} \leq \aleph_{[\lambda_0+\varepsilon,\infty)}(A)$. If $\lambda_0 = 0$, then equation (4.4) is clear, since $[\lambda_0, \lambda_0+\varepsilon)$ is an open subset of $\sigma(\rho(\mathcal{S},A))$. If $\lambda_0 > 0$, let $0 \leq \lambda < \lambda_0$ and $0 < \varepsilon < \lambda_0 - \lambda$. Then $\lambda + \varepsilon \leq \lambda_0$. Since $\lambda_0 = \max\{\lambda \geq 0 : P_{\mathcal{S}} \leq \aleph_{[\lambda,\infty)}(A)\}$, it holds that $P_{\mathcal{S}}\aleph_{[\lambda,\infty)}(A) = P_{\mathcal{S}}\aleph_{[\lambda+\varepsilon,\infty)}(A) = P_{\mathcal{S}}$. Hence

$$P_{\mathcal{S}} = P_{\mathcal{S}}\aleph_{[\lambda,\infty)}(A)P_{\mathcal{S}}\aleph_{[\lambda,\lambda+\varepsilon)}(A) + P_{\mathcal{S}}\aleph_{[\lambda+\varepsilon,\infty)}(A) = P_{\mathcal{S}}\aleph_{[\lambda,\lambda+\varepsilon)}(A) + P_{\mathcal{S}}.$$

Therefore $P_{\mathcal{S}}\aleph_{[\lambda,\lambda+\varepsilon)}(A) = 0$, which proves equation (4.4). □

Examples

Example. Consider the operator $M_x \in L(L^2([0,1]))$ defined by

$$M_x(f)(t) = tf(t),$$

and let $\mathcal{S}$ be the orthogonal complement to the subspace of constant functions. We claim that $\sigma(\rho(\mathcal{S},M_x)) = [0,1]$. Since by Proposition 4.6 $\sigma(\rho(\mathcal{S},M_x)) \subseteq \sigma(M_x)$, it is enough to prove that $(0,1) \in \sigma(\rho(\mathcal{S},M_x))$. Take $r \in (0,1)$. Then, by Theorem 4.1 it holds that

$$\begin{aligned} R(\aleph_{[r,+\infty)}(\rho(\mathcal{S},M_x))) &= R(\aleph_{[r,+\infty)}(M_x)) \cap \mathcal{S} \\ &= \left\{ f \in L^2([0,1]) : f|_{[0,r)} \equiv 0, \text{ and } \int_0^1 f(t)\,dt = 0 \right\} \end{aligned}$$

So, given $\varepsilon > 0$, if we define $f_{r,\varepsilon}(t) = (t-r)\aleph_{[r-\varepsilon,\, r+\varepsilon)}(t)$, then

$$f_{r,\varepsilon} \in R\Big(\aleph_{[r-\varepsilon,\, +\infty)}(\rho\,(\mathcal{S}, M_x))\Big) \quad \text{but} \quad f_{r,\varepsilon} \notin R\Big(\aleph_{[r+\varepsilon,\, +\infty)}(\rho\,(\mathcal{S}, M_x))\Big),$$

which shows, by Remark 4.5, that $r \in \sigma\,(\rho\,(\mathcal{S}, M_x))$.

Example. Let $\mathcal{H} = \ell^2$ and let $\{e_n\}$ be the canonical (orthonormal) basis of ℓ^2. If $w = (1, 2^{-1}, 2^{-2}, \ldots)$, let $\mathcal{S}$ be the orthogonal complement to the subspace generated by w. In $L(\ell^2)^+$ consider the compact operator A defined by

$$A = \sum_{n=1}^{\infty} \frac{1}{n} e_n \otimes e_n$$

where $(x \otimes y)z = \langle z, y\rangle\, x$, for $x, y, z \in \mathcal{H}$. We shall study the spectral decomposition of $\rho\,(\mathcal{S}, A)$. Since $\sigma\,(\rho\,(\mathcal{S}, A)) \subseteq \sigma\,(A)$, the spectrum of $\rho\,(\mathcal{S}, A)$ is also discrete. Actually, $\rho\,(\mathcal{S}, A)$ is compact because $\rho\,(\mathcal{S}, A) \preccurlyeq A$. Let $\lambda_1 \geq \lambda_2 \geq \ldots$ be the eigenvalues of $\rho\,(\mathcal{S}, A)$ arranged in non-increasing order.

By Theorem 4.1, $\lambda_1 < 1$ because $e_1 \notin \mathcal{S}$. However, the subspace $\mathcal{T}$ generated by e_1 and e_2 intersects $\mathcal{S}$, because $\dim \mathcal{S}^\perp = 1$. So, $\lambda_1 = 1/2$. Moreover, by Theorem 4.1,

$$\ker(\rho\,(\mathcal{S}, A) - \frac{1}{2}\ ) = \Big[\ker(A - \frac{1}{2}\ ) \oplus \ker(A - 1)\Big] \cap \mathcal{S} = \mathcal{T} \cap \mathcal{S}.$$

It is easy to deduce that $\ker(\rho\,(\mathcal{S}, A) - \frac{1}{2}\ )$ is the subspace generated by $f_1 = e_1 - 2e_2$. Following in a similar way, the subspace generated by e_1, e_2 and e_3 intersects $\mathcal{S}$ and the intersection has dimension two. This implies that $\lambda_2 = 1/3$ with multiplicity one. On the other hand, to find and eigenvector f_2 associated to λ_2, it suffices to look for a vector generated by e_1, e_2 and e_3 and orthogonal to f_1 and w. Take, for instance, $f_2 = e_1 + (1/2)e_2 - (21/2)e_3$. Going on in a similar way, we obtain that

$$\sigma\,(\rho\,(\mathcal{S}, A)) = \{1/n :\ n \geq 2\} \cup \{0\},$$

each eigenvalue has multiplicity one, and the corresponding eigenvectors are:

$$\begin{array}{lclllllll}
f_1 & = & (1, & -2, & 0, & \ldots & & &) \\
f_2 & = & (1, & 1/2, & -5, & 0, & \ldots & &) \\
f_3 & = & (1, & 1/2, & 1/4, & 21/2, & 0, & \ldots &) \\
\vdots & & & & & & \vdots & & \\
f_n & = & (1, & 1/2, & 1/4, & \ldots, & 1/2^{n-1}, & \frac{-(4^n-1)}{3\cdot 2^{n-2}}, & \ldots) \\
\vdots & & & & & & \vdots & &
\end{array}$$

5. The case $\dim \mathcal{S} = 1$

This final section is devoted to the study of $\rho\,(\mathcal{S}, A)$ when $\mathcal{S}$ is one dimensional.

Definition 5.1. Suppose that $\dim \mathcal{S} = 1$ and let $\xi \in \mathcal{S}$ be a unit vector. For every $A \in L(\mathcal{H})^+$ there exist $\lambda, \mu \geq 0$ such that $\rho(\mathcal{S}, A) = \lambda P_{\mathcal{S}}$ and $\Sigma(\mathcal{S}, A) = \mu P_{\mathcal{S}}$. Denote $\rho\,(A, \xi) = \lambda$ and $\Sigma\,(A, \xi) = \mu$.

Remark 5.2. Let $\mathcal{S}$ be the subspace generated by the unit vector $\xi \in \mathcal{H}$. There are several ways for computing $\rho\,(A, \xi)$ in terms of $\rho\,(\mathcal{S}, A)$, and similarly $\Sigma\,(A, \xi)$ in terms of $\Sigma\,(\mathcal{S}, A)$. We mention four of them.

1. By Theorem 4.9,
$$\begin{aligned} \rho\,(A, \xi) &= \min \sigma\,(\rho\,(\mathcal{S}, A)) \\ &= \min \Big\{ \mu \in \sigma\,(A) : \; P_{\mathcal{S}} \aleph_{[\mu, \mu+\varepsilon)}(A) \neq 0 \;\; \forall\, \varepsilon > 0 \Big\} \\ &= \min \Big\{ \mu \in \sigma\,(A) : \; \aleph_{[\mu, \mu+\varepsilon)}(A)\xi \neq 0 \;\; \forall\, \varepsilon > 0 \Big\}. \end{aligned} \tag{5.1}$$
2. By Proposition 4.8
$$\rho\,(A, \xi) = \max\{\lambda \geq 0 : \langle A^n \eta, \eta \rangle \geq \lambda^n |\langle \xi, \eta \rangle|^2 \;, \;\; \forall\, n \in \mathbb{N}, \;\eta \in \mathcal{H}\}.$$
3. Also $\rho\,(A, \xi) = \|\rho\,(\mathcal{S}, A)\,\xi\| = \langle \rho\,(\mathcal{S}, A)\,\xi, \xi \rangle$. Similar formulae hold for $\Sigma\,(A, \xi)$.
4. By Proposition 4.6, $\rho\,(A, \xi) \in \sigma\,(A)$. Moreover, by Theorem 4.1 (or Theorem 4.9),
$$\rho\,(A, \xi) = \max\{\lambda \in \sigma\,(A) : \xi \in R(\aleph_{[\lambda, \infty)}(A))\}. \tag{5.2}$$

The following result relates the spectral short of an operator to one dimensional subspaces and the spectral order.

Proposition 5.3. *Let $A,\ B \in L(\mathcal{H})^+$. Then $A \preccurlyeq B$ if and only if $\rho\,(A, \xi) \leq \rho\,(B, \xi)$ for every unit vector $\xi \in \mathcal{H}$.*

Proof. One implication follows from Corollary 3.6. On the other hand, suppose that $\rho\,(A, \xi) \leq \rho\,(B, \xi)$ for every unit vector $\xi \in \mathcal{H}$. Given $\lambda \geq 0$ such that $\aleph_{[\lambda, \infty)}(A) \neq 0$, let $\zeta \in R(\aleph_{[\lambda, \infty)}(A))$. By equation (5.2), $\lambda \leq \rho\,(A, \zeta)$. Since $\rho\,(A, \zeta) \leq \rho\,(B, \zeta)$, we have that $\zeta \in R(\aleph_{[\lambda, \infty)}(B))$. Hence, for every $\lambda \geq 0$, $R(\aleph_{[\lambda, \infty)}(A)) \subseteq R(\aleph_{[\lambda, \infty)}(B))$. By Theorem 2.6, we deduce that $A \preccurlyeq B$. ☐

Proposition 5.4. *Let $A \in L(\mathcal{H})^+$ and let $\mathcal{S}$ be the subspace of $\mathcal{H}$ generated by the unit vector ξ. If A is invertible, then for $m \in \mathbb{N}$,*
$$\Sigma\left(A^{2m}, \xi\right)^{1/2m} = \|A^{-m}\xi\|^{-1/m} = \langle A^{-2m}\xi, \xi \rangle^{-1/2m}, \tag{5.3}$$
and
$$\rho\,(A, \xi) = \lim_{m \to \infty} \|A^{-m}\xi\|^{-1/m} = \inf_{m \in \mathbb{N}} \|A^{-m}\xi\|^{-1/m} \tag{5.4}$$
If $R(A)$ is closed, then:

1. *If $\xi \notin R(A)$, then $\rho\,(A, \xi) = 0$.*
2. *If $\xi \in R(A)$ and $B = A^\dagger$, then*
$$\rho\,(A, \xi) = \lim_{m \to \infty} \|B^m \xi\|^{-1/m} = \inf_{m \in \mathbb{N}} \|B^m \xi\|^{-1/m}.$$

Proof. Using Theorem 4.9, the closed range case easily reduces to the invertible case, by considering A as acting on $R(A)$, because $A^\dagger$ acts on $R(A)$ as the inverse of A. Note that, if $R(A)$ is closed, then there exists $\varepsilon > 0$ such that $\aleph_{[0,\varepsilon)}(A) = P_{N(A)}$. Therefore $\xi \notin R(A)$ implies that $P_{\mathcal{S}}\aleph_{[0,\varepsilon)}(A) \neq 0$, and, by Remark 5.2, we get $\rho\,(A,\xi) = 0$.

Suppose that A is invertible. For $m \in \mathbb{N}$, denote by $\eta_m A^{-m/2}\xi$. Fix $m \in \mathbb{N}$. By Theorem 2.2, if $\mathcal{M}_m = A^{-m/2}(\mathcal{S})$, then $\Sigma\,(\mathcal{S}, A^m) = A^{m/2}P_{\mathcal{M}_m}A^{m/2}$, and

$$\Sigma\,(A^m,\xi) = \|\Sigma\,(\mathcal{S}, A^m)\,\xi\| = \|A^{m/2}P_{\mathcal{M}_m}A^{m/2}\xi\|.$$

Note that $\mathcal{M}_m$ is the subspace generated by η_m, so $P_{\mathcal{M}_m}\rho = \|\eta_m\|^{-2}\langle\rho,\eta_m\rangle\eta_m$, $\rho \in \mathcal{H}$. Then

$$\Sigma\,(A^m,\xi) = \|A^{m/2}P_{\mathcal{M}_m}A^{m/2}\ \xi\| \quad \left\|A^{m/2}\Big(\|\eta_m\|^{-2}\langle A^{m/2}\ \xi,\eta_m\rangle\eta_m\Big)\right\|$$

$$\|\eta_m\|^{-2}\|\langle\xi,\xi\rangle\ \xi\| = \|\eta_m\|^{-2}.$$

Therefore $\Sigma\left(A^{2m},\xi\right) = \|A^{-m}\xi\|^{-2}$, so that

$$\Sigma\left(A^{2m},\xi\right)^{1/2m} = \|A^{-m}\xi\|^{-1/m}\ , \quad m \in \mathbb{N}.$$

Equation (5.4) follows using Remark 5.2 and the definition of $\rho\,(\mathcal{S}, A)$. □

Remark 5.5. Equation (5.3) and, consequently, Proposition 5.4, can also be deduced from the following formula: for every invertible $B \in L(\mathcal{H})^+$ and $\xi \in \mathcal{H}$ with $\|\xi\| = 1$,

$$\Sigma\,(B,\xi) = \langle B^{-1}\xi,\xi\rangle^{-1}.$$

This formula is the one dimensional version of the characterization of Schur complements in terms of the block representation of the inverse of an operator (see [10] Lemma 4.7 or, for a matrix version, Horn-Johnson book [6]).

Let $A \in L(\mathcal{H})^+$. Consider the set

$$\sigma_{sh}\,(A) = \big\{\rho\,(A,\xi) : \|\xi\| = 1\big\}.$$

By Proposition 4.6, we have that $\sigma_{sh}\,(A) \subseteq \sigma\,(A)$. If $\dim\mathcal{H} < \infty$, it was shown in [3] (see also [5]) that $\sigma_{sh}\,(A) = \sigma\,(A)$. We shall see that this property fails in general in the infinite dimensional case. First we fix some notations:

1. For $B \in L(\mathcal{H})^+$ we denote

$$\begin{aligned}\sigma_+\,(A) &= \big\{\lambda \in \sigma\,(A) : \exists\ (\mu_n)_{n\in\mathbb{N}}\ in\ \sigma\,(A)\ \ with\ \mu_n > \lambda\ and\ \mu_n \searrow_{n\to\infty} \lambda\big\}\\ &= \big\{\lambda \in \sigma\,(A) : \forall\ \varepsilon > 0\ ,\ \aleph_{(\lambda,\lambda+\varepsilon)}(A) \neq 0\big\},\end{aligned}$$

   i.e. those points $\lambda \in \sigma\,(A)$ which are limit point of $\sigma\,(A) \setminus \{\lambda\}$ from the right.
2. $\sigma_{pt}(A) = \big\{\lambda \in \sigma\,(A) : N(A - \lambda I) \neq \{0\}\big\}$, the point spectrum of A.

Proposition 5.6. *Let $A \in L(\mathcal{H})^+$. Then*

$$\sigma_{sh}\,(A) = \sigma_+\,(A) \cup \sigma_{pt}(A) = \big\{\lambda \in \sigma\,(A) : \forall\ \varepsilon > 0\ ,\ \aleph_{[\lambda,\lambda+\varepsilon)}(A) \neq 0\big\}.$$

In particular, $\sigma_{sh}\,(A)$ is dense in $\sigma\,(A)$.

Proof. Let $\lambda \in \sigma(A)$ and let $(\mu_n)_{n\in\mathbb{N}}$ be a sequence in $\sigma(A)$ such that $\mu_n \searrow_{n\to\infty} \lambda$. Denote by $\lambda_0 = \mu_1 + 1$ and $\lambda_n = \frac{1}{2}(\mu_{n+1} + \mu_n)$, $n \in \mathbb{N}$. Note that, since $\mu_n \in (\lambda_n, \lambda_{n-1})$, then $\aleph_{(\lambda_n,\lambda_{n-1})}(A) \neq 0$. We take, for every $n \in \mathbb{N}$, an unit vector $\xi_n \in R(\aleph_{(\lambda_n,\lambda_{n-1})}(A))$. Consider the unit vector

$$\xi = \sum_{n\in\mathbb{N}} \frac{\xi_n}{2^n} .$$

From formula (5.2) and the construction of ξ, it is clear by that $\rho(A,\xi) = \lambda$, because $\lambda = \inf_n \mu_n = \inf_n \lambda_n$. If $\lambda \in \sigma_{pt}(A)$, just take $\xi \in N(A - \lambda I)$ and clearly $\rho(A,\xi)\,\Sigma(A,\xi) = \lambda$.

Now suppose that $\lambda \in \sigma(A)$ but $\lambda \notin \sigma_+(A) \cup \sigma_{pt}(A)$. This means that there exists $\varepsilon > 0$ such that $\aleph_{[\lambda,\lambda+\varepsilon)}(A) = 0$. Therefore, for any unit vector ξ, it is impossible that

$$\lambda = \max\{\mu \in \sigma(A) : \xi \in R(\aleph_{[\mu,\infty)}(A))\},$$

because if $\xi \in R(\aleph_{[\lambda,\infty)}(A))$, then $\xi \in R(\aleph_{[\lambda+\varepsilon,\infty)}(A))$. □

Remark 5.7. If $A \in L(\mathcal{H})^+$ is not invertible, then $0 \in \sigma(A)$. If 0 is an isolated point of $\sigma(A)$ then A has closed range. So that, $N(A) \neq \{0\}$. Otherwise $\aleph_{(0,\varepsilon)}(A) \neq 0$ for every $\varepsilon > 0$. This shows that $0 \in \sigma_{sh}(A)$. More generally, for $A \in L(\mathcal{H})^+$, it holds that $\lambda_{min}(A) = \min \sigma(A) \in \sigma_{sh}(A)$. On the other hand, by Proposition 5.6, $\|A\| \in \sigma_{sh}(A)$ if and only if $\|A\|$ is an eigenvalue of A.

Remark 5.8. For $A \in L(\mathcal{H})^+$, we shall denote by $R_0(A)$ the subspace

$$R_0(A) = \bigcup_{\lambda>0} R(\aleph_{[\lambda,\infty)}(A)).$$

If $R(A)$ is closed, then $R_0(A) = R(A)$, because 0 is an isolated point of $\sigma(A)$. If $R(A)$ is not closed, then, $R_0(A)$ is properly included in $R(A)$, but it is still a dense subspace of $\overline{R(A)}$. We are interested in this subspace because, by formula (5.2), if $\xi \in \mathcal{H}$ an unit vector, then $\rho(A,\xi) \neq 0$ if and only if $\xi \in R_0(A)$.

5.1. Kolmogorov complexity

Given an invertible matrix $A \in L(\mathbb{C}^m)^+$ and a unit vector $\xi \in \mathbb{C}^m$, J. I. Fujii and M. Fujii [5] define the Kolmogorov complexity:

$$K(A,\xi) = \lim_{n\to\infty} \frac{\log(\langle A^n\xi,\xi\rangle)}{n} = \log \lim_{n\to\infty} \langle A^n\xi,\xi\rangle^{1/n} . \tag{5.5}$$

Using formula (5.3), we can see that the limit is, in fact, a supremum; and we have the identity

$$K(A,\xi) = \log \rho\left(A^{-1/2},\xi\right)^{-2} = \log \rho\left(A^{-1},\xi\right)^{-1} . \tag{5.6}$$

Using formulae (5.1) and (5.2), we get

$$\begin{aligned} \exp K(A,\xi) &= \min\left\{\lambda \in \sigma(A) : \xi \in R(\aleph_{(-\infty,\lambda]}(A))\right\} \\ &= \max\left\{\mu \in \sigma(A) :\ \aleph_{(\mu-\varepsilon,\mu]}(A)\xi \neq 0 \ \ \forall\, \varepsilon > 0\right\} . \end{aligned} \tag{5.7}$$

With these identities in mind we generalize the notion of Kolmogorov complexity in two directions: firstly, we define it for infinite dimensional Hilbert spaces; secondly, we remove the hypothesis of invertibility of A. Note that the own notion of spectral shorted operator is, in some sense, a generalization of the Kolmogorov complexity relative to arbitrary (not necesarily one dimensional) closed subspaces of a Hilbert space $\mathcal{H}$.

If $\mathcal{H}$ is a Hilbert space and $A \in L(\mathcal{H})^+$ is invertible, then we just have to define $K(A, \xi)$ as in equation (5.6) or, equivalently (5.7). It is easy to see that this is equivalent to define it as in the finite dimensional setting, as in (5.5). We should mention that some of the properties of $K(A, \xi)$ proved by J. I. Fujii and M. Fujii fail if $\mathcal{H}$ is infinite dimensional. As an example, the identity

$$\sigma(A) = \big\{ \exp(K(A, \xi)) : \|\xi\| = 1 \big\}.$$

fails in general.

Definition 5.9. Given $\xi \in \mathcal{H}$ and $A \in L(\mathcal{H})^+$, define

$$k(A, \xi) = \lim_{n\to\infty} \langle A^n \xi, \xi \rangle^{1/n}.$$

Observe that $k(A, \xi) = \exp K(A, \xi)$ in the cases where $K(A, \xi)$ is defined.

Remark 5.10. Let $\xi \in \mathcal{H}$ and $A \in L(\mathcal{H})^+$. Then:

1. if $\|\xi\| = 1$, then the sequence $\langle A^n \xi, \xi \rangle^{1/n}$ is increasing and $\lim\limits_{n\to\infty} \langle A^n \xi, \xi \rangle^{1/n}$ exists for every $\xi \in \mathcal{H}$,
2. $k(A, \xi) = k(A, a\xi)$ for every $0 \neq a \in \mathbb{C}$,
3. $k(A, \xi) = k\big(A, \aleph_{[\lambda,\infty)}(A)\xi\big)$ for every $\lambda > 0$ such that $\aleph_{[\lambda,\infty)}(A)\xi \neq 0$.

Indeed, by Hölder inequality for states (also by Jensen inequality, see [4]), if $\|\xi\| = 1$, $p \geq 1$ and $1/p + 1/q = 1$, then

$$\langle A^p \xi, \xi \rangle^{1/p} \langle I^q \xi, \xi \rangle^{1/q} = \langle A^p \xi, \xi \rangle^{1/p} \geq \langle A\xi, \xi \rangle.$$

Applying this inequality to A^n with $p = (n+1)/n$ one gets that $\langle A^n \xi, \xi \rangle^{1/n} \leq \left\langle A^{n+1} \xi, \xi \right\rangle^{1/n+1}$.

5.10.2 follows from the fact that $|a|^{2/n} \xrightarrow[n\to\infty]{} 1$. To show 5.10.3, suppose that $\|\xi\| = 1$ and denote by $\xi_1 = \aleph_{[\lambda,\infty)}(A)\xi$ and $\xi_2 = \xi - \xi_1$. Then, since $\aleph_{[\lambda,\infty)}(A)$ commutes with A, for every $n \in \mathbb{N}$,

$$\begin{aligned} \langle A^n \xi_1, \xi_1 \rangle &\leq \langle A^n \xi_1, \xi_1 \rangle + \langle A^n \xi_2, \xi_2 \rangle \, \langle A^n \xi, \xi \rangle \\ &\leq \langle A^n \xi_1, \xi_1 \rangle + \lambda^n \leq (1 + \|\xi_1\|^{-2}) \langle A^n \xi_1, \xi_1 \rangle. \end{aligned}$$

This shows that $k(A, \xi) = k(A, \xi_1)$, since $(1 + \|\xi_1\|^{-2})^{1/n} \xrightarrow[n\to\infty]{} 1$.

Recall that, for $A \in L(\mathcal{H})^+$, we denote by $R_0(A) = \bigcup_{\lambda > 0} R(\aleph_{[\lambda,\infty)}(A))$.

Proposition 5.11. *Let $A \in L(\mathcal{H})^+$ and $\xi \in \mathcal{H}$, $\xi \neq 0$. Then $k(A,\xi) \neq 0$ if and only if $P_{\overline{R(A)}}\, \xi \in R_0(A)$. Moreover, equation (5.7) holds in general:*

$$\begin{aligned} k(A,\xi) &= \min\left\{\lambda \in \sigma(A) : \xi \in R(\aleph_{(-\infty,\lambda]}(A))\right\} \\ &= \max\left\{\mu \in \sigma(A) :\ \aleph_{(\mu-\varepsilon,\mu]}(A)\xi \neq 0\ \ \forall\, \varepsilon > 0\right\} \\ &= \sup\left\{\mu \in \sigma(A) :\ \aleph_{[\mu,\infty)}(A)\xi \neq 0\right\}. \end{aligned} \tag{5.8}$$

Proof. Let $\lambda = \sup\left\{\mu \in \sigma(A) :\ \aleph_{[\mu,\infty)}(A)\xi \neq 0\right\}$. Suppose that $\mu > \lambda$. Then $\xi \in R(\aleph_{(-\infty,\mu]}(A))$, so that $\langle A^n\xi,\xi\rangle \leq \mu^n\|\xi\|^2$ for $n \in \mathbb{N}$, and $k(A,\xi) \leq \mu$. On the other hand, if $\mu < \lambda$ then $\aleph_{[\mu,\infty)}(A)\xi = \xi_1 \neq 0$, and, by Remark 5.10, $k(A,\xi) = k(A,\xi_1) \geq \mu$, since $\langle A^n\xi_1,\xi_1\rangle \geq \mu^n\|\xi_1\|^2$ for every $n \in \mathbb{N}$. This shows that $k(A,\xi) = \lambda$. The other equalities are straightforward, by spectral theory. □

By Proposition 4.6, we have that $\sigma_{sh}(A) \subseteq \sigma(A)$ and, therefore, if A is invertible,

$$\left\{k(A,\xi) : \|\xi\| \neq 0\right\}\left\{\rho\left(A^{-1},\xi\right)^{-1} : \|\xi\| = 1\right\} \subseteq \sigma\left(A^{-1}\right)^{-1} = \sigma(A).$$

As we shall see below, the reverse inclusion fails in general:

Proposition 5.12. *If $A \in L(\mathcal{H})^+$ is inversible, then*

$$\begin{aligned} \left\{k(A,\xi) : \|\xi\| \neq 0\right\} \quad & \sigma_-(A) \cup \sigma_{pt}(A) \\ &= \left\{\lambda \in \sigma(A) : \aleph_{(\lambda+\varepsilon,\lambda]}(A) \neq 0\ ,\ \forall\, \varepsilon > 0\ \right\}, \end{aligned}$$

where $\sigma_-(A)$ is the set of points in $\sigma(A)$ which are limit point of $\sigma(A) \setminus \{\lambda\}$ from the left. The set $\left\{k(A,\xi) : \|\xi\| = 1\right\}$ is dense in $\sigma(A)$.

Proof. It is a consequence of Proposition 5.6 (applied to A^{-1}) and the identity

$$\left\{k(A,\xi) : \|\xi\| \neq 0\right\}\left\{k(A,\xi) : \|\xi\| = 1\right\}\left\{\rho\left(A^{-1},\xi\right)^{-1} : \|\xi\| = 1\right\}. \qquad □$$

Remarks 5.13.

1. Proposition 5.12 is also true for a general $A \in L(\mathcal{H})^+$. The proof is similar to the proof of Proposition 5.6, by using equation (5.8) instead of (5.2).
2. Let $\mathcal{H} = \ell^2(\mathbb{N})$ and $\{e_n :\ n \in \mathbb{N}\}$ be the canonical orthonormal basis of $\mathcal{H}$, and consider the diagonal invertible operators $A, B \in L(\mathcal{H})^+$ defined by

   $$A(e_n) = \left(2 + \frac{1}{n}\right)e_n \quad , \quad B(e_n) = \left(2 - \frac{1}{n}\right)e_n\ , \quad n \in \mathbb{N}.$$

   It is easy to see, using Propositions 5.6 and 5.12, that $2 \notin \left\{k(A,\xi) : \|\xi\| = 1\right\}$ and $2 \notin \sigma_{sh}(B)$.
3. If $C \in L(\mathcal{H})^+$, then $\|C\| \in \left\{k(C,\xi) : \|\xi\|1\right\}$ and $\lambda_{min}(C) \in \sigma_{sh}(C)$. On the other hand, if A and B are as in the previous example, then $\|B\| = 2 \notin \sigma_{sh}(B)$ and $\lambda_{min}(A) = 2 \notin \left\{k(A,\xi) : \|\xi\| = 1\right\}$.

Remark 5.14 (Operators with closed range). Suppose that $A \in L(\mathcal{H})^+$ and $R(A)$ is closed. Then, $k(A,\xi)$ and $\rho(A,\xi)$ can be explicitly computed in terms of $\rho\left(A^\dagger,\xi\right)$. More precisely,

1. If $\xi \in R(A)$ is an unit vector, then, by Proposition 5.4, we can deduce that $k(A, \xi) = \rho\left(A^\dagger, \xi\right)^{-1}$.
2. Let $\xi \in \mathcal{H} \backslash (N(A) \cup R(A))$. By Proposition 5.4, $\rho(A, \xi) = \rho\left(A^\dagger, \xi\right) = 0$. On the other hand, if $P = P_{R(A)}$, then $P\xi \neq 0$ and

$$k(A, \xi) = \lim_{n \to \infty} \langle A^n P\xi, P\xi \rangle^{1/n} \, k\left(A, \frac{P\xi}{\|P\xi\|}\right) = \rho\left(A^\dagger, \frac{P\xi}{\|P\xi\|}\right)^{-1} \neq 0.$$

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