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Local maximal function and weights in a general setting

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Abstract For a proper open set Ω immersed in a metric space with the weak homogeneity property, and given a measure μ doubling on a certain family of balls lying "well inside" of Ω, we introduce a local maximal function and characterize the weights w for which it is bounded on $L^p(\Omega, wd\mu)$ when $1 < p < \infty$ and of weak type $(1, 1)$. We generalize previous known results and we also present an application to interior Sobolev's type estimates for appropriate solutions of the differential equation $\Delta^m u = f$, satisfied in an open proper subset Ω of $\mathbb{R}^n$. Here, the data f belongs to some weighted L^p space that could allow functions to increase polynomially when approaching the boundary of Ω.

1 Introduction

We start by describing the setting in which we are going to prove the main results of this work. Let X be a metric space satisfying the weak homogeneity property, that is, there is a fixed number N such that for any ball $B(x, r)$ there are no more than N points in the ball whose distance from each other is $>r/2$. As it is easy to check, the same property holds if we replace $r/2$ by any other fraction λr with $0 < \lambda < 1$. It is clear that this property of the metric space, sometimes called geometrically doubling,

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implies separability. Also, we will denote by Ω any open proper and non empty subset of X such that all balls contained in Ω are connected sets.

For $0 < \beta < 1$ we consider a family of balls contained in Ω defined by

$$\mathcal{F}_\beta = \{B = B(x_B, r_B) : x_B \in \Omega,\ r_B \le \beta\, d(x_B, \Omega^c)\},$$

where $B(x_B, r_B)$ denotes the ball with center x_B and radius r_B and $d(x_B, \Omega^c)$ the distance from x_B to the complementary set of Ω. Observe that the function $d(\cdot, \Omega^c)$ is >0 over Ω and hence $\mathcal{F}_\beta$ contains balls centered at each point of Ω that are small enough.

Given a Borel measure μ defined on Ω such that $0 < \mu(B) < \infty$ for any ball $B \in \mathcal{F} = \bigcup_{0<\alpha<1} \mathcal{F}_\alpha$, we shall say that μ is doubling on $\mathcal{F}_\beta$ if there is some constant C_β such that for any ball $B \in \mathcal{F}_\beta$

$$\mu(B) \le C_\beta\, \mu\left(\frac{1}{2}B\right), \tag{1.1}$$

where, as usual, λB means the ball with the same center and radius λ-times that of B, for any positive λ. Let us remark that inequality (1.1) also holds if we replace $1/2$ by any other λ, $0 < \lambda < 1$, but with perhaps a different constant, depending on λ. Also notice that the doubling condition on $\mathcal{F}_\beta$ implies, except the trivial case $\mu \equiv \infty$, the finitness of $\mu(B)$ for $B \in \mathcal{F}_\beta$, but that is not necessarily true for all the balls in $\mathcal{F}$.

Given $0 < \beta < 1$ and μ as above, we may define the following maximal functions on Ω

$$M_{\mu,\beta} f(x) = \sup_{x\in B\in\mathcal{F}_\beta} \frac{1}{\mu(B)} \int_B |f|\, d\mu \tag{1.2}$$

for any $f \in L^1_{\text{loc}}(\Omega, d\mu)$ and $x \in \Omega$. Let us point out that here, local integrability means that the function is integrable over any ball such that its closure is contained in Ω.

We note that $M_{\mu,\beta} f$ is point wisely bounded by the Hardy-Littlewood maximal function $M_\mu f$. Hence, in particular, $M_{\mu,\beta}$ shares all the boundedness properties of M_μ. Notice that, however, under our assumptions on μ we can not say much about the latter operator.

It is our purpose in this work to provide necessary and sufficient conditions on weights w, such that the operator $\mathcal{M}_{\mu,\beta}$ turns to be bounded on $L^p(\Omega, w d\mu)$, for $1 < p < \infty$, and of weak type $(1, 1)$ with respect to $w d\mu$, whenever μ is doubling on $\mathcal{F}_\beta$. The conditions that will be imposed on the weights are in the spirit of the A_p-Muchenhoupt classes for the Hardy-Littlewood Maximal function (see [8]) but restricted to balls in the family $\mathcal{F}_\beta$. To be more precise, we introduce the classes of weights $A^\beta_p(d\mu)$ as those weights w defined on Ω satisfying

$$\sup_{B\in\mathcal{F}_\beta} \frac{1}{\mu(B)} \left(\int_B w d\mu\right)^{1/p} \left(\int_B w^{-p'/p} d\mu\right)^{1/p'} = \mathcal{C}_{p,\beta} < \infty, \tag{1.3}$$

where p' denotes the conjugate exponent of p. In the case $p = 1$, the second factor should be understood as the μ-essential supremum of w^{-1} taken over the ball B.

Let us remark that when $X = \mathbb{R}$ and $\Omega = (0, \infty)$, the local maximal operator and the corresponding classes of weights have been considered by Nowak and Stempak in [9] and are the inspiration of our work. Also we cover the situation solved in [6] which, in our setting, corresponds to $X = \mathbb{R}^n$ with the d_∞ metric and $\Omega = \mathbb{R}^n \setminus \{0\}$ with μ the restriction of the Lebesgue measure. There, the authors based their proof on a geometrical lemma built for the specific case under consideration. That lemma allows them to solve the problem by extending a restriction of a local A_p-weight to a global A_p-weight in order to apply the well known boundedness results for the Hardy-Littlewood maximal function. Such technique seems difficult to adapt to our general setting.

Our precise result is the following

Theorem 1.1 *Let X be a metric space with the weak homogeneity property and Ω a proper open subset of X such that the balls contained in Ω are connected sets. Let $0 < \beta < 1$ and μ a Borel measure on Ω satisfying the doubling property on $\mathcal{F}_\beta$. Then, for the associated maximal operator $M_{\mu,\beta}$, we have*

(i) *$M_{\mu,\beta}$ is of weak type (1,1) with respect to $w\,d\mu$, that is, there exists a constant C such that the inequality*

$$\int_{\{M_{\mu,\beta} f(x)>0\}} w\,d\mu \le \frac{C}{\lambda} \int |f|\, w\,d\mu$$

holds for any $\lambda > 0$, if and only if $w \in A_1^\beta(d\mu)$.

(ii) *For $1 < p < \infty$, $M_{\mu,\beta}$ is bounded on $L^p(\Omega, w d\mu)$ if and only if $w \in A_p^\beta(d\mu)$.*

In order to prove that the condition on the weight is sufficient for the weak type $(1, 1)$ we will proceed as in the classical case, starting from Vitali's covering lemma.

Since a dilation of a ball in $\mathcal{F}_\beta$ may not longer belong to that family, we can not apply the doubling property (1.1) in such case. So we change a little bit Vitali's Lemma to adapt it to our situation, namely, replacing the dilation of a ball, when it does not belong to $\mathcal{F}_\beta$, by an appropriate open set.

To overcome the problem of measuring such sets with a measure that is doubling only on $\mathcal{F}_\beta$, we decompose our space Ω into small enough balls, in the spirit of Whitney's Lemma.

Notice that both mentioned lemmas are of a geometric nature so we do not need to have a measure defined on Ω. Nevertheless, to obtain our version of Whitney's Lemma, we shall need the weak homogeneity property in our ambient space X, but the assumption of the connectedness of the balls lying in Ω will not be needed at this stage.

The local Whitney's Lemma will be the clue step to obtain not only the weak type $(1, 1)$ but also an interesting property of measures doubling on $\mathcal{F}_\beta$, namely that they also are doubling on any other family $\mathcal{F}_\gamma$, $0 < \gamma < 1$, but with a constant depending on the parameter γ. This property, in turn, will imply that the classes $A_p^\beta(d\mu)$ are also independent of β, as it was shown by Nowak and Stempak in the case $\Omega = (0, \infty)$ and by Lin and Stempak for $\Omega = \mathbb{R}^n \setminus 0$. In view of this fact, later on, we shall refer to these weights as $A_{p,loc}(d\mu)$.

In order to obtain the strong type results, we will adapt Lerner's argument given in [5] to our situation, strongly using the weak type result for $p = 1$.

Finally, we shall use $A_{p,loc}(d\mu)$ weights to obtain some interior *a priori* weighted Sobolev estimates for solutions of the elliptic differential equation

$$\Delta^m u = f, \tag{1.4}$$

where m is a positive integer and the equation is satisfied inside an open subset Ω immersed in $\mathbb{R}^n$. More precisely, if $\delta(x)$ denotes the distance from the point x to Ω^c, for $1 < p < \infty$, we introduce the following weighted Sobolev spaces of order $2m$ by

$$W^{2m,p}_{\delta,w}(\Omega) = \left\{ f \in L^1_{\text{loc}}(\Omega) : \|f\|_{W^{2m,p}_{\delta,w}} = \sum_{|\beta|\le 2m} \|\delta^{|\beta|} D^\beta f\|_{L^p_w(\Omega)} < \infty \right\},$$

where the derivatives should be understood in the weak sense and $L^p_w(\Omega)$ is the weighted Lebesgue space $L^p(\Omega, w dx)$.

With this notation, under the assumption that w belongs to $A_{p,loc}(d\mu)$, we shall prove that a solution of (1.4) satisfies the interior Sobolev estimate

$$\|u\|_{W^{2m,p}_{\delta,w}(\Omega)} \le C \left(\|u\|_{L^p_w(\Omega)} + \|\delta^{2m} f\|_{L^p_w(\Omega)} \right). \tag{1.5}$$

Let us observe that, as it is easy to check, $w(x) = \delta^\alpha(x)$ belongs to $A_{p,loc}(d\mu)$ for any exponent $\alpha \in \mathbb{R}$. Therefore the data function f appearing on the right hand side of (1.4) could increase polynomially when approaching the boundary of Ω and still we might have some control for the derivatives of the solution up to the order $2m$.

Let us mention that weighted Sobolev estimates up to the boundary have been obtained recently for Muckenhoupt weights and bounded domains Ω with smooth boundary (see [3]).

At this point we would like to mention that during the review process of this article, one of the referees called our attention to recent work by Lin, Stempak and Wang (see [7]), published after our paper was sent for revision. There, they also deal with local maximal functions on measure metric spaces in a quite general setting. In particular, in [7, Sect. 5] a situation resembling ours is considered. Nevertheless, there is not overlapping between their results and ours.

The paper is organized as follows. In the next section we present appropriate versions of the geometric results that will be needed later. Section 3 is devoted to prove the main results regarding the local maximal functions and weights. Finally, in Sect. 4, we use local weights to obtain interior estimates for solutions of the differential Eq. (1.4).

2 Geometric lemmas

Let us remind that for any metric space X, the following Vitali's covering Lemma holds. See for example Theorem 1.2 in [4] for a proof.

Lemma 2.1 (Vitali) *Let X be a metric space and Γ a family of balls with bounded radii, i.e., satisfying* $\sup_{B\in\Gamma} r(B) < \infty$*. Then there exists a disjoint subfamily Λ with the property*

$$\forall B \in \Gamma \quad \exists B' \in \Lambda \text{ such that } B \cap B' \neq \emptyset \text{ and } B \subset 5B' \tag{2.1}$$

Let us observe that if we assume the space X to be separable, then the subfamily Λ must be at most countable.

With this in mind we can easily obtain the following "local" Vitali covering lemma, useful to our purposes. Before stating our version let us introduce some notation.

For $0 < \beta < 1$, given a ball $B \in \mathcal{F}_\beta$, we shall denote by $\tilde{B}$ the set

$$\tilde{B} = 5B, \quad \text{if} \quad 5B \in \mathcal{F}_\beta \quad \text{or} \quad \tilde{B} = \bigcup_{\substack{R\cap B\neq\emptyset \\ R\in\mathcal{F}_\beta}} R, \quad \text{otherwise.} \tag{2.2}$$

In the latter case we shall refer to $\tilde{B}$ as the "cloud" of B and it will be denoted by $\mathcal{N}_\beta(B)$.

Lemma 2.2 (local Vitali) *Let X be a separable metric space and Ω an open proper subset of X. Let $0 < \beta < 1$ and Γ a family of balls belonging to $\mathcal{F}_\beta$ with uniformly bounded radii. Then, there exists a disjoint and at most countable subfamily Λ such that the collection of open sets $\{\tilde{B}\}_{B\in\Lambda}$, with $\tilde{B}$ defined by (2.2), still covers $\bigcup_{B\in\Gamma} B$.*

The proof of the lemma is quite obvious since, applying the original Vitali's Lemma, in view of (2.1), we only need to take care of those balls $B \in \Gamma$ such that $5B$ is not a ball in $\mathcal{F}_\beta$. In such case, using (2.1) again, we know that there is a ball $B' \in \Lambda$ with $B \cap B' \neq \emptyset$ and since now $\tilde{B'} = \mathcal{N}_\beta(B')$, it is clear that B is contained in $\tilde{B'}$.

Now we state and prove a special version of Whitney's Lemma which will be the clue to prove, later on, that the cloud of a "large" ball belonging to $\mathcal{F}_\beta$ has a measure equivalent to that of the initial ball, assuming that the measure is doubling on $\mathcal{F}_\beta$. Even though we only ask X to be a metric space for the previous Lemma, we shall need now the full strength of the assumption made on X at the beginning of the article, that is, we ask X to have the weak homogeneity property.

Lemma 2.3 (Whitney for $\mathcal{F}_\beta$) *Let X be a metric space with the weak homogeneity property and Ω a fixed open and proper subset. Then, given $0 < \beta < 1$, for each a, $0 < a < \beta/80$, there exists a covering $\mathcal{W}_a$ of Ω by balls of $\mathcal{F}_\beta$ with the following properties*

(i) *If $P = B(x_P, r(P)) \in \mathcal{W}_a$, then $10P \in \mathcal{F}_\beta$ and moreover*

$$\frac{1}{2} a\, d(x_P, \Omega^c) \leq r(P) \leq a\, d(x_P, \Omega^c). \tag{2.3}$$

(ii) *If P and P' belong to $\mathcal{W}_a$ and $P \cap P' \neq \emptyset$ then $P' \subset 5P$ and $P \subset 5P'$.*

(iii) *There is a number M, only depending on β and a, such that for any ball $B_0 = B(x_0, r_0) \in \mathcal{F}_\beta$ with $5B_0 \notin \mathcal{F}_\beta$, the cardinality of the set*

$$\mathcal{W}_a(B_0) = \{P \in \mathcal{W}_a : P \cap \mathcal{N}(B_0) \neq \emptyset\}$$

is at most M. Moreover if P_0 is such that $x_0 \in P_0$ with $P_0 \in \mathcal{W}_a(B_0)$ then $P_0 \subset \frac{1}{2}B_0$.

Proof Given $k \in \mathbb{Z}$, we define

$$\Omega_k = \{x \in \Omega : 2^{k-1} \leq d(x, \Omega^c) < 2^k\}.$$

If Ω_k is non empty, we choose a maximal net of points in Ω_k whose distances from each other is at least $a2^{k-1}$. Since X is separable, each net is at most countable. Let us call $\{x_i^k\}_{i \in J_k}$ with $J_k \subset \mathbb{N}$, to the net corresponding to Ω_k. Define

$$\mathcal{W}_a = \{P_i^k = B(x_i^k, a2^{k-1}), i \in J_k,\ k \in \mathbb{Z}\}.$$

Clearly for each fixed k, in view of the maximality of the chosen net, the balls $\{B(x_i^k, a2^{k-1})\}_{i \in J_k}$ cover Ω_k. Also, since $x_i^k \in \Omega_k$ and $a < \beta/10$ we have

$$\frac{a}{2}\, d(x_i^k, \Omega^c) \leq \frac{a}{2} 2^k = r(P_i^k) \leq a\, d(x_i^k, \Omega^c) \leq \frac{\beta}{10} d(x_i^k, \Omega^c),$$

and hence (2.3) holds and $10P_i^k$ belongs to $\mathcal{F}_\beta$, finishing the proof of (i).

Next let us take P and P' two members of $\mathcal{W}_a$ with $P \cap P' \neq \emptyset$. Assume that their centers x_P and $x_{P'}$ belong to Ω_k and to $\Omega_{k'}$ respectively. Choosing $z \in P \cap P'$, we have

$$\begin{aligned} 2^{k-1} &\leq d(x_P, \Omega^c) \leq d(x_{P'}, \Omega^c) + d(x_{P'}, z) + d(z, x_P) \\ &\leq 2^{k'} + r(P') + r(P) = \left(1 + \frac{a}{2}\right) 2^{k'} + a2^{k-1} \end{aligned}$$

and hence

$$(1 - a)2^{k-1} \leq \left(1 + \frac{a}{2}\right) 2^{k'}.$$

Consequently,

$$2^{k-k'} \leq 2\frac{1 + a/2}{1 - a} < 4,$$

since $a < \frac{\beta}{80} < \frac{2}{5}$. Thus, by symmetry, it follows that $|k - k'| < 2$, which implies that either $k' = k - 1$ or $k' = k$ or $k' = k + 1$.

To prove that $P' \subset 5P$ we assume that the worst of the three cases occurs, that is, $k' = k + 1$. In this situation we have $r(P) = a2^{k-1}$ and $r(P') = a2^k = 2r(P)$. Therefore for $y \in P'$ and $z \in P \cap P'$

$$\begin{aligned} d(y, x_P) &\le d(y, x_{P'}) + d(x_{P'}, z) + d(z, x_P) \\ &< r(P') + r(P') + r(P) \le 5r(P) \end{aligned}$$

Hence we have proved $P' \subset 5P$. By the symmetry of the situation it is also true that $P \subset 5P'$.

Now we turn to the proof of (iii). To do that we shall first prove three claims.

Claim 1 *If two balls of the family $\mathcal{F}_\beta$ intersect each other, then their centers lie in nearby bands Ω_j. More precisely, let B and B' in $\mathcal{F}_\beta$ be such that $B \cap B' \neq \emptyset$ and assume that $x_B \in \Omega_k$, then $x_{B'} \in \bigcup_{i=k-m}^{i=k+m} \Omega_i$ for some m, only depending on β.*

In fact, taking z in the intersection of the two balls, we have

$$\begin{aligned} d(x_B, \Omega^c) &\le d(x_{B'}, \Omega^c) + d(x_{B'}, z) + d(z, x_B) \\ &\le d(x_{B'}, \Omega^c) + r(B') + r(B), \end{aligned}$$

and so

$$(1 - \beta) d(x_B, \Omega^c) \le (1 + \beta) d(x_{B'}, \Omega^c)$$

Therefore, by symmetry, we arrive to

$$\frac{1-\beta}{1+\beta} d(x_B, \Omega^c) \le d(x_{B'}, \Omega^c) \le \frac{1+\beta}{1-\beta} d(x_B, \Omega^c) \tag{2.4}$$

Since we assume that $x_B \in \Omega_k$, denoting by m the unique positive integer such that $2^{m-1} \le \frac{1+\beta}{1-\beta} < 2^m$, we have

$$2^{-m+k-1} \le d(x_{B'}, \Omega^c) \le 2^{m+k},$$

and hence $x_{B'} \in \bigcup_{i=k-m}^{i=k+m} \Omega_i$. Noticing that m only depends on β we finish the proof of the claim.

Claim 2 *If $B \in \mathcal{F}_\beta$ with $x_B \in \Omega_k$ then there exists a fixed integer n, only depending on β, such that $B \subset \bigcup_{j=k-n}^{j=k+1} \Omega_j$.*

In fact for $y \in B$

$$d(x_B, \Omega^c) - r(B) < d(y, \Omega^c) < d(x_B, \Omega^c) + r(B),$$

and since $r(B) \le \beta d(x_B, \Omega^c)$ we have

$$(1 - \beta) d(x_B, \Omega^c) < d(y, \Omega^c) < (1 + \beta) d(x_B, \Omega^c).$$

Hence, denoting by n the integer such that $2^{-n} \leq 1-\beta < 2^{-n+1}$, using that $x_B \in \Omega_k$ and that $1+\beta < 2$ we arrive to

$$2^{k-n-1} < d(y, \Omega^c) < 2^{k+1},$$

proving the claim.

Claim 3 *If $P \in \mathcal{W}_a$ and $P \cap \Omega_j \neq \emptyset$, then $x_P \in \Omega_{j-1} \cup \Omega_j \cup \Omega_{j+1}$.*

As above, for $y \in P \cap \Omega_j$, we have

$$d(y, \Omega^c) - r(P) < d(x_P, \Omega^c) < d(y, \Omega^c) + r(P).$$

Then, using (i) and that $y \in \Omega_j$ we obtain

$$\frac{2^{j-1}}{1+a} < d(x_P, \Omega^c) < \frac{2^j}{1-a}$$

Since, in particular, $a < 1/2$, the claim follows.

Now we turn to the proof of (iii).

Let B_0 be a ball $B_0 = B(x_0, r_0) \in \mathcal{F}_\beta$ with $5B_0 \notin \mathcal{F}_\beta$, that is

$$\frac{\beta}{5} d(x_0, \Omega^c) < r_0 \leq \beta d(x_0, \Omega^c),$$

and assume $x_0 \in \Omega_{k_0}$. For any $B = B(x_B, r(B)) \in \mathcal{F}_\beta$ such that $B \cap B_0 \neq \emptyset$, in view of Claims 1 and 2, we have that

$$B \subset \bigcup_{j=k_0-m-n}^{k_0+m+1} \Omega_j,$$

and, consequently, taking the union over such balls B we get

$$\mathcal{N}_\beta(B_0) \subset \bigcup_{j=k_0-m-n}^{k_0+m+1} \Omega_j.$$

Now, if $P \in \mathcal{W}_a$ with $P \cap \mathcal{N}(B_0) \neq \emptyset$, using Claim 3, we may conclude that its center, x_P, belongs to Ω_j for some $k_0 - m - n - 1 \leq j \leq k_0 + m + 2$. From here, for such P, that is $P \in \mathcal{W}_a(B_0)$, there is a ball $B \in \mathcal{F}_\beta$ intersecting P. Taking $z \in P \cap B$ and $u \in B \cap B_0$, we have

$$\begin{aligned} d(x_P, x_0) &\leq d(x_P, z) + d(z, u) + d(u, x_0) \\ &< r(P) + 2r(B) + r_0. \end{aligned} \tag{2.5}$$

To estimate $r(B)$ we use (2.4) and that $5B_0 \notin \mathcal{F}_\beta$ obtaining

$$r(B) \leq \beta\, d(x_B, \Omega^c) \leq \beta \frac{1+\beta}{1-\beta}\, d(x_0, \Omega^c) \leq 5 \frac{1+\beta}{1-\beta}\, r_0.$$

For $r(P)$ we use $x_P \in \bigcup_{j=k_0-m-n-1}^{k_0+m+2} \Omega_j$ to get

$$\begin{aligned} r(P) &= a\, 2^{j-1} \leq a\, 2^{k_0+m+1} \leq\ 4a\, 2^m d(x_0, \Omega^c) \\ &\leq \frac{40a}{\beta} \frac{1+\beta}{1-\beta} r_0 \leq \frac{1}{2} \frac{1+\beta}{1-\beta} r_0, \end{aligned}$$

where we used again that $5B_0 \notin \mathcal{F}_\beta$, the definition of m and $a < \beta/80$. Coming back to (2.5) we obtain

$$d(x_P, x_0) \leq 12 \frac{1+\beta}{1-\beta} r_0 = c_\beta r_0.$$

In this way we can conclude that for each fixed j, with $k_0-m-n-1 \leq j \leq k_0+m+2$,

$$\Delta_j(B_0) = \{x_P,\, P \in \mathcal{W}_a(B_0)\} \cap \Omega_j \subset c_\beta B_0$$

and also, for two elements in $\Delta_j(B_0)$, by the way they were chosen, we have

$$d(x_P, x_{P'}) \geq a\, 2^{j-1} \geq a 2^{k_0-m-n-2} \geq \frac{5a}{\beta}\, 2^{-m-n-2} r_0.$$

Applying the weak homogeneity property of the space X, the cardinality of $\Delta_j(B_0)$ is at most a number N that depends only on a and β, but not on B_0. Let us note that the constant c_β is in fact large enough to make such dilation of B_0 intersect Ω^c, so we are really using that the homogeneity property holds in the whole ambient space X.

Keeping in mind that the range of j contains at most $2m+n+4$ elements, we have proved that

$$\sharp \mathcal{W}_a(B_0) \leq (2m+n+4)\ N =\ M(a, \beta),$$

as required.

To prove the last assertion of (iii), suppose that $P_0 \in \mathcal{W}_a$ is such that $x_0 \in P_0$ and notice that we only need to show that $2r(P_0) < \frac{1}{2} r_0$.

Since $x_0 \in \Omega_{k_0} \cap P_0$, by Claim 3, its center x_{P_0} belongs to $\Omega_{k_0-1} \cup \Omega_{k_0} \cup \Omega_{k_0+1}$ and hence

$$2r(P_0) \leq a\, 2^{k_0+1}.$$

On the other hand

$$r_0 \geq \frac{\beta}{5}\, d(x_0, \Omega^c) \geq \frac{\beta}{5}\, 2^{k_0-1}.$$

Since we are assuming $a < \beta/80$, the desired inequality holds. □

Remark 2.4 As a consequence of (iii), it is easy to see that $\mathcal{W}_a$ possesses the finite overlapping property, that is, no point in Ω belongs to more than a fixed number of members of the covering. Further, that is also true for the covering $\{2P\}_{P\in\mathcal{W}_a}$. In fact, as in the proof of (ii), we can check that $2P \cap 2P' \neq \emptyset$ and $x_P \in \Omega_k$ imply that $x_{P'}$ is in the same or in a neighbour band Ω_j. Hence, as in there, $2P' \subset 10P$. From the weak homogeneity property, the sets $\left\{x_{P'} \in \Omega_j : P' \in \mathcal{W}_a,\ 2P' \cap 2P \neq \emptyset\right\}$, for $k-1 \le j \le k+1$, have at most a fixed cardinal, independent of P. With this we may conclude that for some number N, depending only on a and β, we have

$$\sum_{P\in\mathcal{W}a} \chi_{2P} \le N. \tag{2.6}$$

3 Local maximals and local weights

In this section we shall prove the main results concerning the continuity of the local maximal functions $M_{\mu,\beta}$ on weighted L^p spaces as well as the independence from the parameter β of the classes of local weights defined in Ω. To do so, we are going to need all the assumptions on X and Ω stated at the beginning of the introduction. Also, we assume that we have a Borel measure μ defined on Ω, and satisfying the doubling property on $\mathcal{F}_\beta$ for some fixed value of β.

We begin with a kind of technical lemma that tells us how to carefully measure the clouds of "large" balls in $\mathcal{F}_\beta$. Its proof will strongly rely on the construction and properties of the covering of Ω presented in the previous section.

Lemma 3.1 *Let X be a metric space with the weak homogeneity property, Ω a proper open subset such that all the balls contained in Ω are connected sets and μ a measure with the doubling property on $\mathcal{F}_{\beta/2}$. Then, for any ball $B_0 \in \mathcal{F}_\beta$ with $5B_0 \notin \mathcal{F}_\beta$ we have*

(i) *For the family $\mathcal{W}_a(B_0)$ with $a < \beta/80$ given in Whitney's Lemma*

$$\mu(P) \le C^M\, \mu(P'), \tag{3.1}$$

for any P and P' belonging to $\mathcal{W}_a(B_0)$, where C denotes the doubling constant of μ and M is the geometric constant appearing there.

(ii)

$$\mu\left(\mathcal{N}_\beta(B_0)\right) \le K\, \mu\left(\frac{1}{2}B_0\right), \tag{3.2}$$

for some constant K that only depends on β and the constant of the doubling property of μ.

Proof First observe that since any $B \in \mathcal{F}_\beta$ is connected, the set $\mathcal{N}_\beta(B_0)$ is also connected. Further the larger set $\bigcup_{P\in\mathcal{W}_a(B_0)} P$ is connected too.

Let us fix $P' \in \mathcal{W}_a(B_0)$. We assert that

Claim: For any $P \in \mathcal{W}_a(B_0)$ there is a finite chain joining P with P', that is, a finite subset of $\mathcal{W}_a(B_0)$, say $P_1, \ldots, P_n$, which are all different, with $P_1 = P'$, $P_n = P$ and $P_i \cap P_{i+1} \neq \emptyset$, $i = 1, 2, \ldots, n-1$.

To prove the claim let us define

$$\mathcal{C} = \left\{ P \in \mathcal{W}_a(B_0) : \exists \text{ a finite chain from } P \text{ to } P' \right\}$$

Clearly $P' \in \mathcal{C}$. Let $E = \bigcup_{P \in \mathcal{C}} P$ and $F = \bigcup_{P \in \mathcal{W}_a(B_0) \smallsetminus \mathcal{C}} P$. Observe that both, E and F, are open sets and their union is a connected set. Also $E \cap F = \emptyset$. In fact, let us suppose that there is some $x \in E \cap F$.

From $x \in E$, we have that $x \in P$ with $P \in \mathcal{C}$, that is, there are $P_1, P_2, \ldots P_n$, each one belonging to $\mathcal{W}_a(B_0)$ and different from the others, with $P_1 = P'$, $P_n = P$ and $P_i \cap P_{i+1} \neq \emptyset$. Clearly each of the balls P_i also belongs to $\mathcal{C}$, since the chain $P_1, \ldots, P_i$ does the work.

On the other hand, from $x \in F$, we know that $x \in \tilde{P}$ with $\tilde{P} \notin \mathcal{C}$. However, the chain $P_1, P_2, \ldots, P_n, \tilde{P}$ has all its elements different from each other since, as we noticed, P_i belongs to $\mathcal{C}$ and $\tilde{P}$ does not. Also $P_n \cap \tilde{P} \neq \emptyset$ since $P_n = P$ and $x \in P \cap \tilde{P}$. Therefore $\tilde{P} \in \mathcal{C}$, arriving to a contradiction.

Since E is not empty, the set F must be empty and the claim is proved.

Coming back to the proof of the Lemma, given $P \in \mathcal{W}_a(B_0)$, let $P_1, \ldots, P_n$ be a chain as in the Claim. Since $P_i \cap P_{i+1} \neq \emptyset$ and both belong to $\mathcal{W}_a$, by (ii) of Whitney's Lemma, $P_{i+1} \subset 5P_i$ and since by (i), $10P_i \in \mathcal{F}_\beta$ we have that $5P_i \in \mathcal{F}_{\beta/2}$.

Therefore, by the doubling condition for the measure μ on $\mathcal{F}_{\beta/2}$, we have that for some constant C

$$\mu(P_{i+1}) \leq C\, \mu(P_i).$$

Iterating the above inequality and using that the length of the chain is at most M we get

$$\mu(P) \leq C^M\, \mu(P'),$$

which proves (i).

To show (ii), observe that if x_0 denotes the center of B_0, there is some $P_0 \in \mathcal{W}_a(B_0)$ such that $x_0 \in P_0$, and by (iii) of Whitney's Lemma we know that $P_0 \subset \frac{1}{2}B_0$.

Since

$$\mathcal{N}_\beta(B_0) \subset \bigcup_{P \in \mathcal{W}_a(B_0)} P, \tag{3.3}$$

using (i) with $P' = P_0$, we can conclude that

$$\mu(\mathcal{N}_\beta(B_0)) \leq \sum_{P \in \mathcal{W}_a(B_0)} \mu(P) \leq M\, C^M\, \mu(P_0) \leq M\, C^M\, \mu\left(\frac{1}{2}B_0\right).$$

$\square$

Remark 3.2 As an immediate consequence of the above Lemma, we get that if a measure μ is doubling on $\mathcal{F}_\beta$, then for some constant K, the inequality

$$\mu(\tilde{B}) \leq K\,\mu(B), \tag{3.4}$$

holds for any ball $B \in \mathcal{F}_\beta$, with $\tilde{B}$ defined as in (2.2).

Now we are ready to prove the main result concerning the boundedness properties of the local maximal functions $M_{\mu,\beta}$.

Proof of Theorem 1.1 We first prove the weak type inequality for $p = 1$. Let us assume that μ is a doubling measure on $\mathcal{F}_\beta$ and that w belongs to $A_1^\beta(d\mu)$, that is,

$$\int_B w\,d\mu \leq \mathcal{C}_{1,\beta}\,\mu(B)\,\inf_B\,w$$

holds for any ball in $\mathcal{F}_\beta$, where by "inf" we mean the essential infimum taken with respect to μ. We want to apply Vitali's Lemma, but in order to do that, we have to consider maximal functions over balls in $\mathcal{F}_\beta$ having bounded radii. Namely, for any fixed $R > 0$ we set

$$M^R_{\mu,\beta} f(x) = \sup_{\substack{x\in B\in\mathcal{F}_\beta\\ r(B)\leq R}} \frac{1}{\mu(B)} \int_B |f|\,d\mu$$

and for any $\lambda > 0$, we consider the set

$$E^R_\lambda = \left\{x \in \Omega :\, M^R_{\mu,\beta} f(x) > \lambda\right\}.$$

Notice that the set we want to measure, $E_\lambda = \left\{x \in \Omega :\, M_{\mu,\beta} f(x) > \lambda\right\}$, is the increasing union of the sets E^R_λ.

Now, for each x in the set E^R_λ there must be a ball B_x, with $x \in B_x \in \mathcal{F}_\beta$ and radius $r_x \leq R$, such that

$$\frac{1}{\mu(B_x)} \int_{B_x} |f|\,d\mu > \lambda \tag{3.5}$$

Then we apply Vitali's covering Lemma to the family $\{B_x\}_{x\in E^R_\lambda}$ which clearly covers E^R_λ. Moreover, as we pointed out in the introduction, since X satisfies the weak homogeneity property, X is separable. In this way, we obtain a disjoint and at most countable subfamily of balls, $\{B_i\}_{i\in J}$ with $J \subset \mathbb{N}$, in a way that

$$E^R_\lambda \subset \bigcup_{i\in J} \tilde{B}_i \subset \Omega,$$

where $\tilde{B}_i$ is defined as in (2.2).

Since μ is doubling on $\mathcal{F}_\beta$, so it is the measure associated to $w d\mu$. So we may apply Remark 3.2 to the latter measure to get

$$\begin{aligned}
\int_{E_\lambda^R} w\, d\mu &\le \sum_{i\in J} \int_{\tilde{B}_i} w\, d\mu \\
&\le \frac{K}{\lambda} \sum_{i\in J} \int_{B_i} w\, d\mu \, \frac{1}{\mu(B_i)} \int_{B_i} |f|\, d\mu \\
&\le \frac{K\mathcal{C}_{1,\beta}}{\lambda} \sum_{i\in J} \inf_{B_i} w \int_{B_i} |f|\, d\mu \\
&\le \frac{K\mathcal{C}_{1,\beta}}{\lambda} \sum_{i\in J} \int_{B_i} |f|\, w\, d\mu \\
&\le \frac{K\mathcal{C}_{1,\beta}}{\lambda} \int_{\Omega} |f|\, w\, d\mu,
\end{aligned}$$

where we have used (3.4), (3.5) and that $w \in A_1^\beta(d\mu)$.

Since the sets E_λ^R increase to E_λ when R increases to infinity, we get the desired estimate.

As for the proof of the necessity of the condition $A_1^\beta(d\mu)$ for the weak type $(1, 1)$ inequality, it follows the same steps of the classical case, so we omit it.

Now we turn into the proof of continuity for $1 < p < \infty$. We shall follow closely the argument given in Theorem B of [5] for Muckenhoupt basis $\mathcal{B}$ of $\mathbb{R}^n$. Let then w be a weight in $A_p^\beta(d\mu)$, with μ doubling on $\mathcal{F}_\beta$. We set $\sigma = w^{-\frac{1}{p-1}}$ and we denote by ν_w and by ν_σ the measures associated to $w d\mu$ and $\sigma d\mu$ respectively. As in [5], replacing $\mathcal{B}$ by $\mathcal{F}_\beta$ and $\mathbb{R}^n(dx)$ by $(\Omega, d\mu)$, we arrive to the following inequality

$$\left(M_{\mu,\beta} f\right)^{p-1}(x) \le \mathcal{C}_{p,\beta}\, M_{\nu_w,\beta}\left(M_{\nu_\sigma,\beta}((f\sigma^{-1})^{p-1}\, w^{-1}\right)(x) \tag{3.6}$$

Observe that ν_w as well ν_σ are doubling measures on $\mathcal{F}_\beta$. Therefore we may use what we proved in (i) for $M_{\nu_w,\beta}$ and $M_{\nu_\sigma,\beta}$, and hence these operators are of weak type $(1, 1)$ with respect to the measures ν_w and ν_σ respectively (that is, we use (i) with weight identically one). Since both maximal operators are clearly bounded on $L^\infty(\Omega, d\mu) = L^\infty(\Omega, w d\mu) = L^\infty(\Omega, \sigma d\mu)$, by interpolation, they are bounded on $L^r(\Omega, w d\mu)$ and on $L^r(\Omega, \sigma d\mu)$ respectively, for $1 < r < \infty$. Coming back to (3.6), using the above conclusions for $r = p'$ and $r = p$ respectively, we easily arrive to

$$\|M_{\mu,\beta} f\|_{L^p(w d\mu)} \le C_{p,\beta} \|f\|_{L^p(w d\mu)}.$$

As in (i), the proof of the necessity of the $A_p^\beta(d\mu)$ condition follows the usual pattern, taking $f = \chi_B w^{-\frac{1}{p-1}}$, with $B \in \mathcal{F}_\beta$. $\square$

With the aid of Lemma 3.1, we will also be able to show that certain properties holding for a measure on $\mathcal{F}_\beta$ can be proved to hold on any other class $\mathcal{F}_\alpha$. That is the case of the doubling property. However, let us remark that the corresponding constants may increase to infinity as β approaches 1. This fact will be the main step in order to obtain the independence from β of the classes $A_p^\beta(d\mu)$.

Proposition 3.3 *Let X and Ω be as above. Let μ be a measure defined on the Borel sets of Ω and $0 < \alpha, \beta < 1$. Then we have*

(i) *μ is finite and positive on $\mathcal{F}_\beta$ if and only if is finite and positive on $\mathcal{F}_\alpha$.*
(ii) *μ is doubling on $\mathcal{F}_\beta$ if and only if is doubling on $\mathcal{F}_\alpha$.*

Proof Let us assume $\alpha < \beta$. In this case, since $\mathcal{F}_\alpha \subset \mathcal{F}_\beta$, it is obvious that if any of the conditions in (i) or (ii) holds for $\mathcal{F}_\beta$, it also holds for $\mathcal{F}_\alpha$.

To prove the reciprocals, observe that it is enough to consider the case $\alpha = \beta/2$. Indeed, let k be a positive integer such that $\beta/2^k \leq \alpha$, by iteration and using the trivial part of either (i) or (ii), the assertion follows.

Let us assume that μ is finite and positive on $\mathcal{F}_{\beta/2}$. Since for any ball in $\mathcal{F}_\beta$, the ball $\frac{1}{2}B$ belongs to $\mathcal{F}_{\beta/2}$, it follows that $\mu(B) > 0$. On the other hand, (iii) of Whitney's Lemma tells us that B, moreover $\mathcal{N}_\beta(B)$, can be covered by a finite number of balls P. In addition, by (i) of the same Lemma, $10P \in \mathcal{F}_\beta$, which in particular implies that $P \in \mathcal{F}_{\beta/2}$. Therefore $\mu(B)$ must be finite.

To show the remaining part of (ii), let $B = B(x_B, r(B)) \in \mathcal{F}_\beta$. We want to prove that

$$\mu(B) \leq C_\beta\, \mu\left(\frac{1}{2}B\right),$$

assuming the same holds for balls in $\mathcal{F}_{\beta/2}$.

Clearly, we only need to consider the case when $\frac{\beta}{2}\, d(x_B, \Omega^c) < r(B) \leq \beta\, d(x_B, \Omega^c)$. Since, by assumption, μ is doubling on $\mathcal{F}_{\beta/2}$, we may apply Lemma 3.1 to obtain

$$\mu(B) \leq \mu\left(\mathcal{N}_\beta(B)\right) \leq K\, \mu\left(\frac{1}{2}B\right),$$

completing the proof of the Proposition. □

Given a Borel measure on Ω, positive, finite and doubling on some class $\mathcal{F}_\beta$, in view of the above proposition, we can get rid of the parameter β and we shall refer to it as a local doubling measure.

In the next Theorem we establish that the classes $A_p^\beta(d\mu)$ are also independent of the parameter β.

Theorem 3.4 *Let α, β be two constants such that $0 < \alpha, \beta < 1$. Let μ be a measure that is locally finite, positive and doubling. Then*

$$A_p^\beta(d\mu) \equiv A_p^\alpha(d\mu)$$

for $1 \le p < \infty$.

Proof Let $0 < \alpha < \beta < 1$. Since $\mathcal{F}_\alpha \subset \mathcal{F}_\beta$, we only need to prove that $A_p^\alpha(d\mu) \subset A_p^\beta(d\mu)$.

First assume that $w \in A_p^\alpha(d\mu)$ with $1 < p < \infty$. In particular the measure ν_w associated to $wd\mu$ is doubling on $\mathcal{F}_\alpha$ and hence, by the above Proposition, is also doubling on $\mathcal{F}_\beta$. The same can be said about ν_σ with $\sigma = w^{-\frac{1}{p-1}}$, since σ belongs to $A_{p'}^\alpha(d\mu)$. Let B be a ball in $\mathcal{F}_\beta \setminus \mathcal{F}_\alpha$. Then for the ball γB with $\gamma = \frac{\alpha}{\beta} < 1$ we have

$$\nu_w(B) \le C\, \nu_w(\gamma B)$$

and

$$\nu_\sigma(B) \le C\, \nu_\sigma(\gamma B).$$

Therefore, since $\gamma B \in \mathcal{F}_\alpha$, we obtain

$$\begin{aligned} \nu_w(B)\,(\nu_\sigma(B))^{p-1} &\le C^p \nu_w(\gamma B)\,(\nu_\sigma(\gamma B))^{p-1} \\ &\le \tilde{C}\,(\mu(\gamma B))^p \le \tilde{C}\,(\mu(B))^p\,, \end{aligned}$$

finishing the proof of the case $1 < p < \infty$.

For the case $p = 1$, as in the proof of Proposition 3.3, it is enough to consider $\alpha = \beta/2$. As above, let B be a ball in $\mathcal{F}_\beta \setminus \mathcal{F}_{\beta/2}$. Now we can apply (i) of Lemma 3.1 to the measure ν_w and the ball B. Choosing $P' \in \mathcal{W}_a(B)$ such that $\inf_{P'\cap B} w = \inf_B w$, we get

$$\nu_w(B) \le \sum_{P\in\mathcal{W}_a(B)} \nu_w(P) \le M\, C^M\, \nu_w(P') \le \mathcal{C}_{1,\beta/2} M\, C^M \inf_{P'} w\, \mu(P'),$$

where in the last inequality we used that $w \in A_1^{\beta/2}(d\mu)$ with constant $\mathcal{C}_{1,\beta/2}$ and that $P' \in \mathcal{F}_{\beta/2}$. Finally, applying (ii) of Lemma 3.1 to the measure μ and the fact that $\inf_{P'} w \le \inf_B w$, we get the desired conclusion. $\square$

In view of the above result we may drop the parameter β from the notation of the classes of weights and simply refer to them as $A_{p,\mathrm{loc}}(d\mu)$, $1 \le p < \infty$.

4 An application

Along this section the space X will be $\mathbb{R}^n$ with the Euclidean metric and Ω any proper and connected set with the restriction of the Lebesgue measure. It is clear that all the requirements made in the previous section are fulfilled.

We shall denote by $\delta(x)$ the Euclidean distance from a point x in Ω to Ω^c, and also we shall use the shorter notation $L_w^p(\Omega)$ to mean $L^p(\Omega, wdx)$. Similarly, the classes of weights will be denoted by $A_{p,\mathrm{loc}}(\Omega)$, emphasizing that weights only need to be defined in Ω and understanding that the averages are taken with respect to the Lebesgue measure.

In the main result of this section we prove *a priori* interior estimates for nice solutions of the differential Eq. (1.4) in terms of some weighted Sobolev spaces which take into account the distance to the boundary and that have been defined in the introduction, namely

$$W^{k,p}_{\delta,w}(\Omega) = \left\{ f \in L^1_{\mathrm{loc}}(\Omega) : \|f\|_{W^{k,p}_{\delta,w}} = \sum_{|\beta|\le k} \|\delta^{|\beta|} D^\beta f\|_{L^p_w(\Omega)} < \infty \right\}.$$

The Sobolev's norm in $W^{k,p}_{\delta,w}(\Omega)$ involves derivatives of any order up to k. The first step will be to show appropriate interpolation inequalities, which express that the leading terms in the norm are those derivatives of the lowest and highest order, that is of order zero and k. By abuse of notation, for a positive integer k, we will use D^k to indicate the sum of the absolute values of all derivatives of order k.

Proposition 4.1 *Let* $1 < p < \infty$ *and* $w \in A_{p,loc}(\Omega)$. *For any* $u \in W^{k,p}_{\delta,w}(\Omega)$ *and any* j, $1 \le j \le k-1$ *and* γ *such that* $|\gamma| = j$, *we have*

$$\|\delta^j D^\gamma u\|_{L^p_w(\Omega)} \le C \left(\varepsilon^{-j} \|u\|_{L^p_w(\Omega)} + \varepsilon^{k-j} \|\delta^k D^k u\|_{L^p_w(\Omega)} \right), \tag{4.1}$$

for any real number $0 < \varepsilon < 1$ *and* C *independent of* u *and* ε.

Proof By Sobolev's integral representation (see for example [2]) we know that for any $v \in W^{k,1}_{\mathrm{loc}}(\mathbb{R}^n)$ we have

$$|D^\gamma v(x)| \le C \left(\sigma^{-n-j} \int_{B(x,\sigma)} |v| + \int_{B(x,\sigma)} \frac{|D^k v|}{|x-y|^{n-k+j}} \right),$$

for any $\sigma > 0$.

Let us choose Whitney's covering $\mathcal{W}$ of Ω with $\beta = 1/2$ and $a = 1/200$. For $P = P(x_P, r_P) \in \mathcal{W}$, take a $\mathcal{C}^\infty_0$ function η_P such that *supp* $\eta_P \subset 4P \subset \Omega$, $0 \le \eta_P \le 1$, and $\eta_P \equiv 1$ on $2P$.

We apply now the above inequality to $u\eta_P$ which, by our assumptions, belongs to $W^{k,1}_{\mathrm{loc}}(\mathbb{R}^n)$. Observe that for $x \in P$ and $\sigma \le r_P$ we have $B(x,\sigma) \subset 2P$ and consequently $u\eta_P$ as well as its derivatives coincide with u and its derivatives when integrated over such balls.

Therefore for $x \in P$ and $\sigma \le r_P$, we obtain the above inequality with v replaced by u, namely

$$|D^\gamma u(x)| = |D^\gamma (u\eta_P)(x)| \le C \left(\sigma^{-n-j} \int_{B(x,\sigma)} |u| + \int_{B(x,\sigma)} \frac{|D^k u|}{|x-y|^{n-k+j}} \right). \tag{4.2}$$

Moreover, as it is easy to check from the properties of the covering $\mathcal{W}$, the balls $B(x,\sigma)$, for $x \in P$ and $\sigma \le r_P$, belong to the family $\mathcal{F}_\beta$ for $\beta = 1/2$. In fact, for $x \in P$, using (i) of Whitney's Lemma and the triangle inequality we get

$$d(x,\Omega^c) \ge d(x_P,\Omega^c) - r_P \ge \left(\frac{1}{a} - 1\right) r_P > 2r_P,$$

proving the assertion. That tells us that if we take $x \in P$ and $\sigma \le r_P$, the first term in (4.2) is bounded by $\sigma^{-j} M_{\mathrm{loc}} u(x)$, where by M_{loc} we denote the local maximal function $M_{\beta,\mu}$ of Sect. 3, with $\beta = 1/2$ and μ the Lebesgue measure.

As for the second term, splitting the integral dyadically, we obtain that is bounded by a constant times

$$\sigma^{k-j} \sum_{i=0}^{\infty} 2^{i(j-k)} \frac{1}{|2^{-i}B|} \int\limits_{2^{-i}B(x,\sigma)} |D^k u|.$$

Since for $x \in P$ and $\sigma \le r_P$ all averages involved correspond to balls in $\mathcal{F}_{1/2}$ and $j < k$, the second term in (4.2) is bounded by a constant times $\sigma^{k-j} M_{\mathrm{loc}} D^k u(x)$.

Putting together both estimates and taking $\sigma = \varepsilon r_P$, using that $r_P \simeq \delta(x)$ for $x \in P$, we arrive to

$$|D^\gamma(u)(x)| \le C\left((\varepsilon\delta(x))^{-j} M_{\mathrm{loc}}(u)(x) + (\varepsilon\delta(x))^{k-j} M_{\mathrm{loc}}(D^k u(x)\right). \tag{4.3}$$

Since $\mathcal{W}$ is a covering of Ω and the right hand side of (4.3) no longer depends of P, we obtain that (4.3) holds for any $x \in \Omega$.

Multiplying both sides by $\delta^j(x)$ and taking the $L^p_w(\Omega)$ norm, we arrive to

$$\|\delta^j D^\gamma u\|_{L^p_w(\Omega)} \le C\left(\varepsilon^{-j}\|M_{\mathrm{loc}} u\|_{L^p_w(\Omega)} + \varepsilon^{k-j}\|M_{\mathrm{loc}}(D^k u)\|_{L^p_{w\delta^{k/p}}(\Omega)}\right).$$

Next, we observe that if the weight w belongs to $A_{p,\mathrm{loc}}(\Omega)$ so it does $w\delta^s$, for any real number s. In fact, for any ball B in $\mathcal{F}_{1/2}$ we have that $\delta(x) \simeq \delta(x_B)$, for any $x \in B$ so that (1.3) holds provided it is satisfied by w.

Therefore, an application of the continuity results for $M_{\mathrm{loc}} = M_{\mu,1/2}$, with μ the Lebesgue measure, given in Theorem 1.1, leads to the interpolation inequality (4.1). □

Theorem 4.2 *Let $1 < p < \infty$ and w be a weight belonging to $A_{p,loc}(\Omega)$. Then, for a fixed positive integer m, and any $W^{2m,p}_{\delta,w}$ solution of the elliptic differential equation $\Delta^m u = f$ in Ω, we have*

$$\|u\|_{W^{2m,p}_{\delta,w}(\Omega)} \le C\left(\|u\|_{L^p_w(\Omega)} + \|\delta^{2m} f\|_{L^p_w(\Omega)}\right) \tag{4.4}$$

Proof First we notice that in view of the above Proposition we need to estimate only the highest order derivatives of u.

We shall use the well known localization process with respect to a covering as given by Whitney's Lemma of Sect. 2 for, say, $\beta = 1/2$.

Let $\mathcal{W}$ be such a covering of Ω. For each $P \in \mathcal{W}$ we take a $\mathcal{C}_0^\infty$-function, η_P, supported on $2P$, with $\eta_P \equiv 1$ on P and such that $\|\eta_P\|_\infty \leq 1$ and $\|D^\alpha \eta_P\|_\infty \leq cr_P^{-|\alpha|}$, for any multi-index α.

Next we notice that the function $u\eta_P$ satisfies the equation

$$\Delta^m(u\eta_P) = g_P,$$

with

$$g_P = \eta_P f + u\Delta\eta_P + \sum_{\substack{j=1\\|\alpha|=j\\|\beta|=2m-j}}^{2m-1} c_{\alpha\beta} D^\alpha u D^\beta \eta_P.$$

If $\Gamma(x, y)$ denotes the fundamental solution for Δ^m then for $x \in 2P$

$$u\eta_P(x) = \int (\Gamma(x-y) + v(x, y)) g_P(y)\, dy = h_P(x) + v_P(x), \tag{4.5}$$

where $v(x, y)$ verifies for each fixed $y \in 2P$

$$\begin{cases} \Delta^m v(x, y) & = 0, \qquad \mathrm{x} \in 2P \\ (\frac{\partial}{\partial \nu})^j v(x, y) & = -(\frac{\partial}{\partial \nu})^j \Gamma(x-y) \qquad \mathrm{x} \in \partial(2P),\ 0 \leq j \leq m-1. \end{cases}$$

Using the point wise estimates $|D^\gamma v(x, y)| \leq Cr_P^{-n}$ for $|\gamma| = 2m$ obtained in [3] (see Proposition 3.3, p. 6), Hölder inequality and the fact that $w \in A_{p,loc}$, we easily obtain, for $|\gamma| = 2m$, that

$$\|\mathcal{X}_P D^\gamma v_P\|_{L^p_w} \leq C\|g_P\|_{L^p_w}. \tag{4.6}$$

On the other hand, it is well known that for $|\gamma| = 2m$, $D^\gamma h$ with h the convolution of the fundamental solution with a function $g \in L^p_w$ is given by $T_\gamma g$, with T_γ a Calderón Zygmund singular integral operator on $\mathbb{R}^n$ (see [1]). Now, since g_P, is supported on $2P$, we can look at the operator T_γ acting on functions defined over the space of homogeneous type $2P$ equipped with the Euclidean metric and the restriction of Lebesgue measure. Also, it is easy to check that the weight $w\chi_{2P}$ is in $A_p(2P)$, provided w belongs to $A_{p,loc}(\Omega)$, since P has been chosen such that $10P \in \mathcal{F}_{1/2}$. By the theory of singular integrals on spaces of homogeneous type (see for instance [10]), applied to our operator T_γ, we obtain, for $|\gamma| = 2m$, that

$$\|\mathcal{X}_P D^\gamma h_P\|_{L^p_w} \leq C\|g_P\|_{L^p_w}. \tag{4.7}$$

Hence, from (4.5), (4.6) and (4.7), for $|\gamma| = 2m$, we get that

$$\|\mathcal{X}_P D^\gamma \eta_P u\|_{L^p_w} \le C\|g_P\|_{L^p_w}$$
$$\le C\left(\|\mathcal{X}_{2P} f\|_{L^p_w} + \|\mathcal{X}_{2P} u\|_{L^p_w}\frac{1}{r_P^{2m}} + \sum_{\substack{j=1\\|\alpha|=j}}^{2m-1}\frac{1}{r_P^{2m-j}}\|\mathcal{X}_{2P} D^\alpha u\|_{L^p_w}\right),$$

in view of the assumptions on η_P. Now we observe that for $x \in P$, the function $\eta_P u$ coincides with u and also for $x \in 2P$ we have $r_P \simeq \delta(x)$. Then, the above inequality, after multiplication by r_P^{2m}, can be rewritten as

$$\|\mathcal{X}_P \delta^{2m} D^{2m} u\|_{L^p_w}$$
$$\le C\left(\|\mathcal{X}_{2P}\delta^{2m} f\|_{L^p_w} + \|\mathcal{X}_{2P} u\|_{L^p_w} + \sum_{\substack{j=1\\|\alpha|=j}}^{2m-1}\|\mathcal{X}_{2P}\delta^{-j} D^\alpha u\|_{L^p_w}\right), \tag{4.8}$$

with perhaps a different constant C.

We now use the finite overlapping property of the covering $\{2P\}_{P\in\mathcal{W}}$ quoted in Remark 2.4, which for any measurable function h implies that

$$\|h\|^p_{L^p_w} \le \sum_{P\in\mathcal{W}}\|\mathcal{X}_{2P} h\|^p_{L^p_w} = \left\|\sum_{P\in\mathcal{W}}\mathcal{X}_{2P} h\right\|^p_{L^p_w} \le K\|h\|^p_{L^p_w}.$$

Raising both sides of (4.8) to p, adding over $P \in \mathcal{W}$ and taking the $1/p$-th power we arrive to

$$\|\delta^{2m} D^{2m} u\|_{L^p_w}$$
$$\le C\left(\|\delta^{2m} f\|_{L^p_w} + \|u\|_{L^p_w} + \sum_{\substack{j=1\\|\alpha|=j}}^{2m-1}\|\delta^{-j} D^\alpha u\|_{L^p_w}\right),$$

Now, for the last term on the right hand side of the above inequality, we use Proposition 4.1 to get

$$\|\delta^{2m} D^{2m} u\|_{L^p_w} \le \tilde{C}\left(\|\delta^{2m} f\|_{L^p_w} + \varepsilon^{-1}\|u\|_{L^p_w} + \varepsilon\|\delta^{2m} D^{2m} u\|_{L^p_w}\right).$$

Finally, choosing ε such that $\tilde{C}\varepsilon \le 1/2$, subtracting the last term, which is finite from our assumptions on u, we get the desired estimate. □

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