INSTITUTE OF PHYSICS PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND GENERAL

J. Phys. A: Math. Gen. **34** (2001) 4143–4154 www.iop.org/Journals/ja PII: S0305-4470(01)21163-9

# Hamiltonian self-adjoint extensions for (2 + 1)-dimensional Dirac particles

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Received 16 January 2001, in final form 16 March 2001

**Abstract**

We study the stationary problem of a charged Dirac particle in (2+1) dimensions in the presence of a uniform magnetic field $B$ and a singular magnetic tube of flux $\Phi = 2\pi\kappa/e$. The rotational invariance of this configuration implies that the subspaces of definite angular momentum $l + 1/2$ are invariant under the action of the Hamiltonian $H$. We show that for $\kappa - l \geqslant 1$ or $\kappa - l \leqslant 0$ the restriction of $H$ to these subspaces, $H_l$, is essentially self-adjoint, while for $0 < \kappa - l < 1$ $H_l$ admits a one-parameter family of self-adjoint extensions (SAEs). In the latter case, the functions in the domain of $H_l$ are singular (but square integrable) at the origin, their behaviour being dictated by the value of the parameter $\gamma$ that identifies the SAE. We also determine the spectrum of the Hamiltonian as a function of $\kappa$ and $\gamma$, as well as its closure.

PACS numbers: 0230T, 0230S, 0230J, 0365P

Mathematics Subject Classification: 81Q10, 35P05

## 1. Introduction

In quantum mechanics, observables are realized in terms of self-adjoint operators on a Hilbert space. It is for these operators that the spectral theorem holds [1]. In particular, the dynamics of a quantum system should be given by a unitary group whose generator, the Hamiltonian $H$ (usually a differential operator acting on an appropriate space of square integrable functions), must be self-adjoint.

In general, physical considerations lead to a *formal* expression for the Hamiltonian, although they can leave its domain of definition not completely specified. Usually, one can choose a dense subspace of the Hilbert space on which $H$ is well defined and *symmetric*, but not necessarily *self-adjoint*.

In these conditions, the question is posed of determining whether the expression found for $H$ has a unique self-adjoint extension (SAE) in the Hilbert space (i.e. whether $H$ is *essentially self-adjoint*), or whether it admits different SAEs (differing in the physics they describe) and, in this case, which one corresponds to the physical system under consideration.

  

A situation of practical interest in which the Hamiltonian admits nontrivial SAEs corresponds to the movement of charged particles under the influence of a Bohm–Aharonov singular magnetic flux tube [2], such as fermions in the presence of cosmic strings [3] or nonrelativistic spinless quantum particles interacting with a thin solenoid [4]. In [3–6], this problem has been analysed by means of von Neumann's theory of deficiency subspaces [1].

This kind of situation has also been studied as a limit of a smeared flux, using a $\delta$-function shell magnetic field [7, 8] or uniform magnetic fields confined to a finite tube [9, 10], and a punctured plane [11, 12], which leads to the consideration of boundary conditions at a finite radius, both spectral and local. The study of charged particle states bounded to flux tubes has also been of interest [13–17].

The presence of a $\delta$-like magnetic field has also been considered in connection with vacuum polarization effects in [18], to model the presence of a pointlike impurity in a bidimensional system [19] and more recently to describe a nonrelativistic electron in the presence of a uniform electromagnetic field and a singular vortex, as a step toward its application to the quantum Hall effect [20]. This configuration can also be relevant to the description of quasiparticles in unconventional superconductors [21, 22].

It is the aim of this paper to study the behaviour of a Dirac electron of mass $M$ and charge $e$ constrained to live in a $(2+1)$-dimensional space, in the presence of a constant magnetic field $B$ and a singular magnetic flux tube $\Phi = 2\pi\kappa/e$ passing through the origin. In so doing, we will use von Neumann's theory of deficiency indices to determine the existence of nontrivial SAE for the Hamiltonian, a problem that, as far as we know, has not yet been solved.

The rotational symmetry of the problem allows for studying the action of the Hamiltonian (a differential operator $H$ defined on an appropriately restricted set of smooth functions) in each invariant subspace characterized by a definite angular momentum $l + 1/2$. We find that the restriction of $H$ to the subspaces with $\kappa - l \geqslant 1$ or $\kappa - l \leqslant 0$, $H_l$, is essentially self-adjoint, while for $0 < \kappa - l < 1$ the operator $H_l$ admits a one-parameter family of SAEs. In the latter case, the functions in the extended domain of $H_l$ become singular (though square integrable) at the origin, their behaviour being dictated by the value of the parameter $\gamma$ that identifies the SAE.

Finally, we also determine the spectrum of the Hamiltonian as a function of $\kappa$ and $\gamma$.

## 2. Formulation of the problem

Let us consider a Dirac particle of mass $M$ and charge $e$ in a $(2 + 1)$-dimensional spacetime, in the presence of a uniform magnetic field $B$ and a singular magnetic flux tube $\Phi = 2\pi\kappa/e$ passing through the origin (i.e. the flux originated in a magnetic field which is null at each point of the plane except at the origin, and whose flux through every curve enclosing the origin is finite).

The wavefunction of this particle is a two-component spinor $\psi$ satisfying the Dirac equation (we adopt the fundamental units for which $\hbar = 1 = c$),

$$(\mathrm{i}\,\not{D} - M)\psi = 0 \tag{1}$$

where the covariant derivative[1] is $\not{D} = \not{\partial} - \mathrm{i}e\,\not{A}$.

[1] We choose the following representation of the $\gamma$-matrices:

$$\gamma^0 = \sigma^3 \qquad \gamma^1 = -\mathrm{i}\sigma^2 \qquad \gamma^2 = \mathrm{i}\sigma^1 \tag{2}$$

where the $\sigma^i$, $i = 1, 2, 3$, are the Pauli matrices. In a three-dimensional space-time, a nonequivalent representation is obtained by changing the sign of the matrices, $\gamma^\mu \to -\gamma^\mu$, but this amounts to changing the sign of the parameter $M$, which therefore can be considered to take real values.

We choose the following expression for the vector potential leading to the magnetic field under consideration:

$$\vec{A} = \left(\frac{\Omega r}{e} + \frac{\kappa}{er}\right)\hat{e}_\theta \tag{3}$$

where $\Omega = eB/2$ has units of squared mass and $\hat{e}_\theta$ is the unit vector orthogonal to the radial direction.

Accordingly, we obtain for the Dirac Hamiltonian $H_{\mathrm{D}} = \sqrt{\Omega}H$, where $H$ is the dimensionless differential operator

$$H = \begin{bmatrix} m & \mathrm{i}\mathrm{e}^{-\mathrm{i}\theta}\left(\partial_x - \frac{\mathrm{i}}{x}\partial_\theta - x - \frac{\kappa}{x}\right) \\ -\mathrm{i}\mathrm{e}^{\mathrm{i}\theta}\left(-\partial_x - \frac{\mathrm{i}}{x}\partial_\theta - x - \frac{\kappa}{x}\right) & -m \end{bmatrix} \tag{4}$$

expressed in polar coordinates ($x = \sqrt{\Omega}r, \theta$), with $m = M/\sqrt{\Omega}$ the particle mass in units of $\Omega^{1/2}$.

Since $H$ commutes with the angular momentum operator, $J = -\mathrm{i}\partial_\theta + \sigma^3/2$, the subspaces spanned by the two-component spinors of the form

$$\psi(x,\theta) = \begin{pmatrix} \mathrm{e}^{\mathrm{i}l\theta}\phi_{(x)} \\ \mathrm{e}^{\mathrm{i}(l+1)\theta}\chi_{(x)} \end{pmatrix} \in L_2(\mathbb{R}^2, x\,\mathrm{d}x\,\mathrm{d}\theta) \qquad l \in \mathbb{Z} \tag{5}$$

are left invariant by the action of $H$. The restriction of $H$ to each subspace characterized by $l$, $H_l$, can be cast into the form

$$H_l = \begin{pmatrix} m & \mathrm{i}\left(\frac{\mathrm{d}}{\mathrm{d}x} + \frac{1-\alpha}{x} - x\right) \\ \mathrm{i}\left(\frac{\mathrm{d}}{\mathrm{d}x} + \frac{\alpha}{x} + x\right) & -m \end{pmatrix} \tag{6}$$

with $\alpha = \kappa - l$, when acting on two-component functions of the radial coordinate,

$$\psi(x) = \begin{pmatrix} \phi_{(x)} \\ \chi_{(x)} \end{pmatrix} \tag{7}$$

where $\phi_{(x)}, \chi_{(x)} \in L_2(\mathbb{R}^+, 2\pi x\,\mathrm{d}x)$.

In order to ensure that $H_l$ be symmetric and well defined we can restrict its domain to

$$\mathcal{D}(H_l) \equiv C_0^\infty(\mathbb{R}^+) \tag{8}$$

the subspace of functions with compact support away from the origin and continuous derivatives of all orders, which is dense in $L_2(\mathbb{R}^+, 2\pi x\,\mathrm{d}x)$.

To determine whether $H_l$ so defined is (essentially) self-adjoint we must compute its deficiency indices in the Hilbert space $L_2(R^+, 2\pi\, x\,\mathrm{d}x)$, i.e. the dimensions of the characteristic subspaces $\mathcal{K}_\pm$ of its adjoint, $H_l^\dagger$, corresponding to eigenvalues $\pm\mathrm{i}$,

$$n_\pm = \dim \mathcal{K}_\pm. \tag{9}$$

In the following we shall show that $H_l$ admits SAEs for $0 < \kappa - l < 1$, being essentially self-adjoint for the other angular momentum subspaces.

## 3. Self-adjoint extensions

In order to determine the deficiency indices of the operator $H_l$ defined in the previous section, we must determine the deficiency subspaces $\mathcal{K}_\pm$.

Let us recall that the domain of $H_l^\dagger$, $\mathcal{D}(H_l^\dagger)$, is the set of functions $f(x) = \binom{f_1(x)}{f_2(x)} \in L_2(\boldsymbol{R}^+, 2\pi x\,\mathrm{d}x)$, for which functions $g(x) = \binom{g_1(x)}{g_2(x)} \in L_2(\mathbb{R}^+, 2\pi x\,\mathrm{d}x)$ exist, such that

$$(f, H_l\psi) = (g, \psi) \tag{10}$$

for any $\psi \in \mathcal{D}(H_l)$. The adjoint $H_l^\dagger$ is defined by $g = H_l^\dagger f$.

Taking into account equation (8) and the expression for $H_l$, equation (6), one can easily see that, away from the origin, the first weak derivative of $f(x)$ is locally in $L_2(\boldsymbol{R}^+, 2\pi x\,\mathrm{d}x)$. Therefore, by Sobolev's lemma (see [1]), $f(x)$ is absolutely continuous. This allows for an integration by parts in equation (10), which gives

$$\begin{pmatrix} g_1 \\ g_2 \end{pmatrix} = \begin{pmatrix} m & \mathrm{i}\left(\frac{\mathrm{d}}{\mathrm{d}x} + \frac{1-\alpha}{x} - x\right) \\ \mathrm{i}\left(\frac{\mathrm{d}}{\mathrm{d}x} + \frac{\alpha}{x} + x\right) & -m \end{pmatrix} \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}. \tag{11}$$

In conclusion, $H_l^\dagger$ acts as a differential operator in the same way as $H_l$ in equation (6), but on a larger domain $\mathcal{D}(H_l^\dagger)(\supset\mathcal{D}(H_l))$, consisting of the subspace of functions of $L_2(\mathbb{R}^+, 2\pi x\,\mathrm{d}x)$ which are absolutely continuous in $\mathbb{R}^+/\{0\}$.

In accordance with appendix A, we must now determine the subspaces $\mathcal{K}_\pm$ by looking for linearly independent eigenfunctions of the operator $H_l^\dagger$ corresponding to the eigenvalues $\pm\mathrm{i}$, $\psi^\pm_{(x)}$. Taking into account equation (11), it is easily seen from

$$H_l^\dagger \psi^\pm_{(x)} = \pm\mathrm{i}\psi^\pm_{(x)} \tag{12}$$

that the first derivative of $\psi^\pm_{(x)}$ is absolutely continuous, as well as its derivatives of all orders. Thus, $\psi^\pm_{(x)} \in \mathcal{C}^\infty \bigcap L_2(\mathbb{R}^+, 2\pi x\,\mathrm{d}x)$, and the eigenvalue problem equation (12) reduces to a classical ordinary differential equation.

Then, equation (12) leads to the following system of coupled differential equations for the components, $\phi_\pm$ and $\chi_\pm$, of the eigenfunctions $\psi^\pm$:

$$\mathrm{i}\frac{\mathrm{d}\chi_\pm}{\mathrm{d}x} - \mathrm{i}\left(\frac{\alpha-1}{x} + x\right)\chi_\pm = (\pm\mathrm{i} - m)\phi_\pm \tag{13}$$

$$\mathrm{i}\frac{\mathrm{d}\phi_\pm}{\mathrm{d}x} + \mathrm{i}\left(\frac{\alpha}{x} + x\right)\phi_\pm = (\pm\mathrm{i} + m)\chi_\pm. \tag{14}$$

Replacing $\chi_\pm$ from equation (14) in (13), we obtain for the other component

$$\phi''_\pm + \frac{1}{x}\phi'_\pm - \left(\left(\frac{\alpha}{x}\right)^2 + x^2 - 2(1-\alpha) + m^2 + 1\right)\phi_\pm = 0. \tag{15}$$

Making the substitution

$$\phi_\pm = \mathrm{e}^{-\frac{x^2}{2}} x^{-\alpha} F(x^2) \tag{16}$$

we obtain Kummer's equation [23]

$$x^2 \frac{\mathrm{d}^2 F}{\mathrm{d}(x^2)^2}(x^2) + \left[b - x^2\right]\frac{\mathrm{d}F}{\mathrm{d}(x^2)}(x^2) - a\,F(x^2) = 0 \tag{17}$$

with $a = \frac{m^2+1}{4} > 0$ and $b = 1 - \alpha = 1 - \kappa + l$.

This equation has two linearly independent solutions [23], $M(a, b, x^2)$ and $U(a, b, x^2)$, only the latter of which leads to $\phi_\pm \in L_2((\delta, \infty), 2\pi x\,\mathrm{d}x)$, with $\delta > 0$. On the other hand, the condition $\phi_\pm \in L_2((0, \delta), 2\pi x\,\mathrm{d}x)$ requires $0 < b < 2$ (see [23], p 508).

Moreover, the condition that the second component (determined by equation (14)) satisfies $\chi_\pm \in L_2(\mathbb{R}^+, 2\pi x\,\mathrm{d}x)$ imposes $0 < b < 1$. This requires[2] that $\kappa \notin \mathbb{Z}$, and selects the subspace for which $l$ is the integer part of $\kappa$, $\kappa - 1 < l < \kappa$, as the only one where nontrivial SAEs exist.

Thus, for $l \neq [\kappa]$, $H_l$ is essentially self-adjoint, admitting a unique SAE given by the closure of its graph (see appendix B).

[2] Notice that if $\kappa \in \mathbb{Z}$, the presence of the singular flux through the origin amounts to a shift in the value of the orbital angular momentum (as can be seen from equation (6)), without any further consequence, $H_l$ being essentially self-adjoint. For brevity, we will not further consider this case in what follows.

On the other hand, for $l = [\kappa]$ we have found one-dimensional subspaces $\mathcal{K}_\pm$, generated by the solutions of equation (12), $\psi^\pm$, given in components by

$$\phi_\pm = \mathrm{e}^{-\frac{x^2}{2}} x^{-\alpha} U\left(a = \frac{m^2+1}{4}; b = 1-\alpha; x^2\right) \tag{18}$$

$$\chi_\pm = \left[\frac{-\mathrm{i}m \mp 1}{2}\right] \mathrm{e}^{-\frac{x^2}{2}} x^{-\alpha+1} U\left(a = \frac{m^2+5}{4}; b = 2-\alpha; x^2\right). \tag{19}$$

Therefore, $n_+ = 1 = n_-$, and $H_{[\kappa]}$ admits a one-parameter ($\gamma$) family of (essentially) SAEs [1], $H^\gamma_{[\kappa]}$, which, as explained in the appendix A, are in a one-to-one correspondence with the isometries $\mathcal{U}_\gamma$ from $\mathcal{K}_+$ onto $\mathcal{K}_-$:

$$\mathcal{U}_\gamma \psi^+ = \mathrm{e}^{\mathrm{i}\gamma} \psi^- \tag{20}$$

with $-\pi < \gamma \leqslant \pi$.

The functions $\psi_{(x)}$ in the domain of $H^\gamma_{[\kappa]}$ are of the form

$$\psi = \psi_0 + c(\psi^+ + \mathrm{e}^{\mathrm{i}\gamma}\psi^-) \tag{21}$$

where $\psi_0 \in C_0^\infty(\mathbb{R}^+)$ and $c \in \boldsymbol{C}$, the action of $H^\gamma_{[\kappa]}$ being defined by

$$H^\gamma_{[\kappa]}\psi \equiv H_{[\kappa]}\psi_0 + c\,\mathrm{i}(\psi^+ - \mathrm{e}^{\mathrm{i}\gamma}\psi^-). \tag{22}$$

In appendix B, it is shown that the functions in the closure of the graph of $H_{[\kappa]}$ are continuous and vanishing for $x \to 0^+$. Therefore, the behaviour at the origin of the functions in the domain of the closure of $H^\gamma_{[\kappa]}$, $\mathcal{D}(\overline{H^\gamma_{[\kappa]}})$, is determined by the behaviour of $\psi^{(\gamma)} \equiv \psi^+ + \mathrm{e}^{\mathrm{i}\gamma}\psi^-$, whose components satisfy

$$\phi^{(\gamma)} = (1+\mathrm{e}^{\mathrm{i}\gamma}) \frac{\Gamma(\alpha)}{\Gamma(\alpha + \frac{m^2+1}{4})}\, x^{-\alpha} + \mathrm{O}(x^\alpha) \tag{23}$$

$$\chi^{(\gamma)} = \frac{-\mathrm{i}}{2}\left[m(1+\mathrm{e}^{\mathrm{i}\gamma}) - \mathrm{i}(1-\mathrm{e}^{\mathrm{i}\gamma})\right] \frac{\Gamma(1-\alpha)}{\Gamma(\frac{m^2+5}{4})} x^{-1+\alpha} + \mathrm{O}(x^{1-\alpha}). \tag{24}$$

This allows for the following characterization of the *boundary conditions* that the functions $\psi = \begin{pmatrix}\phi \\ \chi\end{pmatrix} \in \mathcal{D}(\overline{H^\gamma_{[\kappa]}})$ satisfy:

$$\lim_{x\to 0^+} \left\{x\left[\phi\,\chi^{(\gamma)} - \chi\,\phi^{(\gamma)}\right]\right\} = 0. \tag{25}$$

We will use this condition in the next section to determine the spectrum of $\overline{H^\gamma_{[\kappa]}}$.

## 4. Spectrum of $\overline{H^\gamma_{[\kappa]}}$

In this section, making use of the *boundary condition* deduced in equation (25), we will determine the eigenfunctions and eigenvalues of $\overline{H^\gamma_{[\kappa]}}$. So we must solve the eigenvalue problem

$$\overline{H^\gamma_{[\kappa]}}\psi = \lambda\psi. \tag{26}$$

Notice that, since $\overline{H^\gamma_{[\kappa]}}$ is the restriction of $H^\dagger_{[\kappa]}$ to $\mathcal{D}(\overline{H^\gamma_{[\kappa]}}) \subset \mathcal{D}(\overline{H^\dagger_{[\kappa]}})$, both operators are realized by the same differential operator (given in equation (6), with $l$ replaced by $[\kappa]$). On the basis of an argument similar to the one following equation (12), we conclude that we are looking for $\mathcal{C}^\infty$ solutions of an ordinary differential equation. In terms of the components $\phi$ and $\chi$, we obtain the pair of coupled differential equations

$$\mathrm{i}\chi' - \mathrm{i}\left(\frac{\alpha-1}{x}+x\right)\chi = (\lambda - m)\phi \tag{27}$$

$$\mathrm{i}\phi' + \mathrm{i}\left(\frac{\alpha}{x}+x\right)\phi = (\lambda + m)\chi. \tag{28}$$

Once again, the substitution given in equation (16) (now with $0 < \alpha < 1$) leads to Kummer's equation for $F(x^2)$, equation (17), with $a = \frac{m^2-\lambda^2}{4}$ and $b = 1-\alpha$. The requirement that $\phi$ and $\chi$ belong to $L_2(\mathbb{R}^+, 2\pi x\,\mathrm{d}x)$ selects as the unique solution

$$\phi_\lambda = \mathrm{e}^{-x^2/2}x^{-\alpha}U(a = (m^2-\lambda^2)/4; b = 1-\alpha; x^2) \tag{29}$$

$$\chi_\lambda = \frac{\mathrm{i}}{2}(\lambda - m)\,\mathrm{e}^{-x^2/2}x^{1-\alpha}U(a = 1+(m^2-\lambda^2)/4; b = 2-\alpha; x^2) \tag{30}$$

behaving, for $x \to 0^+$, as

$$\phi_\lambda = x^{-\alpha}\frac{\Gamma(\alpha)}{\Gamma(\alpha+\frac{m^2-\lambda^2}{4})} + \mathrm{O}(x^\alpha) \tag{31}$$

$$\chi_\lambda = \frac{\mathrm{i}(\lambda-m)}{2}x^{-1+\alpha}\frac{\Gamma(1-\alpha)}{\Gamma(1+\frac{m^2-\lambda^2}{4})} + \mathrm{O}(x^{1-\alpha}). \tag{32}$$

So, the condition expressed in equation (25) implies

$$\begin{aligned}\frac{\Gamma(\alpha)}{\Gamma(\alpha+\frac{m^2-\lambda^2}{4})}&\left[m(1+\mathrm{e}^{\mathrm{i}\gamma}) - \mathrm{i}(1-\mathrm{e}^{\mathrm{i}\gamma})\right]\frac{\Gamma(1-\alpha)}{\Gamma(\frac{m^2+5}{4})}\\ &= -(\lambda-m)\frac{\Gamma(1-\alpha)}{\Gamma(1+\frac{m^2-\lambda^2}{4})}(1+\mathrm{e}^{\mathrm{i}\gamma})\frac{\Gamma(\alpha)}{\Gamma(\alpha+\frac{m^2+1}{4})}\end{aligned} \tag{33}$$

which can also be written as

$$\begin{aligned}G(\lambda) &\equiv (\lambda-m)\left(\frac{\Gamma(\alpha+m^2/4-\lambda^2/4)}{\Gamma(1+m^2/4-\lambda^2/4)}\right)\\ &= (\tan(\gamma/2)-m)\left(\frac{\Gamma(\alpha+m^2/4+1/4)}{\Gamma(m^2/4+5/4)}\right) \equiv \beta(\gamma).\end{aligned} \tag{34}$$

This is a transcendental equation determining the eigenvalues of $\overline{H_{[\kappa]}^\gamma}$. The whole dependence on $\lambda$ is contained in $G(\lambda)$, on the lhs. This function has simple zeros at $\lambda = m$ and $\lambda = \pm\sqrt{4+m^2+4n}$, and simple poles at $\lambda = \pm\sqrt{4\alpha+m^2+4n}$, for $n = 0, 1, 2, \ldots$ (see figure 1).

On the rhs of equation (34), $\beta(\gamma)$ is a constant depending only on $m$, $\alpha$ and the parameter $\gamma$ characterizing the SAE of $H_{[\kappa]}$. It can take all real values with $\gamma$ ranging from $-\pi$ to $\pi$, being $\beta(\gamma) > 0$ for $\gamma_0 < \gamma < \pi$ and $\beta(\gamma) < 0$ for $-\pi < \gamma < \gamma_0$, where $\gamma_0 = 2\arctan(m)$.

It is evident from figure 1 that the spectrum of $\overline{H_{[\kappa]}^\gamma}$ does depend on $\gamma$. If $\gamma_0 < \gamma < \pi$, the eigenvalues lie between a zero of $G(\lambda)$ and the nearest pole on its right: for $\lambda > m$

$$\sqrt{m^2+4(N+1)} < \lambda_N < \sqrt{m^2+4(\alpha+N+1)} \tag{35}$$

with $N = 0, 1, 2, \ldots$ and, for $\lambda < m$,

$$-\sqrt{m^2-4N} < \lambda_N < -\sqrt{m^2+4(\alpha-N-1)} \tag{36}$$

with $N = -1, -2, -3, \ldots$.

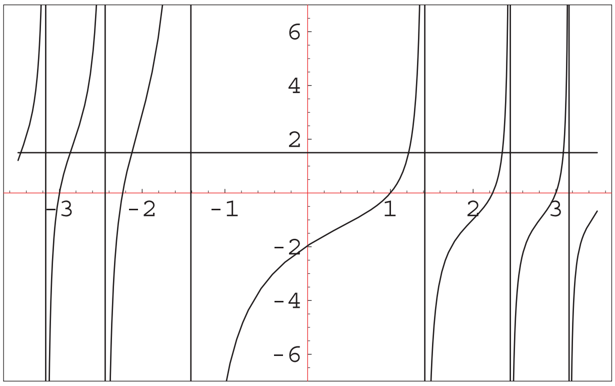


**Figure 1.** Graphs of $G(\lambda)$ for $m = 1$ and $\alpha = 1/4$. The horizontal line corresponds to a positive value of $\beta(\gamma)$.

For $-\pi < \gamma < \gamma_0$ the eigenvalues are bounded on the left by a pole and on the right by the nearest zero of $G(\lambda)$: for $\lambda > m$

$$\sqrt{m^2 + 4(\alpha + N - 1)} < \lambda_N < \sqrt{m^2 + 4N} \tag{37}$$

with $N = 1, 2, 3, \ldots,$

$$-\sqrt{m^2 + 4\alpha} < \lambda_0 < m \tag{38}$$

and, for $\lambda < 0$,

$$-\sqrt{m^2 + 4(\alpha - N)} < \lambda_N < -\sqrt{m^2 - 4N} \tag{39}$$

with $N = -1, -2, -3, \ldots.$

Notice that there is only one level with $|\lambda| < \sqrt{m^2 + 4\alpha}$. Moreover, the spectrum of $\overline{H^{\gamma}_{[\kappa]}}$ is symmetric with respect to the origin only for $\gamma = \pi$ and (except for the eigenvalue $\lambda_0 = m$) for $\gamma = \gamma_0$.

## 5. Spectrum of $\overline{H_L}$ for $L \neq [\kappa]$

In this section we complete the description of the Hamiltonian spectrum by computing the eigenfunctions and eigenvalues of $\overline{H_l}$ for $l \neq [\kappa]$.

As we saw in section 3, in the present case $H_l$ is essentially self-adjoint, admitting a unique SAE given by the closure of its graph. According to appendix B, the vectors in $\mathcal{D}(\overline{H_l})$ are absolutely continuous functions vanishing at the origin.

We are looking for solutions of the system given by equations (27) and (28) in this domain. Once again, by an argument similar to the one employed in section 3, one can see that the eigenvectors belong to $\mathcal{C}^\infty \bigcap L_2(\mathbb{R}^+, 2\pi x\, \mathrm{d}x)$.

Following the same steps as in section 4, one obtains the solutions in terms of Kummer's functions. It is convenient to write them in terms of the following pair of linearly independent solutions of equation (17):

$$F_1(x^2) = M(a; b; x^2) \tag{40}$$

$$F_2(x^2) = x^{2\alpha} M(1 + a - b; 2 - b; x^2) \tag{41}$$

where $a = \frac{m^2 - \lambda^2}{4}$ and $b = 1 - \alpha$, with $\alpha = \kappa - l$ ($\notin \mathbb{Z}$—see footnote 2). We will consider the cases $l < [\kappa]$ and $l > [\kappa]$ separately.

(i) $l < [\kappa]$

For $l < [\kappa]$ $(\alpha = \kappa - l > 1)$, only $F_2(x^2)$ leads to functions

$$\phi_\lambda = \mathrm{e}^{-x^2/2} x^\alpha M\left(\frac{m^2-\lambda^2}{4} + \alpha; 1+\alpha; x^2\right) \tag{42}$$

$$\chi_\lambda = \left[\frac{2\mathrm{i}}{(m+\lambda)}\right] \mathrm{e}^{-x^2/2} x^{-1+\alpha}\left[\alpha M\left(\frac{m^2-\lambda^2}{4} + \alpha; 1+\alpha; x^2\right) + \frac{m^2-\lambda^2+4\alpha}{4(1+\alpha)} x^2 M\left(\frac{m^2-\lambda^2}{4} + \alpha + 1; 2+\alpha; x^2\right)\right] \tag{43}$$

which are in $L_2((0, \delta > 0), 2\pi x\,\mathrm{d}x)$. Moreover, the condition $\phi_\lambda, \chi_\lambda \in L_2((\delta, \infty), 2\pi x\,\mathrm{d}x)$ requires that $M(1+a-b; 2-b; x^2)$ reduces to a polynomial, which occurs only when

$$1 + a - b = \kappa - l + \frac{m^2-\lambda^2}{4} = -n \tag{44}$$

with $n = 0, 1, 2, \ldots$. So, the eigenvalues are given by

$$\lambda = \pm 2\sqrt{m^2/4 + \kappa + N} N = -l, -l+1, -l+2, \ldots. \tag{45}$$

Notice that both the eigenfunctions and eigenvalues depend on the singular flux $\kappa$.

(ii) $l > [\kappa]$

For $l > [\kappa]$ $(\alpha = \kappa - l < 0)$, only $F_1(x^2)$ leads to functions

$$\phi_\lambda = \mathrm{e}^{-x^2/2} x^{-\alpha} M\left(\frac{m^2-\lambda^2}{4}; 1-\alpha; x^2\right) \tag{46}$$

$$\chi_\lambda = \left[\frac{\mathrm{i}(m-\lambda)}{2(1-\alpha)}\right] \mathrm{e}^{-x^2/2} x^{1-\alpha} M\left(\frac{m^2-\lambda^2}{4} + 1; 2-\alpha; x^2\right) \tag{47}$$

which are in $L_2((0, \delta > 0), 2\pi x\,\mathrm{d}x)$. Once again, the condition $\phi_\lambda, \chi_\lambda \in L_2((\delta, \infty), 2\pi x\,\mathrm{d}x)$ requires that $M(a; b; x^2)$ reduces to a polynomial, which now occurs when

$$a = \frac{m^2-\lambda^2}{4} = -n \tag{48}$$

with $n = 0, 1, 2, \ldots$. This time, the eigenvalues are given by

$$\lambda = \pm 2\sqrt{m^2/4 + N} \qquad N = 0, 1, 2, \ldots. \tag{49}$$

In the present case the eigenfunctions do depend on the singular flux, but the eigenvalues are independent of $\kappa$.

Finally, notice that in both cases ($l < [\kappa]$ and $l > [\kappa]$) the eigenfunctions obtained vanish at the origin, thus belonging to the domains $\mathcal{D}(\overline{H_l})$ of the corresponding operator.

## Appendix A. Self-adjoint extensions of unbounded operators

In this appendix we briefly review the theory of deficiency indices of von Neumann (for an extended presentation of the subject, see [1]). We first recall the definition of the adjoint of a given linear operator.

Let $A$ be a linear operator defined on a dense subspace $\mathcal{D}(A)$ of a Hilbert space $H$. The domain of definition of the adjoint operator $A^\dagger$, $\mathcal{D}(A^\dagger)$, is the set of vectors $\psi \in H$ making the inner product $(\psi, A\phi)$ continuous in $\phi \in \mathcal{D}(A)$. By virtue of the Riesz–Fischer theorem, for any such $\psi$ there exists a unique vector $\chi \in H$ satisfying $(\psi, A\phi) = (\chi, \phi)$, $\forall \phi \in \mathcal{D}(A)$. One defines $A^\dagger\psi \equiv \chi$.

A linear operator $A$ is symmetric if

$$(\phi_1, A\,\phi_2) = (A\,\phi_1, \phi_2) \qquad \forall \phi_1, \phi_2 \in \mathcal{D}(A). \tag{A.1}$$

A linear operator $A$ is self-adjoint if it coincides with its adjoint $A^\dagger$, i.e. if $\mathcal{D}(A^\dagger) = \mathcal{D}(A)$ and

$$A^\dagger\phi = A\phi \qquad \forall \phi \in \mathcal{D}(A). \tag{A.2}$$

To establish the conditions a closed[3] symmetric operator must satisfy to be self-adjoint, a few definitions are in order. Let $\mathcal{K}_\pm = \mathrm{Ker}(A^\dagger \mp \mathrm{i})$ be the characteristic subspaces of $A^\dagger$ corresponding to the $\pm\mathrm{i}$ eigenvalues respectively. The *deficiency indices* of the operator $A$, $n_\pm$, are defined as the dimensions of the subspaces $\mathcal{K}_\pm$.

It is worth recalling that a closed symmetric operator is self-adjoint if and only if its deficiency indices are zero [1]. However, if the deficiency indices are not zero but equal the operator admits a family of SAEs whose construction can be carried out by means of the following theorem [1]: *let A be a closed symmetric operator whose deficiency indices $n_\pm$ are equal; then it admits a family of SAEs which are in a one-to-one correspondence with the unitary maps from $\mathcal{K}_+$ onto $\mathcal{K}_-$.*

In fact, let $\mathcal{U}$ be such an isometry, then the corresponding SAE $A_\mathcal{U}$ has domain $\mathcal{D}(A_\mathcal{U}) = \{\psi : \psi = \phi + \phi_+ + \mathcal{U}(\phi_+)\}$, where $\phi \in \mathcal{D}(A)$, and $\phi_+ \in \mathcal{K}_+$. The action of the extension $A_\mathcal{U}$ is given by

$$A_\mathcal{U}(\phi + \phi_+ + \mathcal{U}(\phi_+)) = A(\phi) + \mathrm{i}\phi_+ - \mathrm{i}\mathcal{U}(\phi_+). \tag{A.3}$$

This provides a method for constructing the SAE of closed symmetric operators with equal deficiency indices by identifying each possible unitary map from $\mathcal{K}_+$ onto $\mathcal{K}_-$.

## Appendix B. Closure of $H_L$

In this appendix we will study the closure $\overline{H_l}$ of the operator in equation (6),

$$H_l = \begin{pmatrix} m & \mathrm{i}\left(\frac{\mathrm{d}}{\mathrm{d}x} + \frac{1-\alpha}{x} - x\right) \\ \mathrm{i}\left(\frac{\mathrm{d}}{\mathrm{d}x} + \frac{\alpha}{x} + x\right) & -m \end{pmatrix} \tag{B.1}$$

defined on $\mathcal{D}(H_l) = C_0^\infty(\mathbb{R}^+)$, a dense subspace of $L_2(\mathbb{R}^+, 2\pi x\,\mathrm{d}x)$. It will be shown that the functions in the domain of definition of $\overline{H_l}$ are continuous near the origin, and vanishing for $x \to 0^+$.

In order to obtain $\mathcal{D}(\overline{H_l})$ we must add to the domain of $H_l$ the limit points of the Cauchy sequences in $\mathcal{D}(H_l)$ whose images by $H_l$ are also Cauchy sequences.

So, let us consider a Cauchy sequence $\{\psi_n\}_{n\in\mathbb{N}}$ with $\psi_n \in \mathcal{D}(H_l)$, $\forall n \in \mathbb{N}$, and such that $\{H_l\psi_n\}_{n\in\mathbb{N}}$ is also a Cauchy sequence. Therefore, given $\varepsilon > 0$,

$$\|\psi_n - \psi_m\|^2 < \varepsilon \tag{B.2}$$

$$\|H_l(\psi_n - \psi_m)\|^2 < \varepsilon \tag{B.3}$$

for $n, m$ sufficiently large. Making use of equation (B.1), it is easily seen that

$$\|H_l(\psi_n - \psi_m)\|^2 = \int_0^\infty \left(|\phi'|^2 + p_{(x)}|\phi|^2 + |\chi'|^2 + q_{(x)}|\chi|^2\right) 2\pi x\,\mathrm{d}x \tag{B.4}$$

[3] Recall that an operator is closed if its graph is a closed subset of $H \times H$. Every symmetric operator defined on a dense set is closable, i.e. has a closed symmetric extension.

where we have denoted by $\phi$ and $\chi$ respectively the upper and lower component of $(\psi_n - \psi_m)$, while the functions $p(x)$, $q(x)$ are given by

$$p(x) = \left[\left(\frac{\alpha}{x} + x\right)^2 + m^2 - 2\right] \tag{B.5}$$

$$q(x) = \left[\left(\frac{1-\alpha}{x} + x\right)^2 + m^2 + 2\right] \tag{B.6}$$

and are $\mathrm{O}(x^{-2})$ for $x \to 0^+$ (since we are taking $\alpha \notin \mathbb{Z}$—see footnote 2). It is not hard to see that both $p(x)$ and $q(x)$ are positive in the interval $[0, \delta]$ for some positive $\delta$. Only $p(x)$ can change its sign in an interval $(x_1, x_2)$ (depending on $\alpha$ and $m$), with $0 < \delta < x_1 < x_2 < \infty$. Notice that the integrand of equation (B.4) (obtained through an integration by parts) could take negative values only in $(x_1, x_2)$, as a consequence of the term $p(x)|\phi(x)|^2$.

Moreover, for $\delta$ small enough, we can choose $K > 0$ such that $p(x) > K/x^2$. Taking into account equations (B.2) and (B.4), for a given $\varepsilon > 0$, we can write

$$\int_0^\delta |\phi'|^2 x \,\mathrm{d}x < \varepsilon \qquad \int_0^\delta \frac{|\phi|^2}{x} \,\mathrm{d}x < \varepsilon \tag{B.7}$$

and

$$\int_0^\delta |\chi'|^2 x \,\mathrm{d}x < \varepsilon \qquad \int_0^\delta \frac{|\chi|^2}{x} \,\mathrm{d}x < \varepsilon \tag{B.8}$$

if $n, m$ are large enough. Therefore,

$$\left\{\sqrt{x}\,\psi_n'(x)\right\} \qquad \text{and} \qquad \left\{\psi_n(x)/\sqrt{x}\right\} \tag{B.9}$$

are Cauchy sequences in $L_2(0, \delta)$ (with respect to the usual Lebesgue measure), as well as the sum

$$\left\{\sqrt{x}\,\psi_n'(x) + \psi_n(x)/(2\sqrt{x})\right\} = \left\{\left[\sqrt{x}\,\psi_n(x)\right]'\right\}. \tag{B.10}$$

Let us call $\Phi(x) = \lim_{n\to\infty}\left[\sqrt{x}\,\psi_n(x)\right]' \in L_2(0, \delta)$, and denote its primitive by

$$\sqrt{x}\,\Psi(x) \equiv \int_0^x \Phi(y)\,\mathrm{d}y \tag{B.11}$$

which is an absolutely continuous function [1] in $(0, \delta)$. In particular, $\Psi(x)$ is continuous in $(0, \delta)$.

On the basis of

$$\begin{aligned} |\sqrt{x}(\Psi(x) - \psi_n(x))| &= \left|\int_0^x \left[\Phi(y) - \left(\sqrt{y}\,\psi_n(y)\right)'\right] \mathrm{d}y\right| \\ &\leqslant \sqrt{\int_0^\delta \left|\Phi(y) - \left(\sqrt{y}\psi_n(y)\right)'\right|^2 \mathrm{d}y}\sqrt{\int_0^\delta 1\,\mathrm{d}y} \longrightarrow 0 \qquad \text{for} \quad n \to \infty \end{aligned} \tag{B.12}$$

we conclude that the sequence $\{\sqrt{x}\,\psi_n(x)\}$ converges uniformly to $\sqrt{x}\,\Psi(x)$ in $(0, \delta)$, and consequently also in the metric of $L_2(0, \delta)$,

$$\lim_{n\to\infty}\left\{\sqrt{x}\,\psi_n(x)\right\} = \sqrt{x}\,\Psi(x). \tag{B.13}$$

(Notice that $\Psi(x)$ is the limit of $\{\psi_n(x)\}$ in $L_2[(0, \delta), x\,\mathrm{d}x]$.)

In addition, it is straightforward to show that

$$\frac{\Psi(x)}{\sqrt{x}} = \lim_{n\to\infty}\left\{\frac{\psi_n(x)}{\sqrt{x}}\right\}. \tag{B.14}$$

Then, we conclude from equations (B.11) and (B.14) that

$$\lim_{n\to\infty}\left\{\sqrt{x}\psi_n'(x)\right\} = \sqrt{x}\Psi'(x) \tag{B.15}$$

in the metric of $L_2(0, \delta)$.

Therefore, we can write

$$\int_0^\delta |\Psi'|^2 x\,\mathrm{d}x < \infty \qquad \int_0^\delta \frac{|\Psi|^2}{x}\,\mathrm{d}x < \infty. \tag{B.16}$$

This implies that $\Psi'(x)\cdot\Psi(x) = 1/2\,(\Psi(x)\cdot\Psi(x))' \in L_1(0, \delta)$.

On the other hand, the components of $\Psi(x)$, $\Psi_\alpha(x)$ with $\alpha = 1, 2$, are absolutely continuous functions in $(\epsilon, \delta)$, for $\epsilon < \delta$, by virtue of equation (B.11). In consequence

$$\int_\epsilon^\delta \left(\Psi_\alpha^2(x)\right)'\,\mathrm{d}x = \Psi_\alpha^2(\delta) - \Psi_\alpha^2(\epsilon). \tag{B.17}$$

In this expression we can take the $\lim_{\epsilon\to 0^+}$, proving that the continuous function $\Psi_\alpha^2(x)$ has a well defined limit for $x \to 0^+$. Moreover, on account of equation (B.16), this limit must be zero.

As a consequence of the previous results, we conclude that the behaviour near the origin of the functions in $\mathcal{D}(\overline{H_{[\kappa]}^{\gamma}})$ is dominated by the functions in $\mathcal{K}_\pm$ (see equations (23) and (24)).

On the other hand, since the restriction of the Hamiltonian to the subspaces with $l \neq [\kappa]$ is, as already mentioned, essentially self-adjoint, the behaviour of the functions at the origin is dictated by its closure, therefore being continuous and satisfying the boundary condition

$$\lim_{x\to 0^+}\psi(x) = 0. \tag{B.18}$$

## Acknowledgments

The authors thank M A Muschietti and E M Santangelo for useful discussions and comments. HF thanks P Giacconi and R Soldati for calling his attention to this problem. This work was partially supported by ANPCyT (PICT'97 no 00039) and CONICET (PIP no 0459) of Argentina.

## References

[1] Reed M and Simon B (ed) 1980 *Methods of Modern Mathematical Physics* vols 1, 2 (New York: Academic)
[2] Aharonov Y and Bohm D 1959 *Phys. Rev.* **115** 485
[3] de Sousa Gerbert Ph 1989 *Phys. Rev.* D **40** 1346
[4] Adami R and Teta A 1998 *Lett. Math. Phys.* **43** 43
[5] Sitenko Yu A 1999 *Phys. At. Nucl.* **62** 1084
[6] Sitenko Yu A 2000 *Ann. Phys., NY* **282** 167
[7] Hagen C R 1990 *Phys. Rev. Lett.* **64** 503
[8] Alford M G, March-Russell J and Wilczek F 1989 *Nucl. Phys.* B **328** 140
[9] Thienel H P 2000 *Ann. Phys.* **280** 140–62
(Thienel H P 1998 *Preprint* quant-ph/9809047)
[10] Cavalcanti R M 2000 Comment on quantum mechanics of an electron in a homogeneous magnetic field and a singular magnetic flux tube *Preprint* quant-ph/0003148
[11] Polychronakos A P 1986 *Nucl. Phys.* B **278** 207
Polychronakos A P 1987 *Nucl. Phys.* B **283** 268
[12] Beneventano C G, De Francia M and Santangelo E M 1999 *Int. J. Mod. Phys.* A **14** 4749
[13] Bentosela F, Cavalcanti R M, Exner P and Zagrebnov V A 1999 *J. Phys. A: Math. Gen.* **32** 3029
[14] Bordag M and Voropaev S 1993 *J. Phys. A: Math. Gen.* **26** 7637
[15] Cavalcanti R M and de Carvalho C A A 1998 *J. Phys. A: Math. Gen.* **31** 7061

[16] Cavalcanti R M, Fraga E S and de Carvalho C A A 1997 *Phys. Rev.* B **56** 9243

[17] Moroz A *Phys. Rev.* A **53** 669

[18] Flekkøy E G and Leinaas J M 1991 *Int. J. Mod. Phys.* A **6** 5327

[19] Desbois J, Ouvry S and Teixer C 1997 *Nucl. Phys.* B **500** 486

[20] Giacconi P and Soldati R 2000 *J. Phys. A: Math. Gen.* **33** 5193

[21] Franz M and Tešanović Z 2000 *Phys. Rev. Lett.* **84** 554

[22] Vafek O, Melikyan A, Franz M and Tešanović Z 2000 Quasiparticles and vortices in unconventional superconductors *Preprint* cond-mat/0007296 (*Phys. Rev.* B at press)

[23] Abramowitz M and Stegun I (ed) 1970 *Handbook of Mathematical Functions* (New York: Dover)