# Muckenhoupt weights with singularities on closed lower dimensional sets in spaces of homogeneous type ☆

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A R T I C L E   I N F O

A B S T R A C T

We give sufficient conditions on a real number $\beta$ and on a closed set $F$ in a general space of homogeneous type $(X, d, \mu)$ in such a way that $\mu(B(x, d(x, F)))^{\beta}$ becomes a Muckenhoupt weight. In order to prove our result, we modify the underlying space so that it becomes 1-Ahlfors regular.

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## 1. Introduction

In some sense, Muckenhoupt weights are set to test the stability of harmonic analysis. The original reference for the theory is [14]. See also [10] or [16]. Muckenhoupt weights are the densities of the measures which preserve the boundedness properties of the basic operators of harmonic analysis: Hardy–Littlewood maximal and Calderón–Zygmund singular integral operators.

Muckenhoupt weights are also used extensively in the theory of partial differential equations, since they can be applied in different problems substituting the Lebesgue measure in the Euclidean space. Sobolev spaces without weights occur as spaces of solutions for elliptic and parabolic partial differential equations. For degenerate partial differential equations, i.e., equations with various types of singularities in the coefficients, it is natural to look for solutions in *weighted* Sobolev spaces. Weighted Sobolev spaces with Muckenhoupt weights are of particular interest in the study of the solutions of degenerate elliptic equations, since weighted imbedding theorems and Poincaré type inequalities hold (see [9]).

Muckenhoupt weights play also some important roles in more classical problems of partial differential equations. Such is the case of the Dirichlet boundary value problem

$$-\triangle u + u = f \quad \text{on } \Omega,$$
$$u = g \quad \text{in } F,$$

where $\Omega$ is a domain in $\mathbb{R}^n$ with boundary $F$. The behavior of the source $f$ near the boundary of $\Omega$ may cause non-solvability of this problem in a classical non-weighted Sobolev space. So we can ask whether this problem has a weak solution in a weighted Sobolev space, with an adequate weight function such that these difficulties might be avoided. For example, if $f$ has an unbounded growth near $F$, we should search for a weight which vanishes there. This is the case of the power-type weights, which are of the form $d^\beta(x, F)$, where $d(x, F)$ denotes the distance from the point $x$ to the set $F$. So the singularity ($\beta < 0$) or degenerations ($\beta > 0$) can appear on the boundary of $\Omega$ as well as in the interior of the domain.

From the above remarks, it seems natural to seek conditions on a set $F$ in such a way that $d^\beta(x, F)$ belongs to a Muckenhoupt class, for adequate values of $\beta$. For the Euclidean case, the results in [8] show that in a domain in $\mathbb{R}^n$ whose boundary has dimension $n-1$, if the domain is smooth enough then $d^\beta(x, \partial\Omega) \in A_p(\mathbb{R}^n)$ for $-1 < \beta < p - 1$. In [7] this result has been generalized to some $s$-dimensional compact sets $F$ in $\mathbb{R}^n$ with $0 \leqslant s < n$. They proved that $d^\beta(x, F) \in A_p(\mathbb{R}^n)$ for $-(n - s) < \beta < (n - s)(p - 1)$. In [1] the authors extend this result to a general metric measure space satisfying an Ahlfors condition. A fundamental tool for this extension is that adequate powers of the maximal Hardy–Littlewood operator belong to the $A_1$-Muckenhoupt class. This result was extended to a general space of homogeneous type in [1], following the lines given in [6,10,16] for the Euclidean case, with some technical modifications.

The basic theory of Muckenhoupt weights related to the boundedness of the Hardy–Littlewood maximal operator and singular integrals has been extended to some very general environments. In particular, to the setting of space of homogeneous type.

Ahlfors spaces are a particular case of non-atomic space of homogeneous type. Nevertheless, a measure can be doubling but not Ahlfors of any order. In this note we obtain a class of power-type weights in a general space of homogeneous type $(X, d, \mu)$. In particular the general setting allows atoms and coexistence of several dimensions. More precisely, we give sufficient conditions on a closed subset $F$ of $X$ in such a way that $\mu(B(x, d(x, F)))^\beta$ becomes a Muckenhoupt weight for adequate values of $\beta$.

## 2. Definitions and statement of the main result

A **quasi-metric** on a set $X$ is a non-negative symmetric function $d$ defined on $X \times X$ such that $d(x, y) = 0$ if and only if $x = y$, and there exists a constant $K \geqslant 1$ such that the inequality

$$d(x, y) \leqslant K\big(d(x, z) + d(z, y)\big)$$

holds for every $x, y, z \in X$. We will refer to $K$ as the triangle constant for $d$. As it is shown in [12], $d$ induces a metrizable topology on $X$.

A Borel measure $\mu$ defined on the $d$-balls $B(x, r) = \{y \in X \colon d(x, y) < r\}$ is said to be non-trivial if $0 < \mu(B(x, r)) < \infty$ for every $x \in X$ and every $r > 0$. A non-trivial measure $\mu$ is said to be **doubling** if for some constant $A \geqslant 1$ we have the inequality

$$\mu\big(B(x, 2r)\big) \leqslant A\mu\big(B(x, r)\big),$$

for every $x \in X$ and every $r > 0$. When $\mu$ is a doubling measure, $(X, d, \mu)$ is said to be a **space of homogeneous type** (see [5]).

We say that a point $x$ in a space of homogeneous type $(X, d, \mu)$ is an **atom** if $\mu(\{x\}) > 0$. When $\mu(\{x\}) = 0$ for every $x \in X$ we say that $(X, d, \mu)$ is a **non-atomic** space. Macías and Segovia proved in [12] that a point is an atom if and only if it is topologically isolated, and that the set of such points is at most countable.

One of the main interests regarding the structure of space of homogeneous type is due to the fact that several problems in harmonic analysis can be extended to those settings. In particular, the theory of Muckenhoupt weights and their relation with the Hardy–Littlewood maximal operator has been consider in the literature [14,4,3,2]. Let $(X, d, \mu)$ be a space of homogeneous type. For a given locally integrable function $f$, the **Hardy–Littlewood maximal operator** is given by

$$\mathcal{M}_\mu f(x) = \sup \frac{1}{\mu(B)} \int_B |f| \, d\mu,$$

where the supremum is taken over the family of the $d$-balls $B$ containing $x$.

The definition of the Hardy–Littlewood maximal operator can be extended to a non-negative Borel measure $\nu$ such that every ball has finite $\nu$-measure by

$$\mathcal{M}_\mu \nu(x) = \sup \frac{\nu(B)}{\mu(B)},$$

where the supremum is taken over the family of the $d$-balls $B$ containing $x$. Since $\mu$ is doubling, $\mathcal{M}_\mu \nu(x)$ is equivalent to its centered version, i.e.

$$\mathcal{M}_\mu \nu(x) = \sup_{r>0} \frac{\nu(B(x,r))}{\mu(B(x,r))}.$$

If $(X, d, \mu)$ is a space of homogeneous type and $1 < p < \infty$, the Muckenhoupt class $A_p(X, d, \mu)$ is defined as the set of all **weights** (non-trivial, non-negative, measurable and locally integrable functions) $w$ defined on $X$ for which there exists a constant $C$ such that the inequality

$$\left( \frac{1}{\mu(B)} \int_B w \, d\mu \right) \left( \frac{1}{\mu(B)} \int_B w^{-\frac{1}{p-1}} \, d\mu \right)^{p-1} \leqslant C$$

holds for every $d$-ball $B$ in $X$. For $p = 1$, we say that $w \in A_1(X, d, \mu)$ if there exists a constant $C$ such that

$$\frac{1}{\mu(B)} \int_B w \, d\mu \leqslant C w(x)$$

holds for every $d$-ball $B$ in $X$ and almost every $x \in B$. Set $A_\infty(X, d, \mu) = \bigcup_{p \geqslant 1} A_p(X, d, \mu)$. These functions $w$ are known with the name of **Muckenhoupt weights**. A basic property of Muckenhoupt weights is that $w \, d\mu$, with $w \in A_\infty(X, d, \mu)$, is also a doubling measure on $X$. Notice that if $w_0$ and $w_1$ belong to $A_1$, then $w := w_0 w_1^{1-p} \in A_p$. It is a classical result in the theory of Muckenhoupt weights that every weight in $A_p(X, d, \mu)$ can be factorized in this way (see [11]).

In this note we aim to produce weights with singularities on a closed set $F$, of the form

$$w(x) = \mu\big(B\big(x, d(x, F)\big)\big)^\beta,$$

under certain dimensional conditions on $F$, for some positive and negative values of $\beta$. Here $d(x, F) = \inf\{d(x, y) : y \in F\}$. We shall provide an interval about 0 for $\beta$, such that $w(x)$ is an $A_p$-Muckenhoupt weight.

We start by defining a particular type of $s$-dimensional set in a general space of homogeneous type. Later on we shall prove that this concept coincides with the $s$-Ahlfors condition with respect to the normalized quasi-distance defined by Macías and Segovia in [12].

To illustrate the definition, let us consider the one dimensional character of a line $L$ or any smooth curve in the plane. First notice that for any point $x \in L$, if the area of the disc $B = B(x,t)$ is less than $r$, then the length of $B \cap L$ is bounded above by a constant times $\sqrt{r}$. Of course the above condition is not sufficient for a set $L$ to be a one-dimensional. Indeed a singleton $L = \{x_0\}$ satisfies this property. To recover the idea of $L$ as an one dimensional set, we observe that for every $x \in L$ and for $r$ small enough, there exists a disc $B$ containing $x$ with area less than $r$ and with length of $B \cap L$ bounded below by $\sqrt{r}$.

We shall say that a closed subset $F$ of $X$ is **$s$-dimensional with respect to $\mu$**, $s \geqslant 0$, if there exist a Borel measure $\nu$ supported on $F$ and three constants $c_1, c_2, c_3 > 0$ such that for every $x \in F$ and every $0 < r < \operatorname{diam}(F)$ the following two conditions are satisfied;

(1) if $t$ is a positive number for which $\mu(B(x,t)) < r$, then $\nu(B(x,t)) \leqslant c_1 r^s$;
(2) there exists a $d$-ball $B$ containing $x$ with $\mu(B) < c_2 r$ and $\nu(B) \geqslant c_3 r^s$.

If $F$ is unbounded and the above conditions hold for every $0 < r < r_0$, where $r_0$ is a positive number less than $\operatorname{diam}(F)$, we say that $F$ is **locally $s$-dimensional with respect to $\mu$**.

**Remark 1.** From [15, Prop. 1.5] and Theorem 9 in this note, we can conclude that if $F$ is an $s$-dimensional set with respect to $\mu$ in a non-atomic space of homogeneous type $(X, d, \mu)$, then $\dim_H(F) = s$. Here $\dim_H$ denotes the Hausdorff dimension *relative to $\mu$*.

The main result in this note is contained in the next statement.

**Theorem 1.** *Let $(X, d, \mu)$ be a space of homogeneous type and let $F \subseteq X$ be $s$-dimensional with respect to $\mu$, with $0 \leqslant s < 1$. If no atom of $X$ belongs to $F$, then*

$$w(x) = \mu\big(B\big(x, d(x, F)\big)\big)^{\gamma(s-1)}$$

*belongs to $A_1(X, d, \mu)$ for every $0 \leqslant \gamma < 1$. Consequently $\mu(B(x, d(x, F)))^\beta \in A_p(X, d, \mu)$ for $-(1-s) < \beta < (1-s)(p-1)$ and $1 \leqslant p < \infty$.*

The paper is organized as follows. In Section 3 we associate to a given space of homogeneous type a 1-Ahlfors space. In order to achieve this, we shall use the normalization introduced by Macías and Segovia in [12] and an *ad hoc* procedure to substitute atoms by continua. We would like to observe that the normalization in [12] generally does not produce 1-Ahlfors spaces since it does not eliminate atoms. In this sense is that we distinguish among normal and 1-Ahlfors spaces. On the other hand, in order to compare the Muckenhoupt classes in the original space with the corresponding classes in the new normalized structure, for the sake of completeness we prove a result that is probably known on the equivalence of the Hardy–Littlewood maximal operators of each setting. In Section 4 we apply [1, Thm. 7] to the 1-Ahlfors space defined in the previous section to obtain an $A_1$-Muckenhoupt weight in $(X, d, \mu)$. The mentioned theorem is indeed a particular instance of Theorem 1 above, when $(X, d, \mu)$ is an $\alpha$-Ahlfors space. We complete the proof of Theorem 1 rewriting the obtained weight in terms of the original metric $d$. Section 5 contains some examples and particular cases.

## 3. A 1-Ahlfors space associated to a space of homogeneous type

Let $(X, d, \mu)$ be a given space of homogeneous type. The aim of this section is to associate to the given space a 1-Ahlfors space. We start by recalling some terminology. A quasi-metric measure space $(X, d, \mu)$ is said to be an **$\alpha$-normal space** if there exists a constant $c \geqslant 1$ such that

$$c^{-1} r^\alpha \leqslant \mu\big(B_d(x, r)\big) \leqslant c r^\alpha, \tag{3.1}$$

for every $x \in X$ and every $\mu(\{x\}) < r < \mu(X)$. We shall refer to $c$ as **the constant for the $\alpha$-normality** of $(X, d, \mu)$. We shall say that $(X, d, \mu)$ is **normal** when it is 1-normal.

A non-atomic $\alpha$-normal space is also known as **$\alpha$-Ahlfors space**. We will refer to the triangle constant $K$ and the constants $c$ and $\alpha$ in (3.1) as the **geometric constants** of the space. The most classical example of $n$-normal (and $n$-Ahlfors) space is the Euclidean space $(\mathbb{R}^n, |\cdot|, \lambda)$, where $|\cdot|$ denotes the usual distance and $\lambda$ the Lebesgue measure on $\mathbb{R}^n$.

We shall say that a closed subset $F$ of $X$ is **locally $s$-Ahlfors with measure $\nu$** in $(X, d)$ if $\nu$ is a Borel measure supported on $F$ such that (3.1) holds, with $\nu$ instead of $\mu$, for every $x \in F$ and every $0 < r < r_0$, for some positive $r_0$.

In [1, Prop. 1] it is proved that the concepts of $s$-Ahlfors and locally $s$-Ahlfors coincide when the set $F$ is bounded. More precisely, there is proved that if $F$ is a bounded subset of $X$ and is locally $s$-Ahlfors with measure $\nu$, then $(F, d, \nu)$ is an $s$-Ahlfors space.

It is easy to see that each $\alpha$-normal space is a space of homogeneous type with doubling constant which depends only on $c$ and $\alpha$. Nevertheless a measure can be doubling but not $\alpha$-normal for any $\alpha > 0$. The examples can even be obtained in the interval $[0, 1]$ for measures that are absolutely continuous with respect to Lebesgue measure. Indeed, $d\mu(x) = w(x)\,dx$ with $w(x) = x^{-1/2}$ is a doubling measure on the interval $[0, 1]$, but $\mu$ is not $\alpha$-normal for any $\alpha > 0$. This is a consequence of the fact that, for small $\varepsilon > 0$, $\int_0^\varepsilon w\,dx \simeq \sqrt{\varepsilon}$ while $\int_{1-\varepsilon}^1 w\,dx \simeq \varepsilon$. However, Macías and Segovia give in [12] an explicit construction of a quasi-metric $\delta$ on the space of homogeneous type $(X, d, \mu)$ in such a way that the new structure $(X, \delta, \mu)$ becomes a normal space, and the topologies induced on $X$ by $d$ and $\delta$ coincide. This quasi-metric is defined as

$$\delta(x, y) = \inf\{\mu(B)\colon B \text{ is a } d\text{-ball with } x, y \in B\}$$

if $x \neq y$, and $\delta(x, y) = 0$ if $x = y$. By the definition of $\delta$ we have that for every $x \in X$ and every $r > 0$, if $\mu(\{x\}) \geqslant r$ then $B_\delta(x, r) = \{x\}$, where $B_\delta(x, r) := \{y \in X\colon \delta(x, y) < r\}$. It will be also useful to notice that in the proof of the above mentioned result of Macías and Segovia it is proved that

$$B_\delta(x, r) = \bigcup\{B\colon B \text{ is a } d\text{-ball with } x \in B \text{ and } \mu(B) < r\},$$

for every $x \in X$ and every $r > \mu(\{x\})$. Throughout this paper $\delta$ shall denote this quasi-metric.

We shall prove that $A_1(X, \delta, \mu) = A_1(X, d, \mu)$. Actually, we have the following result.

**Proposition 2.** *The Hardy–Littlewood maximal operators on $(X, d, \mu)$ and $(X, \delta, \mu)$ are equivalent.*

To see this, it is enough to observe that if $d_1$ and $d_2$ are two quasi-metrics on $X$ satisfying that there exists a constant $C$ such that for every $d_1$-ball $B_1$ there exists a $d_2$-ball $B_2$ including $B_1$ with $\mu(B_2) \leqslant C\mu(B_1)$, then $M_{d_1}f(x) \leqslant CM_{d_2}f(x)$. Here we have used $M_{d_i}$ to denote the Hardy–Littlewood maximal operators on $(X, d_i, \mu)$. Then, the fact that $M_\delta$ and $M_d$ are equivalent operators is a consequence of the following lemma.

**Lemma 3.** *Let $(X, d, \mu)$ be a space of homogeneous type with triangular constant $K$ and doubling constant $A$. Set $\delta$ the quasi-metric defined by Macías and Segovia and let $c$ be the constant for the 1-normality of $(X, \delta, \mu)$. Then*

(1) *for every $d$-ball $B_d$ there exists a $\delta$-ball $B_\delta$ such that $B_d \subseteq B_\delta$ and $\mu(B_\delta) \leqslant 2c\mu(B_d)$;*

(2) *for every $\delta$-ball $B_\delta$ there exists a $d$-ball $B_d$ such that $B_\delta \subseteq B_d$ and $\mu(B_d) \leqslant cA^m\mu(B_\delta)$, with $m$ a positive integer satisfying $m \geqslant \log_2(5K^2)$.*

**Proof.** To prove (1), let us fix a $d$-ball $B_d$. Set $r^*$ to denote $\mu(B_d)$, and let $x_0$ be the center of $B_d$. Since $2r^* > \mu(\{x_0\})$ we have that

$$B_\delta(x_0, 2r^*) = \bigcup \{B \colon B \text{ is a } d\text{-ball with } x_0 \in B \text{ and } \mu(B) < 2r^*\},$$

and then $B_d \subseteq B_\delta(x_0, 2r^*)$. Since $(X, \delta, \mu)$ is normal, if $2r^* < \mu(X)$ we also have that

$$\mu\big(B_\delta(x_0, 2r^*)\big) \leqslant c2r^* = 2c\mu(B_d).$$

Otherwise, if $2r^* \geqslant \mu(X)$ then $X$ is bounded and $B_\delta := B_\delta(x_0, 2\operatorname{diam}_\delta(X))$ works, since $B_d \subseteq B_\delta = X$ and $\mu(B_\delta) = \mu(X) \leqslant 2r^* = 2\mu(B_d)$. Here $\operatorname{diam}_\delta(X)$ denotes the value $\sup\{\delta(x, y) \colon x, y \in X\}$.

In order to show (2), let us fix a $\delta$-ball $B_\delta(z, t)$. Notice first that if $t \leqslant \mu(\{z\})$, then $B_\delta(z, t) = \{z\}$. In other words, $z$ is an isolated point in $(X, \delta)$. Since the topologies induced by $d$ and $\delta$ on $X$ coincide, $z$ is an isolated point in $(X, d)$. Then there exists $r^* > 0$ such that $B(z, r^*) = \{z\}$ and (2) holds taking $B_d = B(z, r^*)$. For the case $t > \mu(\{z\})$, we claim that there exist $x_0 \in X$ and $r^* > 0$ such that $\mu(B(x_0, r^*)) < t$ and $B_\delta(z, t) \subseteq B(x_0, 5K^2 r^*)$. If this statement holds, we have that

$$\mu\big(B(x_0, 5K^2 r^*)\big) \leqslant A^m \mu\big(B(x_0, r^*)\big) \leqslant A^m t \leqslant cA^m \mu\big(B_\delta(z, t)\big).$$

To prove the claim, recall that

$$B_\delta(z, t) = \bigcup \{B \colon B \text{ is a } d\text{-ball with } z \in B \text{ and } \mu(B) < t\}.$$

Let $s = \sup\{r > 0 \colon B \text{ is a } d\text{-ball with radius } r, \ z \in B \text{ and } \mu(B) < t\}$. If $s < \infty$, there exist $\frac{s}{2} < r^* \leqslant s$ and a $d$-ball $B$ with ratio $r_0$ such that $z \in B$ and $\mu(B) < t$. Let $x_0$ be the center of $B$. Then $B_\delta(z, t) \subseteq B(x_0, 5K^2 r^*)$. In fact, for any $d$-ball $B(w, r)$ containing $z$ with measure less than $t$, we have that $r \leqslant s < 2r^*$. Therefore, if $y \in B(w, r)$, we have that

$$d(y, x_0) \leqslant K^2\big(d(y, w) + d(w, z) + d(z, x_0)\big) < K^2\big(2r + r^*\big) < 5K^2 r^*.$$

Finally, if $s = \infty$ then $\mu(X) < \infty$ so that $X$ is bounded. Then we fix any $x_0$ in $X$ and take $r^* = 2\operatorname{diam}(X)$.  $\square$

From [Proposition 2](#) we obtain that the $A_p$ classes are invariant under normalization. More precisely, we have the following result.

**Corollary 4.** $w \in A_p(X, d, \mu)$ if and only if $w \in A_p(X, \delta, \mu)$.

As we already mentioned, since the topologies induced by $d$ and $\delta$ on $X$ coincide, a point $x$ is an atom in $(X, d, \mu)$ if and only if it is an atom in $(X, \delta, \mu)$. In other words, $\delta$ does not remove atoms. We shall proceed to construct a new space $(Z, \rho, \bar{\mu})$ induced by $(X, d, \mu)$. For this purpose, let us consider the space $Y = X \times \mathbb{R}$ equipped with the quasi-metric defined by

$$\rho \colon Y \times Y \longrightarrow \mathbb{R}_0^+$$
$$\rho\big((x, t), (y, s)\big) = \max\big\{\delta(x, y), |t - s|\big\}.$$

Let $Z \subseteq Y$ be defined as

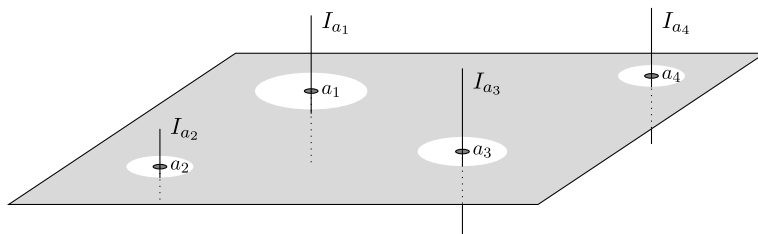$$Z = X_0 \cup \bigcup_{a \in \mathcal{A}} \big(\{a\} \times I_a\big),$$

**Fig. 1.** The set $Z$ with $\mathcal{A} = \{a_1, a_2, a_3, a_4\}$.

where $\mathcal{A}$ denotes the set of all atoms of $X$, $X_0 = \{(x, 0): \; x \in X \backslash \mathcal{A}\}$, and $I_a = (-\mu(\{a\})/2, \mu(\{a\})/2)$ (see Fig. 1). Given a Borel subset $E$ of $Z$, we define

$$\bar{\mu}(E) = \mu(E \cap X_0) + \sum_{a \in \mathcal{A}} \lambda\big((\{a\} \times I_a) \cap E\big),$$

where $\lambda$ denotes the Lebesgue measure on $\mathbb{R}$.

**Theorem 5.** *$(Z, \rho, \bar{\mu})$ is a non-atomic normal space. In other words, $(Z, \rho, \bar{\mu})$ is a 1-Ahlfors space.*

**Proof.** Fix $(x, s) \in Z$ and $r > 0$, and set $B_\rho((x, s), r) = \{(y, t) \in Z: \; \rho((x, s), (y, t)) < r\}$. Since

$$B_\rho\big((x, s), r\big) \cap X_0 = \big(B_\delta(x, r) \times (s - r, s + r)\big) \cap \big((X \backslash \mathcal{A}) \times \{0\}\big)$$

$$= \begin{cases} (B_\delta(x, r) \backslash \mathcal{A}) \times \{0\}, & |s| < r, \\ \emptyset, & |s| \geqslant r, \end{cases}$$

we have that

$$\bar{\mu}\big(B_\rho((x, s), r)\big) = \mu\big(B_\delta(x, r) \backslash \mathcal{A}\big) \mathcal{X}_{(-r, r)}(s) + \sum_{a \in \mathcal{A} \cap B_\delta(x, r)} \Lambda(r, a)$$

$$= \mu\big(B_\delta(x, r) \backslash \mathcal{A}\big) \mathcal{X}_{(-r, r)}(s) + \sum_{\substack{a \in \mathcal{A} \cap B_\delta(x, r): \\ r \leqslant \mu(\{a\})}} \Lambda(r, a) + \sum_{\substack{a \in \mathcal{A} \cap B_\delta(x, r): \\ r > \mu(\{a\})}} \Lambda(r, a),$$

where $\mathcal{X}_E$ denotes the characteristic function of $E$ and $\Lambda(r, a) = |(s - r, s + r) \cap I_a|$. We will first analyze the last term in the right hand side of above equality. Take $a \in \mathcal{A} \cap B_\delta(x, r)$ such that $r > \mu(\{a\})$. In this case we have that $I_a \subseteq (s - r, s + r)$ so that $|(s - r, s + r) \cap I_a| = \mu(\{a\})$. To see this inclusion, let $t \in I_a$, or in other words, $-\mu(\{a\})/2 < t < \mu(\{a\})/2$. If $x$ is not an atom in $X$, then $s = 0$ and clearly $t \in (-r, r)$, since $\mu(\{a\}) < r$. If $x$ is an atom, we have that $-\mu(\{x\})/2 < s < \mu(\{x\})/2$. Then if $\mu(\{x\}) \leqslant r$, it is clear that $-r < t - s < r$. So that $t \in (s - r, s + r)$. Otherwise, if $\mu(\{x\}) > r$, then $B_\delta(x, r) = \{x\}$ and therefore $a = x$. Hence $r < \mu(\{a\}) < r$, which is a contradiction. Then

$$\bar{\mu}\big(B_\rho((x, s), r)\big) = \mu\big(B_\delta(x, r) \backslash \mathcal{A}\big) \mathcal{X}_{(-r, r)}(s) + \sum_{\substack{a \in \mathcal{A} \cap B_\delta(x, r): \\ r \leqslant \mu(\{a\})}} \big|(s - r, s + r) \cap I_a\big|$$

$$+ \mu\big(\{a \in \mathcal{A} \cap B_\delta(x, r): \; r > \mu(\{a\})\}\big).$$

In order to analyze the second term, let

$$\mathcal{A}_r(x) := \big\{a \in \mathcal{A} \cap B_\delta(x, r): \; r \leqslant \mu(\{a\})\big\}.$$

If $a \in \mathcal{A}_r(x)$, then $x \in B_\delta(a, r) = \{a\}$ and consequently we have $a = x$. Hence $\mathcal{A}_r(x) = \emptyset$ if $x \notin \mathcal{A}$ or if $r > \mu(\{x\})$, and $\mathcal{A}_r(x) = \{x\} = B_\delta(x, r)$ if $x \in \mathcal{A}$ and $r \leqslant \mu(\{x\})$. Then, if we define

$$\mathcal{A}_r := \{a \in \mathcal{A} \colon r \leqslant \mu(\{a\})\},$$

we have that $\mathcal{A}_r(x) = \{x\} = B_\delta(x, r)$ if $x \in \mathcal{A}_r$, and $\mathcal{A}_r(x) = \emptyset$ otherwise, so that

$$\sum_{\substack{a \in \mathcal{A} \cap B_\delta(x, r): \\ r \leqslant \mu(\{a\})}} \left| (s - r, s + r) \cap I_a \right| = \begin{cases} \left| (s - r, s + r) \cap I_x \right|, & x \in \mathcal{A}_r, \\ 0, & x \notin \mathcal{A}_r. \end{cases}$$

Hence,

$$\bar{\mu}\big(B_\rho((x, s), r)\big) = \mu\big(B_\delta(x, r) \backslash \mathcal{A}\big) \mathcal{X}_{(-r, r)}(s) + \left| (s - r, s + r) \cap I_x \right| \mathcal{X}_{\mathcal{A}_r}(x)$$
$$+ \mu\big(\{a \in \mathcal{A} \cap B_\delta(x, r): r > \mu(\{a\})\}\big).$$

We shall now see that if $x \in \mathcal{A}_r$, then $|(s - r, s + r) \cap I_x| \simeq r$. Indeed, it is clear that $|(s - r, s + r) \cap I_x| \leqslant |(s - r, s + r)| = 2r$. On the other hand, notice that $(s - r, s + r) \cap I_x$ includes at least one of the intervals $(s - r/2, s)$ or $(s, s + r/2)$, both with length equal to $r/2$. This follows from the fact that $x \in \mathcal{A}_r$ implies $-\mu(\{x\})/2 < s < \mu(\{x\})/2$ and $r \leqslant \mu(\{x\})$, so it is enough consider the cases $s < 0$ and $s \geqslant 0$ to obtain the respective inclusions. So that

$$\left| (s - r, s + r) \cap I_x \right| \simeq r, \quad \text{if } x \in \mathcal{A}_r. \tag{3.2}$$

We are now in position to show that $\bar{\mu}(B_\rho((x, s), r)) \simeq r$. To do this, let us first consider the possibility $|s| < r$. In this case we have that

$$\mu\big(B_\delta(x, r) \backslash \mathcal{A}\big) \mathcal{X}_{(-r, r)}(s) + \mu\big(\{a \in \mathcal{A} \cap B_\delta(x, r): r > \mu(\{a\})\}\big)$$

is equal to $\mu(B_\delta(x, r) \backslash \mathcal{A}_r(x))$, and since $\mathcal{A}_r(x) = B_\delta(x, r)$ if $x \in \mathcal{A}_r$, and $\mathcal{A}_r(x) = \emptyset$ otherwise, we obtain

$$\bar{\mu}\big(B_\rho((x, s), r)\big) = \mu\big(B_\delta(x, r)\big) \mathcal{X}_{\mathcal{A}_r^c}(x) + \left| (s - r, s + r) \cap I_x \right| \mathcal{X}_{\mathcal{A}_r}(x).$$

From (3.2) and the fact that $(X, \delta, \mu)$ is normal we have that

$$\bar{\mu}\big(B_\rho((x, s), r)\big) \simeq r \mathcal{X}_{\mathcal{A}_r^c}(x) + r \mathcal{X}_{\mathcal{A}_r}(x).$$

Otherwise, if $|s| \geqslant r$ then $x$ is an atom and $|s| < \mu(\{x\})/2$, so that $r \leqslant \mu(\{x\})$ and $\mathcal{X}_{\mathcal{A}_r}(x) = 1$. Then

$$\bar{\mu}\big(B_\rho((x, s), r)\big) \simeq r + \mu\big(\{a \in \mathcal{A} \cap B_\delta(x, r): r > \mu(\{a\})\}\big).$$

But in this case $B_\delta(x, r) = \{x\}$, so that $\{a \in \mathcal{A} \cap B_\delta(x, r): r > \mu(\{a\})\} = \emptyset$. In brief,

$$\bar{\mu}\big(B_\rho((x, s), r)\big) \simeq \begin{cases} r \mathcal{X}_{\mathcal{A}_r^c}(x) + r \mathcal{X}_{\mathcal{A}_r}(x), & |s| < r, \\ r, & |s| \geqslant r. \end{cases}$$

We can conclude that $(Z, \rho, \bar{\mu})$ is a normal space, and since has not isolated points, it will be non-atomic. □

To every function $f : X \to \mathbb{R}$ we have a canonically associated function $\bar{f} : Z \to \mathbb{R}$ defined by $\bar{f}((x, 0)) = f(x)$ if $x \notin \mathcal{A}$, and $\bar{f}((x, t)) = f(x)$ for every $x \in \mathcal{A}$ and every $t \in I_x$. Hence the space $L^p(X, \mu)$ can be

identified with the subspace of those functions in $L^p(Z, \bar{\mu})$ which are constant on each $a \times I_a$, with $a \in \mathcal{A}$. An analogous point of view can be applied to the Muckenhoupt classes. Notice also that if $M^X$ and $M^Z$ denote the Hardy–Littlewood maximal operator of each setting, we have that $M^Z \bar{f}(x, t) = M^X f(x)$, for every $(x, t) \in Z$.

## 4. Proof of Theorem 1

Let $(X, d, \mu)$ be a given space of homogeneous type, and let $(Z, \rho, \bar{\mu})$ and $\mathcal{A}$ be as in previous section. Let $F$ be a given closed $s$-Ahlfors set in $(X, \delta)$ for some $0 \leqslant s < 1$. Then $F_0 := \{(y, 0): y \in F\}$ is an $s$-Ahlfors set in $(Z, \rho)$. Indeed, since $F$ is $s$-Ahlfors in $(X, \delta)$ then there is no point of $\mathcal{A}$ in $F$, so that $(B_\delta(y, r) \cap F)_0 = B_\rho((y, 0), r) \cap F_0$ for every $y \in F$ and every $r > 0$. Since $(Z, \rho, \bar{\mu})$ is a 1-Ahlfors space, if we define

$$w(z) := \rho(z, F_0)^{\gamma(s-1)} \tag{4.1}$$

on $Z$, then $w \in A_1(Z, \rho, \bar{\mu})$ for every $0 \leqslant \gamma < 1$ (see [1, Thm. 7]). The first purpose of this section is to prove that this implies $\delta(x, F)^{\gamma(s-1)} \in A_1(X, d, \mu)$ for every $0 \leqslant \gamma < 1$. Consequently $\delta(x, F)^\beta \in A_p(X, d, \mu)$ for $-(1-s) < \beta < (1-s)(p-1)$.

**Lemma 6.** *For every* $z = (x, t) \in Z$, *we have that* $\rho(z, F_0) = \delta(x, F)$. *Consequently,* $\rho((x, t), F_0) = \rho((x, 0), F_0)$ *for every* $(x, t) \in Z$.

**Proof.** Let us fix $z = (x, t) \in Z$. We shall prove first that $\rho(z, F_0) = \max\{\delta(x, F), |t|\}$. In fact, notice that

$$\rho(z, F_0) = \inf_{y \in F} \{\max\{\delta(x, y), |t|\}\}.$$

Then, if we prove that

$$\inf_{y \in F} \{\max\{f(y), C\}\} = \max\{\inf_{y \in F} f(y), C\},$$

with $f$ a function defined on $F$ and $C$ any constant, the result is obtained taken $f(y) = \delta(x, y)$ and $C = |t|$. In order to prove the inequality, let us denote by $L$ and $R$ the left and right hand side respectively. Assume first that $f(y) > C$ for every $y \in F$. Then $L = \inf_{y \in F} f(y) = R$. On the other hand, if there exists $y_0 \in F$ such that $f(y_0) \leqslant C$, then $L \leqslant C$ and $R = C$. Since also we always have that $L \geqslant C$, we obtain $L = C = R$.

Now, if $x \notin \mathcal{A}$, then $t = 0$ and hence $\max\{\delta(x, F), |t|\} = \delta(x, F)$. On the other hand, for any $x \notin F$ we have that for every $\varepsilon > 0$ there exists $y_\varepsilon \in F$ such that $\delta(x, y_\varepsilon) < \delta(x, F) + \varepsilon$. Also there exists a $d$-ball $B$ containing $x$ and $y_\varepsilon$ such that $\mu(B) < \delta(x, y_\varepsilon) + \varepsilon$. Then

$$\mu(\{x\}) \leqslant \mu(B) < \delta(x, F) + 2\varepsilon.$$

Making $\varepsilon$ tend to zero we obtain that $\mu(\{x\}) \leqslant \delta(x, F)$. Then, if $x \in \mathcal{A}$ we have that $|t| \leqslant \mu(\{x\})/2 \leqslant \delta(x, F)$, so that $\max\{\delta(x, F), |t|\} = \delta(x, F)$. $\quad\square$

**Lemma 7.** $\delta(x, F)^{\gamma(s-1)} \in A_1(X, \delta, \mu)$.

**Proof.** Fix a $\delta$-ball $B = B_\delta(x_0, r)$ and let $\tilde{B} = B_\rho((x_0, 0), r)$. If $w$ is the weight defined in (4.1), then we have that

$$\int_{\tilde{B}} w(x, t) \, d\bar{\mu} \leqslant C \bar{\mu}(\tilde{B}) w(x_0, 0),$$

for some constant $C$ which does not depend on $x_0$ or $r$. Then, since $\int_{\mathcal{A}} f(x)\, d\mu(x) = \sum_{a \in \mathcal{A}} f(a)\mu(\{a\})$ and $\int_{I_a} g(a,t)\, dt = g(a,0)\mu(\{a\})$ provided that $g(a,t) = g(a,0)$ for every $t \in I_a$, applying Lemma 6 we have that

$$
\begin{aligned}
\int_B \delta(x,F)^{\gamma(s-1)}\, d\mu(x) &= \int_{B \cap \mathcal{A}^c} \delta(x,F)^{\gamma(s-1)}\, d\mu(x) + \int_{B \cap \mathcal{A}} \delta(x,F)^{\gamma(s-1)}\, d\mu(x) \\
&= \int_{\tilde{B} \cap X_0} w(x,t)\, d\bar{\mu} + \sum_{\substack{a \in \mathcal{A} \\ \delta(a,x_0)<r}} \mu(\{a\})w(a,0) \\
&= \int_{\tilde{B} \cap X_0} w(x,t)\, d\bar{\mu} + \sum_{\substack{a \in \mathcal{A} \\ \delta(a,x_0)<r}} \int_{I_a} w(x,t)\, dt \\
&= \int_{\tilde{B}} w(x,t)\, d\bar{\mu} \\
&\leqslant C\bar{\mu}(\tilde{B})w(x_0,0) \\
&\leqslant \tilde{C}\mu\big(B_\delta(x_0,r)\big)\delta(x,F)^{\gamma(s-1)},
\end{aligned}
$$

where we have used the fact that $(Z,\rho,\bar{\mu})$ and $(X,\delta,\mu)$ are normal spaces. $\quad\square$

From Lemma 7 and Corollary 4, we have that $\delta(x,F)^{\gamma(s-1)} \in A_1(X,d,\mu)$. Then, Theorem 1 will be proved if we show the following two facts. First, that the function $\delta(x,F)$ defined for $x \in X$, is equivalent to the function $\mu(B(x,d(x,F)))$, and finally, that every $s$-dimensional set $F$ with respect to $\mu$ is $s$-Ahlfors in $(X,\delta)$. Moreover, we shall prove that this concepts are equivalent in Theorem 9.

**Lemma 8.** *Let $F$ be a closed subset of $X$. Then*

$$
A^{-1}\delta(x,F) \leqslant \mu\big(B\big(x,d(x,F)\big)\big) \leqslant A^\ell\delta(x,F),
$$

*where $A$ is the doubling constant for $\mu$ and $m$ is a positive integer satisfying $\ell \geqslant \log_2(3K^2)$, with $K$ the triangular constant for $d$.*

**Proof.** If $x \in F$, then $\delta(x,F) = \mu(B(x,d(x,F))) = 0$, so that we can assume $x \notin F$. Since $F$ is closed, there exists $\varepsilon$ such that $0 < \varepsilon < d(x,F)$. For this fixed $\varepsilon$, there exists $y_0 \in F$ such that $d(x,y_0) < d(x,F) + \varepsilon$. Then $d(x,y_0) < 2d(x,F)$. On the other hand, if $B$ is any $d$-ball containing $x$ and $y_0$, we have that $\delta(x,F) \leqslant \delta(x,y_0) \leqslant \mu(B)$. In particular, this inequality holds taking $B = B(x,2d(x,F))$. Hence

$$
\delta(x,F) \leqslant \mu\big(B\big(x,2d(x,F)\big)\big) \leqslant A\mu\big(B\big(x,d(x,F)\big)\big).
$$

To obtain the other inequality, take $x \notin F$ and $\varepsilon > 0$. Let us fix $y_0 \in F$ such that $\delta(x,y_0) < \delta(x,F) + \varepsilon$, and let $B$ be a $d$-ball containing $x$ and $y_0$ such that $\mu(B) < \delta(x,y_0) + \varepsilon$. Then

$$
\mu(B) < \delta(x,F) + 2\varepsilon. \tag{4.2}
$$

Let us use $x_0$ to denote the center of $B$ and $r_0$ to its radius. Then

$$
B\big(x,d(x,y_0)\big) \subseteq B\big(x_0,3K^2 r_0\big), \tag{4.3}
$$

where $K$ is the triangular constant for $d$. In fact, if $y \in B(x,d(x,y_0))$ then

$$
d(y,x) < d(x,y_0) \leqslant K\big(d(x,x_0) + d(x_0,y_0)\big) < 2Kr_0.
$$

Hence

$$d(y, x_0) \leqslant K\big(d(y,x) + d(x,x_0)\big) \leqslant 2K^2 r_0 + K r_0 \leqslant 3K^2 r_0.$$

From (4.3), (4.2) and the doubling condition for $\mu$ we obtain

$$\mu\big(B(x, d(x,F))\big) \leqslant \mu\big(B(x, d(x,y_0))\big) \leqslant A^\ell\big(\delta(x,F) + 2\varepsilon\big).$$

By letting $\varepsilon$ tend to zero we obtain the desired inequality.    □

**Theorem 9.** *Let $(X,d,\mu)$ be a space of homogeneous type and let $F$ be a non-atomic closed subset of $X$. Then $F$ is (locally) $s$-dimensional with respect to $\mu$ if and only if $F$ is (locally) $s$-Ahlfors in $(X,\delta)$.*

**Proof.** Suppose that $F$ is $s$-dimensional with respect to $\mu$ with associated measure $\nu$, and fix $x \in F$ and $0 < r < \operatorname{diam}(F)$. If $B$ is the $d$-ball given by (2) and $c_2 \leqslant 1$, we immediately have that

$$\nu\big(B_\delta(x,r)\big) \geqslant \nu(B) \geqslant c_3 r^s.$$

Otherwise, if $c_2 > 1$ take $B$ the $d$-ball given by (2) for $\tilde{r} = r/c_2$. Then $B$ contains $x$, $\mu(B) < r$ and $\nu(B) \geqslant c_3 r^s / c_2^s$. Hence

$$\nu\big(B_\delta(x,r)\big) \geqslant \nu(B) \geqslant c_3 r^s / c_2^s.$$

The case $\operatorname{diam}(F) \leqslant r < \nu(F)$ only can occur if $F$ is bounded, so that we apply [1, Prop. 1] to obtain the result.

On the other hand, notice that

$$B_\delta(x,r) \subseteq \bigcup_{t \in T} B(x,t),$$

where $T = \{t > 0 : \mu(B(x,t)) \leqslant A^m r\}$, where $m$ is an integer greater than or equal to $1 + \log_2 K$, $K$ denotes the triangular constant for $d$, and $A$ the doubling constant for $\mu$. In fact, if $y$ belongs to $B_\delta(x,r)$ then there exist $z \in X$ and $t > 0$ such that $x,y \in B(z,t)$ and $\mu(B(z,t)) < r$. So that $d(x,y) \leqslant 2Kt$ and $\mu(B(x,2Kt)) \leqslant A^m \mu(B(x,t)) < A^m r$.

Let $t^* = \sup T$. Then

$$\nu\big(B_\delta(x,r)\big) \leqslant \nu\bigg(\bigcup_{t \in T} B(x,t)\bigg) \leqslant \nu\big(B(x,t^*)\big).$$

Notice also that $\mu(B(x,t^*)) \leqslant A^{m+1} r$, so that from property (1) in definition of $s$-dimensional set with respect to $\mu$, we obtain

$$\nu\big(B(x,t^*)\big) \leqslant c_1\big(A^{m+1} r\big)^s = C r^s,$$

provided that $A^{m+1} r < \operatorname{diam}(F)$. This inequality always holds if $F$ is an unbounded set. If $F$ is bounded, we use [1, Prop. 1] to obtain that $F$ is $s$-Ahlfors in $(X,\delta)$.

In the case that $F$ is *locally* $s$-dimensional with respect to $\mu$, we take $r < \tilde{r}_0 := r_0 A^{-m-1}$.

For the converse, fix $x \in F$ and $r > 0$. By hypothesis there exist a Borel measure $\nu$ supported on $F$ and a constant $c \geqslant 1$ such that

$$c^{-1} r^s \leqslant \nu\big(B_\delta(x,r)\big) \leqslant c r^s,$$

provided that $r < \nu(F)$ ($r < r_0$ for the local case). We shall assume then $r < \nu(F)$ (resp. $r < r_0$). Assume that there exists $t > 0$ such that $\mu(B(x,t)) < r$. Then $B(x,t) \subseteq B_\delta(x,r)$ and

$$\nu\big(B(x,t)\big) \leqslant \nu\big(B_\delta(x,r)\big) \leqslant cr^s,$$

so that (1) holds taking $c_1 = c$. To see (2), let $x_0 \in X$ and $r^* > 0$ be such that $\mu(B(x_0,r^*)) < r$ and $B_\delta(x,r) \subseteq B(x_0, 5K^2 r^*)$ (see proof of Lemma 3). Set $B = B(x_0, 5K^2 r^*)$. Then $\mu(B) \leqslant A^m r$ and

$$\nu(B) \geqslant \nu\big(B_\delta(x,r)\big) \geqslant c^{-1} r^s,$$

where $A$ denotes the doubling constant for $\mu$, and $m$ is a positive integer such that $2^m \geqslant 5K^2$. Then (2) holds with $c_3 = c^{-1}$ and $c_2 = A^m$.

It only remains to consider the case $\nu(F) \leqslant r < \operatorname{diam}(F)$. In this case we have that $\nu(F) < \infty$, so that if we prove that also $\operatorname{diam}(F) < \infty$, we can use the argument in [1, Prop. 1] to obtain that $F$ is $s$-dimensional with respect to $\mu$. In order to prove that $\operatorname{diam}(F) < \infty$, notice first that a set $F$ is $d$-bounded if and only if is $\delta$-bounded. Indeed, from the definition of $\delta$ we have that $\operatorname{diam}(F) < \infty$ implies $\operatorname{diam}_\delta(F) < \infty$. Reciprocally, if $F \subseteq B_\delta$ for some $\delta$-ball, then $F$ is contained in the $d$-ball with finite radius constructed in the proof of (2) in Lemma 3. On the other hand, it is well known that a space of homogeneous type is bounded if and only if has finite measure. So that we can conclude that an $s$-Ahlfors set $F$ with measure $\boldsymbol{\nu}$ in $(X,d)$ is bounded with respect to $d$ or $\delta$ if and only if $\nu(F) < \infty$. This completes the proof. $\quad\square$

From the above result and [1, Prop. 1] we can conclude that the concepts of $s$-dimensional with respect to $\mu$ and locally $s$-dimensional with respect to $\mu$ coincide when the set $F$ is bounded.

Finally, we want to point out that when we deal with several sets which have different dimensions, with the additional hypothesis of boundedness we obtain the following result. The proof is analogous to that of Theorem 1, applying [1, Thm. 6] instead of [1, Thm. 7].

**Theorem 10.** *Let $(X, d, \mu)$ be a space of homogeneous type and let $\{F_1, \ldots, F_H\}$ be a family of pairwise disjoint bounded and non-atomic subsets of $X$, such that $F_i$ is locally $s_i$-dimensional with respect to $\mu$, with $0 \leqslant s_i < 1$ for $i = 1, 2, \ldots, H$. Then there exist open sets $U_1, \ldots, U_H$ pairwise disjoint with $U_i$ containing $F_i$ such that*

$$w(x) = \begin{cases} \mu(B(x, d(x, F_i)))^{\gamma(s_i - 1)}, & \text{for } x \in U_i; \\ 1, & \text{for } x \in \left(\bigcup_{i=1}^H U_i\right)^c \end{cases}$$

*belongs to $A_1(X, d, \mu)$ for every $0 \leqslant \gamma < 1$. Consequently,*

$$v(x) = \begin{cases} \mu(B(x, d(x, F_i)))^{\beta_i}, & \text{for } x \in U_i; \\ 1, & \text{for } x \in \left(\bigcup_{i=1}^H U_i\right)^c \end{cases}$$

*belongs to $A_p(X, d, \mu)$ for every $-(1 - s_i) < \beta_i < (1 - s_i)(p - 1)$ and every $1 \leqslant p < \infty$.*

## 5. Examples and particular cases

Let us start this section by showing that when $(X, d, \mu)$ is an $\alpha$-Ahlfors space and $F$ is an $s$-Ahlfors subset of $X$, for some $0 \leqslant s < \alpha$, then we recover the result in [1], which is actually a basic ingredient in the proof of Theorem 1. Indeed, as the next lemma shows, we have that $d(x, F)^{\gamma(s-\alpha)} \in A_1(X, d, \mu)$ for every $0 \leqslant \gamma < 1$.

**Lemma 11.** *Let $(X, d, \mu)$ be an $\alpha$-Ahlfors space. Then*

(1) *the quasi-metrics $\delta$ and $d^\alpha$ are equivalent;*
(2) *if $F$ is (locally) $s$-Ahlfors with measure $\nu$ in $(X, d)$, then $F$ is (locally) $s/\alpha$-Ahlfors with measure $\nu$ in $(X, \delta)$;*
(3) *if $F$ is $s$-Ahlfors in $(X, d)$ for some $0 \leqslant s < \alpha$, then $d(x, F)^{\gamma(s-\alpha)} \in A_1(X, d, \mu)$ for every $0 \leqslant \gamma < 1$.*

**Proof.** The proof of (1) is a straightforward calculation, (2) follows directly from (1), and (3) is a consequence of (2), Lemma 7, Corollary 4 and (1).  □

The next two examples deal with non-Ahlfors spaces.

**Example 1.** Let $X = \mathbb{R}^2$ equipped with the usual distance $d$, and with the measure $\mu$ defined by

$$\mu(E) = \int_E |y|^\beta \, dy,$$

for a fixed $\beta > -2$. Then $(X, d, \mu)$ is a space of homogeneous type since $|x|^\beta \in A_p(\mathbb{R}^2)$ for some $p > 1$, but is not an $\alpha$-Ahlfors space for any $\alpha$. Fix two real numbers $a$ and $b$ with $0 < a < b$ and let $F = \{(x, 0) : a \leqslant x \leqslant b\}$. It is known that $\mu(B(x, r)) \simeq |x|^\beta r^2$, for every $x = (x_1, 0)$ in $F$ and every $0 < r < a/2$.

Let us check that $F$ is $\frac{1}{2}$-Ahlfors in $(\mathbb{R}^2, \delta)$, with the one-dimensional Lebesgue measure $\lambda$ restricted to $F$. For this, observe that since $\delta(x, y) \simeq |x - y|^2 |x|^\beta$ whenever $x$ and $y$ are close to each other and away from the origin, we have that there exists $r_0 > 0$ such that

$$\lambda\big(F \cap B_\delta(x, r)\big) \simeq \lambda\big(F \cap B\big(x, r^{1/2}|x|^{\beta/2}\big)\big) \simeq r^{1/2},$$

for $0 < r < r_0$. Then, applying Theorem 1 we have that

$$w(x) = \mu\big(B\big(x, d(x, F)\big)\big)^\gamma \in A_p(X, d, \mu),$$

for $-\frac{1}{2} < \gamma < \frac{(p-1)}{2}$ and $1 \leqslant p < \infty$. Finally, notice that for $x$ in an enlargement of $F$ (i.e. $x$ such that $d(x, F) < a/2$) we have that

$$w(x) \simeq |x|^{\beta\gamma} d(x, F)^{2\gamma}.$$

**Example 2.** Let $(\mathbb{R}, |\cdot|, \mu)$ with $\mu(E) = \int_E |x|^{-\beta} \, dx$, for some $0 < \beta < 1$. Since $|x|^{-\beta} \in A_1(X, |\cdot|, \lambda)$, we have that $(\mathbb{R}, |\cdot|, \mu)$ is a space of homogeneous type, but is not an $\alpha$-Ahlfors space for any $\alpha$. Let us consider the usual ternary Cantor set $C$ defined on $[0, 1]$, and set $F = f(C)$ with $f(x) = x^{\frac{1}{1-\beta}}$. Then $F$ is an $s$-Ahlfors set in $(X, \delta)$, with $s = \frac{\log 2}{\log 3}$. Indeed, by the definition of $\delta$ and $\mu$ we have that $\delta(x_0, y) = \frac{1}{1-\beta}|x_0^{1-\beta} - y^{1-\beta}|$ if $x_0 \neq y$ and $\delta(x_0, y) = 0$ if $x_0 = y$. Then for $x_0 \in F$ we have

$$
\begin{aligned}
F \cap B_\delta(x_0, r) &= \big\{y \in F : \delta(x_0, y) < r\big\} \\
&= \big\{y \in F : \big|x_0^{1-\beta} - y^{1-\beta}\big| < r(1-\beta)\big\} \\
&= f\big(\{\tilde{y} \in C : |\tilde{x}_0 - \tilde{y}| < r(1-\beta)\}\big) \\
&= f\big(C \cap B\big(\tilde{x}_0, r(1-\beta)\big)\big),
\end{aligned}
$$

where $\tilde{x}_0 = f^{-1}(x_0) \in C$. Since it is well known that $C$ is an $s$-Ahlfors set in $(X, d)$ with the $s$-dimensional Hausdorff measure (see for example [13]), we have that $F$ is an $s$-Ahlfors set in $(X, \delta)$ with the pullback of

the $s$-dimensional Hausdorff measure. Then, applying Theorem 1 we have that

$$\mu\big(B\big(x, d(x, F)\big)\big)^{\beta} \in A_p(X, d, \mu)$$

for $-(1-s) < \beta < (1-s)(p-1)$ and $1 \leqslant p < \infty$.

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