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FREE THREE-VALUED CLOSURE ŁUKASIEWICZ ALGEBRAS

A b s t r a c t. In this paper, the structure of finitely generated free objects in the variety of three-valued closure Łukasiewicz algebras is determined. We describe their indecomposable factors and we give their cardinality.

1. Introduction and Preliminaries

A *Łukasiewicz algebra of order* n, or an n*-valued Łukasiewicz algebra*, is an algebra $\langle L, \wedge, \vee, \sim, \varphi_1, \varphi_2, \ldots, \varphi_{n-1}, 0, 1\rangle$, n integer, $n \geq 2$, of type $(2, 2, 1, 1, 1, \ldots, 1, 0, 0)$, where $\langle L, \wedge, \vee, \sim, 0, 1\rangle$ is a De Morgan algebra, and $\varphi_1, \varphi_2, \ldots, \varphi_{n-1}$ are lattice homomorphisms satisfying: $\varphi_i x \vee \sim \varphi_i x = 1$,

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$\varphi_i\varphi_j x = \varphi_j x$, $\varphi_i \sim x = \sim \varphi_{n-i}x$, $\varphi_1 x \leq \varphi_2 x \leq \ldots \leq \varphi_{n-1}x$, $x \leq \varphi_{n-1}x$, $x \wedge \sim \varphi_i x \wedge \varphi_{i+1} y \leq y$ for all $i < n-1$. Sometimes we will refer to these algebras simply as Łukasiewicz algebras, if there is not risk of confusion.

The notion of Łukasiewicz algebra of order n was introduced by Gr. C. Moisil, and was developed and investigated further by several authors. Three- and four-valued Łukasiewicz algebras are an algebraic counterpart of Łukasiewicz logics. However, this is not so in the general case. This is the reason why many authors use the name "Moisil algebras" instead of "Łukasiewicz algebras", or, at least, "Łukasiewicz-Moisil algebras".

We assume that the reader is familiar with the theory of n-valued Łukasiewicz algebras. For the basic properties, the reader is referred to [4], [7] and [8].

The class of Łukasiewicz algebras of order n form a variety which we will denote $\mathcal{L}_n$. For $L \in \mathcal{L}_n$, we denote $B(L)$ the Boolean algebra of all complemented elements in L. It is known that $x \in B(L)$ if and only if $\varphi_i x = x$, for every i. Since for every $i = 1, \ldots, n-1$, $\varphi_i(L) = \{x \in L : \varphi_i x = x\}$, it follows that $B(L) = \varphi_i(L)$, for every i. It is also known that a Boolean algebra is a Łukasiewicz algebra of order n if we define $\sim x$ as the boolean complement of x and $\varphi_i x = x$ for all i.

Closure Łukasiewicz algebras have been studied in [3] and [7]. A *closure Łukasiewicz algebra of order* n is an algebra $\langle L, C\rangle$, where L is a Łukasiewicz algebra of order n and C is a unary operator defined on L fulfilling the following properties:

(C1) $C0 = 0$,

(C2) $Cx \vee x = Cx$,

(C3) $C(x \vee y) = Cx \vee Cy$,

(C4) $CCx = Cx$,

(C5) $C\varphi_i x = \varphi_i Cx$, $1 \leq i \leq n-1$.

The equational class of closure Łukasiewicz algebras of order n will be denoted by $\mathcal{CL}_n$.

An important subvariety of $\mathcal{CL}_n$ is the variety $\mathcal{ML}_n$ of monadic Łukasiewicz algebras [1, 7, 12], characterized within $\mathcal{CL}_n$ by the equation $C(x \wedge Cy) = Cx \wedge Cy$. Another important subvariety of $\mathcal{CL}_n$ is the variety $\mathcal{C}$ of closure Boolean algebras [2, 6, 9]. $\mathcal{C}$ consists of those algebras A in $\mathcal{CL}_n$ that satisfy that for every element $x \in A$, $\sim x$ is the Boolean complement of x.

With the operators C and $\sim$ we can define a new unary operator Q (an interior operator) by $Qx = \sim C \sim x$, for $x \in L$. This operator satisfies the following dual conditions: (Q1) $Q1 = 1$, (Q2) $Qx \wedge x = Qx$, (Q3) $Q(x \wedge y) = Qx \wedge Qy$, (Q4) $QQx = Qx$, (Q5) $Q\varphi_i x = \varphi_i Qx$, $1 \leq i \leq n-1$.

Closure Łukasiewicz algebras can be defined by means of equations (Q1) to (Q5), and in that case, by defining $Cx = \sim Q \sim x$ we obtain the closure operator satisfying equations (C1) to (C5).

The set of *open elements* of L is $Q(L) = \{x \in L : Qx = x\}$, and the set of *closed elements* of L is $C(L) = \{x \in L : Cx = x\}$. $Q(L)$ and $C(L)$ are anti-isomorphic sublattices of L such that $\varphi_i(Q(L)) \subseteq Q(L)$ and $\varphi_i(C(L)) \subseteq C(L)$, $i = 1, \ldots, n-1$. Observe that $x \in Q(L)$ if and only if $\sim x \in C(L)$.

In the closure Boolean algebra $\langle B(L), C\rangle$, the set of open elements is $Q(B(L)) = Q(L) \cap B(L) = \{x \in L : Q\varphi_i x = x\}$.

It is known that the set of open elements of a closure Boolean algebra, in this case $Q(B(L))$, is a Heyting algebra if we define

$$x \mapsto y = Q(\sim x \vee y),$$

for every $x, y \in Q(B(L))$. On the other hand, in any Łukasiewicz algebra L, we can define the implication

$$x \Rightarrow y = \bigwedge_{j=1}^{n-1} (\sim \varphi_j x \vee \varphi_j y) \vee y.$$

With this operation L becomes a Heyting algebra [10].

Lemma 1.1 [3] *For $L \in \mathcal{CL}_n$, the $(0,1)-$sublattice $Q(L)$ is a Heyting algebra if we define the open implication*

$$x \hookrightarrow y = Q(x \Rightarrow y),$$

for $x, y \in Q(L)$.

Let $\mathcal{F}(L)$ denote the set of all filters of an algebra L. A filter $F \in \mathcal{F}(L)$, is a *Stone filter*, if for each $x \in F$ there exists an element $b \in F \cap B(L)$ such that $b \leq x$. Cignoli proved [8] that for Łukasiewicz algebras, the notion of Stone filter is equivalent to that of filter satisfying the property $x \in F$ implies $\varphi_1 x \in F$. We define an *open Stone filter* as a Stone filter F such that $Qx \in F$, whenever $x \in F$.

If $G \subseteq B(L)$ is a filter in $B(L)$ that satisfies the condition $Q(G) \subseteq G$, we say that G is an *open filter* of $B(L)$.

Let $\mathcal{F}_{\varphi_1 Q}(L)$, $\mathcal{F}_Q(B(L))$ and $\mathcal{F}(Q(B(L)))$ respectively denote the lattices of open Stone filters of L, open filters of $B(L)$ and filters of $Q(B(L))$. It is not difficult to see that $\mathcal{F}_{\varphi_1 Q}(L)$ and $\mathcal{F}(Q(B(L)))$ are isomorphic. So, if $Con(L)$ denotes the lattice of congruences of an algebra L, we have:

Theorem 1.2 *Let* $L \in \mathcal{CL}_n$. *Then* $Con(L) \simeq \mathcal{F}_{\varphi_1 Q}(L) \simeq \mathcal{F}_Q(B(L)) \simeq \mathcal{F}(Q(B(L))) \simeq Con(Q(B(L)))$.

In particular, the variety $\mathcal{CL}_n$ is congruence-distributive and has the congruence extension property.

It is known [12] that a closure three-valued Łukasiewicz algebra $\langle L, C\rangle$ is a monadic Łukasiewicz algebra if and only if $\langle B(L), C\rangle$ is a monadic Boolean algebra. This result also holds in the $n-$valued case.

Theorem 1.3 *If* $L \in \mathcal{CL}_n$, *for all* $x, y \in L$ *the following conditions are equivalent:*

(i) $C(x \wedge C\varphi_i y) = Cx \wedge C\varphi_i y$, *for all* $i = 1, \ldots, n-1$.

(ii) $C(x \wedge Cy) = Cx \wedge Cy$.

(iii) $C(L)$ *is a Łukasiewicz subalgebra of* L.

(iv) $C \sim Cx = \sim Cx$.

Proof. (i) $\Rightarrow$ (ii) $\varphi_i(C(x \wedge Cy)) = C(\varphi_i(x \wedge Cy)) = C(\varphi_i x \wedge \varphi_i Cy) = C(\varphi_i x \wedge C\varphi_i y)$, for every $i = 1, \ldots, n-1$. By (i), $C(\varphi_i x \wedge C\varphi_i y) = C\varphi_i x \wedge$

$C\varphi_i y$. Since $C\varphi_i x \wedge C\varphi_i y = \varphi_i Cx \wedge \varphi_i Cy = \varphi_i(Cx \wedge Cy)$, it follows that, $\varphi_i(C(x \wedge Cy)) = \varphi_i(Cx \wedge Cy)$, for $i = 1, \dots, n-1$, so $C(x \wedge Cy) = Cx \wedge Cy$.

(ii) $\Rightarrow$ (iii) By (ii), C is a quantifier, so $\langle L, C\rangle \in \mathcal{ML}_n$ and consequently, $C(L)$ is a Łukasiewicz subalgebra of L.

(iii) $\Rightarrow$ (iv) By (iii), $Cx \in C(L)$ implies $\sim Cx \in C(L)$, so $C \sim Cx = \sim Cx$.

(iv) $\Rightarrow$ (i) $x \leq Cx$ and $y \leq Cy$ imply $x \wedge y \leq Cx \wedge Cy$, thus, $C(x \wedge y) \leq C(Cx \wedge Cy) = Cx \wedge Cy$. Hence, for all $i = 1, \dots, n-1$, $C(x \wedge C\varphi_i y) \leq Cx \wedge CC\varphi_i y = Cx \wedge C\varphi_i y$. For every $i = 1, \dots, n-1$, $x = x \wedge (C\varphi_i y \vee \sim C\varphi_i y) = (x \wedge C\varphi_i y) \vee (x \wedge \sim C\varphi_i y) \leq (x \wedge C\varphi_i y) \vee \sim C\varphi_i y$. Then, $Cx \leq C(x \wedge C\varphi_i y) \vee C \sim C\varphi_i y$, and taking into account (iv), $Cx \leq C(x \wedge C\varphi_i y) \vee \sim C\varphi_i y$. Hence $Cx \wedge C\varphi_i y \leq [C(x \wedge C\varphi_i y) \vee \sim C\varphi_i y] \wedge C\varphi_i y = C(x \wedge C\varphi_i y) \wedge C\varphi_i y \leq C(x \wedge C\varphi_i y)$. □

Suppose that $\langle L, C\rangle \in \mathcal{CL}_n$, and $\langle B(L), C\rangle$ is a monadic Boolean algebra. If $x \in L$, for each $i = 1, \dots, n-1$, $\varphi_i C \sim Cx = C\varphi_i \sim Cx = C \sim \varphi_{n-i} Cx = C \sim C\varphi_{n-i} x = \sim C\varphi_{n-i} x = \sim \varphi_{n-i} Cx = \varphi_i \sim Cx$. Hence, $C \sim Cx = \sim Cx$, so $\langle L, C\rangle \in \mathcal{ML}_n$. Consequently, we have:

Corollary 1.4 *An algebra $\langle L, C\rangle \in \mathcal{CL}_n$, belongs to $\mathcal{ML}_n$ if and only if $\langle B(L), C\rangle$ is a monadic Boolean algebra.*

The following theorems follow immediately from Theorem 1.2.

Theorem 1.5 *An algebra $L \in \mathcal{CL}_n$ is subdirectly irreducible if and only if the Heyting algebra $\langle Q(B(L)), \hookrightarrow\rangle$ is subdirectly irreducible, that is, $Q(B(L)) \simeq A \oplus 1$, for some A Heyting algebra.*

Theorem 1.6 *An algebra $L \in \mathcal{CL}_n$ is indecomposable if and only if $Q(B(L))$ is indecomposable as a Heyting algebra.*

In addition, from Corollary 1.4 we obtain

Theorem 1.7 *The simple objects of the variety $\mathcal{CL}_n$ are the simple monadic Łukasiewicz algebras of order n.*

In what follows, we prove some properties of the subvariety of $\mathcal{CL}_n$ of those closure Łukasiewicz algebras in which the Heyting algebra of open elements $\langle Q(L), \hookrightarrow\rangle$ is a three-valued Heyting algebra. Recall that a three-valued Heyting algebra is a Heyting algebra $\langle A, \rightarrow\rangle$ such that $((x \rightarrow z) \rightarrow y) \rightarrow (((y \rightarrow x) \rightarrow y) \rightarrow y) = 1$, for every $x, y, z \in A$ [11].

The following characterization of the ordered set of prime filters of an algebra in the variety of three-valued Heyting algebras is known.

Theorem 1.8 ([11]). *Let A be a Heyting algebra. Then the following are equivalent:*
(a) A is a three-valued Heyting algebra.
(b) Every prime filter of A is either maximal or minimal, and every prime filter is contained in at most one maximal prime filter.

In the case of closure Boolean algebras, a similar investigation was carried out for the subvariety $\mathcal{C}_T$ of those closure Boolean algebras such that the set of open elements form a three-valued Heyting algebra [9].

Let $L \in \mathcal{CL}_n$ such that $Q(L)$ is a three-valued Heyting algebra. It is proved in [3] that if L is a simple algebra, then it is a simple algebra in $\mathcal{ML}_3$, and if L is a non-simple subdirectly irreducible algebra, then $L \in \mathcal{C}_T$. So, if $L \in \mathcal{CL}_n$ is such that $Q(L)$ is a three-valued Heyting algebra, $L \in \mathcal{CL}_3$. We denote this subvariety by $\mathcal{C}_T\mathcal{L}_3$ and we have that for $L \in \mathcal{CL}_n$, $L \in \mathcal{C}_T\mathcal{L}_3$ if and only if for every $x, y, z \in L$ the following identity holds

$$((Qx \hookrightarrow Qz) \hookrightarrow Qy) \hookrightarrow (((Qy \hookrightarrow Qx) \hookrightarrow Qy) \hookrightarrow Qy) = 1.$$

The following theorem follows immediately from Theorem 1.6 and Theorem 1.8

Corollary 1.9 *The finite indecomposable algebras in $\mathcal{C}_T\mathcal{L}_3$ are the algebras $\langle L, Q\rangle$, where $Q(B(L)) = 0 \oplus B$, for a finite Boolean algebra B.*

Recall that L is called a *centered three-valued Łukasiewicz algebra*, or a *three-valued Post algebra*, if it has a *center*, that is, an element c of L such that $\sim c = c$. The center of L (if it exists) is unique. An *axis* of a three-valued Łukasiewicz algebra is an element e of L such that $\varphi_1 e = 0$ and $\varphi_2 x \leq \varphi_1 x \vee \varphi_2 e$, for all x of L. If the axis of L exists, it is unique. The axis and the center of an algebra $L \in \mathcal{C}_T\mathcal{L}_3$ belong to $C(L)$ (see [3]).

Let $\mathbf{2}$ be the Boolean algebra $\{0, 1\}$ and let $\mathbf{3}$ be the centered Łukasiewicz algebra $\{0, \frac{1}{2}, 1\}$. Let $\mathbf{B}_k$ be the simple monadic Boolean algebra with

k atoms, and let $\mathbf{T}_k = \langle \mathbf{3}^k, C\rangle$ where $C(\mathbf{3}^k) = \{0, c, 1\}$, c the center of $\mathbf{3}^k$ (see [12]).

Lemma 1.10 *Every finite subdirectly irreducible algebra in $\mathcal{ML}_3$ is simple. The finite simple algebras of the variety $\mathcal{ML}_3$ are the algebras $\mathbf{B}_k$, $k \geq 1$ and the algebras $\mathbf{T}_k$, $k \geq 1$.*

Let $\mathbf{B}_{k,l}$ be the closure Boolean algebra with $k + l$ atoms such that $Q(\mathbf{B}_{k,l}) = \{0, a, 1\}$ and there are k atoms preceding a and l atoms preceding $\sim a$, $k \geq 1$, $l \geq 1$.

Lemma 1.11 [2, 9] *The finite simple algebras in the variety $\mathcal{C}_T$ are the algebras $\mathbf{B}_k$ and the finite non-simple subdirectly irreducible algebras in $\mathcal{C}_T$ are the algebras $\mathbf{B}_{k,l}$.*

Then we have the following theorem.

Theorem 1.12 *The finite subdirectly irreducible algebras in $\mathcal{C}_T\mathcal{L}_3$ are the algebras $\mathbf{B}_k$, $\mathbf{T}_k$ and $\mathbf{B}_{k,l}$.*

Lemma 1.13 *If $L = \langle \mathbf{3}^m, Q\rangle$ is an algebra of $\mathcal{C}_T\mathcal{L}_3$, then L is a three-valued monadic Post algebra.*

Proof. Indeed, if $L \notin \mathcal{ML}_3$, by Corollary 1.4, $\langle B(L), Q\rangle$ is not a monadic Boolean algebra, that is $\{0, 1\} \subset Q(B(L)) \subset B(L)$. Let $N = \{b \in Q(B(L)) :\sim b \notin Q(L)\}$ and consider a maximal element m in N. Observe that:

1) $\sim (m \vee Q \sim m) \notin Q(L)$, as $\sim m \notin Q(L)$ and $\sim m =\sim m \wedge (Q \sim m \vee \sim Q \sim m) = Q \sim m \vee \sim (m \vee Q \sim m)$.
2) $Q \sim m = 0$. Indeed, if we suppose $0 < Q \sim m <\sim m$, then $m < m \vee Q \sim m < 1$ and $\sim (m \vee Q \sim m) \notin Q(L)$, contradicting the maximality of m.

Let c be the center of L. We know that $c \in Q(L)$. Consider the element $a = c \vee m \in Q(L)$. Then $((c \hookrightarrow 0) \hookrightarrow a) \hookrightarrow (((a \hookrightarrow c) \hookrightarrow a) \hookrightarrow a) = (0 \hookrightarrow a) \hookrightarrow ((Q(\sim m \vee c) \hookrightarrow a) \hookrightarrow a) = 1 \hookrightarrow a = a < 1$, and consequently $Q(L)$ is not a three-valued Heyting algebra. □

The following result gives the structure of any finite algebra in $\mathcal{C}_T\mathcal{L}_3$. It is crucial in the determination of the n-generated free algebra of the variety.

Theorem 1.14 *If $L \in \mathcal{C}_T\mathcal{L}_3$ is finite, then L is a direct product of a three-valued closure Boolean algebra and a three-valued monadic Post algebra.*

Proof. We know that if $B(L)$ has j atoms, then $L \simeq_{\mathcal{L}_3} \mathbf{2}^n \times \mathbf{3}^m$ ($\simeq_{\mathcal{L}_3}$ means isomorphism as Łukasiewicz algebras), where $n + m = j$. If c is the center of $\mathbf{3}^m$, then $(0, c)$ is the axis of L, thus $\sim (0, c) = (1, c) \in Q(L)$. In addition, $Q(1, 0) = Q\varphi_1(1, c) = \varphi_1 Q(1, c) = (1, 0)$, that is, $(1, 0)$ is an open of L. Let us see that $(0, 1)$ is also an open of L. If $Q(0, 1) = (0, 0)$, taking $a = (1, 0)$ and $b = (1, c)$ we have that $((a \hookrightarrow 0) \hookrightarrow b) \hookrightarrow (((b \hookrightarrow a) \hookrightarrow b) \hookrightarrow b) < 1$. So $Q(0, 1) > (0, 0)$. Suppose that $Q(0, 1) = (0, b) < (0, 1)$. If we take $a = (0, \sim b)$, then $Qa \leq Q(0, 1) \wedge a = 0$. If $\alpha = (1, 0) \vee Q(0, 1)$ and $\beta = (1, c) \vee \alpha$, we get $((\alpha \hookrightarrow 0) \hookrightarrow \beta) \hookrightarrow (((\beta \hookrightarrow \alpha) \hookrightarrow \beta) \hookrightarrow \beta) = (Qa \hookrightarrow \beta) \hookrightarrow ((\alpha \hookrightarrow \beta) \hookrightarrow \beta) = \beta < 1$, which implies $L \notin \mathcal{C}_T\mathcal{L}_3$. So $Q(0, 1) = (0, 1)$.

Thus the filters $F_1 = [(1, 0)), F_2 = [(0, 1)) \in \mathcal{F}_{\varphi_1 Q}(L)$, $\theta_1 = \theta(F_1)$ and $\theta_2 = \theta(F_2)$ is a pair of factor congruences, L/θ_1 is a three-valued closure Boolean algebra and, by the Lemma 1.13, L/θ_2 is a three-valued monadic Post algebra. □

A variety $\mathcal{V}$ has the *Fraser-Horn Property* if there are no skew congruences on any direct product of a finite number of algebras in $\mathcal{V}$; that is, for all $A_1, A_2 \in \mathcal{V}$, every $\theta \in Con(A_1 \times A_2)$ is a product congruence $\theta_1 \times \theta_2$, $\theta_i \in Con(A_i), i = 1, 2$. Every congruence-distributive variety has the Fraser-Horn Property. In particular, the variety $\mathcal{C}_T\mathcal{L}_3$ has the Fraser-Horn Property.

If the congruence lattice of an algebra L has a unique coatom, then L is directly indecomposable. A variety $\mathcal{V}$ has the *Apple Property* if the converse holds as well for all finite algebras; that is, if the finite directly indescomposable algebras in $\mathcal{V}$ are precisely the finite algebras whose congruence lattices have a unique coatom. If L is a finite directly indecomposable algebra in $\mathcal{C}_T\mathcal{L}_3$, then, from Corollary 1.9, $Q(B(L)) = 0 \oplus B$, where B is a finite Boolean algebra. So $\mathcal{F}(Q(B(L)))$ has a unique coatom and thus $Con(Q(B(L)))$, and consequently $Con(L)$, have a unique coatom. Hence the variety $\mathcal{C}_T\mathcal{L}_3$ has the Apple Property.

The Fraser-Horn and Apple Properties, extensively studied in [5], will play an important role in the determination of the n-generated free algebra in the variety $\mathcal{C}_T\mathcal{L}_3$.

2. Finitely generated free algebras

The aim of this section is to explicitly give the structure of $\mathbf{F}(G)=\mathbf{F}_{\mathcal{C}_T\mathcal{L}_3}(G)$ – the free algebra over a finite set G in the variety $\mathcal{C}_T\mathcal{L}_3$.

Since $\mathcal{C}_T\mathcal{L}_3$ is a locally finite variety (see [3]), then the algebra $\mathbf{F}(G)$ is finite, and consequently, every meet-irreducible open Stone filter M_p of $\mathbf{F}(G)$ is generated by a join-irreducible open element p of $B(\mathbf{F}(G))$.

If $\mathcal{V}$ is a variety, the variety $\mathcal{V}_0$ generated by the finite simple algebras in $\mathcal{V}$ is the *prime variety* associated with $\mathcal{V}$.

In [5], Berman and Blok showed that if $\mathcal{V}$ is a locally finite variety with the Fraser-Horn and Apple Properties, and, in addition, it has the property that every subalgebra of a finite simple algebra is a product of simple algebras, then the number of directly indecomposable factors of $\mathbf{F}_{\mathcal{V}_0}(G)$ equals that of $\mathbf{F}_{\mathcal{V}}(G)$. They also proved that if a given finite simple algebra L is a direct factor of the free algebra in $\mathcal{V}_0$, there exists a directly indecomposable factor of $\mathbf{F}_{\mathcal{V}}(G)$ having L as homomorphic image. These results can be applied to the variety $\mathcal{C}_T\mathcal{L}_3$, as this variety has the Fraser-Horn and Apple Properties, and, additionally, every subalgebra of a finite simple algebra is simple.

The prime variety $(\mathcal{C}_T\mathcal{L}_3)_0$ is the variety $\mathcal{ML}_3$ of monadic three-valued Łukasiewicz algebras. It is known ([12]) that the free monadic three-valued Łukasiewicz algebra $\mathbf{F}_{\mathcal{ML}_3}(G)$ is given by

$$\mathbf{F}_{\mathcal{ML}_3}(G) \cong \prod_{j=1}^{|\mathbf{2}^G|} \mathbf{B}_j^{\binom{|\mathbf{2}^G|}{j}} \times \prod_{k=1}^{|\mathbf{3}^G|} \mathbf{T}_k^{\binom{|\mathbf{3}^G|}{k}-\binom{|\mathbf{2}^G|}{k}},$$

where $\binom{|\mathbf{2}^G|}{k} = 0$ if $k > |\mathbf{2}^G|$.

So, from [5], the algebra $\mathbf{F}(G)$ has a factorization as

$$\mathbf{F}(G) \cong \prod_{j=1}^{|\mathbf{2}^G|} \mathbf{A}_j^{\binom{|\mathbf{2}^G|}{j}} \times \prod_{k=1}^{|\mathbf{3}^G|} \mathbf{P}_k^{\binom{|\mathbf{3}^G|}{k}-\binom{|\mathbf{2}^G|}{k}},$$

where each $\mathbf{A}_j$ and each $\mathbf{P}_k$ has as homomorphic image a factor of the free monadic three-valued Łukasiewicz algebra $\mathbf{F}_{\mathcal{ML}_3}(G)$.

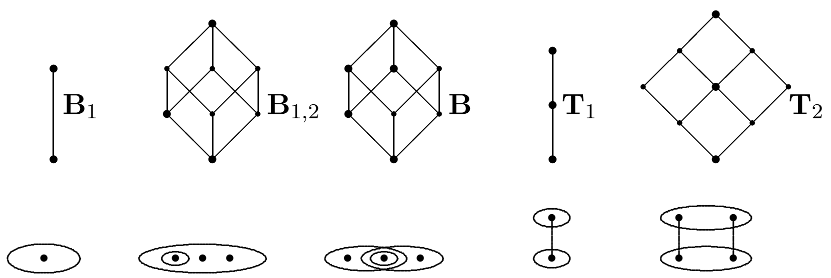
We will now determine the structure of each directly indecomposable factor of $\mathbf{F}(G)$.

For a given finite algebra $L \in \mathcal{C}_T\mathcal{L}_3$, let $\mathcal{J}(Q(L))$ and $\mathcal{J}(Q(B(L)))$ be the set of join-irreducible elements of $Q(L)$ and $Q(B(L))$, respectively. Observe that $\mathcal{J}(Q(B(L))) \subseteq \mathcal{J}(Q(L))$. Indeed, if $b \in \mathcal{J}(Q(B(L)))$ is such that $b = c \vee d$, $c, d \in Q(L)$, then $b = \varphi_1 c \vee \varphi_1 d$, and then $\varphi_1 c = c = b$ or $\varphi_1 d = d = b$, so $b \in \mathcal{J}(Q(L))$. Consider the following sets, where $min(X)$ ($max(X)$) denotes the set of minimal (maximal non minimal) elements of a poset X:

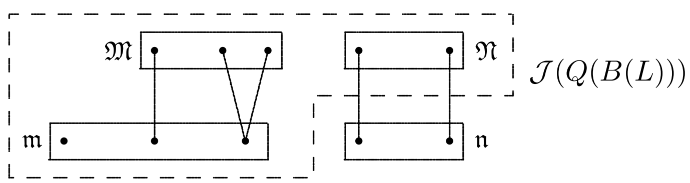
$$\mathfrak{m} = min(\mathcal{J}(Q(L))) \cap B(L), \quad \mathfrak{M} = max(\mathcal{J}(Q(B(L)))),$$

$$\mathfrak{n} = min(\mathcal{J}(Q(L))) \setminus \mathfrak{m}, \quad \text{and} \quad \mathfrak{N} = max(\mathcal{J}(Q(L))) \setminus \mathfrak{M}.$$

As an example, let L be the product $\mathbf{B}_1 \times \mathbf{B}_{1,2} \times \mathbf{B} \times \mathbf{T}_1 \times \mathbf{T}_2$, where the factor algebras are listed in the following figure. The open elements are highlighted and the corresponding dual spaces are given.



Then we have the following situation on $\mathcal{J}(Q(L))$:



In the case of the algebra $\mathbf{F}(G)$ we have $\mathcal{J}(Q(\mathbf{F}(G))) = \sum_{p \in \mathfrak{m} \cup \mathfrak{n}} C_p$, where

$C_p = \{q \in \mathcal{J}(Q(\mathbf{F}(G))) : q \geq p\}$, and $\mathcal{J}(Q(B(\mathbf{F}(G)))) = \sum_{p \in \mathfrak{m}} C_p \cup \mathfrak{N}$. So

$$Q(\mathbf{F}(G)) \cong_{\mathcal{H}} \prod_{p \in \mathfrak{m} \cup \mathfrak{n}} D_p,$$

where D_p is the distributive lattice such that $\mathcal{J}(D_p) \cong C_p$. Observe that if $p \in \mathfrak{n}$, then $D_p \cong \mathbf{3}$. Thus if $p \in \mathfrak{m} \cup \mathfrak{n}$ the elements $p^* = \bigvee_{q \in C_p} q \in Q(B(\mathbf{F}(G)))$ are complemented, the complement coincides with the complement in $B(\mathbf{F}(G))$ and is given by

$$-p^* = \bigvee_{q \in \mathcal{J}(Q(\mathbf{F}(G))) \setminus C_p} q.$$

In particular, $-p^* = \sim p^*$ is open.

We establish the following simple but useful lemma. Let $At(L)$ denote the set of atoms of an algebra L.

Lemma 2.1 *If $x \in At(\mathbf{F}(G))$, then there exists $p \in \mathcal{J}(Q(\mathbf{F}(G)))$ such that $x \leq p$.*

Proof. Let $p \in \mathfrak{m} \cup \mathfrak{n}$. If $x \leq q$ for some $q \in C_p$, then the lemma holds. Suppose that $x \not\leq q$, for every $q \in C_p$. In particular, $x \not\leq p^*$. Then $x \leq \sim p^* = \bigvee_{q \in \mathcal{J}(Q(\mathbf{F}(G))) \setminus C_p} q$. Since x is an atom it follows that $x \leq q$ for some $q \in \mathcal{J}(Q(\mathbf{F}(G))) \setminus C_p$. □

The above lemma shows that the set $P = \{At(p^*)\}_{p \in \mathfrak{m} \cup \mathfrak{n}}$, where $At(p^*) = \{x \in At(\mathbf{F}(G)) : x \leq p^*\}$, is a partition of the set $At(\mathbf{F}(G))$.

Let F_x and I_x respectively denote the principal filter and principal ideal generated by x. Observe that $I_x \in \mathcal{C}_T\mathcal{L}_3$ for $x \in Q(B(\mathbf{F}(G)))$. Then we have the following theorem.

Theorem 2.2

$$\begin{aligned} \mathbf{F}(G) &\cong_{\mathcal{C}_T\mathcal{L}_3} \prod_{p \in \mathfrak{m} \cup \mathfrak{n}} \mathbf{F}(G)/F_{p^*} \cong_{\mathcal{C}_T\mathcal{L}_3} \prod_{p \in \mathfrak{m} \cup \mathfrak{n}} I_{p^*} \\ &\cong_{\mathcal{C}_T\mathcal{L}_3} \prod_{p \in \mathfrak{m}} \mathbf{F}(G)/F_{p^*} \times \prod_{q \in \mathfrak{N}} \mathbf{F}(G)/F_q. \end{aligned}$$

As in [2] we can see that if p, $r \in \mathfrak{m}$ are such that $I_p \cong I_r \cong \mathbf{B}_k$, then $I_{p^*} \cong I_{r^*}$. It is not difficult to see that the algebras $I_{p_k^*}$, $1 \leq k \leq \left|\mathbf{2}^G\right|$, and $I_{q_k^*} = I_{q_k}$, $1 \leq k \leq \left|\mathbf{3}^G\right|$, are the directly indecomposable factors of $\mathbf{F}(G)$. Then

Theorem 2.3 $\mathbf{F}(G) \cong \prod_{k=1}^{\left|\mathbf{2}^G\right|} I_{p_k^*}^{\binom{\left|\mathbf{2}^G\right|}{k}} \times \prod_{k=1}^{\left|\mathbf{3}^G\right|} I_{q_k^*}^{\binom{\left|\mathbf{3}^G\right|}{k}-\binom{\left|\mathbf{2}^G\right|}{k}}$.

Our next objective is to determine the number of elements of $\mathbf{F}(G)$.

Let $p \in \mathcal{J}(Q(\mathbf{F}(G)))$. If $p \in \mathfrak{m}$, then $\mathbf{F}(G)/M_p \cong \mathbf{B}_k$ and thus, there exist k atoms preceding p. If $p \in \mathfrak{M}$, then $\mathbf{F}(G)/M_p \cong \mathbf{B}_{k,l}$. Thus there are $k+l$ atoms preceding p. In addition, k of these atoms precede the only element $q \in \mathfrak{m}$ such that $q \leq p$. If $p \in \mathfrak{N}$, then $\mathbf{F}(G)/M_p \cong \mathbf{T}_k$ and thus, there exist k atoms (not boolean elements) preceding p.

If we put $\mathfrak{m}_k = \{p \in \mathfrak{m} : \mathbf{F}(G)/M_p \cong \mathbf{B}_k\}$, $\mathfrak{M}_{k,l} = \{p \in \mathfrak{M} : \mathbf{F}(G)/M_p \cong \mathbf{B}_{k,l}\}$ and $\mathfrak{N}_k = \{p \in \mathfrak{N} : \mathbf{F}(G)/M_p \cong \mathbf{T}_k\}$, then the number of atoms of the free algebra is ([2] and [12])

$$|At(\mathbf{F}(G))| = \sum_{1\leq k\leq|\mathbf{2}^G|} |\mathfrak{m}_k| k \;+ \sum_{1\leq k\leq|\mathbf{2}^G-1|,1\leq l\leq|\mathbf{2}^G|} |\mathfrak{M}_{k,l}| l + \sum_{1\leq k\leq|\mathbf{3}^G|} |\mathfrak{N}_k| k.$$

If we put $\binom{k}{l} = 0$, whenever $l > k$, $M = \left|\mathbf{2}^G\right|$, and $N = \left|\mathbf{3}^G\right|$, then ([2])

$$|\mathfrak{m}_k| = \binom{M}{k}, \quad 1 \leq k \leq M,$$

and

$$|\mathfrak{M}_{k,l}| = \binom{M}{k}\left(\binom{M}{l} - \binom{k}{l}\right), \quad 1 \leq k \leq M-1, \quad 1 \leq l \leq M.$$

Similarly ([12]),

$$|\mathfrak{N}_k| = \binom{N}{k} - \binom{M}{k}, \quad 1 \leq k \leq N.$$

The following theorem gives the cardinality of $\mathbf{F}(G)$.

Theorem 2.4 $|\mathbf{F}(G)| = 2^{M\cdot(2^{2M-1}-\ 3^{M-1})} \cdot 3^{N\cdot\ 2^{N-1}-M\cdot\ 2^{M-1}}$.

Now we determine the structure of $Q(\mathbf{F}(G))$. If $p_k \in \mathfrak{m}_k$, the closure algebra $I_{p_k^*}$ has

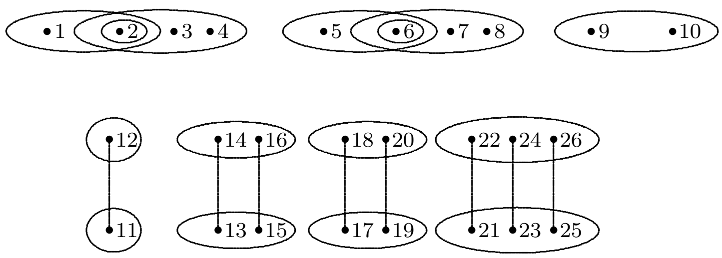
$$k + \sum_{1\le l\le M} l\left(\binom{M}{l} - \binom{k}{l}\right) = M2^{M-1} + k\left(1 - 2^{k-1}\right)$$

atoms. In addition, $Q\left(I_{p_k^*}\right) \cong 0 \oplus \mathbf{2}^{S_k}$, where $\mathbf{2}^{S_k}$ is the Boolean algebra with $S_k = \sum_{1\le l\le M}\left(\binom{M}{l} - \binom{k}{l}\right) = 2^M - 2^k$ atoms. If $q_k \in \mathfrak{N}_k$, then $I_{q_k} \cong T_k$ and $Q(I_{q_k}) \cong \mathbf{3}$.

Thus we conclude

Corollary 2.5 $Q(\mathbf{F}(G)) \cong \prod_{k=1}^{M}\left(0 \oplus \mathbf{2}^{S_k}\right)^{\binom{M}{k}} \times \prod_{k=1}^{N} \mathbf{3}^{\binom{N}{k}-\binom{M}{k}}$.

Example 2.6 Let $\mathbf{F}(1)$ be the free algebra with one generator. Then $\mathbf{F}(1) \cong A^2 \times B \times \mathbf{T}_1 \times \mathbf{T}_2^2 \times \mathbf{T}_3$ where A is the Boolean algebra with four atoms and $Q(A) \cong 0 \oplus \mathbf{2}^2$, and B is the Boolean algebra with two atoms such that $Q(B) \cong \mathbf{2}$.
The dual space X of $\mathbf{F}(1)$ looks like the following diagram:



$\mathbf{F}(1)$ is isomorphic to the family of decreasing subsets of its dual space X. If a is a decreasing subset of X, Qa is the greatest open decreasing subset contained in a, $\varphi_1 a$ is the greatest boolean decreasing subset contained in a, and so on. For example, if

$$\begin{aligned} g &= \{2,3,5,8,9,11,13,17,18,19,21,22,23\}, \text{then} \\ Qg &= \{2,11,17,19\}, \\ \varphi_1 g &= \{2,3,5,8,9,17,18,21,22\}, \\ \sim g &= \{1,4,6,7,10,11,13,15,16,19,23,25,26\} \text{ and} \\ Cg &= \{1,2,3,4,5,7,8,9,10,11,13,15,17,18,19,20,21,22,23,24,25,26\}. \end{aligned}$$

The element g is a generator of $\mathbf{F}(1)$ as the atoms of $\mathbf{F}(1)$ can be obtained from g in the following way:

- $\{1\} = \sim g \wedge C\varphi_1 Qg \wedge Q \sim (g \wedge \sim Qg)$,
- $\{2\} = \varphi_1 Qg$,
- $\{3\} = g \wedge C\varphi_1 Qg \wedge C \sim g$,
- $\{4\} = \sim g \wedge C\,(g \wedge C\varphi_1 Qg \wedge C \sim g)$,
- $\{5\} = g \wedge C\varphi_1 Q \sim g \wedge Q \sim (Cg \wedge \sim g)$,
- $\{6\} = \varphi_1 Q \sim g$,
- $\{7\} = \sim g \wedge C\varphi_1 Q \sim g \wedge Cg$,
- $\{8\} = g \wedge C\,(\sim g \wedge C\varphi_1 Q \sim g \wedge Cg)$,
- $\{9\} = g \wedge \varphi_1 Q\,(g \vee \sim g) \wedge QCg \wedge QC \sim g$,
- $\{10\} = \sim g \wedge \varphi_1 Q\,(g \vee \sim g) \wedge QCg \wedge QC \sim g$,
- $\{11\} = Q\,(g \wedge \sim g)$,
- $\{13\} = g \wedge \varphi_1 \sim Qg \wedge Q \sim g$,
- $\{15\} = Cg \wedge \varphi_1 \sim g \wedge Q \sim g$,
- $\{17\} = \sim Qg \wedge \varphi_1 g \wedge Qg$,
- $\{19\} = Qg \wedge \sim g \wedge \varphi_1 Cg$,
- $\{21\} = \varphi_1\,(g \wedge C \sim g) \wedge \sim Q\,(g \vee \sim g)$,
- $\{23\} = g \wedge \sim g \wedge \varphi_1\,(Cg \wedge C \sim g)$,
- $\{25\} = \varphi_1\,(\sim g \wedge Cg) \wedge \sim Q\,(g \vee \sim g)$.

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