

A FREE BOUNDARY PROBLEM FOR THE $p(x)$ - LAPLACIAN

JULIÁN FERNÁNDEZ BONDER, SANDRA MARTÍNEZ AND NOEMI WOLANSKI

ABSTRACT. We consider the optimization problem of minimizing $\int_{\Omega} |\nabla u|^{p(x)} + \lambda \chi_{\{u>0\}} dx$ in the class of functions $W^{1,p(\cdot)}(\Omega)$ with $u - \varphi_0 \in W_0^{1,p(\cdot)}(\Omega)$, for a given $\varphi_0 \geq 0$ and bounded. $W^{1,p(\cdot)}(\Omega)$ is the class of weakly differentiable functions with $\int_{\Omega} |\nabla u|^{p(x)} dx < \infty$. We prove that every solution u is locally Lipschitz continuous, that it is a solution to a free boundary problem and that the free boundary, $\Omega \cap \partial\{u > 0\}$, is a regular surface.

1. INTRODUCTION

In this paper we study a free boundary problem for the $p(x)$ -Laplacian. The $p(x)$ -Laplacian, is defined as

$$(1.1) \quad \Delta_{p(x)}u = \operatorname{div}(|\nabla u(x)|^{p(x)-2}\nabla u).$$

This operator extends the classical Laplacian ($p(x) \equiv 2$) and the so-called p -Laplacian ($p(x) \equiv p$ with $1 < p < \infty$) and it has been recently used in image processing and in the modeling of electrorheological fluids.

For instance, Chen, Levin and Rao [11] proposed the following model in image processing

$$E(u) = \int_{\Omega} \frac{|\nabla u(x)|^{p(x)}}{p(x)} + |u(x) - I(x)| dx \rightarrow \min$$

where $p(x)$ is a function varying between 1 and 2. It is chosen $p(x)$ next to 1 where there is likely to be edges and next to 2 where it is likely not to be edges.

Observe that the Euler-Lagrange equation associated to E is the $p(x)$ -laplacian.

For the modeling of electrorheological fluids, see [31].

On the other hand, a free boundary problem associated to the $p(x)$ -Laplacian, was studied in [21] namely, the obstacle problem. In that paper, existence and Hölder continuity of minimizers was proved. No further regularity was studied.

To our knowledge, no other free boundary problem associated to this operator has been analyzed up to date.

This paper is devoted to the study of the so-called *Bernoulli free boundary problem*, that is

$$(1.2) \quad \begin{cases} \Delta_{p(x)}u = 0 & \text{in } \{u > 0\} \\ u = 0, |\nabla u| = \lambda^*(x) & \text{on } \partial\{u > 0\}. \end{cases}$$

where λ^* is a given function away from zero and infinity.

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This free boundary problem, in the linear case $p(x) \equiv 2$, was first studied by A. Beurling in [9] for $N = 2$.

Still in the linear setting and for $N \geq 2$, this problem was analyzed by H. Alt and L. Caffarelli in the seminal paper [4]. In that work, the authors prove existence of a weak solution by minimizing the functional

$$u \mapsto \int_{\Omega} \frac{|\nabla u|^2}{2} + \frac{(\lambda^*(x))^2}{2} \chi_{\{u>0\}} dx.$$

Then, they prove local Lipschitz regularity of weak solutions and, when λ^* is C^α , they prove the $C^{1,\alpha}$ regularity of the free boundary up to some negligibly set of possible singularities.

Later, in [5], these results were extended to the quasilinear uniformly elliptic case.

Problem (1.2) with $p(x) \equiv p$ was addressed in [12], where the same approach was applied to obtain similar results in the p -Laplacian case. In that paper, the authors had to deal with the problem of the degeneracy or singularity of the underlying equation.

Recently, the method was further extended in [29], where this free boundary problem for operators with non-standard growth was treated in the setting of Orlicz spaces.

The Bernoulli free boundary problem, appears in many different applications, such as limits of singular perturbation problems of interest in combustion theory (see for instance, [8, 25, 26]) fluid flow e.g. the problem of jets (see for instance [6, 7]) and some shape optimization problems with a volume constrain (see for instance, [2, 3, 18, 19, 24, 28, 30]).

In this work, in order to analyze the Bernoulli free boundary problem (1.2), we follow the same approach as in the previously mentioned works and prove optimal regularity of solutions and $C^{1,\alpha}$ regularity of their free boundaries.

So we consider the following minimization problem: For Ω a smooth bounded domain in \mathbb{R}^N and φ_0 a nonnegative function with $\varphi_0 \in L^\infty(\Omega)$ and $\int_{\Omega} |\nabla \varphi_0|^{p(x)} dx < \infty$, we consider the problem of minimizing the functional,

$$(1.3) \quad \mathcal{J}(u) = \int_{\Omega} \frac{|\nabla u|^{p(x)}}{p(x)} + \lambda(x) \chi_{\{u>0\}} dx$$

in the class of functions

$$\mathcal{K} = \left\{ v \in W^{1,p(x)}(\Omega) : v - \varphi_0 \in W_0^{1,p(x)}(\Omega) \right\}.$$

For the definition of the variable exponent Sobolev spaces, see Appendix A.

In order to state the main results of the paper, we need to introduce some notation and assumptions.

Assumptions on $p(x)$. Throughout this work, we will assume that the function $p(x)$ verifies

$$(1.4) \quad 1 < p_{min} \leq p(x) \leq p_{max} < \infty, \quad x \in \Omega$$

When we are restricted to a ball B_r we use $p_- = p_-(B_r)$ and $p_+ = p_+(B_r)$ to denote the infimum and the supremum of $p(x)$ over B_r .

We also assume that $p(x)$ is continuous up to the boundary and that it has a modulus of continuity $\omega : \mathbb{R} \rightarrow \mathbb{R}$, i.e. $|p(x) - p(y)| \leq \omega(|x - y|)$ if $|x - y|$ is small. At several stages it is necessary to assume that p is log-Hölder continuous. This is, $w(r) = C(\log \frac{1}{r})^{-1}$.

The precise assumptions on the modulus of continuity ω will be clearly stated in each section.

For our main result we need to assume further that $p(x)$ is Lipschitz continuous in Ω . In that case, we denote by L the Lipschitz constant of $p(x)$, namely, $\|\nabla p\|_{L^\infty(\Omega)} \leq L$

Assumptions on $\lambda(x)$. In the firsts sections we will only need $\lambda(x)$ to be bounded away from zero and infinity. We denote $0 < \lambda_1 \leq \lambda(x) \leq \lambda_2 < \infty$ for $x \in \Omega$.

We will assume in the last part that $\lambda(x)$ is Hölder continuous.

Main Results. Our first result gives the existence of a minimizer and, under the assumption of Lipschitz continuity of $p(x)$ and that $p(x) \geq 2$, the Lipschitz regularity of minimizers.

Theorem 1.1. *We prove*

- Assume that $p(x)$ is log-Hölder continuous. Then, there exists a minimizer of \mathcal{J} in \mathcal{K} . Any such minimizer u is nonnegative, bounded and locally Hölder continuous.
- Under the same assumptions, any minimizer u is globally $p(x)$ -subharmonic and

$$\Delta_{p(x)}u = 0 \quad \text{in } \{u > 0\}.$$

- If $p \in C^{0,1}(\Omega)$ then, every minimizer is nondegenerate (see Corollary 4.2).
- If moreover $p(x) \geq 2$ in Ω , then u belongs to $C_{loc}^{0,1}(\Omega)$.

Our second result states that Lipschitz, nondegenerate minimizers of (1.3) are weak solutions to (1.2).

Theorem 1.2. *Assume that $p(x)$ is Hölder continuous and that $\lambda(x)$ is continuous.*

Let u be a nondegenerate, locally Lipschitz continuous minimizer of (1.3). Then, $\{u > 0\}$ has finite perimeter locally in Ω and $\mathcal{H}^{N-1}(\partial\{u > 0\} \setminus \partial_{red}\{u > 0\}) = 0$.

Moreover, for every $x_0 \in \partial_{red}\{u > 0\}$, (this is, for every x_0 where there is an exterior unit normal $\nu(x_0)$ to $\partial\{u > 0\}$ in the measure theoretic sense), u has the following asymptotic development,

$$(1.5) \quad u(x) = \lambda^*(x_0)\langle x - x_0, \nu(x_0) \rangle^- + o(|x - x_0|) \quad \text{as } x \rightarrow x_0$$

where $\lambda^*(x) = \left(\frac{p(x)}{p(x)-1} \lambda(x)\right)^{1/p(x)}$.

Finally, for every $\phi \in C_0^\infty(\Omega)$, there holds

$$-\int_{\{u>0\}} |\nabla u|^{p(x)-2} \nabla u \nabla \phi \, dx = \int_{\partial_{red}\{u>0\}} (\lambda^*(x))^{p(x)-1} \phi \, d\mathcal{H}^{N-1}.$$

That is, u is a weak solution to (1.2) in the sense of distributions.

Now, we arrive at the last result of the paper. Namely, the regularity of the free boundary $\partial\{u > 0\}$ for Lipschitz minimizers of (1.3).

Theorem 1.3. *Let $p(x)$ be Lipschitz continuous, $\lambda(x)$ be Hölder continuous, and u be a locally Lipschitz continuous minimizer of (1.3). Then, for \mathcal{H}^{N-1} -almost every point in the free boundary $\partial\{u > 0\}$ there exists a neighborhood V such that $V \cap \partial\{u > 0\}$ is a $C^{1,\gamma}$ surface, for some $\gamma > 0$.*

Technical comments. We finish this introduction with some comments on the technical difficulties that we have encountered when dealing with the $p(x)$ –Laplacian, highlighting the differences in the arguments with respect to the previous works on Bernoulli-type free boundary problems.

- As mentioned in the Appendix A, the log-Hölder continuity is a key ingredient in order to deal with variable exponent Sobolev spaces. For instance, up to date, this is the minimum requirement in order to have that C^∞ be dense in $L^{p(\cdot)}$. See [15].
- One fundamental tool in the analysis of this free boundary problem is the use of barriers. In order to construct barriers one has to look at the operator in non-divergence form. See [5, 12, 29], etc. In the $p(x)$ –Laplacian case, in order to write the equation in non-divergence form, one has to be able to compute the derivative of $p(x)$. Therefore, the assumption that $p(x)$ be Lipschitz continuous becomes natural. See Remark B.3.
- Probably, the main technical difficulty that we have encountered is the fact that the class of $p(x)$ –harmonic functions is not invariant under the scaling $u(x) \mapsto u(tx)/k$ if $t \neq k$. In [12, 29] this invariance was used in a crucial way in the proof of the Lipschitz continuity of the solutions. See, for instance, the proof of Lemma 3.2 in [12].

In order to overcome this difficulty we went back to the ideas in [4], but we are left with the additional technical assumption that $p(x) \geq 2$ in order to get the Lipschitz continuity of the minimizers.

- As for Harnack’s inequality in the case of $p(x)$ –harmonic functions, the inequality that holds is analogous to Harnack’s inequality for the nonhomogeneous Laplace equation. Moreover, the constant in this inequality is not universal, but depends (in a nontrivial manner) on the L^∞ norm of the solution. Nevertheless, the constant in Harnack’s inequality remains invariant under homogenous scalings of a solution, see Remark B.2.
- We are not aware of the validity of the strong minimum principle for $p(x)$ –harmonic functions (it does not come out of Harnack’s inequality). This property was used at several stages in previous works. In our new arguments we use instead the nondegeneracy of minimizers (see Lemma 4.3), which is valid for any $p(x) > 1$. With this property we can prove, for instance, Corollary 4.3 which is a crucial step to obtain the Lipschitz regularity of minimizers.
- We believe that the hypothesis $p(x) \geq 2$ –that is needed in order to obtain the Lipschitz regularity of minimizers– is purely technical. This assumption is only used in Lemma 4.2. If one is able to prove this lemma for a general $p(x)$, this assumption can be eliminated.
- There is another step where the hypothesis that $p(x)$ be Lipschitz is crucial. Namely, in order to obtain the result on the regularity of the free boundary one needs a differential inequality for a function of the gradient. In this paper we prove that if u is $p(x)$ -harmonic and if $v = |\nabla u|$ is far from zero and infinity, then v is a subsolution of an elliptic equation with principal part in divergence form (see Lemma B.5). In order to prove this result we need to differentiate the equation, and therefore we need $p(x)$ to be Lipschitz.
- As in [4], the hypothesis $\lambda(x)$ Hölder continuous is needed in the proof of the regularity of the free boundary in Section 8. Note that this is a natural assumption if one expects the $C^{1,\alpha}$ regularity of the free boundary to imply the C^α continuity of ∇u up to $\partial\{u > 0\}$.

Outline of the paper. First, in Section 2, under the assumption of log-Hölder continuity of $p(x)$, by using standard variational arguments, we prove the existence of a minimizer for \mathcal{J} in the class \mathcal{K} . Then, we show that every minimizer is $p(x)$ –subharmonic and bounded.

In Section 3, we analyze the regularity properties of minimizers and prove, under minimal assumptions on $p(x)$, that minimizers are Hölder continuous (Theorem 3.2). As a consequence, we deduce that u is $p(x)$ -harmonic in $\{u > 0\}$.

In Section 4, we further analyze the regularity of minimizers. This time, under the assumption that $p(x)$ is (locally) Lipschitz continuous, we first prove that u is nondegenerate near a free boundary point (Corollary 4.1). Then, assuming further that $p(x) \geq 2$, we prove that u is locally Lipschitz continuous (Theorem 4.1). This completes the proof of Theorem 1.1.

In Sections 5, 6 and 7 we assume that u is a Lipschitz non-degenerate minimizer and that p is locally Hölder continuous.

In Section 5, we begin the proof of Theorem 1.2 and show the positive density of $\{u > 0\}$ and $\{u = 0\}$ at every free boundary point (Theorem 5.1).

In Section 6 we study the measure $\Lambda = \Delta_{p(x)}u$ and prove that it is absolutely continuous with respect to $\mathcal{H}^{N-1} \llcorner \partial\{u > 0\}$. Then, we deduce that almost every point on the free boundary belongs to the reduced free boundary (Lemma 6.2).

In Section 7, we finish the proof of Theorem 1.2 by proving the asymptotic development of u near a free boundary point in the reduced boundary.

Finally, Section 8 is devoted to the proof of Theorem 1.3.

We finish this paper with a couple of appendices with some previous and some new results about $p(x)$ -harmonic and subharmonic functions, that can be of independent interest.

2. THE MINIMIZATION PROBLEM

In this section we look for minimizers of the functional \mathcal{J} . We begin by discussing the existence of extremals. Next, we prove that any minimizer is a subsolution to the equation $\mathcal{L}u = 0$ and finally, we prove that $0 \leq u \leq \sup \varphi_0$.

Theorem 2.1. *Let $p \in C(\overline{\Omega})$ and $0 < \lambda_1 \leq \lambda(x) \leq \lambda_2 < \infty$. If $\mathcal{J}(\varphi_0) < \infty$ there exists a minimizer of \mathcal{J} .*

Proof. The proof of existence is similar to the one in [29]. Since we are dealing with the Sobolev variable exponent, we write it down for the reader's convenience.

Take a minimizing sequence $(u_n) \subset \mathcal{K}$, then $\mathcal{J}(u_n)$ is bounded, so $\int_{\Omega} |\nabla u_n|^{p(x)}$ and $|\{u_n > 0\}|$ are bounded. As $u_n = \varphi_0$ in $\partial\Omega$, we have by Remark A.1 that $\|\nabla u_n - \nabla \varphi_0\|_{p(x)} \leq C$ and by Lemma A.1 we also have $\|u_n - \varphi_0\|_{p(x)} \leq C$. Therefore, by Theorem A.1 there exists a subsequence (that we still call u_n) and a function $u_0 \in W^{1,p(\cdot)}(\Omega)$ such that

$$u_n \rightharpoonup u_0 \quad \text{weakly in } W^{1,p(\cdot)}(\Omega),$$

and by Theorem A.2

$$u_n \rightharpoonup u_0 \quad \text{weakly in } W^{1,p_{\min}}(\Omega).$$

Now, by the compactness of the immersion $W^{1,p_{\min}}(\Omega) \hookrightarrow L^{p_{\min}}(\Omega)$ we have that, for a subsequence that we still denote u_n ,

$$u_n \rightarrow u_0 \quad \text{a.e. } \Omega.$$

As \mathcal{K} is convex and closed, it is weakly closed, so $u_0 \in \mathcal{K}$.

Moreover,

$$\begin{aligned} \{u_0 > 0\} &\subset \liminf_{n \rightarrow \infty} \{u_n > 0\} \quad \text{so that} \\ \chi_{\{u_0 > 0\}} &\leq \liminf_{n \rightarrow \infty} \chi_{\{u_n > 0\}}. \end{aligned}$$

On the other hand,

$$\int_{\Omega} \frac{|\nabla u_0|^{p(x)}}{p(x)} dx \leq \liminf_{n \rightarrow \infty} \int_{\Omega} \frac{|\nabla u_n|^{p(x)}}{p(x)} dx.$$

In fact,

$$(2.6) \quad \int_{\Omega} \frac{|\nabla u_n|^{p(x)}}{p(x)} dx \geq \int_{\Omega} \frac{|\nabla u_0|^{p(x)}}{p(x)} dx + \int_{\Omega} |\nabla u_0|^{p(x)-2} \nabla u_0 \cdot (\nabla u_n - \nabla u_0) dx.$$

Recall that ∇u_n converges weakly to ∇u_0 in $L^{p(\cdot)}(\Omega)$. Now, since $|\nabla u_0|^{p(x)-1} \in L^{p'(\cdot)}(\Omega)$, by Theorem A.1 and passing to the limit in (2.6) we get

$$\liminf_{n \rightarrow \infty} \int_{\Omega} \frac{|\nabla u_n|^{p(x)}}{p(x)} dx \geq \int_{\Omega} \frac{|\nabla u_0|^{p(x)}}{p(x)} dx.$$

Hence

$$\mathcal{J}(u_0) \leq \liminf_{n \rightarrow \infty} \mathcal{J}(u_n) = \inf_{v \in \mathcal{K}} \mathcal{J}(v).$$

Therefore, u_0 is a minimizer of \mathcal{J} in \mathcal{K} . □

Lemma 2.1. *Let $1 < p(x) < \infty$ and $0 \leq \lambda(x) < \infty$. Let u be a local minimizer of \mathcal{J} . Then, u is $p(x)$ -subharmonic.*

Proof. Let $\varepsilon > 0$ and $0 \leq \xi \in C_0^\infty$. Using the minimality of u we have

$$\begin{aligned} 0 &\leq \frac{1}{\varepsilon} (\mathcal{J}(u - \varepsilon \xi) - \mathcal{J}(u)) \leq \frac{1}{\varepsilon} \int_{\Omega} \frac{|\nabla u - \varepsilon \nabla \xi|^p}{p} - \frac{|\nabla u|^p}{p} dx \\ &\leq - \int_{\Omega} |\nabla u - \varepsilon \nabla \xi|^{p-2} \nabla(u - \varepsilon \nabla \xi) \nabla \xi dx \end{aligned}$$

and if we take $\varepsilon \rightarrow 0$ we obtain

$$0 \leq - \int_{\Omega} |\nabla u|^{p-2} \nabla u \nabla \xi dx$$

□

Lemma 2.2. *Let p be log-Hölder continuous, $1 < p(x) < \infty$, $0 \leq \lambda(x) < \infty$ and u a minimizer of \mathcal{J} in \mathcal{K} . Then, $0 \leq u \leq \sup_{\Omega} \varphi_0$.*

Proof. The proof follows as in Lemma 1.5 in [5] once we show that the functions $\min(M - u, 0)$ and $\min(u, 0)$ are in $W_0^{1,p(\cdot)}(\Omega)$, where $M = \sup_{\Omega} \varphi_0$.

But this fact follows from Corollary 3.6 and Theorem 3.7 in [14]. □

3. HOLDER CONTINUITY

In this section we study the regularity of the minimizers of \mathcal{J} .

As a first step, we prove that minimizers are Hölder continuous provided the function p is log-Hölder continuous. We use ideas from [1] and [12].

Theorem 3.1. *Assume p has modulus of continuity $\omega(r) = C \log(\frac{1}{r})^{-1}$. Then, for every $0 < \gamma_0 < 1$, there exist $r_0(\gamma_0, p_{min})$ and $\rho_0 = \rho_0(r_0, \gamma_0)$ such that, if u is a minimizer of \mathcal{J} in B_{r_0} then, $u \in C^{\gamma_0}(B_{\rho_0})$. Moreover, if $M > 0$ is such that $\|u\|_{L^\infty(B_{r_0})} \leq M$ there exists $C = C(N, p_{min}, p_{max}, \omega(r), \lambda_2, M, \gamma_0)$ such that $\|u\|_{C^{\gamma_0}(\overline{B_{\rho_0}})} \leq C$.*

Proof. We will prove that there exist r_0 and ρ_0 as in the statement such that, if $\rho \leq \rho_0$ and $\|u\|_{L^\infty(B_{r_0})} \leq M$ then,

$$(3.7) \quad \left(\int_{B_\rho} |\nabla u|^{p_-} dx \right)^{1/p_-} \leq C \rho^{\gamma_0 - 1}$$

where $p_- = p_-(B_{r_0})$.

In fact, let $0 < r \leq r_0$ and v be the solution of

$$(3.8) \quad \Delta_{p(x)} v = 0 \quad \text{in } B_r, \quad v - u \in W_0^{1,p}(B_r).$$

Let $u^s(x) = su(x) + (1-s)v(x)$. By using that v is a solution of (3.8) we get,

$$(3.9) \quad \int_{B_r} \frac{|\nabla u|^p}{p} - \frac{|\nabla v|^p}{p} dx = \int_0^1 \frac{ds}{s} \int_{B_r} \left(|\nabla u^s|^{p-2} \nabla u^s - |\nabla v|^{p-2} \nabla v \right) \cdot \nabla (u^s - v) dx.$$

By a standard inequality (see Remark B.1) we have that,

$$(3.10) \quad \int_{B_r} \frac{|\nabla u|^p}{p} - \frac{|\nabla v|^p}{p} dx \geq C \left(\int_{B_r \cap \{p \geq 2\}} |\nabla u - \nabla v|^p dx + \int_{B_r \cap \{p < 2\}} |\nabla u - \nabla v|^2 (|\nabla u| + |\nabla v|)^{p-2} dx \right),$$

where $C = C(p_{min}, p_{max}, N)$.

Therefore, by the minimality of u , we have (if $A_1 = B_r \cap \{p < 2\}$ and $A_2 = B_r \cap \{p \geq 2\}$)

$$(3.11) \quad \int_{A_2} |\nabla u - \nabla v|^p dx \leq C \lambda_2 r^N$$

$$(3.12) \quad \int_{A_1} |\nabla u - \nabla v|^2 (|\nabla u| + |\nabla v|)^{p-2} dx \leq C \lambda_2 r^N$$

Let $\varepsilon > 0$. Take $\rho = r^{1+\varepsilon}$ and suppose that $r^\varepsilon \leq 1/2$. Take η to be chosen later. Then, by Young inequality and the definition of A_1 we obtain,

$$(3.13) \quad \begin{aligned} \int_{A_1 \cap B_\rho} |\nabla u - \nabla v|^p dx &\leq \frac{C}{\eta} \int_{A_1 \cap B_r} (|\nabla u| + |\nabla v|)^{p-2} |\nabla u - \nabla v|^2 dx \\ &\quad + C \eta \int_{B_\rho \cap A_1} (|\nabla u| + |\nabla v|)^p dx \\ &\leq \frac{C}{\eta} r^N + C \eta \int_{B_\rho \cap A_1} (|\nabla u| + |\nabla v|)^p dx. \end{aligned}$$

Therefore, by (3.11) and (3.13), we get,

$$(3.14) \quad \int_{B_\rho} |\nabla u - \nabla v|^p dx \leq \frac{C}{\eta} r^N + C\eta \int_{B_\rho \cap A_1} (|\nabla u| + |\nabla v|)^p dx.$$

where $C = C(\lambda_2, N, p_{min}, p_{max})$.

Since, $|\nabla u|^q \leq C(|\nabla u - \nabla v|^q + |\nabla v|^q)$, for any $q > 1$, we have by (3.14), choosing η small that

$$(3.15) \quad \int_{B_\rho} |\nabla u|^p dx \leq Cr^N + C \int_{B_\rho} |\nabla v|^p dx.$$

where $C = C(\lambda_2, N, p_{min}, p_{max})$.

On the other hand, we have by Lemma B.1

$$(3.16) \quad \int_{B_{r/2}} |\nabla v|^p dx \leq C \int_{B_{3r/4}} \left| \frac{v - \{v\}_{3r/4}}{r} \right|^p dx.$$

By the regularity of solutions, (see [1]) we have that for any $0 < \gamma < 1$,

$$(3.17) \quad |v - \{v\}_{\frac{3}{4}r}| \leq C(\gamma, \|v\|_{L^\infty(B_r)}, \omega(r)) r^\gamma$$

Therefore,

$$(3.18) \quad \int_{B_\rho} |\nabla u|^p dx \leq Cr^N + C(\gamma, \|v\|_{L^\infty(B_r)}, \omega(r), p_{min}, p_{max}, N, \lambda_2) r^{N-(1-\gamma)p_+}.$$

Since $v - u \in W_0^{1,p(\cdot)}(B_r)$, the same proof as that of Lemma 2.2 shows that $\|v\|_{L^\infty(B_r)} \leq \|u\|_{L^\infty(B_r)}$. On the other hand, since u is a subsolution, by comparison we have $0 \leq u \leq v$ and then

$$(3.19) \quad \|v\|_{L^\infty(B_r)} = \|u\|_{L^\infty(B_r)}$$

This means that the constant C depends on $p_{min}, p_{max}, N, \lambda_2, \gamma, \omega(r)$ and $\|u\|_{L^\infty(B_r)}$.

Let $0 < \gamma_0 < 1$ and let $\varepsilon > 0$ and $0 < \gamma < 1$ such that

$$\frac{N}{p_{min}} \frac{\varepsilon}{1 + \varepsilon} + (1 - \gamma) = 1 - \gamma_0.$$

Let $r_0 > 0$ such that

$$\frac{p_+(B_{r_0})}{p_-(B_{r_0})} \leq 1 + \varepsilon.$$

From now on we denote, $p_- = p_-(B_{r_0})$ and $p_+ = p_+(B_{r_0})$.

Then,

$$\rho^{\frac{-(N\varepsilon + (1-\gamma)p_+)}{1+\varepsilon}} \leq \rho^{\frac{-N\varepsilon}{1+\varepsilon} - (1-\gamma)p_-}.$$

Then we have by (3.18), (3.19) and by our election of γ and ρ that,

$$\begin{aligned}
\int_{B_\rho} |\nabla u|^{p_-} dx &\leq \int_{B_\rho} |\nabla u|^p dx + \frac{1}{|B_\rho|} \int_{B_\rho \cap \{|\nabla u| < 1\}} |\nabla u|^{p_-} dx \\
&\leq \int_{B_\rho} |\nabla u|^p dx + 1 \\
&\leq 1 + C \left(\frac{r}{\rho}\right)^N + Cr^{-(1-\gamma)p_+} \left(\frac{r}{\rho}\right)^N \\
&\leq 1 + Cr^{-\varepsilon N} + Cr^{-\varepsilon N - (1-\gamma)p_+} = 1 + C\rho^{\frac{-N\varepsilon}{1+\varepsilon}} + C\rho^{\frac{-N\varepsilon - p_+(1-\gamma)}{1+\varepsilon}} \\
&\leq C\rho^{\frac{-N\varepsilon - p_+(1-\gamma)}{1+\varepsilon}} \leq C\rho^{\frac{-N\varepsilon}{1+\varepsilon} - (1-\gamma)p_-}
\end{aligned}$$

where $C(\gamma_0, N, M, \omega(r), \lambda_2, p_{min}, p_{max})$.

Let r_0 as before for this choice of ε and small enough so that $r_0^\varepsilon \leq 1/2$. Then, if $\rho \leq \rho_0 = r_0^{1+\varepsilon} \leq 1$,

$$(3.20) \quad \int_{B_\rho} |\nabla u|^{p_-} dx \leq C\rho^{-[\frac{N\varepsilon}{(1+\varepsilon)p_-} + (1-\gamma)]p_-} \leq C\rho^{-[\frac{N\varepsilon}{(1+\varepsilon)p_{min}} + (1-\gamma)]p_-} = C\rho^{-(1-\gamma_0)p_-}$$

This is, if $\rho \leq \rho_0 = r_0^{1+\varepsilon}$

$$\left(\int_{B_\rho} |\nabla u|^{p_-} dx\right)^{1/p_-} \leq C\rho^{\gamma_0 - 1}.$$

Therefore (3.7) holds.

Applying Morrey's Theorem, see e.g. [27], Theorem 1.53, we conclude that, $u \in C^{\gamma_0}(B_{\rho_0})$ and $\|u\|_{C^{\gamma_0}(\overline{B_{\rho_0}})} \leq C$ with $C(\gamma_0, M, N, \omega(r), \lambda_2, p_{min}, p_{max})$.

□

Thus, we have the following,

Theorem 3.2. *Assume p has modulus of continuity $\omega(r) = C \log(\frac{1}{r})^{-1}$. Then, for every $0 < \gamma_0 < 1$, any minimizer u belongs to $C^{\gamma_0}(\Omega)$. Moreover, let $\Omega' \subset \subset \Omega$ and $M = \|u\|_{L^\infty(\Omega)}$. There exists, $C = C(N, \Omega', p_{min}, p_{max}, \omega(r), \lambda_2, M, \gamma_0)$ such that $\|u\|_{C^{\gamma_0}(\overline{\Omega'})} \leq C$*

Then, we have that u is continuous. Therefore, $\{u > 0\}$ is open. We can prove the following property for minimizers.

Proposition 3.1. *Assume p has modulus of continuity $\omega(r) = C \log(\frac{1}{r})^{-1}$. Let u be a minimizer of \mathcal{J} in \mathcal{K} . Then, u is $p(x)$ -harmonic in $\{u > 0\}$.*

Proof. Let $B \subset \{u > 0\}$ be a ball and let v such that

$$\Delta_{p(x)} v = 0 \quad \text{in } B, \quad v - u \in W_0^{1,p}(B).$$

Since $u > 0$ in B we get, proceeding as in (3.9) and (3.10),

$$\begin{aligned}
0 &\geq \int_B \frac{|\nabla u|^p}{p} - \frac{|\nabla v|^p}{p} dx + \lambda(x)\chi_{B \cap \{u > 0\}} - \lambda(x)\chi_{B \cap \{v > 0\}} \geq \int_B \frac{|\nabla u|^p}{p} - \frac{|\nabla v|^p}{p} dx \\
&\geq C \left(\int_{A_1} (|\nabla u| + |\nabla v|)^{p-2} |\nabla u - \nabla v|^2 dx + \int_{A_2} |\nabla u - \nabla v|^p dx \right)
\end{aligned}$$

Therefore,

$$\int_{A_1} (|\nabla u| + \nabla v)^{p-2} |\nabla u - \nabla v|^2 dx = 0.$$

Thus, $(|\nabla u| + \nabla v)^{p-2} |\nabla u - \nabla v|^2 = 0$ in A_1 and, by the definition of A_1 , we conclude that $|\nabla u - \nabla v| = 0$ in this set.

On the other hand, we also have

$$\int_{A_2} |\nabla u - \nabla v|^p dx = 0$$

so that $|\nabla u - \nabla v| = 0$ everywhere in B .

Hence, as $u - v \in W_0^{1,p}(B)$ we have that $u = v$. Thus, $\Delta_{p(x)}u = 0$ in B . □

4. LIPSCHITZ CONTINUITY

In this section we prove the Lipschitz continuity and the non degeneracy of the minimizers. We assume throughout this section that $p(x)$ is Lipschitz continuous. We take ideas from [4].

Lemma 4.1. *Let p be Lipschitz continuous. Let u be a minimizer in $B_r(x_0) \subset\subset \Omega$ and v a solution to*

$$\Delta_p v = 0 \quad \text{in } B_r(x_0), \quad v - u \in W_0^{1,p}(B_r(x_0)).$$

Then, there exist $r_0 = r_0(p_{max}, p_{min}, L, \|u\|_\infty)$ and $C = C(p_{max}, p_{min}, N)$ such that for every $\varepsilon > 0$ there exists $M_\varepsilon = M(\varepsilon, p_{max}, p_{min}, L, \|u\|_\infty)$ so that if $M \geq M_\varepsilon$ and $r \leq r_0$,

$$\int_{B_r(x_0)} |\nabla(u - v)|^{p(x)} dx \geq C |B_r(x_0) \cap \{u = 0\}| M^{(1-\varepsilon)p^-}.$$

where

$$M = \frac{1}{r} \sup_{B_{\frac{3}{4}r}} u.$$

Proof. First observe that if we take $u_r(x) = \frac{1}{r}u(x_0 + rx)$, $v_r(x) = \frac{1}{r}v(x_0 + rx)$ and $p_r(x) = p(x_0 + rx)$ then,

$$\int_{B_1} |\nabla(u_r - v_r)|^{p_r(x)} dx = r^{-N} \int_{B_r(x_0)} |\nabla(u - v)|^{p(x)} dy,$$

$$|B_1 \cap \{u_r = 0\}| = r^{-N} |B_r(x_0) \cap \{u = 0\}|,$$

$$\sup_{B_{3/4}(0)} u_r = \frac{1}{r} \sup_{B_{\frac{3}{4}r}(x_0)} u,$$

and $\|\nabla p_r(x)\|_\infty = r \|\nabla p(x_0 + rx)\|_\infty$. Since $\|\nabla p_r(x)\|$ is small, if r is small, we will assume that $r = 1$ and $\|\nabla p\|_{L^\infty(B_1)} \leq \delta$ with δ as small as needed (by taking r_0 small enough).

So that, from now on we assume that $x_0 = 0$ and $r = 1$.

For $|z| \leq \frac{1}{2}$ we consider the change of variables from B_1 into itself such that z becomes the new origin. We call $u_z(x) = u((1 - |x|)z + x)$, $v_z(x) = v((1 - |x|)z + x)$, $p_z(x) = p((1 - |x|)z + x)$. Observe that this change of variables leaves the boundary fixed. Define,

$$r_\xi = \inf \left\{ r / \frac{1}{8} \leq r \leq 1 \quad \text{and} \quad u_z(r\xi) = 0 \right\},$$

if this set is nonempty.

Now, for almost every $\xi \in \partial B_1$ we have

$$(4.21) \quad v_z(r_\xi \xi) = \int_{r_\xi}^1 \frac{d}{dr} (u_z - v_z)(r\xi) dr \leq \int_{r_\xi}^1 |\nabla(u_z - v_z)(r\xi)| dr.$$

Let us assume that the following inequality holds: There exist $\delta_0 > 0$ and M_ε such that if $\delta \leq \delta_0$ and $M \geq M_\varepsilon$,

$$(4.22) \quad v_z(r_\xi \xi) \geq (1 - r_\xi)M^{1-\varepsilon}$$

where $M = \sup_{B_{3/4}(0)} u$.

Let $B = M^{1-\varepsilon}$. Then, for any $\eta > 0$ we have,

$$(4.23) \quad v_z(r_\xi \xi) \leq \int_{r_\xi}^1 \frac{|\nabla(u_z - v_z)(r\xi)|^{p_z(r\xi)}}{(\eta B)^{p_z(r\xi)-1} p_z(r\xi)} dr + \int_{r_\xi}^1 \frac{p_z(r\xi) - 1}{p_z(r\xi)} B \eta dr,$$

and using (4.22) and (4.23) with η small we have,

$$(4.24) \quad \int_{r_\xi}^1 \frac{|\nabla(u_z - v_z)(r\xi)|^{p_z(r\xi)}}{(B)^{p_z(r\xi)-1} p_z(r\xi)} dr \geq C(p_+, p_-)(1 - r_\xi)B.$$

Therefore,

$$(4.25) \quad \int_{r_\xi}^1 |\nabla(u_z - v_z)(r\xi)|^{p_z(r\xi)} dr \geq C(p_+, p_-)(1 - r_\xi)B^{p_-}$$

Then, using (4.25), integrating first over ∂B_1 and then over $|z| \leq 1/2$ we obtain as in [4],

$$\int_{B_1} |\nabla(u - v)|^{p(x)} dx \geq C|B_1 \cap \{u = 0\}|B^{p_-}.$$

So we have the desired result.

Therefore, we only have to prove (4.22). Observe that, since $|z| \leq 1/2$, it is enough to prove that $v(x) \geq M^{1-\varepsilon}(1 - |x|)$ if M is large enough.

If $|x| \leq 3/4$, by Remark B.2 we have

$$v(x) \geq C_1 \left(\sup_{B_{3/4}(0)} v - 3/4 \right) \geq C_1 \left(\sup_{B_{3/4}(0)} u - 3/4 \right) = C_1(M - 3/4) \geq \frac{C_1}{2}M$$

if $M \geq 2$, with C_1 depending on p_{min}, p_{max}, L and $\|v\|_\infty$ (the bound of v before rescaling, see Remark B.2, that equals the bound of u before rescaling).

If $|x| \geq 3/4$ we prove by a comparison argument that inequality (4.22) also holds. In fact, we know

$$v \geq C_2 M \text{ in } \overline{B_{3/4}}.$$

Take $w(x) = \theta M(e^{-\mu|x|^2} - e^{-\mu})$, where θ is such that $w \leq v$ on $\partial B_{3/4}$. Let μ_0 and ε_0 as in Lemma B.4. Then if $\mu \geq \mu_0$ and $\delta \leq \varepsilon_0$, there holds that

$$\begin{cases} C(\mu, \theta, M, p)\Delta_{p(x)}w \geq \bar{C}_1(\mu - \bar{C}_2\|\nabla p\|_\infty|\log(\theta M)|) \geq \bar{C}_1(\mu - \bar{C}_2\delta|\log(\theta M)|) & \text{in } B_1 \setminus B_{3/4}, \\ w \leq C_2M & \text{on } \partial B_{3/4}, \\ w = 0 & \text{on } \partial B_1. \end{cases}$$

Recall that $\theta = \frac{C_2}{e^{-\mu 9/16} - e^{-\mu}}$. Thus, if μ is large enough (depending on C_2), there holds that $|\log \theta| \leq 2\mu$.

Therefore,

$$C(\mu, \theta, M, p) \Delta_{p(x)} w_\mu \geq \bar{C}_1(\mu - \bar{C}_2 2\delta\mu - \bar{C}_2 \delta |\log M|).$$

Let $\delta_0 > 0$. Assume further that $\delta \leq \delta_0 < \frac{1}{4}(\bar{C}_2)^{-1}$ and take $\mu = 2\bar{C}_2 \delta_0 |\log M|$. Then,

$$\begin{cases} \Delta_{p(x)} w \geq 0 & \text{in } B_1 \setminus B_{3/4}, \\ w \leq C_2 M & \text{on } \partial B_{3/4}, \\ w = 0 & \text{on } \partial B_1. \end{cases}$$

Thus, $w \leq v$ in $B_1 \setminus \overline{B_{3/4}}$. Now,

$$\begin{aligned} w &\geq \theta e^{-\mu} \mu (1 - |x|) M \geq C_2 M e^{-7/16\mu} \mu (1 - |x|) \\ &\geq C_3 M^{1-C\delta_0} (1 - |x|) \delta_0 \log M = C_3 M^{1-C\delta_0} (1 - |x|) \log M^{\delta_0} \end{aligned}$$

if $M \geq 1$.

Given $\varepsilon > 0$, assume further that $C\delta_0 \leq \varepsilon$ and then, that M is large enough (that is $M \geq M_\varepsilon$) so that $C_3 \log M^{\delta_0} \geq 1$. Then,

$$w \geq M^{1-\varepsilon} (1 - |x|) \quad \text{in } B_1 \setminus \overline{B_{3/4}},$$

so that

$$v \geq M^{1-\varepsilon} (1 - |x|) \quad \text{in } B_1 \setminus \overline{B_{3/4}}.$$

Recalling the estimate inside the ball $B_{3/4}$ we get, as $M \geq 1$,

$$v \geq C_2 M \geq M^{1-\varepsilon} (1 - |x|) \quad \text{in } B_{3/4},$$

if M is large enough, and (4.22) is proved. \square

Lemma 4.2. *Let $p \in Lip(\Omega)$, $p \geq 2$ in Ω , There exist $r_0 = r_0(p_{max}, p_{min}, L, \|u\|_\infty)$ and $C_{max} = C_{max}(p_{max}, p_{min}, L, \lambda_2, M)$ such that if $r \leq r_0$, each local minimizer u with $\|u\|_\infty \leq M$ has the following property: If $B_{\frac{3}{4}r}(x_0) \subset\subset \Omega$,*

$$\frac{1}{r} \sup_{B_{\frac{3}{4}r}(x_0)} u \geq C_{max} \quad \text{implies} \quad \Delta_{p(x)} u = 0 \quad \text{in } B_r(x_0)$$

Proof. Take v as in the previous lemma. By a standard inequality we have if $p(x) \geq 2$ (see [12]),

$$\begin{aligned} \lambda_2 |B_r(x_0) \cap \{u = 0\}| &\geq \lambda_2 (|B_r(x_0) \cap \{v > 0\}| - |B_r(x_0) \cap \{u > 0\}|) \geq \\ &\geq \int_{B_r(x_0)} \frac{|\nabla u|^{p(x)}}{p} - \frac{|\nabla v|^{p(x)}}{p} dx \geq C \int_{B_r(x_0)} |\nabla u - \nabla v|^{p(x)} dx. \end{aligned}$$

If C_{max} is large enough, by the previous lemma we get,

$$|\{u = 0\} \cap B_r(x_0)| \geq C |\{u = 0\} \cap B_r(x_0)| \left(\frac{1}{r} \sup_{B_{3/4r}(x_0)} u \right)^{p-(1-\varepsilon)}.$$

Therefore, if C_{max} is big enough we have that $|\{u = 0\} \cap B_r| = 0$ and we obtain the desired result since

$$\int_{B_r} |\nabla u - \nabla v|^{p(x)} dx \leq C|\{u = 0\} \cap B_r|$$

so that, $u = v$ in B_r . \square

Lemma 4.3. *Let $p \in Lip(\Omega)$. For any $0 < \kappa < 1$ There exist $r_0, c_\kappa > 0$ such that if $r \leq r_0$ and $B_r(x_0) \subset\subset \Omega$,*

$$\frac{1}{r} \sup_{B_r(x_0)} u \leq c_\kappa \quad \text{implies} \quad u = 0 \quad \text{in } B_{\kappa r}(x_0).$$

Here r_0 depends on $\kappa, p_{min}, p_{max}, L$ and N and c_κ depends also on λ_1 .

Proof. We may suppose that $r = 1$ and that B_r is centered at zero, (if not, we take the rescaled function $\tilde{u} = \frac{u(x_0+rx)}{r}$). Moreover, by taking $r \leq r_0$ we may assume that $\|\nabla p\|_{L^\infty(B_1)} \leq \delta$.

Let $\varepsilon := \sup_{B_{\sqrt{\kappa}}} u$. Choose v as

$$v = \begin{cases} \frac{\varepsilon}{c_\mu} (e^{-\mu|x|^2} - e^{-\mu\kappa^2}) & \text{in } B_{\sqrt{\kappa}} \setminus B_\kappa, \\ 0 & \text{in } B_\kappa, \end{cases}$$

where $c_\mu = (e^{-\mu\kappa} - e^{-\mu\kappa^2}) < 0$.

By Lemma B.4 we have if μ is large enough,

$$\begin{aligned} \frac{-c_\mu}{\varepsilon\mu} e^{\mu|x|^2} |\nabla v|^{2-p} \Delta_{p(x)}(-v) &\geq C_1(\mu - C_2\|\nabla p\|_\infty |\log \frac{\varepsilon}{-c_\mu}|) \\ &\geq C_1(\mu - C_2\|\nabla p\|_\infty |\log(-c_\mu)| - C_2\|\nabla p\|_\infty |\log \varepsilon|). \end{aligned}$$

If $\mu \geq \frac{\log 2}{(\kappa(1-\kappa))}$ we have,

$$e^{-\mu\kappa} \leq e^{-\mu\kappa}(e^{-\mu\kappa(\kappa-1)} - 1) = e^{-\mu\kappa^2} - e^{-\mu\kappa} = -c_\mu \leq e^{-\mu\kappa^2} < 1.$$

Then, $0 > \log(-c_\mu) \geq -\mu\kappa$. Therefore,

$$\frac{-c_\mu}{\varepsilon\mu} e^{\mu|x|^2} |\nabla v|^{2-p} \Delta_{p(x)}(-v) \geq C_1((1 - C_2\kappa\|\nabla p\|_\infty)\mu - C_2\|\nabla p\|_\infty |\log \varepsilon|).$$

If $\delta \leq \frac{1}{2C_2\kappa}$ we have,

$$\frac{-c_\mu}{\varepsilon\mu} e^{\mu|x|^2} |\nabla v|^{2-p} \Delta_{p(x)}(-v) \geq C_1\left(\frac{\mu}{2} - C_2\delta |\log \varepsilon|\right) \geq 0,$$

if we choose $\mu \geq 2C_2\delta |\log \varepsilon|$.

Hence, if $r \leq r_0 := \frac{1}{2LC_2\kappa}$ so that $\|\nabla p\|_\infty \leq \delta_0 = \frac{1}{2C_2\kappa}$, and we take $\mu = 2C_2\delta_0 |\log \varepsilon| = \kappa^{-1} |\log \varepsilon|$,

$$\Delta_{p(x)} v < 0 \quad \text{in } B_{\sqrt{\kappa}} \setminus \overline{B_\kappa}.$$

By construction $v \geq u$ on $\partial B_{\sqrt{\kappa}}$. Thus, if we take

$$w = \begin{cases} \min(u, v) & \text{in } B_{\sqrt{\kappa}}, \\ u & \text{in } \Omega \setminus B_{\sqrt{\kappa}}, \end{cases}$$

we find that w is an admissible function for the minimizing problem. Thus, using the convexity we find that

$$\begin{aligned}
& \int_{B_\kappa} \frac{|\nabla u|^{p(x)}}{p(x)} + \lambda \chi_{B_\kappa \cap \{u > 0\}} dx \\
&= \mathcal{J}(u) - \int_{\Omega \setminus B_\kappa} \frac{|\nabla u|^{p(x)}}{p(x)} dx + \int_{\Omega} (\lambda \chi_{B_\kappa \cap \{u > 0\}} - \lambda \chi_{\Omega \cap \{u > 0\}}) dx \\
&\leq \mathcal{J}(w) - \int_{\Omega \setminus B_\kappa} \frac{|\nabla u|^{p(x)}}{p(x)} dx + \int_{\Omega} (\lambda \chi_{B_\kappa \cap \{u > 0\}} - \lambda \chi_{\Omega \cap \{u > 0\}}) dx \\
&\leq \int_{B_{\sqrt{\kappa}} \setminus B_\kappa} \frac{|\nabla w|^{p(x)}}{p(x)} dx - \int_{B_{\sqrt{\kappa}} \setminus B_\kappa} \frac{|\nabla u|^{p(x)}}{p(x)} dx \\
&\leq \int_{B_{\sqrt{\kappa}} \setminus B_\kappa} |\nabla w|^{p(x)-2} \nabla w (\nabla w - \nabla u) dx = - \int_{B_{\sqrt{\kappa}} \setminus B_\kappa} |\nabla w|^{p(x)-2} \nabla w \nabla (u - v)^+ dx \\
&= - \int_{(B_{\sqrt{\kappa}} \setminus B_\kappa)} |\nabla v|^{p(x)-2} \nabla v \nabla (u - v)^+ dx
\end{aligned}$$

and as v is a classical supersolution we have,

$$\int_{B_\kappa} \frac{|\nabla u|^{p(x)}}{p(x)} + \lambda \chi_{B_\kappa \cap \{u > 0\}} dx \leq \int_{\partial B_\kappa} |\nabla v|^{p(x)-1} u d\mathcal{H}^{N-1}.$$

On the other hand, if $\mu \geq \frac{\log 2}{\kappa(1-\kappa)}$, then $1 - e^{-\mu\kappa(1-\kappa)} \geq 1/2$ and therefore, v satisfies

$$|\nabla v|_{|x|=\kappa} = \frac{2\kappa\varepsilon\mu e^{-\mu\kappa^2}}{e^{-\mu\kappa^2} - e^{-\mu\kappa}} = \frac{2\varepsilon\kappa\mu}{1 - e^{-\mu\kappa(1-\kappa)}} \leq 4\kappa\mu\varepsilon = 4\varepsilon |\log \varepsilon|$$

Thus,

$$\int_{B_\kappa} \frac{|\nabla u|^{p(x)}}{p(x)} + \lambda \chi_{B_\kappa \cap \{u > 0\}} dx \leq C(p)(\varepsilon |\log \varepsilon|)^{p-1} \int_{\partial B_\kappa} u d\mathcal{H}^{N-1}.$$

By Sobolev's trace inequality we have,

$$\begin{aligned}
\int_{\partial B_\kappa} u &\leq C(N, \kappa) \int_{B_\kappa} |\nabla u| + u dx \\
&\leq C(N, p, \kappa) \left(\int_{B_\kappa} |\nabla u|^{p(x)} + |B_\kappa \cap \{u > 0\}| + \int_{B_\kappa} u dx \right) \\
&\leq C(N, \kappa, p, \lambda_1)(1 + \varepsilon) \left(\int_{B_\kappa} \frac{|\nabla u|^{p(x)}}{p(x)} + \lambda_1 |\{u > 0\} \cap B_\kappa| \right)
\end{aligned}$$

where in the last inequality we are using that $\int_{B_\kappa} u dx \leq \varepsilon |\{u > 0\} \cap B_\kappa|$. Therefore,

$$\int_{B_\kappa} \frac{|\nabla u|^{p(x)}}{p(x)} dx + \lambda_1 |B_\kappa \cap \{u > 0\}| \leq C(\varepsilon |\log \varepsilon|)^{p-1} \left(\int_{B_\kappa} \frac{|\nabla u|^{p(x)}}{p(x)} dx + \lambda_1 |B_\kappa \cap \{u > 0\}| \right),$$

where $C = C(N, \kappa, p, \lambda_1)$.

So that, if ε is small enough

$$\int_{B_\kappa} \frac{|\nabla u|^{p(x)}}{p(x)} dx + \lambda_1 |B_\kappa \cap \{u > 0\}| = 0.$$

In particular, $u = 0$ in B_κ and the result follows. \square

As a corollary we have,

Corollary 4.1. *Let $p \in Lip(\Omega)$. There exist $r_0, C_{min} > 0$ such that if $r \leq r_0$ and $B_r(x_0) \subset\subset \Omega$,*

$$\frac{1}{r} \sup_{B_{\frac{3}{4}r}(x_0)} u \leq C_{min} \quad \text{implies} \quad u = 0 \quad \text{in } B_{r/2}(x_0).$$

Here r_0 depends on p_{min}, p_{max}, L and N and C_{min} depends also on λ_1 .

Corollary 4.1 states that, if p is Lipschitz, then any minimizer is non-degenerate, i.e,

Corollary 4.2. *Let $p \in Lip(\Omega)$. Let $D \subset\subset \Omega$, $x_0 \in D \cap \partial\{u > 0\}$. Then*

$$\sup_{B_r(x_0)} u \geq C_{min}r, \quad \text{if } r \leq r_0$$

where C_{min} is the constant in Corollary 4.1 and r_0 depends also on D .

Corollary 4.3. *Let $p \in Lip(\Omega)$ and $p \geq 2$. Let $D \subset\subset \Omega$, $x_0 \in D \cap \partial\{u > 0\}$. Then*

$$\sup_{B_{\frac{3}{4}r}(x_0)} u \leq C_{max}r, \quad \text{if } r \leq r_0$$

where C_{max} is the constant in Lemma 4.2 and r_0 depends also on D .

Proof. Assume by contradiction that the inequality is false. Then, by Lemma 4.2, $\Delta_{p(x)}u = 0$ in $B_r(x_0)$. Therefore, by the regularity results in [1], $\nabla u \in C^\alpha(\overline{B_r(x_0)})$ and, since $u \geq 0$ and $u(x_0) = 0$, there holds that $\nabla u(x_0) = 0$. Thus, $|\nabla u(x)| \leq C\rho^\alpha$ in $\overline{B_\rho}$ if $\rho \leq 3r/4$. From here we have that $u(x) \leq C\rho^{1+\alpha}$ in $\overline{B_\rho}$ if $\rho \leq 3r/4$.

On the other hand, by Corollary 4.2, $\frac{4}{3}C_{min}\rho \leq \sup_{B_\rho(x_0)} u \leq C\rho^{1+\alpha}$ if ρ is small, which is a contradiction. \square

Now we can prove the local Lipschitz continuity of minimizers of \mathcal{J} when $p \in Lip(\Omega)$ and $p \geq 2$.

Theorem 4.1. *Let $p \in Lip(\Omega)$ and $p \geq 2$. Let u be a minimizer of \mathcal{J} in \mathcal{K} . Then, u is locally Lipschitz continuous in Ω . Moreover, for any connected open subset $D \subset\subset \Omega$ containing free boundary points, the Lipschitz constant of u in D is estimated by a constant C depending only on $N, p_{max}, p_{min}, L, dist(D, \partial\Omega), \|u\|_{L^\infty(\Omega)}, \|\nabla u\|_{L^1(\Omega)}, \lambda_1$ and λ_2 .*

Proof. The proof follows as in [4], from Corollary 4.3 and the gradient estimate

$$|\nabla u(y)| \leq C \left(1 + \frac{1}{r} \sup_{B_r(y)} u \right)^{\frac{p_+(B_r)}{p_-(B_r)}}$$

that holds if $\Delta_{p(x)}u = 0$ in $B_r(y)$ (see Lemma B.3). \square

5. LINEAR GROWTH – POSITIVE DENSITY

Throughout this section we will assume that u is a locally Lipschitz, non-degenerate (i.e. satisfying the conclusions of Corollary 4.2) minimizer, and we also assume that p is Hölder continuous.

Theorem 5.1. *Suppose that p is Hölder continuous, u is Lipschitz with constant C_{Lip} and non-degenerate with constants c and r_0 . For any domain $D \subset\subset \Omega$ there exists a constant \tilde{c} , with $0 < \tilde{c} < 1$ depending on N, C_{Lip}, c, D and the Hölder modulus of continuity of p , such that, for any minimizer u and for every $B_r \subset \Omega$, centered at the free boundary with $r \leq r_0$ we have,*

$$\tilde{c} \leq \frac{|B_r \cap \{u > 0\}|}{|B_r|} \leq 1 - \tilde{c}$$

Proof. First, by the non-degeneracy we have that there exists $y \in B_r$ such that $u(y) \geq cr$ so that,

$$\sup_{B_{\kappa r}(y)} u \geq u(y) \geq cr.$$

Therefore,

$$\frac{1}{\kappa r} \sup_{B_{\kappa r}(y)} u \geq \frac{c}{\kappa}.$$

Now, if κ is small enough, we have

$$\frac{1}{\kappa r} \sup_{B_{\kappa r}(y)} u > C_{Lip}.$$

Using the fact that u is Lipschitz with constant C_{Lip} we find that $u > 0$ in $B_{\kappa r}$, where $\kappa = \kappa(C_{Lip}, c)$. Thus,

$$\frac{|B_r \cap \{u > 0\}|}{|B_r|} \geq \frac{|B_{\kappa r}|}{|B_r|} = \kappa^N.$$

In order to prove the other inequality, we may assume that $r = 1$ and the ball is centered at the origin. Let us suppose by contradiction that there exists a sequence of minimizers u_k in B_1 , corresponding to powers $p_{min} \leq p_k(x) \leq p_{max}$ with the same Hölder modulus of continuity $\omega(r)$, u_k Lipschitz with constant C_{Lip} and non-degenerate with constant c such that, $0 \in \partial\{u_k > 0\}$ and $|\{u_k = 0\} \cap B_1| = \varepsilon_k \rightarrow 0$. Let us take $v_k \in W^{1,p_k(x)}(B_{1/2})$ such that,

$$(5.26) \quad \Delta_{p_k(x)} v_k = 0 \quad \text{in } B_{1/2}, \quad v_k - u_k \in W_0^{1,p_k(x)}(B_{1/2}).$$

We have, by the arguments leading to (3.11), (3.12),

$$\begin{aligned} \int_{\{p_k \geq 2\} \cap B_{1/2}} |\nabla u_k - \nabla v_k|^{p_k(x)} dx &\leq C\varepsilon_k \quad \text{and} \\ \int_{\{p_k < 2\} \cap B_{1/2}} (|\nabla u_k| + |\nabla v_k|)^{p_k(x)-2} |\nabla u_k - \nabla v_k|^2 dx &\leq C\varepsilon_k. \end{aligned}$$

Now, since $|\nabla u_k| \leq C_{Lip}$ for every k ,

$$\begin{aligned} & \int_{\{p_k < 2\} \cap B_{1/2}} (|\nabla u_k| + |\nabla v_k|)^{p_k(x)-2} |\nabla u_k - \nabla v_k|^2 dx \geq \\ & \int_{\{p_k < 2\} \cap B_{1/2}} (2|\nabla u_k| + |\nabla u_k - \nabla v_k|)^{p_k(x)-2} |\nabla u_k - \nabla v_k|^2 dx \geq \\ & C \int_{\{p_k < 2, |\nabla u_k - \nabla v_k| \leq |\nabla u_k|\} \cap B_{1/2}} |\nabla u_k - \nabla v_k|^2 dx + \\ & C \int_{\{p_k < 2, |\nabla u_k - \nabla v_k| > |\nabla u_k|\} \cap B_{1/2}} |\nabla u_k - \nabla v_k|^{p_k(x)} dx \end{aligned}$$

On the other hand, using again that $|\nabla u_k| \leq C_{Lip}$ for every k , assuming that $p_- < 2$,

$$\begin{aligned} & \int_{\{p_k < 2, |\nabla u_k - \nabla v_k| \leq |\nabla u_k|\} \cap B_{1/2}} |\nabla u_k - \nabla v_k|^{p_k(x)} dx \leq \\ & C \int_{\{p_k < 2, |\nabla u_k - \nabla v_k| \leq |\nabla u_k|\} \cap B_{1/2}} |\nabla u_k - \nabla v_k|^{p_-} dx \leq \\ & C \left(\int_{\{p_k < 2, |\nabla u_k - \nabla v_k| \leq |\nabla u_k|\} \cap B_{1/2}} |\nabla u_k - \nabla v_k|^2 dx \right)^{p_-/2} \end{aligned}$$

If $p_- \geq 2$,

$$\begin{aligned} & \int_{\{p_k < 2, |\nabla u_k - \nabla v_k| \leq |\nabla u_k|\} \cap B_{1/2}} |\nabla u_k - \nabla v_k|^{p_k(x)} dx \leq \\ & C \int_{\{p_k < 2, |\nabla u_k - \nabla v_k| \leq |\nabla u_k|\} \cap B_{1/2}} |\nabla u_k - \nabla v_k|^2 dx \leq C \varepsilon_k. \end{aligned}$$

Summing up we get,

$$(5.27) \quad \int_{B_{1/2}} |\nabla u_k - \nabla v_k|^{p_k(x)} dx \leq C \max\{\varepsilon_k, \varepsilon_k^{p_-/2}\}.$$

On the other hand, since $\|u_k\|_\infty \leq C_{Lip}$ and $\|v_k\|_{C^{1,\alpha}(B_\rho)} \leq C(N, \rho, p_+, p_-, \omega(r), \|u_k\|_{L^\infty(B_{1/2})})$ (see (3.19) for the bound of $\|v_k\|_{L^\infty(B_{1/2})}$ and [1] for the regularity of v_k), there holds that, for a subsequence, $v_k \rightarrow v_0$ and $\nabla v_k \rightarrow \nabla v_0$ uniformly on compact subsets of $B_{1/2}$.

Finally, since $\|u_k\|_{Lip(B_{1/2})} \leq C_{Lip}$ we have, for a subsequence, $u_k \rightarrow u_0$ uniformly in $B_{1/2}$.

Let $w_k = u_k - v_k$. Then, $w_k \rightarrow u_0 - v_0$ uniformly on compact subsets of $B_{1/2}$. Let us see that $u_0 = v_0$.

In fact, by (5.27) we have that $\|\nabla w_k\|_{L^{p_k(x)}} \rightarrow 0$. Since $w_k \in W_0^{1,p_k(x)}(B_{1/2})$, by Poincaré inequality we get that $\|w_k\|_{L^{p_k(x)}(B_{1/2})} \rightarrow 0$. By Theorem A.2 there holds that $w_k \rightarrow 0$ in $L^{p_-}(B_{1/2})$ and, for a subsequence, $w_k \rightarrow 0$ almost everywhere. Thus, $u_0 = v_0$.

Since, the p_k 's are uniformly Hölder continuous and are uniformly bounded, there exists p_0 such that (for a subsequence) $p_k \rightarrow p_0$ uniformly in $B_{1/2}$.

Now, recall that $v_k \rightarrow u_0$ in $C^1(B_{1/2})$. Then, $\Delta_{p_0(x)} u_0 = 0$ in $B_{1/2}$.

As $u_k \rightarrow u_0$ uniformly in $B_{1/2}$ and are uniformly non-degenerate we get, $\sup_{B_s} u_0 \geq cs$ for s small. But $u_0(0) = \lim u_k(0) = 0$. By the same argument as that in Corollary 4.3 we arrive at a contradiction. \square

Remark 5.1. Theorem 5.1 implies that the free boundary has Lebesgue measure zero. In fact, in order to prove this statement, it is enough to use the first inequality in Theorem 5.1, as this estimate says that the set of Lebesgue points of $\chi_{\{u>0\}}$ in $\partial\{u > 0\} \cap D$ is empty. On the other hand almost every point $x_0 \in \partial\{u > 0\} \cap D$ is a Lebesgue point, therefore $|\partial\{u > 0\} \cap D| = 0$.

6. THE MEASURE $\Lambda = \Delta_{p(x)}u$

We still assume that u is a non-degenerate, locally Lipschitz minimizer.

In this section we assume that p is Hölder continuous. First, we prove that $\{u > 0\} \cap \Omega$ is locally of finite perimeter. Then, we study the measure $\Lambda = \Delta_{p(x)}u$ and prove that it is absolutely continuous with respect to the \mathcal{H}^{N-1} measure restricted to the free boundary. This result gives rise to a representation theorem for the measure Λ . Finally, we prove that almost every point in the free boundary belongs to the reduced free boundary.

Theorem 6.1. *For every $\varphi \in C_0^\infty(\Omega)$ such that $\text{supp}(\varphi) \subset \{u > 0\}$,*

$$(6.28) \quad \int_{\Omega} |\nabla u|^{p(x)-2} \nabla u \nabla \varphi = 0.$$

Moreover, the application

$$\Lambda(\varphi) := - \int_{\Omega} |\nabla u|^{p(x)-2} \nabla u \nabla \varphi \, dx$$

from $C_0^\infty(\Omega)$ into \mathbb{R} defines a nonnegative Radon measure $\Lambda = \Delta_{p(x)}u$ with support on $\Omega \cap \partial\{u > 0\}$.

Proof. We know that u is $p(x)$ -subharmonic. Then, by the Riesz Representation Theorem, there exists a nonnegative Radon measure Λ , such that $\Delta_{p(x)}u = \Lambda$. And, as $\Delta_{p(x)}u = 0$ in $\{u > 0\}$, for any $\varphi \in C_0^\infty(\Omega \setminus \partial\{u > 0\})$ there holds that $\Lambda(\varphi) = 0$ and the result follows. \square

Now we want to prove that $\Omega \cap \partial\{u > 0\}$, has finite $N - 1$ dimensional Hausdorff measure. First, we need the following lemma,

Lemma 6.1. *Let u_k be a sequence of minimizers in B_1 corresponding to powers $p_k(x)$ and coefficients $\lambda_k(x)$ with $1 < p_{\min} \leq p_k(x) \leq p_{\max} < \infty$, $0 < \lambda_1 \leq \lambda_k(x) \leq \lambda_2 < \infty$, and all the p_k 's with the same modulus of continuity $\omega(r)$. Assume $u_k \rightarrow u_0$ uniformly in B_1 , $|\nabla u_k| \leq C_{Lip}$ in B_1 , and that the u_k 's are non-degenerate in B_1 with constants c_0 and r_0 . Then,*

- (1) $\partial\{u_k > 0\} \rightarrow \partial\{u_0 > 0\}$ locally in Hausdorff distance,
- (2) $\chi_{\{u_k > 0\}} \rightarrow \chi_{\{u_0 > 0\}}$ in $L^1(B_1)$,
- (3) If $0 \in \partial\{u_k > 0\}$, then $0 \in \partial\{u_0 > 0\}$.

Proof. The proof follows as in pp. 19–20 of [5]. \square

Now, we prove the following theorem,

Theorem 6.2. *For any domain $D \subset\subset \Omega$ there exist constants c, C , depending on $N, C_{Lip}, c_0, r_0, p_{min}, p_{max}, \lambda_1, \lambda_2, \omega(r)$ and D such that, for any minimizer u with $|\nabla u| \leq C_{Lip}$, non-degenerate with constants c_0 and r_0 , and for every $B_r \subset \Omega$, centered on the free boundary with $r \leq r_0$, we have*

$$cr^{N-1} \leq \int_{B_r} d\Lambda \leq Cr^{N-1}$$

Proof. The ideas are similar to the ones for the case p constant in [12], with modifications similar to the ones in the proof of Theorem 5.1. \square

Therefore, we have the following representation theorem

Theorem 6.3 (Representation Theorem). *Let u be a non-degenerate, locally Lipschitz continuous minimizer. Then,*

- (1) $\mathcal{H}^{N-1}(D \cap \partial\{u > 0\}) < \infty$ for every $D \subset\subset \Omega$.
- (2) There exists a Borel function q_u such that

$$\Delta_{p(x)}u = q_u \mathcal{H}^{N-1} \llcorner \partial\{u > 0\}.$$

i.e

$$-\int_{\Omega} |\nabla u|^{p(x)-2} \nabla u \nabla \varphi \, dx = \int_{\Omega \cap \partial\{u > 0\}} q_u \varphi \, d\mathcal{H}^{N-1}, \quad \forall \varphi \in C_0^\infty(\Omega).$$

- (3) For $D \subset\subset \Omega$ there are constants $0 < c \leq C < \infty$ such that for $B_r(x) \subset D$ and $x \in \partial\{u > 0\}$,

$$c \leq q_u(x) \leq C, \quad cr^{N-1} \leq \mathcal{H}^{N-1}(B_r(x) \cap \partial\{u > 0\}) \leq Cr^{N-1}.$$

Proof. It follows as in Theorem 4.5 in [4]. \square

Remark 6.1. As u satisfies the conclusions of Theorem 6.3, the set $\Omega \cap \{u > 0\}$ has finite perimeter locally in Ω (see [17] 4.5.11). That is, $\mu_u := -\nabla \chi_{\{u > 0\}}$ is a Borel measure, and the total variation $|\mu_u|$ is a Radon measure. We define the reduced boundary as in [17], 4.5.5. (see also [16]) by, $\partial_{red}\{u > 0\} := \{x \in \Omega \cap \partial\{u > 0\} / |\nu_u(x)| = 1\}$, where $\nu_u(x)$ is a unit vector with

$$(6.29) \quad \int_{B_r(x)} |\chi_{\{u > 0\}} - \chi_{\{y / \langle y-x, \nu_u(x) \rangle < 0\}}| = o(r^N)$$

for $r \rightarrow 0$, if such a vector exists, and $\nu_u(x) = 0$ otherwise. By the results in [17] Theorem 4.5.6 we have,

$$\mu_u = \nu_u \mathcal{H}^{N-1} \llcorner \partial_{red}\{u > 0\}.$$

Lemma 6.2. $\mathcal{H}^{N-1}(\partial\{u > 0\} \setminus \partial_{red}\{u > 0\}) = 0$.

Proof. This is a consequence of the density property of Theorem 5.1 and Theorem 4.5.6 (3) of [17]. \square

7. ASYMPTOTIC DEVELOPMENT AND IDENTIFICATION OF THE FUNCTION q_u

In this section we still assume that u is a non-degenerate, locally Lipschitz continuous minimizer, p is Hölder continuous and, moreover we assume that λ is continuous.

We prove some properties of blow up sequences of minimizers and prove that any limit of a blow up sequence is a minimizer. Then, we find an asymptotic development of minimizers near points in their reduced free boundary. Finally, we identify the function q_u for almost every point in the reduced free boundary.

We first prove some properties of blow up sequences,

Definition 7.1. *Let $B_{\rho_k}(x_k) \subset D \subset\subset \Omega$ be a sequence of balls with $\rho_k \rightarrow 0$, $x_k \rightarrow x_0 \in \Omega$ and $u(x_k) = 0$. Let*

$$u_k(x) := \frac{1}{\rho_k} u(x_k + \rho_k x).$$

We call u_k a blow-up sequence with respect to $B_{\rho_k}(x_k)$.

Since u is locally Lipschitz continuous, there exists a blow-up limit $u_0 : \mathbb{R}^N \rightarrow \mathbb{R}$ such that, for a subsequence,

$$\begin{aligned} u_k &\rightarrow u_0 \quad \text{in } C_{\text{loc}}^\alpha(\mathbb{R}^N) \quad \text{for every } 0 < \alpha < 1, \\ \nabla u_k &\rightarrow \nabla u_0 \quad * \text{-weakly in } L_{\text{loc}}^\infty(\mathbb{R}^N), \end{aligned}$$

and u_0 is Lipschitz in \mathbb{R}^N with constant C_{Lip} .

Lemma 7.1. *If u is a non-degenerate, locally Lipschitz continuous minimizer then,*

- (1) $\partial\{u_k > 0\} \rightarrow \partial\{u_0 > 0\}$ locally in Hausdorff distance,
- (2) $\chi_{\{u_k > 0\}} \rightarrow \chi_{\{u_0 > 0\}}$ in $L_{\text{loc}}^1(\mathbb{R}^N)$,
- (3) $\nabla u_k \rightarrow \nabla u_0$ uniformly in compact subsets of $\{u_0 > 0\}$,
- (4) $\nabla u_k \rightarrow \nabla u_0$ a.e in Ω ,
- (5) If $x_k \in \partial\{u > 0\}$, then $0 \in \partial\{u_0 > 0\}$
- (6) $\Delta_{p(x_0)} u_0 = 0$ in $\{u_0 > 0\}$
- (7) u_0 is Lipschitz continuous and non-degenerate with the same constants C_{Lip} and c_0 as u .

Proof. (1), (2) and (5) follow from Lemma 6.1. For the proof of (3) and (4) we use that ∇u_k are uniformly Hölder continuous in compact subsets of $\{u_0 > 0\}$ and ideas similar to those in pp. 19-20 in [5]. (6) follows from (3) and the fact that $\Delta_{P_k(x)} u_k = 0$ in $\{u_k > 0\}$ with $p_k(x) = p(x_0 + \rho_k x) \rightarrow p(x_0)$ uniformly in compact sets of \mathbb{R}^N . (7) follows immediately from the uniform convergence of u_k and the fact that they are all non-degenerate with constant c_0 . \square

Lemma 7.2. *Let u be a non-degenerate, locally Lipschitz continuous minimizer with $u(x_m) = 0$, $x_m \rightarrow x_0 \in \Omega$. Then, any blow up limit u_0 respect to $B_{\rho_m}(x_m)$ is a minimizer of \mathcal{J} corresponding to $p \equiv p(x_0)$ and $\lambda \equiv \lambda(x_0)$ in any ball.*

Proof. See [4]. \square

In the sequel we will call $\lambda^*(x) = \left(\frac{p(x)}{p(x)-1} \lambda(x)\right)^{1/p(x)}$.

We have,

Lemma 7.3. *Let u be a non-degenerate, Lipschitz continuous, local minimizer in \mathbb{R}^N corresponding to $p(x) \equiv p_0$ and $\lambda(x) \equiv \lambda_0$, such that $u = \lambda_1 \langle x, \nu_0 \rangle^-$ in B_R , with $R > 0$, $0 < \lambda_1 < \infty$ and ν_0 a unit vector. Then, $\lambda_1 = \left(\frac{p_0}{p_0-1} \lambda_0\right)^{1/p_0}$.*

Proof. See [29]. □

Lemma 7.4. *Let u be a locally Lipschitz, non-degenerate local minimizer in B_1 with power $p(x)$ Hölder and coefficient $\lambda(x)$ continuous. Let $x_0 \in \partial\{u > 0\} \cap B_1$ such that there exists a ball B contained in $B_1 \cap \{u = 0\}$ touching $\partial\{u > 0\}$ at the point x_0 . Then,*

$$\limsup_{\substack{x \rightarrow x_0 \\ u(x) > 0}} \frac{u(x)}{\text{dist}(x, B)} = \lambda^*(x_0).$$

Proof. See, for instance [29] for the idea of the proof. Here we use Lemmas 7.2 and 7.3. □

Lemma 7.5. *Let $u \in \mathcal{K}$ be a minimizer. Then, for every $x_0 \in \Omega \cap \partial\{u > 0\}$*

$$(7.30) \quad \limsup_{\substack{x \rightarrow x_0 \\ u(x) > 0}} |\nabla u(x)| = \lambda^*(x_0).$$

Proof. For the idea of the proof see, for instance [29]. Here we use Lemmas 7.2 and 7.3. □

Theorem 7.1. *Let u be a minimizer, then for \mathcal{H}^{N-1} -a.e $x_0 \in \partial\{u > 0\}$, the following properties hold,*

$$q_u(x_0) = \lambda^*(x_0)^{p(x_0)-1}$$

and

$$(7.31) \quad u(x) = \lambda^*(x_0) \langle x - x_0, \nu_u(x_0) \rangle^- + o(|x - x_0|)$$

Proof. In order to prove (7.31) we follow the ideas of [29] using Lemma 7.1 items (6) and (7) and Lemmas 7.2, 7.3 and 7.5. □

8. REGULARITY OF THE FREE BOUNDARY

In this section we assume that λ is Hölder continuous and p Lipschitz with constant L , and therefore the corresponding λ^* will also be Hölder continuous. We denote by C^* the constant of Hölder continuity of λ^* and by α^* its Hölder exponent.

We prove the regularity of the free boundary of a minimizer u in a neighborhood of every “flat” free boundary point. In particular, we prove the regularity in a neighborhood of every point in $\partial_{red}\{u > 0\}$ where u has the asymptotic development (7.31). Then, if u is a minimizer, $\partial_{red}\{u > 0\}$ is smooth and the remainder of the free boundary has \mathcal{H}^{N-1} -measure zero.

First, we recall some definitions and then, we point out the only significant differences with the proofs in [12] with p constant. The rest of the proof of the regularity then follows as sections 6, 7, 8 and 9 of [12].

Remark 8.1. In [12], Sections 6, 7 and 8 the authors use the fact that when $|\nabla u| \geq c$, u satisfies a linear nondivergence uniformly elliptic equation, $Tu = 0$. In our case we have that when $|\nabla u| \geq c$, u is a solution of the equation defined in (B.7). As in those sections the authors only use the fact that this operator is linear and uniformly elliptic, then the results of those sections in [12] extend to our case without any change.

For the reader's convenience, we sketch here the proof of the regularity of the free boundary by a series of steps and we write down the proofs in those cases in which we have to make modifications.

8.1. Flatness and nondegeneracy of the gradient.

Definition 8.1 (Flat free boundary points). *Let $0 < \sigma_+, \sigma_- \leq 1$ and $\tau > 0$. We say that u is of class*

$$F(\sigma_+, \sigma_-; \tau) \quad \text{in } B_\rho = B_\rho(0) \quad \text{with power } p(x)$$

if u is a local minimizer of \mathcal{J} in B_ρ with power $p(x)$,

(1) $0 \in \partial\{u > 0\}$ and

$$\begin{aligned} u &= 0 && \text{for } x_N \geq \sigma_+\rho, \\ u(x) &\geq -\lambda^*(0)(x_N + \sigma_-\rho) && \text{for } x_N \leq -\sigma_-\rho. \end{aligned}$$

(2) $|\nabla u| \leq \lambda^*(0)(1 + \tau)$ in B_ρ .

If the origin is replaced by x_0 and the direction e_N by the unit vector ν we say that u is of class $F(\sigma_+, \sigma_-; \tau)$ in $B_\rho(x_0)$ in direction ν .

Theorem 8.1. *Let p be Lipschitz continuous, $1 < p_{\min} \leq p(x) \leq p_{\max} < \infty$, λ Hölder continuous with $0 < \lambda_1 \leq \lambda(x) \leq \lambda_2 < \infty$ and modulus of continuity $\omega_\lambda(r) = C_* r^{\alpha^*}$. Then, there exist $\sigma_0 > 0$ and $C_0 > 0$ such that if $C_* \rho^{\alpha^*} \leq \lambda^*(0)\sigma$ and $0 < \sigma < \sigma_0$,*

$$u \in F(\sigma, 1; \sigma) \text{ in } B_1 \text{ with power } p(x) \text{ and } |\nabla p| \leq C\sigma \text{ in } B_\rho \text{ implies}$$

$$u \in F(2\sigma, C_0\sigma; \sigma) \text{ in } B_{\rho/2}.$$

Proof. By rescaling, we may assume that $\rho = 1$ and $\text{osc}_{B_1} \lambda^* \leq C_* \rho^{\alpha^*}$.

Then, we proceed as in [12], Lemmas 6.5, 6.6 and Theorem 6.3. One of the differences in our case is that λ^* is not a constant. Moreover, we cannot assume that $\lambda^*(0) = 1$. First, we construct, for $\kappa > 0$, a barrier v as a solution to

$$\begin{cases} \Delta_{p(x)} v = 0 & \text{in } D \setminus B_r(\xi) \\ v = 0 & \text{on } \partial D \setminus B_1 \\ v = \lambda^*(0)(1 + \sigma)(\sigma - x_N) & \text{on } \partial D \cap B_1 \\ v = -\lambda^*(0)(1 - \kappa\sigma)x_N & \text{on } \partial B_r(\xi) \end{cases}$$

Here the set D is constructed as in [12]. As in that paper, we want to prove that there exists $x_\xi \in \partial B_r(\xi)$ such that $v(x_\xi) \geq u(x_\xi)$ if κ is large enough.

By contradiction, by Lemma 7.4, if $v \leq u$ on $\partial B_r(\xi)$ there holds that

$$|\nabla v(z)| \geq \lambda^*(z)$$

where $z \in \partial D \cap B_{1/2} \cap \partial\{u > 0\}$.

Then,

$$|\nabla v(z)| \geq \lambda^*(0)(1 - \sigma).$$

In order to get the contradiction we need the following estimate:

$$|\nabla v(z)| \leq \lambda^*(0)(1 + C\sigma - c\kappa\sigma).$$

For that purpose, we proceed again as in [12] by constructing a barrier for v of the form $v_1 - \kappa\sigma v_2$ where v_1 and v_2 are the same functions of [12], Claim 6.8. One can check, as in [12], that v_1 is a supersolution and v_2 is a subsolution to an elliptic equation in nondivergence form in such a way that $\Delta_{p(x)}(v_1 - \kappa\sigma v_2) \leq 0$. The difference in our case is that this equation has first order terms. But these terms are bounded by $L\sigma \log 2$ since by construction $\frac{1}{2} \leq |\nabla(v_1 - \kappa\sigma v_2)| \leq 2$.

In this way the results corresponding to Lemmas 6.5 and 6.6 in [12] are proved. In order to finish the proof of the theorem we proceed as in [12], Theorem 6.3. We consider the function

$$w(x) = \lambda^*(0)(1 + \sigma)(\sigma - x_N) - u(x) \geq 0 \quad \text{in } B_{2r}(\xi)$$

and prove that $w(x_\xi) \leq C\sigma$ where $x_\xi \in \partial B_r(\xi)$ is such $v(x_\xi) \geq u(x_\xi)$. Finally, in order to apply Harnack inequality to get $w(x) \leq C\sigma$ in $B_r(\xi)$ we observe that w satisfies

$$|\mathbb{L}w| \leq C\sigma \quad \text{in } B_{2r}(\xi)$$

where \mathbb{L} is the linear operator given in (B.6) such that $\mathbb{L}u = 0$ (observe that at this stage we already know that $|\nabla u| \geq \lambda^*(0)/2$ in $B_{4r}(\xi)$). \square

Theorem 8.2. *Let p be Lipschitz continuous, $1 < p_{min} \leq p(x) \leq p_{max} < \infty$, λ Hölder continuous with $0 < \lambda_1 \leq \lambda(x) \leq \lambda_2 < \infty$ and modulus of continuity $\omega_\lambda(r) = C_* r^{\alpha^*}$. For every $\delta > 0$ there exist $\sigma_\delta > 0$ and $C_\delta > 0$ such that if $C_* \rho^{\alpha^*} \leq \lambda^*(0)\sigma$, $0 < \sigma < \sigma_\delta$,*

$$u \in F(\sigma, 1; \sigma) \text{ in } B_\rho \text{ with power } p(x) \text{ and } |\nabla p| \leq C\sigma \text{ implies}$$

$$|\nabla u| \geq \lambda^*(0)(1 - \delta) \text{ in } B_{\rho/2} \cap \{x_N \leq -C_\delta\sigma\}.$$

Proof. The proof follows as Theorem 6.4 in [12]. \square

8.2. Nonhomogeneous blow-up.

Lemma 8.1. *Let $u_k \in F(\sigma_k, \sigma_k; \tau_k) \in B_{\rho_k}$ with power $p_k(x)$ and coefficient $\lambda_k(x)$ such that $|\nabla p_k| \leq L$, $1 < p_{min} \leq p_k(x) \leq p_{max} < \infty$, λ_k^* Hölder with exponent α^* and constant C_* , $0 < \lambda_1 \leq \lambda_k(x) \leq \lambda_2 < \infty$. Assume $\sigma_k \rightarrow 0$, $\tau_k \sigma_k^{-2} \rightarrow 0$ and $\rho_k^{\alpha^*} \leq \rho_0 \tau_k$ with $\rho_0 > 0$. For $y \in B'_1$, set*

$$\begin{aligned} f_k^+(y) &= \sup\{h : (\rho_k y, \sigma_k \rho_k h) \in \partial\{u_k > 0\}\}, \\ f_k^-(y) &= \inf\{h : (\rho_k y, \sigma_k \rho_k h) \in \partial\{u_k > 0\}\}. \end{aligned}$$

Then, for a subsequence,

$$(1) \quad f(y) = \limsup_{\substack{z \rightarrow y \\ k \rightarrow \infty}} f_k^+(z) = \liminf_{\substack{z \rightarrow y \\ k \rightarrow \infty}} f_k^-(z) \quad \text{for all } y \in B'_1.$$

Further, $f_k^+ \rightarrow f$, $f_k^- \rightarrow f$ uniformly, $f(0) = 0$, $|f| \leq 1$ and f is continuous.

$$(2) \quad f \text{ is subharmonic.}$$

Proof. (1) is the analogue of Lemma 5.3 in [5]. The proof is based on Theorem 6.3 and is identical to the one of Lemma 7.3 in [4].

For the reader's convenience, we write down the proof of (2) that is a little bit different from the one in [4] since we do not have the homogeneity of the operator so that we need to keep track of the coefficient $\lambda_k^*(0)$. Also, our assumption in this and the ongoing sections is that $\lambda(x)$ is Hölder continuous as compared to the assumption in [4].

We assume by taking $\tilde{u}_k(x) = u_k(\rho_k x)/\rho_k$, $\tilde{p}_k(x) = p_k(\rho_k x)$ and $\tilde{\lambda}_k(x) = \lambda_k(\rho_k x)$ that $u_k \in F(\sigma_k, \sigma_k; \tau_k)$ in B_1 with power \tilde{p}_k and coefficient $\tilde{\lambda}_k$. We drop the tildes but recall that now $|\nabla p_k| \leq L\rho_k$ and $|\lambda_k^*(x) - \lambda_k^*(0)| \leq C^* \rho_k^{\alpha^*} |x|^{\alpha^*}$.

Observe that by the Hölder continuity of the original λ_k^* we have that,

$$(8.32) \quad \lambda_k^*(x) \geq \lambda_k^*(0) - C^* \rho_k^{\alpha^*} = \lambda_k^*(0)(1 - C \rho_k^{\alpha^*})$$

Let us assume, by contradiction, that there is a ball $B'_\rho(y_0) \subset B_1$ and a harmonic function g in a neighborhood of this ball, such that

$$g > f \text{ on } \partial B'_\rho(y_0) \quad \text{and} \quad f(y_0) > g(y_0).$$

Let,

$$Z^+ = \{x \in B_1 / x = (y, h), y \in B'_\rho(y_0), h > \sigma_k g(y)\},$$

and similarly Z_0 and Z^- . As in Lemma 7.5 in [4], using the same test function and the Representation Theorem 6.3 we arrive at,

$$(8.33) \quad \int_{\{u_k > 0\} \cap Z_0} |\nabla u_k|^{p_k-2} \nabla u_k \cdot \nu d\mathcal{H}^{N-1} = \int_{\partial_{red}\{u_k > 0\} \cap Z^+} q_{u_k}(x) d\mathcal{H}^{N-1}.$$

As $u_k \in F(\sigma_k, \sigma_k; \tau_k)$ we have that $|\nabla u_k| \leq \lambda_k^*(0)(1 + \tau_k)$ and, by Theorem 7.1, there holds that $q_{u_k}(x) = \lambda_k^*(x)^{p_k(x)-1}$ for \mathcal{H}^{N-1} - a.e point in $\partial_{red}\{u_k > 0\}$. Therefore,

$$(8.34) \quad \int_{\{u_k > 0\} \cap Z_0} |\nabla u_k|^{p_k-2} \nabla u_k \cdot \nu d\mathcal{H}^{N-1} = \int_{\partial_{red}\{u_k > 0\} \cap Z^+} \lambda_k^{*p_k-1} d\mathcal{H}^{N-1}.$$

Applying the estimate (8.32) to (8.34) and, assuming for simplicity that $\lambda_k^*(0) \geq 1$ we have,

$$\begin{aligned} \lambda_k^*(0)^{p_k^- - 1} (1 - C \rho_k^{\alpha^*})^{p_k^+ - 1} \mathcal{H}^{N-1}(\partial_{red}\{u_k > 0\} \cap Z^+) \\ \leq \int_{\{u_k > 0\} \cap Z_0} |\nabla u_k|^{p_k-2} \nabla u_k \cdot \nu d\mathcal{H}^{N-1} \\ \leq \lambda_k^*(0)^{p_k^+ - 1} (1 + \tau_k)^{p_k^+ - 1} \mathcal{H}^{N-1}(\{u_k > 0\} \cap Z_0) \end{aligned}$$

Then we have,

$$(8.35) \quad \mathcal{H}^{N-1}(\partial_{red}\{u_k > 0\} \cap Z^+) \leq \lambda_k^*(0)^{p_k^+ - p_k^-} \left(\frac{1 + \tau_k}{1 - C^* \rho_k^{\alpha^*}} \right)^{p_k^+ - 1} \mathcal{H}^{N-1}(\{u_k > 0\} \cap Z_0).$$

On the other hand, by the excess area estimate in Lemma 7.5 in [4] we have that,

$$\mathcal{H}^{N-1}(\partial_{red} E_k \cap Z) \geq \mathcal{H}^{N-1}(Z_0) + c\sigma_k^2,$$

where $Z = B'_\rho(y_0) \times \mathbb{R}$ and $E_k = \{u_k > 0\} \cup Z^-$.

We also have,

$$\mathcal{H}^{N-1}(\partial_{red}E_k \cap Z) \leq \mathcal{H}^{N-1}(Z^+ \cap \partial_{red}\{u_k > 0\}) + \mathcal{H}^{N-1}(Z_0 \cap \{u_k = 0\}).$$

Using these two inequalities and the fact that $\mathcal{H}^{N-1}(Z_0 \cap \partial\{u_k > 0\}) = 0$ (if this is not true we replace g by $g + c_0$ for a small constant c_0) we have that,

$$(8.36) \quad \mathcal{H}^{N-1}(\partial_{red}\{u_k > 0\} \cap Z^+) \geq \mathcal{H}^{N-1}(Z_0 \cap \{u_k > 0\}) + c\sigma_k^2.$$

Finally by (8.35) and (8.36) we have that,

$$\mathcal{H}^{N-1}(\{u_k > 0\} \cap Z_0) + c\sigma_k^2 \leq \lambda_k^*(0)^{p_k^+ - p_k^-} \left(\frac{1 + \tau_k}{1 - C^* \rho_k^{\alpha^*}} \right)^{p_k^+ - 1} \mathcal{H}^{N-1}(\{u_k > 0\} \cap Z_0).$$

Therefore, since $p_k^+ - p_k^- \leq L\rho_k$,

$$c\sigma_k^2 \leq \left[\lambda_k^*(0)^{p_k^+ - p_k^-} \left(\frac{1 + \tau_k}{1 - C^* \rho_k^{\alpha^*}} \right)^{p_k^+ - 1} - 1 \right] \mathcal{H}^{N-1}(\{u_k > 0\} \cap Z_0) \leq C(\tau_k + \rho_k^{\alpha^*}).$$

Observe that if $\lambda_k^*(0) \leq 1$, we arrive at the same estimate.

Finally, since $\rho^{\alpha^*} \leq \rho_0 \tau_k$ this contradicts the fact that $\frac{\tau_k}{\sigma_k} \rightarrow 0$ as $k \rightarrow \infty$. \square

Lemma 8.2. *There exists a positive constant $C = C(N)$ such that, for any $y \in B'_r/2$,*

$$\int_0^{1/4} \frac{1}{r^2} \left(\int_{\partial B'_r(y)} f - f(y) \right) \leq C_1.$$

Proof. It follows as Lemma 8.3 in [12], by Remark 8.1 and Theorem 8.2.

The only difference is that the functions $w_k = (u_k(y, h) + h)/\sigma_k$ verify a second order elliptic equation in non-divergence form with Hölder principal coefficients and bounded first order coefficients.

As in the proof of Lemma 8.1, since $|\nabla p_k| \rightarrow 0$, these first order coefficients converge to 0 and so, by the $W^{2,q}$ regularity estimates of [20], Chapter 9, we can pass to the limit to discover that $w_k \rightarrow w$ and w satisfies a second order elliptic equation in non-divergence form with constant coefficients with only principal part.

After that, the proof follows without any change as that of Lemma 8.3 in [12]. \square

With these two lemmas we have by Lemma 7.7 and Lemma 7.8 in [4],

Lemma 8.3. (1) *f is Lipschitz in $\bar{B}'_{1/4}$ with Lipschitz constant depending on C_1 and N .*

(2) *There exists a constant $C = C(N) > 0$ and for $0 < \theta < 1$, there exists $c_\theta = c(\theta, N) > 0$, such that we can find a ball B'_r and a vector $l \in \mathbb{R}^{N-1}$ with*

$$c_\theta \leq r \leq \theta, \quad |l| \leq C, \quad \text{and } f(y) \leq l \cdot y + \frac{\theta}{2}r \quad \text{for } |y| \leq r.$$

And, as in Lemma 7.9 in [4] we have,

Lemma 8.4. *Let θ, C, c_θ as in Lemma 8.3. There exists a positive constants σ_θ , such that*

$$(8.37) \quad u \in F(\sigma, \sigma; \tau) \text{ in } B_\rho \text{ in direction } \nu$$

with $\sigma \leq \sigma_\theta$, $\tau \leq \sigma_\theta \sigma^2$ and $\rho^{\alpha^} \leq \rho_0 \tau$, implies*

$$u \in F(\theta\sigma, 1; \tau) \text{ in } B_{\bar{\rho}} \text{ in direction } \bar{\nu}$$

for some $\bar{\rho}$ and $\bar{\nu}$ with $c_\theta \rho \leq \bar{\rho} \leq \theta \rho$ and $|\bar{\nu} - \nu| \leq C\sigma$, where $\sigma_\theta = \sigma_\theta(\theta, N)$.

Lemma 8.5. *Given $0 < \theta < 1$, there exist positive constants σ_θ , c_θ and C such that*

$$(8.38) \quad u \in F(\sigma, 1; \tau) \text{ in } B_\rho \text{ in direction } \nu$$

with $\sigma \leq \sigma_\theta, \tau \leq \sigma_\theta \sigma^2$ and $\rho^{\alpha^*} \leq \rho_0 \tau$, then

$$u \in F(\theta\sigma, \theta\sigma; \theta^2\tau) \text{ in } B_{\bar{\rho}} \text{ in direction } \bar{\nu}$$

for some $\bar{\rho}$ and $\bar{\nu}$ with $c_\theta \rho \leq \bar{\rho} \leq \frac{1}{4}\rho$ and $|\bar{\nu} - \nu| \leq C\sigma$, where $c_\theta = c_\theta(\theta, N)$, $\sigma_\theta = \sigma_\theta(\theta, N)$.

Proof. We obtain the improvement of the value τ inductively. If σ_θ is small enough, we can apply Theorem 8.1 and obtain

$$u \in F(C\sigma, C\sigma; \tau) \text{ in } B_{\rho/2} \text{ in direction } \nu.$$

Then for $0 < \theta_1 \leq \frac{1}{2}$ we can apply Lemma 8.4, if again σ_θ is small, and we obtain

$$(8.39) \quad u \in F(C\theta_1\sigma, C\sigma; \tau) \text{ in } B_{r_1} \text{ in direction } \nu_1$$

for some r_1, ν_1 with

$$c_{\theta_1}\rho \leq 2r_1 \leq \theta_1\rho, \text{ and } |\nu_1 - \nu| \leq C\sigma.$$

In order to improve τ , we consider the functions $U_\varepsilon = (|\nabla u| - \lambda^*(0) - \varepsilon)^+$ and $U_0 = (|\nabla u| - \lambda^*(0))^+$ in B_{2r_1} . By Lemma 7.5, we know that U_ε vanishes in a neighborhood of the free boundary. Since $U_\varepsilon > 0$ implies $|\nabla u| > \lambda^*(0) + \varepsilon$, the closure of $\{U_\varepsilon > 0\}$ is contained in $\{|\nabla u| > \lambda^*(0) + \varepsilon/2\}$.

Since $|\nabla u|$ is bounded from above in B_{2r_1} , and from below in the set $\{|\nabla u| > \lambda^*(0) + \varepsilon/2\}$ the hypotheses of Lemma B.5 are satisfied, and we have that $v = |\nabla u|$ satisfies,

$$-\operatorname{div} D\nabla v + B\nabla v \leq \operatorname{div} H \quad \text{in } \{|\nabla u| > \lambda^*(0) + \varepsilon/2\}$$

Hence U_ε satisfies

$$-\operatorname{div} D\nabla U_\varepsilon + B\nabla U_\varepsilon \leq \operatorname{div} H \quad \text{in } \{U_\varepsilon > 0\}$$

Extending the operator by a uniformly elliptic operator with principal part in divergence form with ellipticity constant β and H by \tilde{H} with $\|\tilde{H}\|_\infty \leq C\|H\|_\infty$ we get,

$$\begin{cases} -\operatorname{div} \tilde{D}\nabla U_\varepsilon + \tilde{B}\nabla U_\varepsilon \leq \operatorname{div} \tilde{H} & \text{in } B_{2r_1} \\ U_\varepsilon \leq \lambda^*(0)\tau & \text{on } \partial B_{2r_1} \end{cases}$$

and $r_1 \leq \theta_1 \rho \leq \rho/4$. Then, $U_\varepsilon \leq \lambda^*(0)\tau + C(N, L, \beta)r_1\|H\|_\infty$. Let $S = \lambda^*(0)\tau + Cr_1\|H\|_\infty$.

Let $W = S - U_\varepsilon \geq 0$. Then

$$-\operatorname{div} \tilde{D}\nabla W + \tilde{B}\nabla W \geq \operatorname{div} \tilde{H} \quad \text{in } B_{2r_1}.$$

By the weak Harnack inequality (see [20] Theorem 8.18) we have that, if $1 < q < \frac{N-2}{2}$ then,

$$\frac{1}{r_1^{N/q}} \|W\|_{L^q(B_{2r_1})} \leq C_1 [\inf_{B_{r_1}} W + \|H\|_{L^\infty(B_{2r_1})} r_1].$$

In $\{u = 0\}$, $W = S$. Moreover $u = 0$ in $B_{r_1/4}(\frac{3}{4}r_1\bar{\nu})$ since $\theta_1 \leq 1/4$. Therefore,

$$S \leq C_2 [S - \sup_{B_{r_1}} U_\varepsilon + \|H\|_{L^\infty(B_{2r_1})} r_1].$$

Then,

$$\sup_{B_{r_1}} U_\varepsilon \leq \left(1 - \frac{1}{C_2}\right)S + C_3 r_1 \leq \delta \lambda^*(0)\tau + C_4 r_1$$

with $0 < \delta < 1$. And we have,

$$\sup_{B_{r_1}} |\nabla u| \leq \lambda^*(0)(1 + \delta\tau) + C_4 r_1$$

with $C_4 = C_4(N, \lambda^*(0), L, \beta)$.

Since $r_1 \leq \theta_1 \rho$ and $\rho \leq \rho^{\alpha^*} \leq \rho_0 \tau$, we have

$$\sup_{B_{r_1}} |\nabla u| \leq \lambda^*(0)(1 + \delta\tau) + C_4 r_1 \leq \lambda^*(0)(1 + \delta\tau + C\theta_1 \rho) \leq \lambda^*(0)(1 + \tau(\delta + C\theta_1 \rho_0)).$$

Let us choose θ_1 such that $C\rho_0\theta_1 + \delta < 1$. Take $\theta_0 = \max\{\theta_1^{\alpha^*/2}, (\delta + C\rho_0\theta_1)^{1/2}\}$.

We have

$$u \in F(\theta_0 \sigma, 1; \theta_0^2 \tau) \text{ in } B_{r_1} \text{ in direction } \nu_1.$$

Moreover, $r_1^{\alpha^*} \leq \theta_1^{\alpha^*} \rho^{\alpha^*} \leq \theta_0^2 \rho^{\alpha^*} \leq \rho_0 \theta_0^2 \tau$. We also have, $\theta_0 \sigma \leq \theta_0 \sigma_{\theta_1} \leq \sigma_{\theta_1}$ and $\theta_0^2 \tau \leq \theta_0^2 \sigma_{\theta_1} \sigma^2 = \sigma_{\theta_1} (\theta_0 \sigma)^2$.

Then, we can repeat this argument a finite number of times, and we obtain

$$u \in F(\theta_0^m \sigma, 1; \theta_0^{2m} \tau) \text{ in } B_{r_1 \dots r_m} \text{ in direction } \nu_m,$$

with

$$c_{\theta_j} \leq 2r_j \leq \theta_j, \text{ and } |\nu_m - \nu| \leq \frac{C}{1 - \theta} \sigma.$$

Finally we choose m large enough such that $\theta_0^m \leq \theta$, we have that

$$u \in F(\theta \sigma, 1; \theta^2 \tau) \text{ in } B_{r_1 \dots r_m} \text{ in direction } \nu_m,$$

and using Theorem 8.1 we have if $\sigma \leq \sigma_{\theta_1, \rho_0}$, $\tau \leq \sigma_{\theta_1, \rho_0} \sigma^2$ and $\rho^{\alpha^*} \leq \rho_0 \tau$ the desired result. \square

8.3. Smoothness of the free boundary.

Theorem 8.3. *Suppose that u is a minimizer of \mathcal{J} in \mathcal{K} and $D \subset\subset \Omega$. Assume p is Lipschitz and λ is Hölder. Then, there exist positive constants $\bar{\sigma}_0$, C and γ such that if*

$$u \in F(\sigma, 1; \infty) \text{ in } B_\rho(x_0) \subset D \text{ in direction } \nu$$

with $\sigma \leq \bar{\sigma}_0$, $\rho \leq \bar{\rho}_0(\bar{\sigma}_0, \sigma)$, then

$$B_{\rho/4}(x_0) \cap \partial\{u > 0\} \text{ is a } C^{1,\gamma} \text{ surface.}$$

More precisely, a graph in direction ν of a $C^{1,\gamma}$ function and, for any x_1, x_2 on this surface

$$|\nu(x_1) - \nu(x_2)| \leq C\sigma \left| \frac{x_1 - x_2}{\rho} \right|^\gamma$$

Proof. See Theorem 9.3 in [29]. \square

Remark 8.2. By the nondegeneracy (Corollary 4.1) and by (7.31), we have that for $x_0 \in \partial_{red}\{u > 0\}$ we have that $u \in F(\sigma_\rho, 1; \infty)$ in $B_\rho(x_0)$ in direction $\nu_u(x_0)$, with $\sigma_\rho \rightarrow 0$ as $\rho \rightarrow 0$. Hence, applying Theorem 8.3 we have,

Theorem 8.4. *Let u be a local minimizer of \mathcal{J} in \mathcal{K} with power $p \in \text{Lip}$ and coefficient $\lambda \in C^\alpha$. Then, for any $x_0 \in \partial_{\text{red}}\{u > 0\}$ there exist $r > 0$ and $0 < \gamma < 1$ such that $B_r(x_0) \cap \partial\{u > 0\}$ is a $C^{1,\gamma}$ surface. Thus, for every $D \subset\subset \Omega$ there exists $0 < \gamma < 1$ such that $D \cap \partial_{\text{red}}\{u > 0\}$ is a $C^{1,\gamma}$ surface and moreover, $\mathcal{H}^{N-1}(\partial\{u > 0\} \setminus \partial_{\text{red}}\{u > 0\}) = 0$.*

APPENDIX A. THE SPACES $L^{p(\cdot)}(\Omega)$ AND $W^{1,p(\cdot)}(\Omega)$

Let $p : \Omega \rightarrow [1, \infty)$ be a measurable bounded function, called a variable exponent on Ω and denote $p_{\max} = \text{esssup } p(x)$ and $p_{\min} = \text{essinf } p(x)$. We define the variable exponent Lebesgue space $L^{p(\cdot)}(\Omega)$ to consist of all measurable functions $u : \Omega \rightarrow \mathbb{R}$ for which the modular $\varrho_{p(\cdot)}(u) = \int_{\Omega} |u(x)|^{p(x)} dx$ is finite. We define the Luxemburg norm on this space by

$$\|u\|_{L^{p(\cdot)}(\Omega)} = \|u\|_{p(\cdot)} = \inf\{\lambda > 0 : \varrho_{p(\cdot)}(u/\lambda) \leq 1\}.$$

This norm makes $L^{p(\cdot)}(\Omega)$ a Banach space.

One central property of these spaces (since p is bounded) is that $\varrho_{p(\cdot)}(u_i) \rightarrow 0$ if and only if $\|u_i\|_{p(\cdot)} \rightarrow 0$, so that the norm and modular topologies coincide.

Remark A.1. Observe that we have the following estimate,

$$\|u\|_{L^{p(\cdot)}(\Omega)} \leq \max \left\{ \left(\int_{\Omega} |u|^{p(x)} dx \right)^{1/p_{\min}}, \left(\int_{\Omega} |u|^{p(x)} dx \right)^{1/p_{\max}} \right\}$$

In fact. If $\int_{\Omega} |u|^{p(x)} dx = 0$ then $u = 0$ a.e and the result follows. If $\int_{\Omega} |u|^{p(x)} dx \neq 0$, take $k = \max \left\{ \left(\int_{\Omega} |u|^{p(x)} dx \right)^{1/p_{\min}}, \left(\int_{\Omega} |u|^{p(x)} dx \right)^{1/p_{\max}} \right\}$. Then we have,

$$\int_{\Omega} \left(\frac{|u|}{k} \right)^{p(x)} dx \leq \max \left\{ \frac{1}{k^{p_{\min}}}, \frac{1}{k^{p_{\max}}} \right\} \int_{\Omega} |u|^{p(x)} dx \leq 1$$

therefore $\|u\|_{L^{p(\cdot)}(\Omega)} \leq k$ and the result follows.

Let $W^{1,p(\cdot)}(\Omega)$ denote the space of measurable functions u such that u and the distributional derivative ∇u are in $L^{p(\cdot)}(\Omega)$. The norm

$$\|u\|_{1,p(\cdot)} := \|u\|_{p(\cdot)} + \|\nabla u\|_{p(\cdot)}$$

makes $W^{1,p(\cdot)}$ a Banach space.

Theorem A.1. *Let $p'(x)$ such that,*

$$\frac{1}{p(x)} + \frac{1}{p'(x)} = 1.$$

Then $L^{p'(\cdot)}(\Omega)$ is the dual of $L^{p(\cdot)}(\Omega)$. Moreover, if $p_{\min} > 1$, $L^{p(\cdot)}(\Omega)$ and $W^{1,p(\cdot)}(\Omega)$ are reflexive.

Theorem A.2. *Let $q(x) \leq p(x)$, then $L^{p(\cdot)}(\Omega) \hookrightarrow L^{q(\cdot)}(\Omega)$ continuously.*

We define the space $W_0^{1,p(\cdot)}(\Omega)$ as the closure of the $C_0^\infty(\Omega)$ in $W^{1,p(\cdot)}(\Omega)$. Then we have the following version of Poincaré's inequality,

Lemma A.1. *If $p(x)$ is continuous in $\overline{\Omega}$, there exists a constant C such that for every $u \in W_0^{1,p(\cdot)}(\Omega)$,*

$$\|u\|_{L^{p(\cdot)}(\Omega)} \leq C \|\nabla u\|_{L^{p(\cdot)}(\Omega)}$$

For the proof of these results, and more about these spaces see [23].

In order to have better properties of these spaces, we need more hypotheses on the regularity of $p(x)$.

We say that p is log-Hölder continuous if there exists a constant C such that

$$|p(x) - p(y)| \leq \frac{C}{|\log |x - y||}$$

if $|x - y| < 1/2$.

It was proved in [32], Theorem 3 that if one assumes that p is log-Hölder continuous then, C^∞ is dense in $W^{1,p(\cdot)}(\Omega)$ (see also [10] and [13]). See [15] for more references on this topic.

APPENDIX B. RESULTS ON $p(x)$ -HARMONIC AND SUBHARMONIC FUNCTIONS

In this section we will give some of the properties of $p(x)$ -harmonic and subharmonic functions. Some of them are known results and others are new. For the reader's convenience we will list all the results, and give the reference when it corresponds. Here $\omega(r)$ is the modulus of continuity of $p(x)$. We will state which is the type of ω that we are considering for each result.

Remark B.1. For any x fixed we have the following inequalities

$$\begin{aligned} |\eta - \xi|^{p(x)} &\leq C(|\eta|^{p(x)-2}\eta - |\xi|^{p(x)-2}\xi)(\eta - \xi) && \text{if } p(x) \geq 2, \\ |\eta - \xi|^2 \left(|\eta| + |\xi|\right)^{p(x)-2} &\leq C(|\eta|^{p(x)-2}\eta - |\xi|^{p(x)-2}\xi)(\eta - \xi) && \text{if } p(x) < 2, \end{aligned}$$

These inequalities say that the function $A(x, q) = |q|^{p(x)-2}q$ is strictly monotone. Then, the comparison principle holds since it follows from the monotonicity of $A(x, q)$.

The following result, a Cacciopoli type inequality, is included in the proof of Lemma 6 in [21],

Lemma B.1. *Assume $p(x)$ is bounded and let $u \in W^{1,p(\cdot)}(\Omega)$ be a nonnegative subsolution of the problem*

$$(B.1) \quad \Delta_{p(x)} u = 0 \text{ in } \Omega.$$

Then, for any $B_r \subset \Omega$

$$\int_{B_{r/2}} |\nabla u|^{p(x)} dx \leq C \int_{B_r} \left(\frac{u}{r}\right)^{p(x)} dx,$$

where $C = C(p_{\min}, p_{\max})$.

Proof. See inequality (5) in the proof of Lemma 6 of [21]. □

Lemma B.2. Assume $\omega(r) = C(\log \frac{1}{r})^{-1}$ and let u be a nonnegative solution of the problem

$$(B.2) \quad \Delta_{p(x)} u = 0 \text{ in } \Omega.$$

Then, there exists a constant C such that

$$\sup_{B_r(x_0)} u \leq C \left(\inf_{B_r(x_0)} u + r \right)$$

for any r with $B_{10r}(x_0) \subset \Omega$. The constant depends on $N, \omega(\cdot), p_{\min}$ and the $L^1(B_r)$ -norm of $|u|^{p(\cdot)}$.

Proof. See Corollary 5.13 in [22]. □

Remark B.2. Assume u is a nonnegative solution of the problem

$$(B.3) \quad \Delta_{p(x)} u = 0 \text{ in } \Omega.$$

Let R, \bar{y} such that $B_{10R}(\bar{y}) \subset \Omega$, $x_0 \in \Omega$ and $r > 0$. Let $\bar{x} = \frac{\bar{y} - x_0}{r}$ and $\bar{u}(x) = \frac{u(x_0 + rx)}{r}$. Then, for any $\rho < R/r$ we have,

$$\sup_{B_\rho(\bar{x})} \bar{u} \leq C \left(\inf_{B_\rho(\bar{x})} \bar{u} + \rho \right),$$

where C is the constant of the previous Lemma. In particular, C may be taken independent of r (it depends on the $L^1(\Omega)$ -norm of $|u|^{p(\cdot)}$).

Proof. Let $|x - \bar{x}| < \rho$ and $y = x_0 + rx$ then $|y - \bar{y}| = r|x - \bar{x}| < r\rho < R$. Since,

$$\sup_{B_{r\rho}(\bar{y})} u \leq C \left(\inf_{B_{r\rho}(\bar{y})} u + r\rho \right)$$

there holds that

$$\sup_{B_{r\rho}(\bar{y})} \frac{u}{r} \leq C \left(\inf_{B_{r\rho}(\bar{y})} \frac{u}{r} + \rho \right).$$

Then,

$$\sup_{B_\rho(\bar{x})} \bar{u} \leq C \left(\inf_{B_\rho(\bar{x})} \bar{u} + \rho \right).$$

□

The following result was proved in Theorem 2.2 in [1],

Theorem B.1. Assume $\omega(r) = C_0 r^{\alpha_0}$ for some $0 < \alpha < 1$, and let u be a solution of the problem

$$(B.4) \quad \Delta_{p(x)} u = 0 \text{ in } \Omega.$$

Then, for any $\Omega' \subset\subset \Omega$ there exists a constant C depending on $\| |u|^{p(x)} \|_{L^1(\Omega)}$, $\| |\nabla u|^{p(x)} \|_{L^1(\Omega)}$, $p_{\min}, p_{\max}, \omega(r)$ and Ω' such that

$$\|u\|_{C^{1,\alpha}(\Omega')} \leq C.$$

The following is a consequence of the $C^{1,\alpha}$ regularity of the solutions and the Cacciopoli inequality

Lemma B.3. Assume $\omega(r) = C_0 r^{\alpha_0}$. Let u be a solution of the problem

$$(B.5) \quad \Delta_{p(x)} u = 0 \text{ in } B_R(y).$$

Then, there exists a constant C depending on $\| |u|^{p(x)} \|_{L^1(B_R(y))}$, $\| |\nabla u|^{p(x)} \|_{L^1(B_R(y))}$, p_{\min}, p_{\max} and $\omega(r)$ such that if $R \leq 1$ we have,

$$|\nabla u(y)| \leq C \left(1 + \frac{1}{R} \sup_{B_R(y)} u \right)^{p_+/p_-}$$

where $p_+ = \sup_{B_R(y)} p$, $p_- = \inf_{B_R(y)} p$.

Proof. By Theorem B.1, we have for $x \in B_{R/2}(y)$,

$$|\nabla u(x) - \nabla u(y)| \leq C |x - y|^\alpha,$$

for some constants $C > 0$ and $0 < \alpha < 1$. Therefore, if $x \in B_{R/2}(y)$

$$|\nabla u(y)| \leq |\nabla u(x)| + CR^\alpha.$$

If $|\nabla u(y)| \geq 1$, $p_- = p_-(B_R(y))$, $p_+ = p_+(B_R(y))$, recalling that $R \leq 1$ we get,

$$|\nabla u(y)|^{p_-} \leq |\nabla u(y)|^{p(x)} \leq C |\nabla u(x)|^{p(x)} + C.$$

Integrating for $x \in B_{R/2}(y)$,

$$|\nabla u(y)|^{p_-} \leq C \left(1 + \int_{B_{R/2}(y)} |\nabla u(x)|^{p(x)} \right).$$

Applying Cacciopoli inequality we have, since $R \leq 1$,

$$\begin{aligned} |\nabla u(y)|^{p_-} &\leq C_1 \left(1 + \int_{B_R(y)} \left(\frac{|u(x)|}{R} \right)^{p(x)} \right) \\ &\leq C_1 \left(2 + \int_{B_R(y)} \left(\frac{|u(x)|}{R} \right)^{p_+} \right) \\ &\leq C \left(1 + \left(\frac{1}{R} \sup_{B_R(y)} u(x) \right)^{p_+} \right). \end{aligned}$$

We obtain the desired result. \square

Remark B.3. In some of the proofs we need to look at the $p(x)$ -Laplacian as an operator in non-divergence form. In those cases we have to assume $p(x)$ Lipschitz so that we can differentiate the function,

$$A(x, q) = |q|^{p(x)-2} q.$$

If we take a function u , with $c_1 \leq |\nabla u| \leq c_2$ differentiating we obtain,

$$\begin{aligned} a_{ij}(x, \nabla u) &= \frac{\partial A_i}{\partial q_j}(x, \nabla u) = |\nabla u|^{p-2} \left(\delta_{ij} + \frac{(p-2)}{|\nabla u|^2} u_{x_i} u_{x_j} \right) \\ \frac{\partial A_i}{\partial x_k}(x, \nabla u) &= |\nabla u|^{p-2} \log |\nabla u| p_{x_k} u_{x_i}. \end{aligned}$$

Then, we have a the following non-divergence form for the $p(x)$ -Laplacian,

$$\Delta_{p(x)} u = \mathbb{L}u$$

where

$$(B.6) \quad \mathbb{L}w := a_{ij}(x, \nabla u) w_{x_i x_j} + |\nabla u|^{p-2} \log |\nabla u| p_{x_i} w_{x_i}.$$

Observe that $a_{ij} = |\nabla u|^{p(x)-2} b_{ij}$ and b_{ij} is uniformly elliptic with constant of ellipticity β , independent of the gradient of u . We call

$$(B.7) \quad Tw := b_{ij}(x, \nabla u) w_{x_i x_j} + \log |\nabla u| p_{x_i} w_{x_i}.$$

The following Lemma is the construction of barriers required in several proofs.

Lemma B.4. *Suppose that $p(x)$ is Lipschitz continuous. Let $w_\mu = M e^{-\mu|x|^2}$, for $M > 0$ and $r_1 \geq |x| \geq r_2 > 0$. Then, there exist $\mu_0, \varepsilon_0 > 0$ such that, if $\mu > \mu_0$ and $\|\nabla p\|_\infty \leq \varepsilon_0$,*

$$\mu^{-1} e^{\mu|x|^2} M^{-1} |\nabla w|^{2-p} \Delta_{p(x)} w_\mu \geq C_1 (\mu - C_2 \|\nabla p\|_\infty |\log M|) \text{ in } B_{r_1} \setminus B_{r_2}.$$

Here C_1, C_2 depend only on $r_2, r_1, p_+, p_-, \mu_0 = \mu_0(p_+, p_-, N, \|\nabla p\|_\infty, r_2, r_1)$ and $\varepsilon_0 = \varepsilon_0(p_+, p_-, r_1, r_2)$.

Proof. First note that by Remark B.3

$$\Delta_{p(x)} w = |\nabla w|^{p-2} \left\{ \frac{(p-2)}{|\nabla w|^2} \sum_{i,j} w_{x_i} w_{x_j} w_{x_i x_j} + \Delta w + \langle \nabla w, \nabla p \rangle \log |\nabla w| \right\}.$$

Computing, we have

$$(B.8) \quad w_{x_i} = -2\mu M x_i e^{-\mu|x|^2}, \quad w_{x_i x_j} = M(4\mu^2 x_i x_j - 2\mu \delta_{ij}) e^{-\mu|x|^2}, \quad |\nabla w| = 2M\mu|x| e^{-\mu|x|^2}.$$

Therefore using (B.8) we obtain,

$$\begin{aligned} & e^{\mu|x|^2} (2M\mu)^{-1} |\nabla w|^{2-p} \Delta_{p(x)} w \\ &= (p-2)(2\mu|x|^2 - 1) + (2\mu|x|^2 - N) - \langle x, \nabla p \rangle (\log(M) + \log(|x|2) + \log \mu) + \mu \langle x, \nabla p \rangle |x|^2 \\ &= (p-1)2\mu|x|^2 + \mu \langle x, \nabla p \rangle |x|^2 - (p-2+N) - \langle x, \nabla p \rangle (\log M + \log \mu + \log(2|x|)) \\ &\geq (2(p_- - 1)r_2^2 - r_1^3 \|\nabla p\|_\infty) \mu - r_1 \|\nabla p\|_\infty |\log \mu| - (p_+ - 2 + N) - r_1 \|\nabla p\|_\infty (\log M + C_{r_1, r_2}) \\ &\geq (2(p_- - 1)r_2^2 - r_1 \|\nabla p\|_\infty (r_1^2 + 1)) \mu - (p_+ - 2 + N) - r_1 \|\nabla p\|_\infty (\log M + C_{r_1, r_2}). \end{aligned}$$

In the last inequality we have used that $\frac{\log \mu}{\mu} \leq 1$ if $\mu \geq 1$.

Let $\varepsilon_0 > 0$ such that

$$2(p_- - 1)r_2^2 - r_1(r_1^2 + 1)\varepsilon_0 \geq \frac{3}{2}(p_- - 1)r_2^2.$$

If $\|\nabla p\|_\infty \leq \varepsilon_0$ we obtain

$$\begin{aligned} & e^{\mu|x|^2} (2M\mu)^{-1} |\nabla w|^{2-p} \Delta_{p(x)} w \geq \\ & \frac{3}{2}(p_- - 1)r_2^2 \mu - (p_+ - 2 + N) - r_1 \|\nabla p\|_\infty (\log M + C_{r_1, r_2}). \end{aligned}$$

Now, if we take $\mu \geq \mu_0 = \mu_0(p_+, p_-, N, r_2, r_1, \|\nabla p\|_{L^\infty})$ we obtain that

$$e^{\mu|x|^2} (2M\mu)^{-1} |\nabla w|^{2-p} \Delta_{p(x)} w \geq C_1 (\mu - C_2 \|\nabla p\|_\infty |\log M|).$$

with C_1, C_2 depending only on p_-, r_1, r_2 .

□

Lemma B.5. *Assume $p(x)$ is Lipschitz. Let u be a solution of the problem*

$$(B.9) \quad \Delta_{p(x)}u = 0 \text{ in } \Omega.$$

with $0 < c_1 \leq |\nabla u| \leq c_2$. Then $v = |\nabla u|$ satisfies,

$$- \operatorname{div} D\nabla v + B\nabla v \leq \operatorname{div} H \quad \text{in } \Omega$$

where,

$$\begin{aligned} D_{ij}(x, \nabla u) &= |\nabla u|^{p-1} \left(\delta_{ij} + \frac{(p-2)}{|\nabla u|^2} u_{x_i} u_{x_j} \right), \\ H(x, \nabla u) &= |\nabla u|^{p-2} \log |\nabla u| \langle \nabla u, \nabla p \rangle \nabla u, \\ B(x, \nabla u) &= |\nabla u|^{p-1} \log |\nabla u| \nabla p. \end{aligned}$$

Proof. Let $\eta \in C_0^\infty(\Omega)$. Then, for each k we have after integration by parts,

$$0 = \int_{\Omega} A(x, \nabla u) \nabla \eta_{x_k} dx = - \int_{\Omega} \frac{\partial A}{\partial x_k}(x, \nabla u) \nabla \eta dx - \int_{\Omega} a_{ij}(x, \nabla u) u_{x_j x_k} \eta_{x_i} dx.$$

Observe that, by approximation, we get that the right hand side vanishes for $\eta \in W^{1,p(\cdot)}(\Omega)$.

Taking $\eta = u_{x_k} \psi$ with $\psi \in C_0^\infty(\Omega)$ we have, by using the ellipticity of a_{ij} (see Remark B.3),

$$\begin{aligned} 0 &= - \int_{\Omega} \frac{\partial A}{\partial x_k}(x, \nabla u) u_{x_k} \nabla \psi dx - \int_{\Omega} \frac{\partial A}{\partial x_k}(x, \nabla u) \nabla u_{x_k} \psi dx \\ &\quad - \int_{\Omega} a_{ij}(x, \nabla u) u_{x_j x_k} u_{x_k x_i} \psi dx - \int_{\Omega} a_{ij}(x, \nabla u) u_{x_j x_k} u_{x_k} \psi_{x_i} dx \\ &\leq - \int_{\Omega} \frac{\partial A}{\partial x_k}(x, \nabla u) u_{x_k} \nabla \psi dx - \int_{\Omega} \frac{\partial A}{\partial x_k}(x, \nabla u) \nabla u_{x_k} \psi dx - \int_{\Omega} a_{ij}(x, \nabla u) u_{x_j x_k} u_{x_k} \psi_{x_i} dx. \end{aligned}$$

Observe that $v_{x_j} = \frac{\nabla u}{|\nabla u|} \nabla u_{x_j} = \frac{u_{x_k}}{|\nabla u|} u_{x_k x_j}$. Taking the sum over k in the last inequality, using Remark B.3 and replacing by v , we have

$$\begin{aligned} - \int_{\Omega} a_{ij}(x, \nabla u) |\nabla u| v_{x_j} \psi_{x_i} dx &\geq \int_{\Omega} \frac{\partial A}{\partial x_k}(x, \nabla u) u_{x_k} \nabla \psi dx + \int_{\Omega} \frac{\partial A}{\partial x_k}(x, \nabla u) \nabla u_{x_k} \psi dx \\ &= \int_{\Omega} |\nabla u|^{p-2} \log |\nabla u| p_{x_k} u_{x_i} u_{x_k} \psi_{x_i} dx + \int_{\Omega} |\nabla u|^{p-2} \log |\nabla u| p_{x_k} u_{x_i} u_{x_k x_i} \psi dx \\ &= \int_{\Omega} |\nabla u|^{p-2} \log |\nabla u| \langle \nabla u, \nabla p \rangle \langle \nabla u, \nabla \psi \rangle dx + \int_{\Omega} |\nabla u|^{p-2} \log |\nabla u| |\nabla u| v_{x_k} p_{x_k} \psi dx \\ &= \int_{\Omega} |\nabla u|^{p-2} \log |\nabla u| \langle \nabla u, \nabla p \rangle \langle \nabla u, \nabla \psi \rangle dx + \int_{\Omega} |\nabla u|^{p-2} \log |\nabla u| |\nabla u| \langle \nabla p, \nabla v \rangle \psi dx. \end{aligned}$$

By our election of D, B and H we have,

$$- \int_{\Omega} D_{ij}(x, \nabla u) v_{x_j} \psi_{x_i} dx \geq \int_{\Omega} H \nabla \psi dx + \int_{\Omega} B \nabla v \psi dx.$$

□

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DEPARTAMENTO DE MATEMÁTICA, FCEYN
UBA (1428) BUENOS AIRES, ARGENTINA.

E-mail address: `jfbonder@dm.uba.ar`, `smartin@dm.uba.ar`, `wolanski@dm.uba.ar`

Web-page: `http://mate.dm.uba.ar/~jfbonder`

Web-page: `http://mate.dm.uba.ar/~wolanski`