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Aimar, H.; **Forzani, L.**; **Toledano, R.**
Hölder regularity of solutions of PDE's: a geometrical view.
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In this paper the authors devote themselves to the study, in the abstract setting of homogeneous spaces X (quasimetric spaces endowed with a measure satisfying the doubling condition), of minimal conditions ensuring that a nonnegative function $u\colon X\to\mathbb{R}^+$ satisfies the Harnack inequality:

$$\sup_Q u\le C\inf_Q u$$

for any $Q\subset X$ where C is a suitable constant that must be independent of the function u. This property, classical for harmonic functions, was found true in important papers by J. Moser [Comm. Pure Appl. Math. **14** (1961), 577–591; MR0159138 (28 #2356)] and M. V. Safonov [Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI) **96** (1980), 272–287, 312; MR0579490 (82b:35045)] for weak solutions to elliptic equations in divergence and nondivergence form, respectively. The authors' starting point is a new proof of the Safonov result due to Caffarelli. Like many of Caffarelli's proofs, this one is both ingenious and geometrical and starts from some critical density properties and certain weak forms of the Harnack inequality; then Caffarelli concluded in his masterly way by using certain extensions of Calderón and Zygmund arguments (see also the papers by L. A. Caffarelli and C. E. Gutiérrez [Amer. J. Math. **119** (1997), no. 2, 423–465; MR1439555 (98e:35060); Trans. Amer. Math. Soc. **348** (1996), no. 3, 1075–1092; MR1321570 (96h:35047)]). The authors ask now a very interesting question: what are the most general properties that permit one to prove that a nonnegative function defined on a homogeneous space enjoys the Harnack inequality property? Their answer is that certain fundamental properties, like the one used by Caffarelli, suffice in general homogeneous spaces (see (i) and (ii) from the introduction). More precisely, in Section 3 and Theorem 3 the authors highlight certain fundamental definitions and properties (typically satisfied by solutions to elliptic equations) that allow one to produce certain geometrical arguments in homogeneous spaces, leading to the Harnack inequality. The proofs are clear and the statements are really general.

Reviewed by *Giuseppe Mingione*

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