Inequalities related to Bourin and Heinz means with a complex parameter*

T. Bottazzi, R. Elencwajg, G. Larotonda and A. Varela †

Abstract

A conjecture posed by S. Hayajneh and F. Kittaneh claims that given A, B positive matrices, $0 \leq t \leq 1$, and any unitarily invariant norm it holds

$$|||A^t B^{1-t} + B^t A^{1-t}||| \leq |||A^t B^{1-t} + A^{1-t} B^t|||.$$

Recently, R. Bhatia proved the inequality for the case of the Frobenius norm and for $t \in [\frac{1}{4}; \frac{3}{4}]$. In this paper, using complex methods we extend this result to complex values of the parameter $t = z$ in the strip $\{z \in \mathbb{C} : \mathrm{Re}(z) \in [\frac{1}{4}; \frac{3}{4}]\}$. We give an elementary proof of the fact that equality holds for some z in the strip if and only if A and B commute. We also show a counterexample to the general conjecture by exhibiting a pair of positive matrices such that the claim does not hold for the uniform norm. Finally, we give a counterexample for a related singular value inequality given by $s_j(A^t B^{1-t} + B^t A^{1-t}) \leq s_j(A + B)$, answering in the negative a question made by K. Audenaert and F. Kittaneh.[1]

1 Introduction

We begin this paper with some notations and definitions. The context here is the algebra of $n \times n$ complex entries matrices, but the proofs adapt well to other (infinite dimensional) settings in operator theory, so let us assume that $\mathcal{A}$ stands for an operator algebra with trace, for instance $\mathcal{A} = M_n(\mathbb{C})$ with its usual trace, or $\mathcal{A} = B_2(H)$, the Hilbert-Schmidt operators acting on a separable complex Hilbert space with the infinite trace, or $\mathcal{A} = (\mathcal{A}, Tr)$ a C^*-algebra with a finite faithful trace.

*2000 MSC. Primary 15A45, 47A30; Secondary 15A42, 47A63.

†All authors supported by Instituto Argentino de Matemática, CONICET and Universidad Nacional de General Sarmiento.

[1]**Keywords and phrases:** Frobenius norm, Heinz mean, matrix inequality, matrix power, positive matrix, trace inequality, unitarily invariant norm.

Definitions 1.1. *Let $|||\cdot|||$ denote an unitarily invariant norm on $\mathcal{A}$, which we assume is equivalent to a symmetric norm, that is*

$$|||XYZ|||| \leq \|X\|_\infty |||Y||||\|Z\|_\infty$$

whenever $Y \in \mathcal{A}$ (from now on $\|\cdot\|_\infty$ will denote the norm of the operator algebra).
For convenience we will use the notation $\tau(X) = Re\ Tr(X)$. Let $|X| = \sqrt{X^*X}$ stand for the modulus of the matrix or operator X, then the (right) polar decomposition of X is given by $X = U|X|$ where U is a unitary such that U maps $Ran|X|$ into $Ran(X)$ and is the identity on $Ran|X|^\perp = Ker(X)$. Note that $\|X\|_2^2 = Tr(X^*X) = Tr[|X|^2]$.

Consider the inequality

$$\tau(A^z B^z A^{1-z} B^{1-z}) \leq \tau(AB), \tag{1}$$

for positive invertible operators $A, B > 0$ in $\mathcal{A}$, and $z \in \mathbb{C}$. We introduce some notation regarding vertical strips in the complex plane: let

$$\mathcal{S}_0 = \{z \in \mathbb{C} : 0 \leq \mathrm{Re}(z) \leq 1\}, \qquad \mathcal{S}_{1/4} = \{z \in \mathbb{C} : 1/4 \leq \mathrm{Re}(z) \leq 3/4\};$$

we will study the validity of (1) in both S_0 and $S_{1/4}$.
Intimately related to the expression above are the inequalities

$$|||b_t(A,B)|||| \leq |||h_t(A,B)||| \tag{2}$$

and

$$|||b_t(A,B)||| \leq |||A+B|||, \tag{3}$$

for positive matrices $A, B \geq 0$ in $\mathcal{A}$, where

$$b_t(A,B) = A^t B^{1-t} + B^t A^{1-t} \quad t \in [0,1];$$

the name b_t is due to Bourin, who conjectured inequality (3) for $n \times n$ matrices in [5], and

$$h_t(A,B) = A^t B^{1-t} + A^{1-t} B^t \quad t \in [0,1]$$

is named after Heinz, and the well-known [7] inequality

$$|||h_t(A,B)||| \leq |||A+B|||$$

carrying his name.

Recently, S. Hayajneh and F. Kittanneh proposed in [6] that the stronger (2) should also be valid in $M_n(\mathbb{C})$; however, numerical computations (see Section 3) show that, at least for *the uniform norm*, this is false.

If we focus on the case $|||X||| = \|X\|_2 = Tr(X^*X)^{1/2}$ (the Frobenius norm in the case of $n \times n$ matrices) and we write $h_t = h_t(A,B)$, $b_t = b_t(A,B)$, then

$$\begin{aligned} Tr|b_t|^2 &= \tau(b_t^* b_t) = \tau(B^{1-t}A^t + A^{1-t}B^t)(A^tB^{1-t} + B^tA^{1-t}) \\ &= \tau(B^{2(1-t)}A^{2t}) + \tau(A^{2(1-t)}B^{2t}) + 2\tau(A^tB^tA^{1-t}B^{1-t}) \end{aligned}$$

where we have repeatedly used the ciclicity of τ (i.e. $\tau(XY) = \tau(YX)$) and the fact that $\tau(Z^*) = \tau(Z)$. Likewise

$$Tr|h_t|^2 = \tau(B^{2(1-t)}A^{2t}) + \tau(A^{2(1-t)}B^{2t}) + 2\tau(AB).$$

Thus, proving that $\|b_t\|_2 \leq \|h_t\|_2$ amounts to prove that

$$\tau(A^tB^tA^{1-t}B^{1-t}) \leq \tau(AB), \tag{4}$$

and in fact, it is clear that both inequalities are equivalent -as remarked in [6]-.

2 Main results

We will divide the problem in regions of the plane (or the line), and then we will also consider the possiblity of attaining the equality; we will see that this is only possible in the trivial case, i.e. when A, B commute. We recall the generalized Hölder inequality, that we will use frequently: let $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} = 1$ for $p, q, r \geq 1$ and X, Y, Z in $\mathcal{A}$, then

$$\tau(XYZ) \leq \|XYZ\|_1 \leq \|X\|_p \|Y\|_q \|Z\|_r.$$

This is just a combination of the usual Hölder inequality together with

$$\|XY\|_s \leq \|X\|_p \|Y\|_q$$

provided $s \geq 1$ and $\frac{1}{p} + \frac{1}{q} = \frac{1}{s}$ (see [8], Theorem 2.8, for more details).

2.1 The inequality in the strip $\mathcal{S}_{1/4}$

We begin with an easy consequence of an inequality due to Araki-Lieb and Thirring.

Lemma 2.1. *If $A, B \geq 0$ and $r \geq 2$, then*

$$\|A^{1/r}B^{1/r}\|_r \leq \tau(AB)^{1/r}.$$

Proof. Note that

$$\|A^{1/r}B^{1/r}\|_r^r = \tau([A^{1/r}B^{1/r}B^{1/r}A^{1/r}]^{r/2}) = \tau([A^{1/r}B^{2/r}A^{1/r}]^{r/2})$$

which, by the inequality of Araki-Lieb and Thierring (see [2], and note that $r/2 \geq 1$) is less or equal than

$$\tau(A^{r/2r}B^{r2/2r}A^{r/2r}) = \tau(A^{1/2}BA^{1/2}),$$

which in turn equals $\tau(AB)$. □

Note that if we exchange the variables $z \mapsto 1-z$ and exchange the role of A, B, it suffices to consider half-strips or half-intervals around $\mathrm{Re}(z) = 1/2$.

Proposition 2.2. *If* $0 < A, B$ *and* $z \in \mathcal{S}_{1/4}$*, then*

$$\tau(A^z B^z A^{1-z} B^{1-z}) \leq \tau(AB).$$

Proof. Let $z = 1/2 + iy$, $y \in \mathbb{R}$ denote any point in vertical line of the complex plane passing through $x = 1/2$. Then

$$\begin{aligned} \tau(A^z B^z A^{1-z} B^{1-z}) &= \tau(A^{iy} A^{1/2} B^{1/2} B^{iy} A^{-iy} A^{1/2} B^{1/2} B^{-iy}) \\ &\leq \tau|A^{iy} A^{1/2} B^{1/2} B^{iy} A^{-iy} A^{1/2} B^{1/2} B^{-iy}| \\ &\leq \|A^{iy} A^{1/2} B^{1/2} B^{iy} A^{-iy}\|_2 \|A^{1/2} B^{1/2} B^{-iy}\|_2 = \|A^{1/2} B^{1/2}\|_2^2 \end{aligned}$$

by the Cauchy-Schwarz inequality and the fact that A^{iy}, B^{iy} are unitary operators. Then by the previous lemma,

$$\tau(A^z B^z A^{1-z} B^{1-z}) \leq \tau(AB)^{2/2} = \tau(AB).$$

Now consider $z = 1/4 + iy$, $y \in \mathbb{R}$, a generic point in the vertical line over $x = 1/4$, then noting that $\frac{1}{4} + \frac{1}{4} + \frac{1}{2} = 1$,

$$\begin{aligned} \tau(A^z B^z A^{1-z} B^{1-z}) &= \tau(B^{1/4} A^{1/4} A^{iy} B^{iy} B^{1/4} A^{1/4} A^{-iy} A^{1/2} B^{1/2} B^{-iy}) \\ &\leq \|B^{1/4} A^{1/4}\|_4^2 \|B^{1/2} A^{1/2}\|_2 \leq \tau(AB)^{2/4+1/2} = \tau(AB), \end{aligned}$$

where we used again the previous Lemma and the generalized Hölder's inequality,

$$\tau(XYZ) \leq \|X\|_p \|Y\|_q \|Z\|_r$$

whenever $p, q, r \geq 1$ and $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} = 1$.
By Hadamard's three-lines theorem, the bound $\tau(AB)$ is valid in the vertical strip $1/4 \leq \mathrm{Re}(z) \leq 1/2$, since it holds in the frontier of the strip. Invoking the symmetry $z \mapsto 1-z$ and exchanging the roles of A, B gives the desired bound on the full strip $\mathcal{S}_{1/4} = \{1/4 \leq \mathrm{Re}(z) \leq 3/4\}$. □

Regarding the inequalities conjectured by Bourin et al., note that we can assume $A, B > 0$: replacing A with $A_\varepsilon = A + \varepsilon$ (and likewise with B), if the inequality (1) is valid for $A_\varepsilon, B_\varepsilon$ then making $\varepsilon \to 0^+$ gives the general result: the following result that we state as corollary was recently obtained by R. Bhatia in [4] and we should also point the reader to the paper by T. Ando, F. Hiai, K. Okubo [1].

Corollary 2.3. *For any* $A, B \geq 0$ *and any* $t \in [1/4, 3/4]$*,*

$$\|A^t B^{1-t} + B^t A^{1-t}\|_2 \leq \|A^t B^{1-t} + A^{1-t} B^t\|_2 \leq \|A + B\|_2.$$

2.2 Inequality becomes equality

Let us consider the special case when the inequality above becomes an equality. We begin with a lemma that we will use in several ocasions, and will be useful when we drop the assumption on nonsingularity of A, B.

Lemma 2.4. *Let $A, B \geq 0$, and assume*

$$\tau(A^{1/2}B^{1/2}A^{1/2}B^{1/2}) = \tau(AB),$$

or

$$\|A^{1/4}B^{1/4}\|_4 = \tau(AB)^{1/4}.$$

In either case, A commutes with B.

Proof. Name $X = A^{1/2}B^{1/2}$, and considering the inner product induced by τ, $\langle X, Y\rangle = \tau(XY^*)$,

$$\langle X, X^*\rangle = \tau(X^2) = \tau(A^{1/2}B^{1/2}A^{1/2}B^{1/2}) = \tau(AB) = \tau(X^*X) = \|X\|_2^2 = \|X\|_2\|X^*\|_2.$$

But Cauchy-Schwarz inequality becomes an equality if and only if $X = \lambda X^*$ for some $\lambda > 0$, and since both operators have equal norm ($= \|A^{1/2}B^{1/2}\|_2$), then $X = X^*$. This means

$$A^{1/2}B^{1/2} = B^{1/2}A^{1/2},$$

and this implies that A commutes with B. On the other hand,

$$\|A^{1/4}B^{1/4}\|_4^4 = \tau((B^{1/4}A^{1/2}B^{1/4})^2) = \tau(A^{1/2}B^{1/2}A^{1/2}B^{1/2}),$$

so what we have is just another way of writing the first equality condition. □

Proposition 2.5. *Let $A, B > 0$ and assume that there is $z_0 \in \mathcal{S}_{1/4}$ such that*

$$\tau(A^{z_0}B^{z_0}A^{1-z_0}B^{1-z_0}) = \tau(AB).$$

Then A commutes with B and $\tau(A^zB^zA^{1-z}B^{1-z}) = \tau(AB)$ for any $z \in \mathbb{C}$.

Proof. First consider the case when equality is reached in an interior point of the strip $\mathcal{S}_{1/4}$. Note that by the maximum modulus principle, this would mean that the function

$$f(z) = \tau(A^zB^zA^{1-z}B^{1-z})$$

is constant in the strip $\mathcal{S}_{1/4}$, in particular equality holds at $z_0 = 1/2$, and by the previous Lemma, A commutes with B.

Now suppose equality is attained in the frontier, for instance at $z_0 = 1/4 + iy$ for some $y \in \mathbb{R}$. Let $X = B^{1/4}A^{1/4}A^{iy}B^{iy}B^{1/4}A^{1/4}$, $Y = B^{1/2}B^{iy}A^{iy}A^{1/2}$. Then, if we go through the proof of Proposition 2.2 again, assuming equality

$$\begin{aligned} \tau(AB) &= \tau(XY^*) = \langle X, Y\rangle \leq \|X\|_2\|Y\|_2 \\ &\leq \|B^{1/4}A^{1/4}\|_4^2\|A^{1/2}B^{1/2}\|_2 \leq \tau(AB). \end{aligned} \tag{5}$$

Arguing as in the previous Lemma, there exists $\lambda > 0$ such that $X = \lambda Y$,

$$B^{1/4}A^{1/4}A^{iy}B^{iy}B^{1/4}A^{1/4} = \lambda B^{1/2}B^{iy}A^{iy}A^{1/2}.$$

Cancelling $B^{1/4}$ on the left and $A^{1/4}$ on the right we obtain

$$A^{1/4}A^{iy}B^{iy}B^{1/4} = \lambda B^{1/4}B^{iy}A^{iy}A^{1/4},$$

but now both elements have the same norm and this shows that $\lambda = 1$; then

$$A^{1/4+iy}B^{1/4+iy} = B^{1/4+iy}A^{1/4+iy},$$

and since $A, B > 0$, the existence of analytic logarithms shows that again A commutes with B. By symmetry, the same argument applies for any $z_0 = 3/4 + iy$ in the other border of the strip. $\square$

Corollary 2.6. *If A does not commute with B, the inequality is strict:*

$$\tau(A^zB^tA^{1-z}B^{1-z}) < \tau(AB),$$

in some open set $\Omega \subset \mathbb{C}$ containing the closed strip $\mathcal{S}_{1/4}$.

If we allow A, B to be non invertible, holomorphy is lost, but nevertheless in the same spirit we have the following result.

Proposition 2.7. *For given $A, B \geq 0$, there exists $\delta = \delta(A, B) > 0$ such that*

$$\tau(A^tB^tA^{1-t}B^{1-t}) \leq \tau(AB)$$

holds in the interval $[1/4 - \delta, 3/4 + \delta]$. If A does not commute with B, the inequality is strict in the whole $(1/4 - \delta, 3/4 + \delta)$.

Proof. If A commutes with B, then the assertion is trivial. If not, arguing as in the last part of the proof of the previous proposition, we must have strict inequality

$$\tau(A^tB^tA^{1-t}B^{1-t}) < \tau(AB)$$

for $t = 1/4$, $t = 3/4$, and then by continuity the inequality extends a bit out of the closed interval $[1/4, 3/4]$.

Consider $t \in (1/4, 1/2)$ and put $X = B^{1/4}A^{1/4}A^{t-1/4}B^{t-1/4}$, $Y = B^{1/4}A^{1/4}A^{3/4-t}B^{3/4-t}$. Note that $\frac{1}{t}, \frac{1}{1-t} \geq 1$ and define $1/p = t - 1/4 \in (0, 1/4)$, $1/q = 3/4 - t \in (1/4, 1/2)$, note also that $1/p + 1/4 = t$, $1/q + 1/4 = 1 - t$. By reiterated use of Hölder's inequality compute

$$\begin{aligned} \tau(A^t B^t A^{1-t} B^{1-t}) &\leq \|XY\|_1 \leq \|X\|_{t^{-1}} \|Y\|_{(1-t)^{-1}} \\ &\leq \|B^{1/4}A^{1/4}\|_4 \|A^{1/p}B^{1/p}\|_p \|B^{1/q}A^{1/q}\|_q \|A^{1/4}B^{1/4}\|_4. \end{aligned}$$

Now apply Lemma 2.1 to each of the four terms (note that $p > 4$ and $q > 2$), and we have[2]

$$\tau(A^t B^t A^{1-t} B^{1-t}) \leq \|B^{1/4}A^{1/4}\|_4 \|A^{1/p}B^{1/p}\|_p \|B^{1/q}A^{1/q}\|_q \|A^{1/4}B^{1/4}\|_4 \leq \tau(AB).$$

If we assume equality of the traces, then

$$\tau(AB) = \|B^{1/4}A^{1/4}\|_4 \|A^{1/p}B^{1/p}\|_p \|B^{1/q}A^{1/q}\|_q \|A^{1/4}B^{1/4}\|_4$$

and in particular, it must be that $\|A^{1/4}B^{1/4}\|_4 = \tau(AB)^{1/4}$, and from Lemma 2.4 we can deduce that A commutes with B. By the symmetry ($t \mapsto 1 - t$) the argument extends to $(1/2, 3/4)$, and again by Lemma 2.4 we already know that A commutes with B if equality is attained at $t = 1/2$. This finishes the proof of the assertion that the inequality is strict in $[1/4, 3/4]$ unless A commutes with B. $\square$

Remark 2.8. *The inequalities in the previous proof give in fact*

$$\tau(|B^{\frac{1}{4}} A^t B^t A^{1-t} B^{\frac{3}{4}-t}|) \leq Tr(AB)$$

for any $t \in [\frac{1}{4}, \frac{3}{4}]$; this is a particular instance of [1, Theorem 2.10].

3 Counterexamples

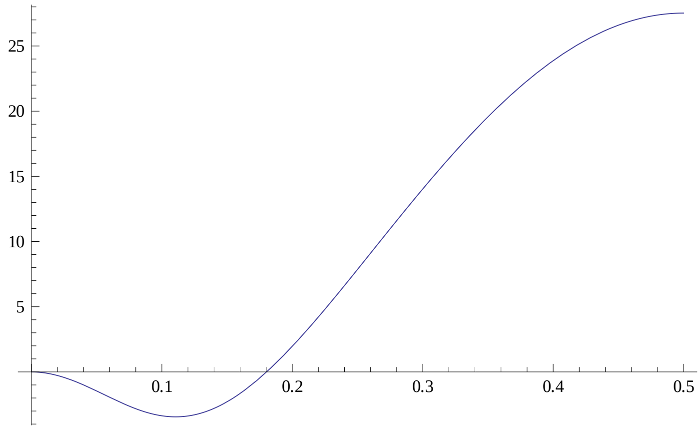
In this section we exhibit specific cases of different kind. In Example 3.1 we choose A, B such that $\|b_t(A, B)\|_\infty > \|h_t(A, B)\|_\infty$, while in Example 3.2, it is shown that the j^{th} singular value of $A + B$ is not always greater than the j^{th} singular value of $b_t(A, B)$. This provides negative answers to [6, Conjecture 1.2] and [3, Problem 4] respectively.

Example 3.1. *Consider the following positive definite matrices*

$$A = \begin{pmatrix} 1141 & 0 & 0 \\ 0 & 204 & 0 \\ 0 & 0 & 1/8 \end{pmatrix} \qquad \textit{and} \qquad B = \begin{pmatrix} 39 & 90 & 43 \\ 90 & 418 & 370 \\ 43 & 370 & 426 \end{pmatrix}.$$

[2] Note that this is another proof of the inequality for real $t \in [\frac{1}{4}, \frac{3}{4}]$.

The following is the graph of $f(t) = -\|b_t(A,B)\|_\infty + \|h_t(A,B)\|_\infty$ *for* $t \in [0, \frac{1}{2}]$*:*



For these matrices $-\|b_t(A,B)\|_\infty + \|h_t(A,B)\|_\infty \simeq -2.3$ *at* $t = .15$.

In [3, Problem 4] K. Audenaert and F. Kittaneh asked if $s_j(b_t(A,B)) \leq s_j(A+B)$ for every j and $0 < t < 1$ (where $s_j(M)$, $j = 1 \ldots n$ denote the singular values of the matrix M arranged in non-increasing order).

Example 3.2. *Consider the following positive definite matrices*

$$A = \begin{pmatrix} 6317 & 0 & 0 \\ 0 & 474 & 0 \\ 0 & 0 & 6 \end{pmatrix} \quad \textit{and} \quad B = \begin{pmatrix} 2078 & 2362 & 2199 \\ 2362 & 3267 & 2585 \\ 2199 & 2585 & 2492 \end{pmatrix}.$$

Then, for $t = \frac{1}{2}$ *we have*

$$s(b_{\frac{1}{2}}(A,B)) = (6826.57,\ 878.499,\ 591.716)$$

and

$$s(A+B) = (10561.4,\ 3629.62,\ 443.017).$$

In particular, $s_3(b_{\frac{1}{2}}(A,B)) > s_3(A+B)$.

References

[1] T. Ando, F. Hiai, K. Okubo. *Trace inequalities for multiple products of two matrices.* Math. Inequal. Appl. 3 (2000), no. 3, 307-318.

[2] H. Araki. *On an inequality of Lieb and Thirring*, Lett. Math. Phys. 19 (1990), pp. 167-170.

[3] K. Audenaert, F. Kittaneh. *Problems and Conjectures in Matrix and Operator Inequalities*, eprint arXiv:1201.5232v3 [math.FA]

[4] R. Bhatia. *Trace inequalities for products of positive definite matrices*, J. Math. Phys. 55 (2014).

[5] J.C. Bourin. *Matrix subadditivity inequalities and block-matrices.* Internat. J. Math. 20 (2009), no. 6, 679–691.

[6] S. Hayajneh, F. Kittaneh. *Lieb-Thirring trace inequalities and a question of Bourin.* J. Math. Phys. 54 (2013), no. 3, 033504, 8 pp.

[7] E. Heinz. *Beiträge zur Störungstheorie der Spektralzerlegung.* (German) Math. Ann. 123, (1951). 415-438.

[8] B. Simon. *Trace ideals and their applications.* Second edition. Mathematical Surveys and Monographs, 120. American Mathematical Society, Providence, RI, 2005.