Theta lifts of Bianchi modular forms and applications to paramodularity

Tobias Berger, Lassina Dembélé, Ariel Pacetti and Mehmet Haluk Şengün

Abstract

We explain how the work of Johnson-Leung and Roberts on lifting Hilbert modular forms for real quadratic fields to Siegel modular forms can be adapted to imaginary quadratic fields. For this, we use archimedean results from Harris, Soudry and Taylor and replace the global arguments of Roberts by the non-vanishing result of Takeda. As an application of our lifting result, we exhibit an abelian surface B defined over $\mathbb{Q}$, which is not a restriction of scalars of an elliptic curve and satisfies the paramodularity Conjecture of Brumer and Kramer.

1. *Introduction*

The following is a special case of [**5**, Conjecture 1.4], known as the Brumer–Kramer paramodularity Conjecture. (For definitions and terminology, we refer the reader to Sections 2 and 3.)

CONJECTURE 1.1.   Let B be an abelian surface defined over $\mathbb{Q}$ of conductor N such that $\mathrm{End}_{\mathbb{Q}}(B) = \mathbb{Z}$. Then, there exists a Siegel newform g of genus 2, weight 2 and paramodular level N such that $L(B,s) = L(g,s)$. Conversely, if g is a Siegel newform of genus 2, weight 2 and paramodular level N, which is not a Gritsenko lift and whose Hecke eigenvalues are integers, then there exists an abelian surface B defined over $\mathbb{Q}$ such that $\mathrm{End}_{\mathbb{Q}}(B) = \mathbb{Z}$ and $L(g,s) = L(B,s)$.

In [**5**], there are examples of abelian surfaces of prime conductor. The first few Euler factors for each of them are shown to match those of a paramodular form in [**27**]. However, none of those surfaces has been proved to be modular.

Let K be a quadratic field, and E be an elliptic curve defined over K which is not a $\mathbb{Q}$-curve. The surface $B_E = \mathrm{Res}_{K/\mathbb{Q}}(E)$ is an abelian surface defined over $\mathbb{Q}$ with $\mathrm{End}_{\mathbb{Q}}(B_E) = \mathbb{Z}$. When K is real quadratic, recent work of Freitas, Le Hung and Siksek in [**14**] shows that there exists a Hilbert newform f of weight 2 such that $L(E,s) = L(f,s)$. The form f can be lifted to a paramodular Siegel cusp form g of genus 2 and weight 2 thanks to work of Johnson-Leung and Roberts [**23**]. A direct consequence of the construction of their theta lift is that $L(B_E,s) = L(g,s)$. In other words, the surface B_E is paramodular. So elliptic curves over real

quadratic fields that are not $\mathbb{Q}$-curves provide a large supply of abelian surfaces which satisfy Conjecture 1.1.

In fact, the above strategy to produce evidence for Conjecture 1.1 was elaborated by Brumer and Kramer themselves. They also speculated that further evidence could be gathered by using abelian surfaces B with trivial endomorphism ring over $\mathbb{Q}$ such that $\mathrm{End}_{\overline{\mathbb{Q}}}(B) \supsetneq \mathbb{Z}$ (see the paragraph following the statement of [5, Conjecture 1.4]). In [9], Dembélé and Kumar provide such numerical evidence. They give explicit examples of paramodular abelian surfaces B defined over $\mathbb{Q}$, which become of GL_2-type over some real quadratic fields. Like with the surfaces B_E above, the Johnson-Leung–Roberts' lift also plays a crucial rôle in their work.

The goal of this paper is twofold. First, we show that one can generalize the construction in [23] to imaginary quadratic fields. For this, one needs to replace the theta correspondence between $\mathrm{GO}(2,2)$ and GSp_4 by the one between $\mathrm{GO}(3,1)$ and GSp_4 used by Harris, Taylor and Soudry in their work on Galois representations associated to cusp forms for GL_2 over imaginary quadratic fields (see [21, 37]). Our analysis is then an adaptation of the one in [23, 29] using archimedean results from [21] and the local–global non-vanishing result of Takeda [35]. Our lift can be seen as a type of Yoshida lift, although, strictly speaking, Yoshida [41] only considered the groups $\mathrm{O}(4)$ and $\mathrm{O}(2,2)$.

Secondly, we combine our lift with explicit computations of Bianchi modular forms to exhibit an abelian surface B defined over $\mathbb{Q}$, which satisfies Conjecture 1.1 but is not a restriction of scalars of an elliptic curve. Because of the difficulties in constructing such examples (see Subsection 5.1) and the paucity of modularity results for GL_2 over imaginary quadratic fields, we had to limit ourselves to one example. However, it should be clear to the reader that our method, which borrows from [9], can be used in principle to generate more cases of the conjecture when further modularity results become available in this case.

As in the real quadratic case, if E is a modular over an imaginary quadratic field that is not a $\mathbb{Q}$-curve, then the associated surface B_E is paramodular by our lifting result. We note that there are several examples of such elliptic curves in the literature (see [10] and references therein).

The paper is organized as follows. In Sections 2 and 3, we recall the necessary background on Bianchi and Siegel modular forms. In Section 4, we prove the existence of our theta lift for any Bianchi modular form of even weight k that is non-base change. In Section 5, we give an example of an abelian surface B defined over $\mathbb{Q}$ which satisfies Conjecture 1.1. We prove the modularity of our surface (over an imaginary quadratic field) using the so-called Faltings–Serre method [34].

2.   Background on Bianchi modular forms

2.1. Bianchi modular forms

Let K be an imaginary quadratic field, and $\mathcal{O}_K$ be its ring of integers. Let $\widehat{\mathbb{Z}}$ be the finite adèles of $\mathbb{Z}$. The adèles of K are given by $\mathbb{A}_K = \mathbb{C} \times \mathbb{A}_K^f$, where $\widehat{\mathcal{O}}_K = \mathcal{O}_K \otimes \widehat{\mathbb{Z}}$ and $\mathbb{A}_K^f = K \otimes \widehat{\mathbb{Z}}$ are the finite adèles of $\mathcal{O}_K$ and K, respectively.

Fix an ideal $\mathfrak{N} \subseteq \mathcal{O}_K$ and define the compact open subgroup

$$U_0(\mathfrak{N}) := \left\{ \gamma \in \mathrm{GL}_2(\widehat{\mathcal{O}}_K) : \gamma \equiv \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \bmod \mathfrak{N} \right\}.$$

Set $\mathcal{K} = \mathcal{K}_\infty \times U_0(\mathfrak{N})$ where $\mathcal{K}_\infty = \mathbb{C}^\times U(2)$.

Let $k \geqslant 2$ be an *even* integer and consider the $\mathrm{GL}_2(\mathbb{C})$-representation $V_k = \mathrm{Sym}^{2k-2}(\mathbb{C}^2)$. Let $\chi : K^\times \backslash \mathbb{A}_K^\times / \mathbb{C}^\times \widehat{\mathcal{O}}_K^\times \to \mathbb{C}^\times$ be an unramified Hecke character with trivial infinity type. For $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{GL}_2(\mathbb{A}_K)$, put $\chi_{\mathfrak{N}}(\gamma) = \prod_{v | \mathfrak{N}} \chi_v(d_v)$. We define the space $S_k(\mathfrak{N}, \chi)$ of *Bianchi cusp forms* of level $U_0(\mathfrak{N})$, central character χ and weight k to be the set of all functions

$f : \mathrm{GL}_2(\mathbb{A}_K) \to V_k$ such that

    (a) $f(\gamma g) = f(g)$ for all $\gamma \in \mathrm{GL}_2(K)$;

    (b) $f(zg) = \chi(z)f(g)$ for all $z \in \mathbb{A}_K^\times$;

    (c) $f(gu) = \chi_{\mathfrak{N}}(u)f(g)u_\infty$ for all $u = (u_\infty, u_f) \in U(2) \times U_0(\mathfrak{N})$;

    (d) f, viewed as a function on $\mathrm{GL}_2(\mathbb{C})$, is an eigenfunction of the complexification (in $\mathfrak{sl}_2(\mathbb{C}) \otimes_\mathbb{R} \mathbb{C}$) of the Casimir operator of $\mathfrak{sl}_2(\mathbb{C})$ with eigenvalue $(k^2 - 2k)/8$;

    (e) for all $g \in \mathrm{GL}_2(\mathbb{A}_K)$, one has

$$\int_{K \backslash \mathbb{A}_K} f\left(\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} g\right) du = 0.$$

We invite the reader to see [16, 22] for details. In the notation of [16], we take

$$\mathbf{n} = (k-2)i + (k-2)c, \mathbf{v} = -((k-2)/2)i - ((k-2)/2)c \text{ in } \mathbb{Z}[\{i, c\}],$$

for $i, c : K \hookrightarrow \mathbb{C}$, to ensure that our forms have trivial action by $\mathbb{C}^\times$. The space $S_k(\mathfrak{N}, \chi)$ is endowed with the so-called Petersson inner product.

2.2. *Newforms and L-series*

For each $\mathfrak{p} \nmid \mathfrak{N}$, write the double coset decomposition

$$U_0(\mathfrak{N}) \begin{pmatrix} \varpi_{\mathfrak{p}} & 0 \\ 0 & 1 \end{pmatrix} U_0(\mathfrak{N}) = \coprod_i x_i U_0(\mathfrak{N}),$$

where $\varpi_{\mathfrak{p}}$ is a uniformizer at $\mathfrak{p} \nmid \mathfrak{N}$. We define the Hecke operator

$$T_{\mathfrak{p}} : S_k(\mathfrak{N}, \chi) \longrightarrow S_k(\mathfrak{N}, \chi)$$

by

$$T_{\mathfrak{p}} f(g) := \sum_i \chi_{\mathfrak{N}}(x_i)^{-1} f(g x_i), \quad g \in \mathrm{GL}_2(\mathbb{A}_K).$$

Similarly, one can define the operator $U_{\mathfrak{p}}$ for $\mathfrak{p} \mid \mathfrak{N}$.

We refer the reader to [39, Section 1.6] for the theory of newforms for Bianchi modular forms. When $f \in S_k(\mathfrak{N}, \chi)$ is a newform, we have

$$T_{\mathfrak{p}} f = a_{\mathfrak{p}} f \quad \text{for all } \mathfrak{p} \nmid \mathfrak{N},$$
$$U_{\mathfrak{p}} f = a_{\mathfrak{p}} f \quad \text{for all } \mathfrak{p} \mid \mathfrak{N}.$$

The L-series attached to f is then defined by

$$L(f, s) := \prod_{\mathfrak{p} \mid \mathfrak{N}} \left(1 - \frac{a_{\mathfrak{p}}}{N(\mathfrak{p})^s}\right)^{-1} \prod_{\mathfrak{p} \nmid \mathfrak{N}} \left(1 - \frac{a_{\mathfrak{p}}}{N(\mathfrak{p})^s} + \frac{\chi(\mathfrak{p})}{N(\mathfrak{p})^{2s-1}}\right)^{-1}.$$

The newform f determines a unique cuspidal irreducible automorphic representation π of $\mathrm{GL}_2(\mathbb{A}_K)$ of level $\mathfrak{N}$, which admits a restricted tensor product (see [13])

$$\pi = \pi_\infty \otimes \bigotimes_{\mathfrak{p} < \infty} \pi_{\mathfrak{p}},$$

such that π_∞ has L-parameter (see Section 4)

$$W_\mathbb{C} = \mathbb{C}^\times \longrightarrow \mathrm{GL}_2(\mathbb{C}) : z \longmapsto \begin{pmatrix} (z/|z|)^{k-1} & 0 \\ 0 & (z/|z|)^{1-k} \end{pmatrix}$$

and the central character of π is χ.

Conversely, any cuspidal automorphic representation π of $\mathrm{GL}_2(\mathbb{A}_K)$ whose infinity component has such an L-parameter corresponds to a newform f of weight k.

2.3. Connections with cohomology

It is standard to pass to the cohomological framework when working with Bianchi automorphic forms. In fact, this is the only approach that is currently suitable for the algorithmic methods that were used to gather the data in Subsection 5.1 (see [8, 28, 40]). So we now give a quick review of this framework.

Let $W_n = \text{Sym}^n(\mathbb{C}^2) \otimes_{\mathbb{C}} \overline{\text{Sym}^n(\mathbb{C}^2)}$ where the action of $\text{GL}_2(\mathbb{C})$ on the second factor is twisted with complex conjugation. Consider the adelic locally symmetric space

$$Y_0(\mathfrak{N}) = \text{GL}_2(K) \backslash \text{GL}_2(\mathbb{A}_K)/\mathcal{K}.$$

The cohomology spaces

$$H^i(Y_0(\mathfrak{N}), \mathcal{W}_n),$$

where $\mathcal{W}_n$ is the local system induced by W_n, carry a natural Hecke action on them. Let $Y_0(\mathfrak{N})^{\text{BS}}$ denote the Borel–Serre compactification of $Y_0(\mathfrak{N})$. This is a compact space with boundary $\partial Y_0(\mathfrak{N})^{\text{BS}}$, made of finitely many 2-tori, that is, homotopy equivalent to $Y_0(\mathfrak{N})$. We define the *cuspidal cohomology* $H^i_{\text{cusp}}(Y_0(\mathfrak{N}), \mathcal{W}_n)$ as the kernel of the restriction map

$$H^i(Y_0(\mathfrak{N})^{\text{BS}}, \mathcal{W}_n) \longrightarrow H^i(\partial Y_0(\mathfrak{N})^{\text{BS}}, \mathcal{W}_n).$$

The Hecke action stabilizes the cuspidal cohomology.

By the strong approximation theorem, the determinant map induces a canonical bijection

$$\text{GL}_2(K) \backslash \text{GL}_2(\mathbb{A}_K^f)/U_0(\mathfrak{N}) \simeq K^{\times} \backslash (\mathbb{A}_K^f)^{\times}/\widehat{\mathcal{O}}_K^{\times} \simeq \text{Cl}(K),$$

where $\text{Cl}(K)$ is the class group of K. Let $\mathfrak{c}_i$, $1 \leqslant i \leqslant h$, be a complete set of representatives for the classes in $\text{Cl}(K)$. For each i, let t_i be a finite idèle which generates $\mathfrak{c}_i$ and set $\Gamma_0(\mathfrak{c}_i, \mathfrak{N}) = \text{GL}_2(K) \cap (\text{GL}_2(\mathbb{C})x_i U_0(\mathfrak{N})x_i^{-1})$ where $x_i = \left(\begin{smallmatrix} 1 & 0 \\ 0 & t_i \end{smallmatrix}\right)$. Then, we have

$$\Gamma_0(\mathfrak{c}_i, \mathfrak{N}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \begin{pmatrix} \mathcal{O}_K & \mathfrak{c}_i^{-1} \\ \mathfrak{c}_i \mathfrak{N} & \mathcal{O}_K \end{pmatrix} : ad - bc \in \mathcal{O}_K^{\times} \right\}$$

and

$$Y_0(\mathfrak{N}) = \bigsqcup_{j=1}^{h} \Gamma_0(\mathfrak{c}_j, \mathfrak{N}) \backslash \mathcal{H}_3,$$

where $\mathcal{H}_3 \simeq \text{GL}_2(\mathbb{C})/\mathcal{K}_{\infty}$ is the hyperbolic 3-space. It follows that the cohomology of the adelic space decomposes as

$$H^i(Y_0(\mathfrak{N}), \mathcal{W}_n) = \bigoplus_{j=1}^{h} H^i(\Gamma_0(\mathfrak{c}_j, \mathfrak{N}) \backslash \mathcal{H}_3, \mathcal{W}_n).$$

The cohomology spaces on the right-hand side lend themselves very well to explicit machine calculations. We refer the reader to [28, 40] for more details on this, and on the Bianchi newforms related to the examples in this paper.

We recall that the diagonal embedding

$$K^{\times} \backslash (\mathbb{A}_K^f)^{\times}/\widehat{\mathcal{O}}_K^{\times} \hookrightarrow \text{GL}_2(K) \backslash \text{GL}_2(\mathbb{A}_K^f)/U_0(\mathfrak{N}),$$

induces an action of $\text{Cl}(K)$ on the $H^i_{\text{cusp}}(Y_0(\mathfrak{N}), \mathcal{W}_n)$. This action is compatible with the Hecke action as it is via the diamond operators. Therefore, these cohomology spaces decompose accordingly.

For $k \geqslant 2$ even, the Generalized Eichler–Shimura Isomorphism (see [20], in our setting [22, Proposition 3.1]) says that

$$H^1_{\text{cusp}}(Y_0(\mathfrak{N}), \mathcal{W}_{k-2}) \simeq \bigoplus_{\chi} S_k(\mathfrak{N}, \chi) \simeq H^2_{\text{cusp}}(Y_0(\mathfrak{N}), \mathcal{W}_{k-2}) \tag{2.1}$$

as Hecke modules, where χ runs over all Hecke characters of trivial infinity type that are unramified everywhere (that is, characters of $\mathrm{Cl}(K)$).

REMARK 2.1. Our Theorem 4.1 only applies to the newforms in $S_k(\mathfrak{N}, \mathbf{1})$, that is, the newforms whose central character is trivial. We refer the reader to [**25**, Sections 2.4 and 2.5; **28**; **40**] for details on how the cohomology corresponding to this space can be computed.

3. Background on Siegel modular forms

3.1. Siegel modular forms

Let

$$J = \begin{pmatrix} 0 & \mathbf{1}_2 \\ -\mathbf{1}_2 & 0 \end{pmatrix} \in M_4(\mathbb{Z}).$$

We recall that the symplectic group of genus 2 is the $\mathbb{Q}$-algebraic group GSp_4 defined by setting

$$\mathrm{GSp}_4(R) := \{\gamma \in \mathrm{GL}_4(R) : \gamma^t J \gamma = \nu(\gamma) J, \ \nu(\gamma) \in R^\times\},$$

for any $\mathbb{Q}$-algebra R. The map $\nu : \mathrm{GSp}_4 \to \mathbb{G}_m$ is called the similitude factor, and its kernel is the symplectic group Sp_4.

Let $\mathrm{GSp}_4(\mathbb{R})^+$ be the subgroup of $\mathrm{GSp}_4(\mathbb{R})$ which consists of the matrices γ such that $\nu(\gamma) > 0$; also let

$$\mathfrak{H}_2 := \{Z \in M_2(\mathbb{C}) : Z = Z^t \text{ and } \mathrm{Im}(Z) > 0\}$$

be the Siegel upper half plane of degree 2. We recall that $\mathrm{GSp}_4(\mathbb{R})^+$ acts on $\mathfrak{H}_2$ by

$$\gamma \cdot Z = (AZ + B)(CZ + D)^{-1}, \quad \gamma := \begin{pmatrix} A & B \\ C & D \end{pmatrix}.$$

Let $k \geqslant 2$ be an even integer, and set $V_{(k,2)} = \mathrm{Sym}^{k-2}(\mathbb{C}^2) \otimes \det^2(\mathbb{C}^2)$; this is a $\mathrm{GL}_2(\mathbb{C})$-representation which we denote by $\rho_{(k,2)}$. The group $\mathrm{GSp}_4(\mathbb{R})^+$ acts on the space of functions $F : \mathfrak{H}_2 \to V_{(k,2)}$ by

$$(F|_{(k,2)}\gamma)(Z) = \nu(\gamma)^{(k+2)/2} \rho_{(k,2)}((CZ + D)^{-1}) F(\gamma \cdot Z).$$

We fix a positive integer N, and consider the paramodular group

$$\Gamma^p(N) := \begin{pmatrix} \mathbb{Z} & N\mathbb{Z} & \mathbb{Z} & \mathbb{Z} \\ \mathbb{Z} & \mathbb{Z} & \mathbb{Z} & N^{-1}\mathbb{Z} \\ \mathbb{Z} & N\mathbb{Z} & \mathbb{Z} & \mathbb{Z} \\ N\mathbb{Z} & N\mathbb{Z} & N\mathbb{Z} & \mathbb{Z} \end{pmatrix} \cap \mathrm{Sp}_4(\mathbb{Q}).$$

A *Siegel modular form* of genus 2, weight $(k, 2)$ and *paramodular* level N is a holomorphic function $F : \mathfrak{H}_2 \to V_{(k,2)}$ such that $F|_{(k,2)}\gamma = F$ for all $\gamma \in \Gamma^p(N)$.

Let F be a Siegel modular form of genus 2, weight $(k, 2)$ and paramodular level N. Then, we see that

$$F(Z + \mathbf{1}_2) = F(Z) \quad \text{for all } Z \in \mathfrak{H}_2.$$

By the Koecher principle [**17**], F admits a Fourier expansion of the form

$$F(Z) = \sum_{Q \geqslant 0} a_Q \, e^{2\pi i \mathrm{Tr}(QZ)},$$

where Q runs over all the 2×2 symmetric matrices in $M_2(\mathbb{Q})$ that are positive semi-definite. We say that F is a *cusp form* if, for all $\gamma \in \mathrm{GSp}_4(\mathbb{Q})$, we have

$$(F|_{(k,2)}\gamma)(Z) = \sum_{Q>0} a_Q^\gamma \, e^{2\pi i \mathrm{Tr}(QZ)}.$$

We denote by $S_{(k,2)}(\Gamma^p(N))$ the space of paramodular Siegel cusp forms of level N and weight $(k, 2)$.

3.2. *Newforms and L-series*

We consider the double coset decompositions

$$\Gamma^p(N) \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & p & \\ & & & p \end{pmatrix} \Gamma^p(N) = \coprod_i \Gamma^p(N)h_i;$$

$$\Gamma^p(N) \begin{pmatrix} p & & & \\ & 1 & & \\ & & p & \\ & & & p^2 \end{pmatrix} \Gamma^p(N) = \coprod_i \Gamma^p(N)h_i';$$

and following [**23**, p. 546] define the Hecke operators

$$T(p)F = p^{(k-4)/2} \sum_i F|_{(k,2)}h_i,$$

$$T(1,p,p,p^2)F = p^{k-4} \sum_i F|_{(k,2)}h_i'.$$

(As noted in [**23**], $T(1, p, p, p^2)$ agrees with the classical $T(p^2)$ for $p \nmid N$. We also note that our Hecke operators are scaled so as to match the definition in [**1**, p. 164].) We refer the reader to [**23**, p. 547] or [**31**] for the definition of the operators U_p for $p \mid N$.

For the paramodular group $\Gamma^p(N)$, the theory of newforms developed in [**30**, **31**] for scalar weights carries over to the vector-valued setting. The old subspace is generated by the images of the level-raising maps of [**30**]. One then defines the new subspace $S_{(k,2)}^{\mathrm{new}}(\Gamma^p(N))$ to be its orthogonal complement with respect to the Petersson inner product for vector-valued forms given in [**1**].

Let $F \in S_{(k,2)}^{\mathrm{new}}(\Gamma^p(N))$ be a newform such that for all p

$$T(p)F = \lambda_p F, \quad T(1,p,p,p^2)F = \mu_p F \quad \text{and} \quad U_p F = \epsilon_p F$$

for $\lambda_p, \mu_p, \epsilon_p \in \overline{\mathbb{Q}}$. By work of [**2**], we can associate an automorphic representation Π to F which admits a restricted tensor product

$$\Pi = \Pi_\infty \otimes \bigotimes_{p<\infty} \Pi_p.$$

The *L*-series attached to F is defined by

$$L(s,F) := \prod_p L_p(s,F),$$

where the local Euler factors $L_p(s, F)$ are obtained as follows:

(a) for $\mathrm{val}_p(N) = 0$, we use the classical Euler factor in [**1**, p. 173]:

$$L(s,F)^{-1} = 1 - \lambda_p p^{-s} + (p\mu_p + p^{k-1} + p^{k-3})p^{-2s} - p^{k-1}\lambda_p p^{-3s} + p^{2k-2}p^{-4s};$$

(b) for $\mathrm{val}_p(N) = 1$, we let

$$L(s,F)^{-1} = 1 - (\lambda_p + p^{k/2-2}\epsilon_p)p^{-s} + (p\mu_p + p^{k-1})p^{-2s} + \epsilon_p p^{3k/2-2}p^{-3s};$$

(c) for $\mathrm{val}_p(N) \geqslant 2$, we let

$$L_p(s,F)^{-1} = 1 - \lambda_p p^{-s} + (p\mu_p + p^{k-1})p^{-2s}.$$

For $\mathrm{val}_p(N) \geqslant 1$, these definitions are motivated by the results of [**31**] and the definitions in [**23**].

4. Theta lifts of Bianchi modular forms

We briefly recall the definition of L-parameters following [**32**, Section 4.1]. Let E be a number field, v be a place of E and E_v be the completion of E at v. Let G be the group GL_2 or GSp_4. Then, the local Langlands correspondence is known for G (see [**6**, **15**, **24**]). It yields a corresponding finite-to-one surjective map

$$L : \Pi(G(E_v)) \longrightarrow \Phi(G(E_v)),$$

where

(i) $\Pi(G(E_v))$ is the set of isomorphism classes of irreducible admissible representations of $G(E_v)$;

(ii) $\Phi(G(E_v))$ is the set of L-parameters for $G(E_v)$, that is, the set of isomorphism classes of admissible homomorphisms $\phi : W'_{E_v} \to {}^L G^0$, where W'_{E_v} is the Weil–Deligne group of E_v and ${}^L G^0$ the dual group of G (which equals $G(\mathbb{C})$ in our cases).

For any irreducible admissible representation π_v of $G(E_v)$, we call $L(\pi_v)$ the L-parameter of π_v.

THEOREM 4.1. *Let $K/\mathbb{Q}$ be an imaginary quadratic field of discriminant D. Let $\mathcal{O}_K$ be the ring of integers of K and $\mathfrak{N} \subseteq \mathcal{O}_K$ be an ideal. Let π be a tempered cuspidal irreducible automorphic representation of $\mathrm{GL}_2(\mathbb{A}_K)$ of level $\mathfrak{N}$ and trivial central character such that π_∞ has L-parameter*

$$W_{\mathbb{C}} = \mathbb{C}^\times \longrightarrow \mathrm{GL}_2(\mathbb{C}) : z \longmapsto \begin{pmatrix} (z/|z|)^{k-1} & 0 \\ 0 & (z/|z|)^{1-k} \end{pmatrix}$$

for some $k \in \mathbb{Z}$, $k \geqslant 2$ even. Assume that π is not Galois-invariant. Then there exists an irreducible cuspidal representation $\Pi = \bigotimes'_v \Pi_v$ of $\mathrm{GSp}_4(\mathbb{A}_{\mathbb{Q}})$ with trivial central character such that

(a) *Π_v is generic for all finite v;*

(b) *Π_∞ is a holomorphic limit of discrete series of Harish-Chandra weight $(k-1,0)$;*

(c) *the following equality of L-parameters holds for all places v:*

$$L(\Pi_v) = \bigoplus_{w|v} \mathrm{Ind}_{W'_{K_w}}^{W'_{\mathbb{Q}_v}} L(\pi_w). \tag{4.1}$$

Consequently, there exists a Siegel newform F of weight $(k,2)$ and paramodular level $N = D^2 N_{K/\mathbb{Q}}(\mathfrak{N})$ with Hecke eigenvalues, epsilon factor and (spinor) L-function determined explicitly by π (and described in the proof below).

REMARK 4.2. The L-parameters for $\mathrm{GSp}_4(\mathbb{Q}_v)$ on the right-hand side of (4.1) are defined as follows.

(i) If v is split, then as in [**23**] (5), $L(\pi_w) \oplus L(\pi_{\overline{w}})(x) = \begin{pmatrix} a_1 & & b_1 & \\ & a_2 & & b_2 \\ c_1 & & d_1 & \\ & c_2 & & d_2 \end{pmatrix}$, where

$$L(\pi_w)(x) = \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} \text{ and } L(\pi_{\overline{w}})(x) = \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix}.$$

(ii) If v is non-split and finite, then as in [23] (6), let $g_0 \in W'_{K_w} \setminus W'_{\mathbb{Q}_v}$ be non-trivial. Then $\mathrm{Ind}_{W'_{K_w}}^{W'_{\mathbb{Q}_v}} L(\pi_w) = \varphi(K_w, \pi_w, \mathbf{1})$, where

(1) if $y \in W'_{K_w}$, then $\varphi(K_w, \pi_w, \mathbf{1})(y) = L(\pi_w)(y) \oplus L(\pi_w)(g_0 y g_0^{-1})$;

(2) $\varphi(K_w, \pi_w, \mathbf{1})(g_0) = \begin{pmatrix} & & & 1 \\ & a_0 & b_0 & \\ & c_0 & d_0 & 1 \\ \end{pmatrix}$, for $L(\pi_w)(g_0^2) = \left(\begin{smallmatrix} a_0 & b_0 \\ c_0 & d_0 \end{smallmatrix} \right)$.

(iii) If $v = \infty$, then $\mathrm{Ind}_{W'_{K_w}}^{W'_{\mathbb{Q}_v}} L(\pi_w) = \varphi(K_w, \pi_w, \mathbf{1})$ as defined in [23] (6) (definition extended to the archimedean case). This L-parameter $L(\Pi_\infty) : W'_{\mathbf{R}} = \mathbb{C}^\times \cup \mathbb{C}^\times j \to \mathrm{GSp}_4(\mathbb{C})$ (where $j^2 = -1$ and $jzj^{-1} = \overline{z}$ for $z \in \mathbb{C}^\times$) can also be described explicitly as follows:

$$j \longmapsto \begin{pmatrix} & & & 1 \\ & & 1 & \\ & (-1)^{k-1} & & \\ (-1)^{k-1} & & & \end{pmatrix},$$

$$z \longmapsto \begin{pmatrix} (z/|z|)^{k-1} & & & \\ & (z/|z|)^{k-1} & & \\ & & (z/|z|)^{1-k} & \\ & & & (z/|z|)^{1-k} \end{pmatrix}, \quad z \in \mathbb{C}^\times.$$

REMARK 4.3.   For odd weights k, one still has a lift to a cuspidal automorphic representation Π of GSp_4, with central character given by the quadratic character $\omega_{K/\mathbb{Q}}$ corresponding to K such that conditions (a) and (b) in the theorem are satisfied. ([22, Lemma 12] also gives a lift with trivial central character, but Π_∞ must then be generic of highest weight $(k, 1)$.) We also note that one can, in fact, lift any non-Galois-invariant π with cyclotomic central character to an irreducible cuspidal representation of $\mathrm{GSp}_4(\mathbb{A}_\mathbb{Q})$ (as discussed in [21, 35]). However, the local calculations of [23] and the paramodular newform theory of [31] apply only for trivial central character, which is the reason why we exclude odd weights in the theorem and impose the condition of a trivial central character.

Proof.   The strategy of the proof is very similar to that of [23], but replaces the theta correspondence between $\mathrm{GO}(2, 2)$ and GSp_4 by that between $\mathrm{GO}(3, 1)$ and GSp_4. There are four steps in the construction of the lift, which are outlined as follows.

(a) As in [21, Section 1], π gives rise to an automorphic representation $\sigma = (\pi, \mathbf{1})$ for the identity component $\mathrm{GSO}(3, 1)$ of $\mathrm{GO}(3, 1)$.

(b) Choosing suitable extensions of the local components of σ promotes this to a representation $\hat{\sigma}$ of $\mathrm{GO}(3, 1)$. (In this step, we follow [29] rather than [21] at the non-archimedean places.)

(c) As in [21] (but using the non-vanishing result of [35]), we then use the theta correspondence between $\mathrm{GO}(3, 1)$ and GSp_4 to lift $\hat{\sigma}$ to the automorphic representation Π of GSp_4 described in the theorem.

(d) To produce the paramodular Siegel modular form, one takes the automorphic form $\Phi = \bigotimes \Phi_v$, where $\Phi_v \in \Pi_v$ for $v \nmid \infty$ are the paramodular newform vectors defined by Roberts and Schmidt [31]. By referring to the classical treatment in [1], we transfer the local non-archimedean calculations of Hecke eigenvalues, epsilon and L-factors of Π in [23] to those of the corresponding vector-valued Siegel modular form on $\mathfrak{H}_2$.

We now give precise details for each of these steps.

(a) Let $X = \{A \in M_2(K) : \overline{A}^t = A\}$ be the space of 2×2 Hermitian matrices over K with quadratic form given by $-\det$. By [21, Proposition 1], $\sigma = (\pi, \mathbf{1})$ defines an irreducible tempered

cuspidal automorphic representation of

$$\mathrm{GSO}(X, \mathbb{A}_{\mathbb{Q}}) \cong (\mathrm{GL}_2(\mathbb{A}_K) \times \mathbb{A}_{\mathbb{Q}}^*)/\{(z\mathrm{Id}_2, \mathrm{N}_{K/\mathbb{Q}}z^{-1}), \ z \in \mathbb{A}_K^*\}.$$

Before we come to Step (b), we review some details of the theta correspondence between $\mathrm{GO}(X, \mathbb{A}_{\mathbb{Q}})$ and $\mathrm{GSp}_4(\mathbb{A}_{\mathbb{Q}})$: Following [21, 29, 35], we consider the extension of the Weil representation for $\mathrm{Sp}_4 \times O(X)$ to the group $R = \{(g, h) \in \mathrm{GSp}_4 \times \mathrm{GO}(X) : \nu(g)\nu(h) = 1\}$ and denote this representation by ω. (As explained in [35, Remark 4.3], there is a difference in the definition of R in [21, 29]. But, since we are working with trivial central characters, this does not matter here.)

Let G be a reductive group over $\mathbb{Q}$, and v be a place of $\mathbb{Q}$. We denote by $\mathrm{Irr}(G(\mathbb{Q}_v))$ the collection of equivalence classes of irreducible smooth admissible representations of $G(\mathbb{Q}_v)$. Let $v \nmid \infty$, $\tau_v \in \mathrm{Irr}(\mathrm{GO}(X, \mathbb{Q}_v))$ and $\Pi_v \in \mathrm{Irr}(\mathrm{GSp}_4(\mathbb{Q}_v))$. We say that τ_v and Π_v correspond (or that τ_v corresponds to Π_v) if there is a non-zero R-homomorphism from ω to $\Pi_v \otimes \tau_v$. If τ_v is tempered and corresponds to Π_v, then Π_v is unique by [35, Lemma 4.1]. In that case, we denote Π_v by $\theta(\tau_v)$. For $v = \infty$, we refer the reader to [35, Lemma 4.2] for the appropriate definitions.

(b) For all finite places $v = p$ of $\mathbb{Q}$, Roberts [29, pp. 277–278] defines extensions of σ_v (certain subrepresentations of $\mathrm{Ind}_{\mathrm{GSO}}^{\mathrm{GO}}\sigma_v$) to representations σ_v^+ of $\mathrm{GO}(X, \mathbb{Q}_p)$ such that $\theta(\sigma_v^+)^\vee$ is the unique generic representation of $\mathrm{GSp}_4(\mathbb{Q}_p)$ with L-parameter satisfying (4.1).

At $v = \infty$, we define σ_v^- to be the representation denoted by $(\pi_{n,s}, \epsilon, \delta)$ for $n = k - 1$, $s = 1 - k$, $\epsilon = 1$ and $\delta = -1$ in [21, p. 394]. By [21, Lemma 12] (see also [36, Proposition 6.5(2)]) and [21, Corollary 3] (which extends to $n = 1$), we know that σ_v^- corresponds to a holomorphic limit of discrete series representation Π_∞ of $\mathrm{GSp}_4(\mathbf{R})$ with Harish-Chandra parameter $(k - 1, 0)$ whose L-parameter satisfies (4.1).

By [35, Proposition 5.4], we obtain an irreducible cuspidal automorphic representation of $\mathrm{GO}(X, \mathbb{A}_{\mathbb{Q}})$ by setting

$$\hat{\sigma} := \bigotimes_{v < \infty} \sigma_v^+ \otimes \sigma_\infty^-.$$

(c) We now consider the global theta lift $\Theta(V_{\hat{\sigma}})$, which is the space generated by the $\mathrm{GSp}_4(\mathbb{A}_{\mathbb{Q}})$ automorphic forms $\theta(f; \varphi)$ for all $f \in V_{\hat{\sigma}}$ and $\varphi \in S(X(\mathbb{A}_{\mathbb{Q}})^2)$. (For details of this definition, we refer the reader to [29, Section 5; 35, p.11].)

We know by the local non-vanishing and [35, Theorem 1.2] that $\Theta(V_{\hat{\sigma}}) \neq 0$. Since $\pi \not\cong \pi^c$, we also know that $\Theta(V_{\hat{\sigma}})$ occurs in the space of cuspforms by Takeda [21, Lemma 5; 35, Theorem 1.3(2)]. Now let Π be an irreducible quotient of $\Theta(V_{\hat{\sigma}})$. Then the representations Π_v and $\hat{\sigma}_v^\vee$ correspond for all places v, so by [29, Theorem 1.8; 35, Lemmas 4.1 and 4.2] we get $\Pi_v = \theta(\hat{\sigma}_v^\vee) = \theta(\hat{\sigma}_v)^\vee$ (where the last equality holds by [29, Proposition 1.10]).

(d) The existence of the Siegel paramodular form is now proved exactly as in [23], but using the argument from [33, Section 3.1] for defining the vector-valued Siegel modular form F.

The Hecke eigenvalues, epsilon and L-factors for the finite part of the automorphic representation Π are identical to those in the main theorem of [23]. (Note that [23] uses the notation π_0 and π instead of our π and Π.)

To match the classical spinor L-factor of [1, p. 173] at unramified places, we see that the shift in the argument of the Euler factors of the local representations Π_v given in [23, Proposition 4.2] to those of F is given by $s \mapsto s - (k - 1)/2$. By the calculations in [23], this means that, for all primes p, we have an equality of Euler factors

$$L_p\left(s + \frac{k-1}{2}, F\right) = L_p(s, \pi) := \prod_{\mathfrak{p} \mid p} L_{\mathfrak{p}}\left(s + \frac{1}{2}, f\right),$$

where the $L_p(s, F)$ were defined in Section 3 and $L_{\mathfrak{p}}(s + \frac{1}{2}, f)$ in Section 2 for the newform f corresponding to π. As in [23], the functional equation of π implies that the completed

L-function satisfies the functional equation

$$\Lambda(k-s, F) = \left(\prod_{p|N} \epsilon_p\right) N^{s-k/2}\Lambda(s, F),$$

where

$$\Lambda(s, F) = (2\pi)^{-2s}\Gamma^2(s)L(s, F). \qquad \square$$

REMARK 4.4.   We end this section with two remarks.

(a) Arthur's multiplicity formula and [**26**, Theorem 2.2] imply that the multiplicity of Π in the space of cuspforms is 1. This has been proved in [**29**, Theorem 8.6] in the real quadratic case using the multiplicity preservation of the theta correspondence [**29**, Proposition 5.3].

(b) An alternative construction of the lift of automorphic representations has been proved by Chan [**7**] using trace formulas (under some local conditions on π). Recently, Mok [**26**] has also described how to obtain this lift from Arthur's endoscopic classification, based on work of Chan and Gan which relates Arthur's local correspondence with that of Gan–Takeda.

5.   Application to paramodularity

In this section, we use our lifting result to prove the following theorem.

THEOREM 5.1.   Let $C'/\mathbb{Q}$ be the curve defined by

$$y^2 = 31x^6 + 952x^5 - 5764x^4 - 3750x^3 + 5272x^2 - 7060x + 4783,$$

and B the Jacobian of C'. Then, B is a paramodular abelian surface of conductor 223^2 in the sense that it satisfies Conjecture 1.1.

We obtain Theorem 5.1 as a consequence of the following statement.

THEOREM 5.2.   Let $K = \mathbb{Q}(\sqrt{-223})$ and $w = (1 + \sqrt{-223})/2$, and consider the curve

$$C: \ y^2 + Q(x)y = P(x),$$

where

$$P := -8x^6 + (54w - 27)x^5 + 9103x^4 + (-14\,200w + 7100)x^3 - 697\,185x^2$$
$$+ (326\,468w - 163\,234)x + 3\,539\,399,$$
$$Q := x^3 + (2w - 1)x^2 - x.$$

Let $A = \mathrm{Jac}(C)$ be the Jacobian of C. Then we have the following.

(a) The curve C is a global minimal model for the base change of C' to K, and it has everywhere good reduction.

(b) The surface A has real multiplication by $\mathbb{Z}[\sqrt{2}]$, and there exists a Bianchi newform f of level (1) and weight 2, and trivial central character such that $f^\sigma = f^\tau$ and

$$L(A, s) = L(f, s)L(f^\tau, s),$$

where $\langle \sigma \rangle = \mathrm{Gal}(K/\mathbb{Q})$ and $\langle \tau \rangle = \mathrm{Gal}(\mathbb{Q}(\sqrt{2})/\mathbb{Q})$.

REMARK 5.3.   Theorem 5.2(b) implies that A is modular in the sense that its L-series is given by the product of those of the forms f and f^τ. So, this is an instance of the Eichler–Shimura Conjecture [**38**, Conjecture 3] in dimension 2. We think that this is the first non-trivial

such example over an imaginary quadratic field. (There is a significant amount of numerical data going back to [**8**, **19**] which supports this conjecture in the case of elliptic curves.)

The rest of this section is dedicated to proving Theorem 5.2. But first, we show how to derive Theorem 5.1 from it.

Proof. The equality $f^\sigma = f^\tau$ implies that the form f cannot be a base change. Moreover, as the abelian surface A is modular by f, the cuspidal irreducible automorphic representation associated to f is tempered. So by Theorem 4.1, it admits a lift g of weight 2 to $\mathrm{GSp}_4 / \mathbb{Q}$ with paramodular level 223^2. By construction, the identity $L(A, s) = L(f, s)L(f^\tau, s)$ implies that $L(g, s) = L(B, s)$. So B is paramodular. □

5.1. *The abelian surface*

As we mentioned in Remark 5.3, the surface A satisfies the Eichler–Shimura Conjecture. It was in fact located via explicit computations of Bianchi modular forms. More specifically, we used the extensive data provided in [**28**].

To simplify notation, let $S_2(\mathbf{1}) := S_2(\mathcal{O}_K, \mathbf{1})$ be the space of Bianchi cusp forms of weight 2, level $\mathfrak{N} = (1)$ and central character $\chi = \mathbf{1}$. Let $S_2^{\mathrm{bc}}(\mathbf{1})$ be the subspace of $S_2(\mathbf{1})$ which consists of twists of those Bianchi cusp forms which arise from classical elliptic newforms via base change, and of those Bianchi cusp forms which are CM (see [**12**]), and $S_2^{\mathrm{bc}}(\mathbf{1})^\perp$ its orthogonal complement with respect to the Petersson inner product. We will call the newforms in $S_2^{\mathrm{bc}}(\mathbf{1})^\perp$ *genuine*. Given a newform f, we let $\mathcal{O}_f = \mathbb{Z}[a_\mathfrak{p}(f) : \mathfrak{p} \subset \mathcal{O}_K]$ denote the order generated by the Hecke eigenvalues of f and L the field of fractions of $\mathcal{O}_f$.

Of all the 186 imaginary quadratic fields K in [**28**] (including all those 153 for which $|D| < 500$), there are *only* 6 for which $S_2^{\mathrm{bc}}(\mathbf{1}) \subsetneq S_2(\mathbf{1})$. (Note that our $S_2(\mathbf{1})$ corresponds to $S_0(1)^+$ of [**28**] by the arguments in [**25**, Sections 2.4 and 2.5].) In each case, $S_2^{\mathrm{bc}}(\mathbf{1})^\perp$ is an irreducible Hecke module of dimension 2, except for $|D| = 643$ when there are two newforms, with rational Hecke eigenvalues, that are $\mathrm{Gal}(K/\mathbb{Q})$-conjugate. Table 1 provides a summary of these data.

REMARK 5.4. The index entries in Table 1 are based on finite sets of Hecke eigenvalues. Since there is no analogue of Sturm's bound for Bianchi modular forms, the last row entries are not proved to be correct. However, we strongly expect them to reflect the truth.

Let f be any of the genuine Bianchi newforms listed in Table 1. Let σ and τ denote the non-trivial elements of $\mathrm{Gal}(K/\mathbb{Q})$ and $\mathrm{Gal}(L/\mathbb{Q})$, respectively. Let f^τ be $\mathrm{Gal}(L/\mathbb{Q})$-conjugate of f, which is determined by the relation

$$a_\mathfrak{p}(f^\tau) = \tau(a_\mathfrak{p}(f)) \quad \text{for all primes } \mathfrak{p},$$

and f^σ be the $\mathrm{Gal}(K/\mathbb{Q})$-conjugate of f which is determined by

$$a_\mathfrak{p}(f^\sigma) = a_{\sigma(\mathfrak{p})}(f) \quad \text{for all primes } \mathfrak{p}.$$

Except for the discriminant $|D| = 643$, dimension considerations show that $f^\tau = f^\sigma$, so we have

$$a_{\sigma(\mathfrak{p})}(f) = \tau(a_\mathfrak{p}(f)) \quad \text{for all primes } \mathfrak{p}.$$

A refinement of the Eichler–Shimura Conjecture implies that there exists an abelian surface A/K with $\mathcal{O}_f \subset \mathrm{End}_K(A)$ (that is, A has real multiplication by $\mathcal{O}_f$) such that

$$L(A, s) = L(f, s)L(f^\tau, s).$$

TABLE 1. *Genuine Bianchi newforms of weight* 2 *over* $\mathbb{Q}(\sqrt{D})$ *found in* [**28**].

$|D|$	223	415	455	571	643	1003
$\dim S_2^{\mathrm{bc}}(\mathbf{1})^{\perp}$	2	2	2	2	$1+1$	2
L	$\mathbb{Q}(\sqrt{2})$	$\mathbb{Q}(\sqrt{3})$	$\mathbb{Q}(\sqrt{5})$	$\mathbb{Q}(\sqrt{5})$	$\mathbb{Q}$	$\mathbb{Q}(\sqrt{7})$
$[\mathcal{O}_L : \mathcal{O}_f]$	1	11	2	2	1	1

There are only two pairs $(|D|, \mathrm{Disc}(L))$, where provably the Hecke eigenvalues of the newform f generate the ring of integers of L, namely $(223, 8)$ and $(1003, 28)$. For the first pair, the (conjectured) abelian surface attached to the form is principally polarizable (see [**18**, Corollary 2.12 and Proposition 3.11]) in contrast to the second pair, for which this does not seem to be the case. So we will only focus on the first pair, for which we found the associated abelian surface. In that case, we have $\mathcal{O}_f = \mathbb{Z}[\sqrt{2}]$ (see Table 2 for the Hecke eigenvalues of the form, which we computed using Yasaki's algorithm [**40**] implemented in Magma [**4**]). Our current approach does not allow us to find the remaining surfaces. (We elaborate on this in Remark 5.5.)

For the discriminant $|D| = 223$, since A is principally polarized and has real multiplication by $\mathcal{O}_L = \mathbb{Z}[\sqrt{2}]$, it corresponds to a K-rational point on the Humbert surface $Y_{-}(8)$ of discriminant 8. In their recent paper [**11**], Elkies and Kumar give an explicit rational model for $Y_{-}(8)$ as a double-cover of the weighted projective space $\mathbf{P}^2_{r,s}$. We look for A using this model. In fact, the same heuristic as in [**9**, Proposition 6 and Remark 5] show that A must be the base change of a surface B defined over $\mathbb{Q}$. Indeed, our newform satisfies the identity $f^\tau = f^\sigma$. So by Theorem 4.1, it admits a lift to a classical Siegel newform g of genus 2, weight 2 and level D^2, with integer coefficients. Moreover, g is not a Gritsenko lift. So, assuming Conjecture 1.1, g corresponds to an abelian surface B over $\mathbb{Q}$ such that $\mathrm{End}_{\mathbb{Q}}(B) = \mathbb{Z}$ and $L(B, s) = L(g, s)$. It follows that $A = B \otimes_{\mathbb{Q}} K$.

We recall that, as a double-cover of $\mathbf{P}^2_{r,s}$, $Y_{-}(8)$ is given by the equation

$$z^2 = 2(16rs^2 + 32r^2s - 40rs - s + 16r^3 + 24r^2 + 12r + 2).$$

The Igusa–Clebsch invariants as a point in $\mathbf{P}^3_{(1:2:3:5)}$ are given by

$$(I_2 : I_4 : I_6 : I_{10}) = \left(-\frac{24B_1}{A_1} : -12A : \frac{96AB_1 - 36A_1B}{A_1} : -4A_1B_2 \right),$$

where

$$
\begin{aligned}
A_1 &= 2rs^2, \\
A &= -(9rs + 4r^2 + 4r + 1)/3, \\
B_1 &= (rs^2(3s + 8r - 2))/3, \\
B &= -(54r^2s + 81rs - 16r^3 - 24r^2 - 12r - 2)/27, \\
B_2 &= r^2.
\end{aligned}
$$

The surface B corresponds to the parameters $s = -2$ and $r = 8$. It is the Jacobian of the curve

$$C' : y^2 = 31x^6 + 952x^5 - 5764x^4 - 3750x^3 + 5272x^2 - 7060x + 4783,$$

whose discriminant is $-2^{20}223^{15}$. The curve $C' \otimes_{\mathbb{Q}} K$ admits the global minimal model C listed in Theorem 5.2. One verifies that the curve C has discriminant 1 and integer Igusa–Clebsch

TABLE 2. *Arithmetic data associated to the genuine Bianchi newform f of weight 2 and level (1) over $\mathbb{Q}(\sqrt{-223})$.*

$N\mathfrak{p}$	$\mathfrak{p}$	$a_{\mathfrak{p}}(f)$	$(x^2 - a_{\mathfrak{p}}(f)x + N(\mathfrak{p}))(x^2 - \tau(a_{\mathfrak{p}}(f))x + N(\mathfrak{p}))$
2	$[2, w+1]$	$e-1$	$x^4 + 2x^3 + 3x^2 + 4x + 4$
2	$[2, w+2]$	$-e-1$	$x^4 + 2x^3 + 3x^2 + 4x + 4$
7	$[7, 2w+5]$	$-e+2$	$x^4 - 4x^3 + 16x^2 - 28x + 49$
7	$[7, 2w]$	$e+2$	$x^4 - 4x^3 + 16x^2 - 28x + 49$
9	$[3]$	-3	$x^4 + 6x^3 + 27x^2 + 54x + 81$
17	$[17, 2w+9]$	$-2e-1$	$x^4 + 2x^3 + 27x^2 + 34x + 289$
17	$[17, 2w+6]$	$2e-1$	$x^4 + 2x^3 + 27x^2 + 34x + 289$
19	$[19, 2w+8]$	$e-4$	$x^4 + 8x^3 + 52x^2 + 152x + 361$
19	$[19, 2w+9]$	$-e-4$	$x^4 + 8x^3 + 52x^2 + 152x + 361$
25	$[5]$	0	$x^4 + 50x^2 + 625$
29	$[29, 2w+25]$	$2e+3$	$x^4 - 6x^3 + 59x^2 - 174x + 841$
29	$[29, 2w+2]$	$-2e+3$	$x^4 - 6x^3 + 59x^2 - 174x + 841$
31	$[31, 2w+4]$	$4e-2$	$x^4 + 4x^3 + 34x^2 + 124x + 961$
31	$[31, 2w+25]$	$-4e-2$	$x^4 + 4x^3 + 34x^2 + 124x + 961$

Here, $w = (1 + \sqrt{-223})/2$ and $e = \sqrt{2}$.

invariants

$$I_2 = -24, \quad I_4 = -540, \quad I_6 = 4968, \quad I_{10} = 4096.$$

So $A = \mathrm{Jac}(C)$ has everywhere good reduction.

REMARK 5.5. For the discriminants $|D| = 415, 1003$, the class number of L is 2. The same heuristics as in [**9**, Remark 8] suggest that the surface A is likely not principally polarized. Further, for $|D| = 415, 455, 571$, A does not have real multiplication by the maximal order in L. So finding these surfaces will require working with more general Humbert surfaces for which no explicit models are yet available.

5.2. Proof of modularity

The above discussion already proves Theorem 5.2(a). We will now show that A is modular, hence completing the proof of the theorem. In fact, we already have strong evidence that this is the case. Indeed, let $\mathfrak{p}$ be a prime, and $L_{\mathfrak{p}}(A, s)$ (respectively, $L_{\mathfrak{p}}(f, s)$ and $L_{\mathfrak{p}}(f^\tau, s)$) be the Euler factor of A (respectively, f and f^τ) at $\mathfrak{p}$. As a built-in of our search method, we already know that for each prime $\mathfrak{p}$ listed in Table 2, we have

$$L_{\mathfrak{p}}(A, s) = L_{\mathfrak{p}}(f, s)L_{\mathfrak{p}}(f^\tau, s) = Q_{\mathfrak{p}}(N\mathfrak{p}^{-s})^{-1},$$

where $Q_{\mathfrak{p}}(x) := (1 - a_{\mathfrak{p}}(f)x + N(\mathfrak{p})x^2)(1 - \tau(a_{\mathfrak{p}}(f))x + N(\mathfrak{p})x^2)$.

Let $\lambda \subset \mathcal{O}_L$ be a prime ideal. We recall that the λ-adic Tate module of A is given by

$$T_\lambda(A) := \varprojlim_n A[\lambda^n] \simeq \mathcal{O}_{L,\lambda} \times \mathcal{O}_{L,\lambda},$$

where $\mathcal{O}_{L,\lambda}$ is the completion of $\mathcal{O}_L$ at λ, and

$$A[\lambda^n] = \{x \in A(\overline{\mathbb{Q}}) : \alpha x = 0 \ \forall \alpha \in \lambda^n\}.$$

This is naturally endowed with an action of $\mathrm{Gal}(\overline{\mathbb{Q}}/K)$ giving rise to the λ-adic representation

$$\rho_{A,\lambda} : \mathrm{Gal}(\overline{\mathbb{Q}}/K) \longrightarrow \mathrm{GL}_2(\mathcal{O}_{L,\lambda}).$$

In the rest of this section, we will drop the reference to the prime in our notation as we are only interested in the prime $\lambda = \lambda_2$ (above 2) for which $\mathcal{O}_{L,\lambda_2} = \mathbb{Z}_2[\sqrt{2}]$. So our aim is to show that the λ_2-adic Tate module

$$\rho_A : \mathrm{Gal}(\overline{\mathbb{Q}}/K) \longrightarrow \mathrm{GL}_2(\mathbb{Z}_2[\sqrt{2}])$$

is isomorphic to the representation

$$\rho_f : \mathrm{Gal}(\overline{\mathbb{Q}}/K) \longrightarrow \mathrm{GL}_2(\overline{\mathbb{Z}}_2)$$

associated to f by work of Taylor *et al.* [**3**, **21**, **26**, **37**]. But, in order to do so, we must first determine the coefficient field of ρ_f.

LEMMA 5.6.   *The coefficient field of ρ_f is $\mathbb{Q}_2(\sqrt{2})$, so we have*

$$\rho_f : \mathrm{Gal}(\overline{\mathbb{Q}}/K) \longrightarrow \mathrm{GL}_2(\mathbb{Z}_2[\sqrt{2}]).$$

Proof.   By construction, the image of ρ_f lies in an extension of $\mathbb{Q}_2$ of degree at most 4. The prime 2 is split in K, and the eigenvalues of the Frobenii at the primes above it are distinct and do not add up to zero (see Table 2), so by [**37**, Corollary 1] we can take the coefficient field L_f to be the one given by the polynomial $x^4 + 2x^3 + 3x^2 + 4x + 4$. But, there are two split primes above 2 in L_f, and the completion of L_f at either of them is isomorphic to $\mathbb{Q}_2(\sqrt{2})$. So the image of ρ_f is in fact contained in $\mathrm{GL}_2(\mathbb{Z}_2[\sqrt{2}])$.   □

From Lemma 5.6, we now have

$$\rho_A, \rho_f : \mathrm{Gal}(\overline{\mathbb{Q}}/K) \longrightarrow \mathrm{GL}_2(\mathbb{Z}_2[\sqrt{2}]).$$

We denote their reductions modulo 2 by $\bar{\rho}_A$ and $\bar{\rho}_f$, respectively. We will show that $\rho_A \simeq \rho_f$ by making use of the following version of the so-called Faltings–Serre criterion [**10**, **34**].

THEOREM 5.7 (Faltings–Serre).   *Let*

$$\rho_1, \rho_2 : \mathrm{Gal}(\overline{\mathbb{Q}}/K) \longrightarrow \mathrm{GL}_2(\overline{\mathbb{Z}}_2)$$

be two continuous representations, whose reductions modulo 2 are $\bar{\rho}_1$ and $\bar{\rho}_2$. Suppose that

  (i) $\det(\rho_1) = \det(\rho_2)$;
 (ii) ρ_1 and ρ_2 are unramified outside a finite set of primes S;
(iii) $\bar{\rho}_1$ and $\bar{\rho}_2$ are absolutely irreducible and isomorphic.

Then there exists a computable finite set of primes T such that $\rho_1 \simeq \rho_2$ if and only if

$$\mathrm{Tr}(\rho_1(\mathrm{Frob}_{\mathfrak{p}})) = \mathrm{Tr}(\rho_2(\mathrm{Frob}_{\mathfrak{p}}))$$

for all $\mathfrak{p} \in T$.

There is an explicit description of the set T, which we recall here for the sake of completeness (see [**10**] for more details on this). Assume that the image lies in $\mathrm{GL}_2(\mathbb{F}_2)$, which will be the case in our example.

Let M be the fixed field of $\mathrm{Im}(\bar{\rho}_1)$, the residual image of ρ_1. (We have the same fixed field for $\mathrm{Im}(\bar{\rho}_2)$ since $\bar{\rho}_1$ and $\bar{\rho}_2$ are isomorphic.) Let $M_2^0(\mathbb{F}_2)$ be the set of all trace zero 2×2

matrices with coefficients in $\mathbb{F}_2$. We note that this is a 2-group of order 8, and we consider the set of all extensions $\widetilde{M}$ of M that are unramified outside S, such that $\widetilde{M}$ is Galois over K and $\mathrm{Gal}(\widetilde{M}/K) \simeq M_2^0(\mathbb{F}_2) \rtimes \mathrm{Im}(\overline{\rho}_1)$. Each such $\widetilde{M}$ is a compositum of quadratic extensions of M, so there is a canonical isomorphism $\varphi_{\widetilde{M}} : \mathrm{Gal}(\widetilde{M}/K) \simeq M_2^0(\mathbb{F}_2) \rtimes \mathrm{Im}(\overline{\rho}_1)$. For algorithmic purpose, this set is determined explicitly using class field theory (see [**10**, Lemma 5.6]). For each $\widetilde{M}$, we then find a prime ideal $\mathfrak{p}_{\widetilde{M}} \subset \mathcal{O}_K$ such that $\varphi_{\widetilde{M}}(\mathrm{Frob}_{\mathfrak{p}_{\widetilde{M}}}) = (A, B)$ with $\mathrm{Tr}(AB) \neq 0$. The set $\{\mathfrak{p}_{\widetilde{M}}\}$ thus obtained has the desired properties.

Proof of Theorem 5.2(b). We will use Theorem 5.7 to show that $\rho_A \simeq \rho_f$; hence that A is modular.

We note that ρ_A and ρ_f are unramified away from λ_2 (for ρ_f this uses the local–global compatibility results in [**26**]) so they satisfy (ii) with $S = \{\mathfrak{p} \mid 2\}$. Condition (i) is satisfied by ρ_f since f has trivial Nebentypus. It is satisfied by ρ_A by basic properties of Tate modules. So we only need to check (iii), find the set T and show that

$$\mathrm{Tr}(\rho_A(\mathrm{Frob}_{\mathfrak{p}})) = \mathrm{Tr}(\rho_f(\mathrm{Frob}_{\mathfrak{p}})) \quad \text{for all } \mathfrak{p} \in T.$$

Since $\mathcal{O}_L/\lambda_2 = \mathbb{F}_2$, $\mathrm{Im}(\overline{\rho}_f)$ and $\mathrm{Im}(\overline{\rho}_A)$ are both contained in $\mathrm{GL}_2(\mathbb{F}_2) \simeq S_3$, where S_3 is the symmetric group on three elements.

CLAIM 5.8. $\mathrm{Im}(\overline{\rho}_A) = \mathrm{Im}(\overline{\rho}_f) = \mathrm{GL}_2(\mathbb{F}_2)$. In particular, $\overline{\rho}_A$ and $\overline{\rho}_f$ are absolutely irreducible.

First, recall that $\mathrm{Tr}(\overline{\rho}_f(\mathrm{Frob}_{\mathfrak{p}})) = a_{\mathfrak{p}}(f) \bmod \lambda_2$, and that since ρ_f is unramified away from 2, $\overline{\rho}_f(\mathrm{Frob}_{\mathfrak{p}})$ is either trivial or has order 3 for every odd prime $\mathfrak{p}$. In particular, for $\mathfrak{p} = (3)$, $\mathrm{Tr}(\overline{\rho}_f(\mathrm{Frob}_{\mathfrak{p}})) = -3 = 1 \in \mathbb{F}_2$ implies that $\overline{\rho}_f(\mathrm{Frob}_{\mathfrak{p}})$ has order 3. Next, for $\mathfrak{p} \mid 2$, the representation ρ_f is ordinary at $\mathfrak{p}$, so the restriction of $\overline{\rho}_f$ to the decomposition group $D_{\mathfrak{p}}$ is of the form

$$\overline{\rho}_f|_{D_{\mathfrak{p}}} \sim \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \mod \lambda_2.$$

If $\mathrm{Im}(\overline{\rho}_f|_{D_{\mathfrak{p}}})$ was trivial for both primes $\mathfrak{p} \mid 2$, then $\overline{\rho}_f$ would be unramified at 2 (and hence everywhere). In that case, the fixed field of $\ker(\overline{\rho}_f)$ would be an unramified cubic extension of K, which is impossible since the class number of K is 7. So $\mathrm{Im}(\overline{\rho}_f)$ contains an element of order 2, hence $\mathrm{Im}(\overline{\rho}_f) = \mathrm{GL}_2(\mathbb{F}_2)$. Similarly, one shows that $\mathrm{Im}(\overline{\rho}_A) = \mathrm{GL}_2(\mathbb{F}_2)$.
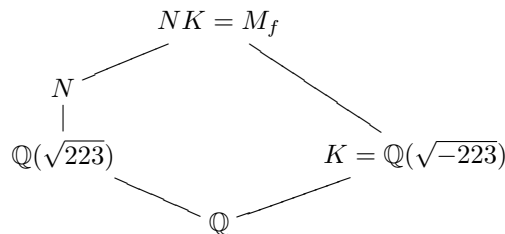
Next, let M_f and M_A be the Galois extensions of K cut out by $\overline{\rho}_f$ and $\overline{\rho}_A$, respectively. Then M_A and M_f are S_3-extensions of K ramified at λ_2 only. To check (iii), we will show that there is a unique such extension. This will force the isomorphism $M_f \simeq M_A$.

Let N_A and N_f be the respective normal closures of M_A and M_f over $\mathbb{Q}$. Since $f^{\sigma} = f \bmod \lambda_2$, $\overline{\rho}_f$ is a base change, so $\mathrm{Gal}(N_f/\mathbb{Q}) \simeq \mathbb{Z}/2\mathbb{Z} \times S_3$. Similarly, we have $\mathrm{Gal}(N_A/\mathbb{Q}) \simeq \mathbb{Z}/2\mathbb{Z} \times S_3$. So each extension comes from an S_3-extension of $\mathbb{Q}$ which is unramified outside $\{2, 223\}$. Now we will show that there is a *unique* such S_3-extension $N/\mathbb{Q}$.

First, we observe that there are exactly seven possible quadratic subfields, given by $\mathbb{Q}(\sqrt{d})$ with $d = -1, 2, -2, 223, -223, 2 \cdot 223, -2 \cdot 223$. From Table 2, we know that $\mathrm{Tr}(\overline{\rho}_f(\mathrm{Frob}_{\mathfrak{p}}))$ is odd for the primes above $3, 17, 29$. The prime 3 is inert for $d = -1, 2, -223$ and $2 \cdot 223$. The prime 29 is also inert for $d = -2$ and $-2 \cdot 223$. So we deduce that the only possible quadratic subfield is $\mathbb{Q}(\sqrt{223})$.

We now turn to the cubic extension $N/\mathbb{Q}(\sqrt{223})$. Since our representation $\overline{\rho}_f$ is unramified at 223, this extension cannot be ramified at the prime above 223, so it is only ramified at λ_2.

To see this, consider the following diagram:

$$
\begin{array}{c}
NK = M_f \\
\diagup \qquad \diagdown \\
N \qquad\qquad\qquad\qquad\qquad \\
\mid \qquad\qquad\qquad\qquad\qquad \\
\mathbb{Q}(\sqrt{223}) \qquad\qquad K = \mathbb{Q}(\sqrt{-223}) \\
\diagdown \qquad \diagup \\
\mathbb{Q}
\end{array}
$$

and note that on the right the ramification index of 223 is 2. So the same holds on the left. Again using class field theory, we see that there is a unique such extension $N/\mathbb{Q}$ given by the polynomial

$$
x^6 - 2x^5 - 29x^4 + 90x^3 - 58x^2 - 8x + 8.
$$

So $\bar{\rho}_A \simeq \bar{\rho}_f$ and $\{\rho_A, \rho_f\}$ satisfy all three hypotheses of Theorem 5.7.

We conclude that ρ_f and ρ_A are isomorphic using the same recipe as in [10]. For this, we need to compare the traces of the Frobenii at the primes in $\mathcal{O}_K$ that lie above the rational primes in $\{3, 5, 7, 19, 29, 31\}$. Since we already know that the traces for these particular primes match, we are done. □

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Tobias Berger
School of Mathematics and Statistics
University of Sheffield
Hicks Building
Hounsfield Road
Sheffield
S3 7RH
United Kingdom

t.t.berger@sheffield.ac.uk

Ariel Pacetti
Departamento de Matemática
Universidad de Buenos Aires
Ciudad Universitaria and IMAS
1428 Buenos Aires
Argentina

apacetti@dm.uba.ar

Lassina Dembélé
Mathematics Institute
University of Warwick
Zeeman Building
Coventry
CV4 7AL
United Kingdom

l.dembele@warwick.ac.uk

Mehmet Haluk Şengün
School of Mathematics and Statistics
University of Sheffield
Hicks Building
Sheffield
S3 7RH
United Kingdom

M.Sengun@sheffield.ac.uk
http://haluksengun.staff.sheffield.ac.uk