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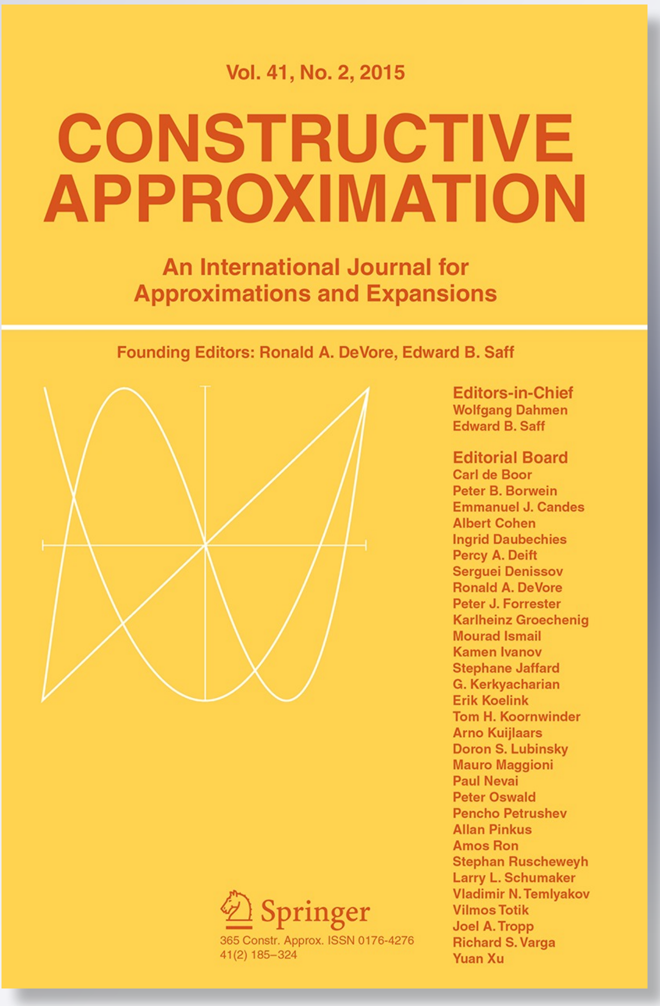

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Improvement of Besov Regularity for Solutions of the Fractional Laplacian

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Abstract We prove a mean value formula for weak solutions of $\operatorname{div}(|y|^a \operatorname{grad} u) = 0$ in $\mathbb{R}^{n+1} = \{(x, y) : x \in \mathbb{R}^n, y \in \mathbb{R}\}$, $-1 < a < 1$, and balls centered at points of the form $(x, 0)$. We obtain an explicit nonlocal kernel for the mean value formula for solutions of $(-\triangle)^s f = 0$ on a domain D of $\mathbb{R}^n$. When D is Lipschitz, we prove a Besov type regularity improvement for the solutions of $(-\triangle)^s f = 0$.

Keywords Degenerate elliptic equations · Fractional Laplacian · Mean value formula · Besov spaces · Gradient estimates

Mathematics Subject Classification 26A33 · 35J70 · 35B65 · 46E35

1 Introduction

For many years, fractional powers of $-\triangle$ have been the object of study. In the Euclidean space $\mathbb{R}^n$, the most elementary way to introduce the nonlocal operator $(-\triangle)^s$ for $0 < s < 1$, is provided by the Fourier transform. In fact, for a test function g of the Schwartz class, $(-\triangle)^s g$ is given by

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$$\widehat{(-\triangle)^s g} = |\xi|^s \widehat{g}$$

in terms of Fourier transforms. The homogeneity properties of the Fourier transform allow us to show that $(-\triangle)^s$ is a convolution operator with the distribution

$$\text{p.v.} \int_{\mathbb{R}^n} \frac{g(x) - g(0)}{|x|^{n+2s}} dx = \lim_{\epsilon \to 0} \int_{|x| \geq \epsilon} \frac{g(x) - g(0)}{|x|^{n+2s}} dx.$$

Hence for g a Schwartz function,

$$(-\triangle)^s g(x) = C \text{ p.v.} \int_{\mathbb{R}^n} \frac{g(y) - g(x)}{|x - y|^{n+2s}} dy,$$

where C is a constant that depends on n and s. By duality $(-\triangle)^s$ can be defined for functions in $L^1\left(\mathbb{R}^n, \frac{dx}{(1+|x|)^{n+2s}}\right)$. See [11].

For our purposes, the best approach is to regard $(-\triangle)^s$ as an operator applying Dirichlet data into Neumann data. For $s = \frac{1}{2}$ the idea is now classical. In [2] L. Caffarelli and L. Silvestre show how every fractional power of $-\triangle$ in $\mathbb{R}^n$ can be obtained as Dirichlet to Neumann type operators in the extended domain $\mathbb{R}^{n+1}_+ = \{(x, y) : x \in \mathbb{R}^n,\ y > 0\}$. This result allowed a better approach to the analysis of PDE's that involves $(-\triangle)^s$. The operator in the extended domain is given by div $(y^a \text{grad}\, u)$, where $a \in (-1, 1)$, $u = u(x, y)$, $x \in \mathbb{R}^n$, $y \in \mathbb{R}^+$, and *div* and *grad* are the standard divergence and gradient operators in $\mathbb{R}^{n+1}_+$. The exponent a is related to the fractional power of the Laplacian $(-\triangle)^s$ through $2s = 1 - a$. We shall write L_a to denote the operator $L_a v = \text{div}\,(|y|^a \text{grad}\, v)$ acting on functions v defined on $\mathbb{R}^{n+1}$. Notice that when $a = 0$, the operator L_a is the Laplacian in $\mathbb{R}^{n+1}$ and $s = \frac{1}{2}$. The theory of Hölder regularity of solutions through Harnack's inequalities is one of the several results in [2]. This theory has been extended in [13] to other second-order partial differential operators including the harmonic oscillator. In Sect. 3 of [5], some of the equivalent different approaches to $(-\triangle)^s$ are proved in detail.

Since for $a \in (-1, 1)$ the weight $w(x, y) = |y|^a$ belongs to the Muckenhoupt class $A_2(\mathbb{R}^{n+1})$, the regularity theory developed by Fabes, Kenig, and Serapioni in [7] can be applied. The fact that w is in $A_2(\mathbb{R}^{n+1})$ follows easily from the fact that it is a product of the weight which is constant and equal to one in $\mathbb{R}^n$ times the $A_2(\mathbb{R})$ weight $|y|^a$ for $a \in (-1, 1)$. In particular, Harnack's inequality and Hölder regularity of solutions are available.

It seems to be clear that when $a \neq 0$, the weight $w(x, y) = |y|^a$ introduces a bias which prevents us from expecting mean values on spherical objects in $\mathbb{R}^{n+1}$. Except at $y = 0$, where the symmetry of w with respect to the hyperplane $y = 0$ may bring back to spheres their classical role. In [6], some generalizations of classical mean value formulas are also considered.

By choosing adequate test functions, we shall prove the mean value formula, for balls centered at the hyperplane $y = 0$, for weak solutions v of $L_a v = 0$.

The considerations above would only allow mean values for solutions with balls centered at such small sets as the hyperplane $y = 0$ of $\mathbb{R}^{n+1}$. But it turns out that this suffices to get mean value formulas for solutions of $(-\triangle)^s f = 0$.

In [11] a mean value formula is proved as Proposition 2.2.13, see also [9]. In order to obtain improved results for the Besov regularity of solutions of $(-\triangle)^s f = 0$ in the spirit of [3] and [1], our formula seems to be more suitable because we can get explicit estimates for the gradients of the mean value kernel. Regarding Besov regularity of harmonic functions, see also [8].

The paper is organized in three sections. In the first one, we prove mean value formulas for solutions of $L_a u = 0$ at the points on the hyperplane $y = 0$ of $\mathbb{R}^{n+1}$. The second section is devoted to applying the result in Sect. 2 in order to obtain a nonlocal mean value formula for solutions of $(-\triangle)^s f = 0$ on domains of $\mathbb{R}^n$. Finally, in Sect. 4, we use the results above to obtain a Besov regularity improvement for solutions of $(-\triangle)^s f = 0$ in Lipschitz domains of $\mathbb{R}^n$.

2 Mean Value Formula for Solutions of $L_a u = 0$

Let D be a domain in $\mathbb{R}^n$. Let Ω be the open set in $\mathbb{R}^{n+1}$ given by $\Omega = D \times (-d, d)$ with d the diameter of D. Notice that for $x \in D$ and $r > 0$ such that $B(x, r) \subset D$, then $S((x, 0), r) \subset \Omega$, where B denotes balls in $\mathbb{R}^n$ and S denotes the balls in $\mathbb{R}^{n+1}$. By $H^1(|y|^a)$ we denote the Sobolev space of those functions in $L^2(|y|^a\, dxdy)$ for which ∇f belongs to $L^2(|y|^a\, dxdy)$.

The main result of this section is contained in the next statement. As in [2] we shall use X to denote the points (x, y) in $\mathbb{R}^{n+1}$ with $x \in \mathbb{R}^n$ and $y \in \mathbb{R}$.

Theorem 1 *Let v be a weak solution of $L_a v = 0$ in Ω. In other words, v belongs to $H^1(|y|^a)$ and*

$$\iint_\Omega \nabla v \cdot \nabla \psi |y|^a\, dxdy = 0$$

for each test function ψ supported in Ω. Let $\varphi(X) = \eta(|X|)$, $\eta \in C_0^\infty(\mathbb{R}^+)$ supported in the interval $\left[\frac{1}{4}, \frac{3}{4}\right]$ and $\iint_{\mathbb{R}^{n+1}} \varphi(X)|y|^a\, dX = 1$ be given. If $x \in D$ and $0 < r < \delta(x) = \inf\{|x - z| : z \in \partial D\}$, then

$$v(x, 0) = \iint_\Omega \varphi_r(x - z, -y) v(z, y)|y|^a dzdy,$$

with

$$\varphi_r(X) = \frac{1}{r^{n+1+a}} \varphi\left(\frac{X}{r}\right).$$

Proof Set $A = \int_0^\infty \rho\eta(\rho)d\rho$ and $\zeta(t) = \int_0^t \rho\eta(\rho)d\rho - A$. Notice that $\zeta(t) \equiv 0$ for $t \geq \frac{3}{4}$ and $\zeta(t) \equiv -A$ for $0 \leq t \leq \frac{1}{4}$. The function $\psi(X) = \zeta(|X|)$ is, then, in $C^\infty(\mathbb{R}^{n+1})$ and has compact support in the ball $S((0, 0), 1)$. It is easy to check that $\nabla\psi(X) = \varphi(X)X$. Now take $x \in D$ and $0 < r < \delta(x)$. Set $\varphi_r(Z) = r^{-n-1-a}\varphi(r^{-1}Z)$, $Z \in \mathbb{R}^{n+1}$, and define

$$\Phi_x(r) = \iint_\Omega \varphi_r(X-Z)v(Z)|y|^a dZ,$$

where $X = (x, 0)$, $Z = (z, y)$, $dZ = dzdy$, and v is a weak solution of $L_a v = 0$ in Ω. As usual, we aim to prove that $\Phi_x(r)$ is a constant function of r and that $\lim_{r\to 0}\Phi_x(r) = v(X)$. From Theorem 2.3.12 in [7] with $w(Z) = |y|^a$, which belongs to the Muckenhoupt class $A_2(\mathbb{R}^{n+1})$ when $-1 < a < 1$, we know that v is Hölder continuous on each compact subset of Ω. Then the convergence $\Phi_x(r) \to v(X) = v(x, 0)$ as $r \to 0$ follows from the fact that

$$\iint \varphi_r(Z)|y|^a dZ = \frac{1}{r^{a+1+n}} \iint \varphi\left(\frac{z}{r}, \frac{y}{r}\right)|y|^a dzdy = 1.$$

In order to prove that $\Phi_x(r)$ is constant as a function of r, we shall take its derivative with respect to r for fixed x. Notice first that

$$\Phi_x(r) = \iint_{S((0,0),1)} \varphi(Z)v(X-rZ)|y|^a dzdy.$$

Since $\nabla v \in L^2(|y|^a dX)$, we have

$$\begin{aligned}
\frac{d}{dr}\Phi_x(r) &= -\iint_{S((0,0),1)} \varphi(Z)\nabla v(X-rZ)\cdot Z|y|^a dZ\\
&= -\iint_{S((0,0),1)} \nabla v(X-rZ)\cdot\nabla\psi(Z)|y|^a dZ\\
&= -\frac{1}{r^{a+1+n}}\iint_\Omega \nabla v(Z)\cdot\nabla\psi\left(\frac{X-Z}{r}\right)|y|^a dZ\\
&= \iint_\Omega \nabla v(Z)\cdot\nabla\left[\frac{1}{r^{n+a}}\psi\left(\frac{X-Z}{r}\right)\right]|y|^a dZ,
\end{aligned}$$

which vanishes since $\frac{1}{r^{n+a}}\psi\left(\frac{X-Z}{r}\right)$ as a function of Z is a test function for the fact that v solves $L_a v = 0$ in Ω. □

3 Mean Value Formula for Solutions of $(-\Delta)^s f = 0$

In this section we shall use the results and we shall closely follow the notation in [2]. Take $f \in L^1\left(\mathbb{R}^n, \frac{dx}{(1+|x|)^{n+2s}}\right)$ with $(-\triangle)^s f = 0$ on the domain $D \subset \mathbb{R}^n$. Then, with $u(x, y) = \left(P_y^a * f\right)(x)$ and $P_y^a(x) = Cy^{1-a}\left(|x|^2 + y^2\right)^{-\frac{n+1-a}{2}}$, the function

$$v(x, y) = \begin{cases} u(x, y) & \text{in } D\times\mathbb{R}^+,\\ u(x, -y) & \text{in } D\times\mathbb{R}^-, \end{cases}$$

is a weak solution of $L_a v = 0$ in $D\times\mathbb{R}$. This follows from Lemma 4.1 and formula (3.1) in [2], since the reflection is possible because $(-\triangle)^s f$ vanishes on D and this condition is equivalent to $\lim_{y\to 0} y^a u_y = 0$ in D. In particular, from Theorem 2.3.12 in [7], v is Hölder continuous in $D \times \mathbb{R}$. Theorem 1 guarantees that, for $0 < r < \delta(x)$ and $x \in D$,

$$f(x) = u(x,0) = v(x,0) = \iint \varphi_r(X - Z) v(Z) |y|^a dZ, \tag{3.1}$$

where, as before, $X = (x, 0)$ and $Z = (z, y)$. On the other hand, the definitions of v and u provide the formula

$$v(Z) = v(z, y) = \left(P^a_{|y|} * f\right)(z). \tag{3.2}$$

Replacing (3.2) in (3.1), provided that the interchange of the order of integration holds, we obtain the main result of this section.

Theorem 2 *Let $0 < s < 1$ be given. Assume that D is an open set in $\mathbb{R}^n$ on which $(-\triangle)^s f = 0$. Then for every $x \in D$ and every $0 < r < \delta(x)$, we have that $f(x) = (\Phi_r * f)(x)$, where $\Phi_r(x) = r^{-n}\Phi\left(\frac{x}{r}\right)$, $\Phi(x) = \int_{y\in\mathbb{R}} \int_{z\in\mathbb{R}^n} \varphi(z, -y) P^a_{|y|}(x - z)|y|^a\, dzdy$, $\varphi_r(x, y) = r^{-(n+1+a)}\varphi\left(\frac{x}{r}, \frac{y}{r}\right)$, φ is a $C^\infty(\mathbb{R}^{n+1})$ radial function supported in the unit ball of $\mathbb{R}^{n+1}$ with $\iint_{\mathbb{R}^{n+1}} \varphi(x, y)|y|^a\, dxdy = 1$ and P^a_y is a constant times $y^{1-a}\left(|x|^2 + y^2\right)^{-\frac{n+1-a}{2}}$.*

Proof Inserting (3.2) into (3.1), we have

$$\begin{aligned}
f(x) = v(x,0) &= \iint \varphi_r(x - z, -y) v(z, y)\ |y|^a dzdy \\
&= \iint \varphi_r(x - z, y)(P^a_{|y|} * f)(z)|y|^a dzdy \\
&= \int_{y\in\mathbb{R}} \int_{z\in\mathbb{R}^n} \varphi_r(x - z, -y) \left(\int_{\bar{z}\in\mathbb{R}^n} P^a_{|y|}(z - \bar{z}) f(\bar{z})\, d\bar{z}\right) |y|^a dzdy \\
&= \int_{\bar{z}\in\mathbb{R}^n} \left(\int_{y\in\mathbb{R}} \int_{z\in\mathbb{R}^n} \varphi_r(x - z, -y) P^a_{|y|}(z - \bar{z})|y|^a dzdy\right) f(\bar{z})d\bar{z} \\
&= \int_{\bar{z}\in\mathbb{R}^n} \Phi_r(x, \bar{z}) f(\bar{z})d\bar{z},
\end{aligned}$$

with $\Phi_r(x, \bar{z}) = \int_{y\in\mathbb{R}} \int_{z\in\mathbb{R}^n} \varphi_r(x - z, -y) P^a_{|y|}(z - \bar{z})\, |y|^a\, dzdy$. The last equality in the above formula follows from the fact that $\frac{f(\bar{z})}{(1+|\bar{z}|^2)^{\frac{n+1-a}{2}}}$ is integrable in $\mathbb{R}^n$, since

$$\int_{y\in\mathbb{R}} \int_{z\in\mathbb{R}^n} |\varphi(x - z, -y)| P^a_{|y|}(z - \bar{z})|y|^a dzdy \le \frac{C}{(1 + |\bar{z}|^2)^{\frac{n+1-a}{2}}}$$

for some positive constant C. In fact, on one hand,

$$\begin{aligned}\int_{y\in\mathbb{R}}\int_{z\in\mathbb{R}^n}|\varphi(x-z,-y)|P^a_{|y|}(z-\bar{z})|y|^a dzdy\\ \leq \int_{-1}^{1}\|\varphi(x-\cdot,y)\|_{L^\infty}\left\|P^a_{|y|}(\cdot-\bar{z})\right\|_{L^1}|y|^a\,dy\leq C;\end{aligned}\tag{3.3}$$

on the other, for $|\bar{z}-x|>2$, we have

$$\begin{aligned}\int_{y\in\mathbb{R}}\int_{z\in\mathbb{R}^n}|\varphi(x-z,-y)|P^a_{|y|}(z-\bar{z})|y|^a dzdy\\ \leq C\iint_{S((x,0),1)}\frac{|y|}{(y^2+|z-\bar{z}|^2)^{\frac{n+1-a}{2}}}dzdy\\ \leq \frac{C}{|x-\bar{z}|^{n+1-a}}.\end{aligned}\tag{3.4}$$

So $\Phi_r(x,\bar{z})\leq\frac{C(r)}{(1+|x-\bar{z}|)^{n+1-a}}\leq\frac{C(x,r)}{(1+|x|)^{n+1-a}}$, hence $\int\Phi_r(x,\bar{z})f(\bar{z})d\bar{z}$ is absolutely convergent. It remains to prove that $\Phi_r(x,\bar{z})=\frac{1}{r^n}\Phi(\frac{x-\bar{z}}{r})$ with $\Phi(x)=\int_{y\in\mathbb{R}}\int_{z\in\mathbb{R}^n}\varphi(z,-y)P^a_{|y|}(x-z)|y|^a dzdy$. Let us compute $\Phi(\frac{x-\bar{z}}{r})$ changing variables. First in $\mathbb{R}^n$ with $\nu=x-rz$, then in $\mathbb{R}$ with $t=ry$,

$$\begin{aligned}\Phi\left(\frac{x-\bar{z}}{r}\right)&=\int_{y\in\mathbb{R}}\int_{z\in\mathbb{R}^n}\varphi(z,-y)P^a_{|y|}\left(\frac{x-\bar{z}-rz}{r}\right)|y|^a dzdy\\ &=\int_{y\in\mathbb{R}}\int_{\nu\in\mathbb{R}^n}\frac{1}{r^n}\varphi\left(\frac{x-\nu}{r},-y\right)P^a_{|y|}\left(\frac{\nu-\bar{z}}{r}\right)|y|^a d\nu dy\\ &=\int_{t\in\mathbb{R}}\int_{\nu\in\mathbb{R}^n}\frac{1}{r^{n+1+a}}\varphi\left(\frac{x-\nu}{r},-\frac{t}{r}\right)P^a_{\left|\frac{t}{r}\right|}\left(\frac{\nu-\bar{z}}{r}\right)|t|^a d\nu dt\\ &=r^n\int_{t\in\mathbb{R}}\int_{\nu\in\mathbb{R}^n}\varphi_r(x-\nu,-t)P^a_{|t|}(\nu-\bar{z})|t|^a d\nu dt\\ &=r^n\Phi_r(x,\bar{z}),\end{aligned}$$

as desired. □

We collect in the next result some basic properties of the mean value kernel Φ.

Proposition 3 *The function Φ defined in the statement of Theorem 2 satisfies the following properties:*

(a) $\Phi(x)$ *is radial;*
(b) $(1+|x|)^{n+1-a}\,|\Phi(x)|$ *is bounded;*
(c) $\int_{\mathbb{R}^n}\Phi(x)dx=1$*;*
(d) $\sup_{r>0}|(\Phi_r*f)(x)|\leq cMf(x)$*, where M is the Hardy-Littlewood maximal operator in $\mathbb{R}^n$;*
(e) if $\Psi^i(x)=\frac{\partial\Phi}{\partial x_i}(x)$*, then* $\Psi^i(0)=0$ *and* $\int\Psi^i(x)\,dx=0$*;*

(f) for some constant $C > 0$, $\left|\Psi^i(x)\right| \leq \frac{C}{|x|^{n+2-a}}$ *for* $|x| > 2$*;*
(g) $\left|\nabla\Psi^i\right|$ *is bounded on* $\mathbb{R}^n$ *for every* $i = 1, \ldots, n$*.*

Proof Let ρ be a rotation of $\mathbb{R}^n$; then

$$\begin{aligned}
\Phi(\rho x) &= \int_{y\in\mathbb{R}}\int_{z\in\mathbb{R}^n} \varphi(z,-y)P^a_{|y|}(\rho x - z)|y|^a dzdy \\
&= \int_{y\in\mathbb{R}}\int_{z\in\mathbb{R}^n} \varphi(\rho^{-1}z,-y)P^a_{|y|}(\rho^{-1}(\rho x - z))|y|^a dzdy \\
&= \int_{y\in\mathbb{R}}\int_{z\in\mathbb{R}^n} \varphi(\rho^{-1}z,-y)P^a_{|y|}(x - \rho^{-1}z)|y|^a dzdy \\
&= \int_{y\in\mathbb{R}}\int_{\bar{z}\in\mathbb{R}^n} \varphi(\bar{z},-y)P^a_{|y|}(x - \bar{z})|y|^a d\bar{z}dy \\
&= \Phi(x),
\end{aligned}$$

which proves *(a)*. Part *(b)* has already been proved in (3.3) and (3.4). By taking $f \equiv 1$ in Theorem 2, we get *(c)*. From *(a)* and *(c)*, the estimate of the maximal operator is a classical result (see [12]). Item *(e)* follows from the fact that Φ is radial and smooth and from *(c)*.

Let us now show that $|\Psi^i(x)| \leq \frac{C}{|x|^{n+2-a}}$ for $|x| > 2$. In fact,

$$\begin{aligned}
|\Psi^i(x)| =& 2\left|\int_0^\infty \int_{z\in\mathbb{R}^n} \frac{\partial\varphi}{\partial x_i}(z,y)P^a_y(x-z)y^a dzdy\right| \\
=& 2\left|\int_0^1 \int_{z\in B(0,1)} \varphi(z,y)\frac{\partial}{\partial x_i}\left(P^a_y(x-z)y^a\right)dzdy\right| \\
\leq& C\int_0^1 \int_{z\in B(0,1)} |\varphi(z,y)|\frac{1}{|x-z|^{n+2-a}}dzdy \\
\leq& \frac{C}{(|x|-1)^{n+2-a}}\int_0^1 \int_{z\in B(0,1)} |\varphi(z,y)|dzdy \\
\leq& \frac{C}{|x|^{n+2-a}}.
\end{aligned}$$

By taking the derivatives of the function φ, the proof of *(g)* proceeds as in (3.3). □

4 Maximal Estimates for Gradients of Solutions of $(-\Delta)^s f = 0$ in Open Domains and the Improvement of Besov Regularity

The mean value formula proved in Sect. 3 for solutions of $(-\triangle)^s f = 0$ in an open domain D of $\mathbb{R}^n$ can be used to obtain improvement of Besov regularity of f. Here we illustrate how Theorem 2 can be used to get a result in the lines introduced by Dahlke and DeVore for harmonic functions. We shall prove the following result.

Theorem 4 *Let D be a bounded Lipschitz domain in $\mathbb{R}^n$. Let $0 < s < 1$. Let $1 < p < \infty$ and $0 < \lambda < \frac{n-1}{n}$ be given. Assume that $f \in B_p^\lambda(\mathbb{R}^n)$ and that $(-\triangle)^s f = 0$ on D. Then $f \in B_\tau^\alpha(D)$ with $\frac{1}{\tau} = \frac{1}{p} + \frac{\alpha}{n}$ and $0 < \alpha < \lambda \frac{n}{n-1}$.*

Here $B_p^\lambda(\mathbb{R}^n)$ and $B_\tau^\alpha(D)$ denote the standard Besov spaces on $\mathbb{R}^n$ and on D with $p = q$ for the usual notation $B_{p,q}^\lambda$ of this scale. Among the several descriptions of these spaces, the best suited for our purposes is the characterization through wavelet coefficients [10].

It is worth noticing that in contrast to the local cases associated with the harmonic functions in [3] and the temperatures in [1], now the B_p^λ regularity is required on the whole space $\mathbb{R}^n$ and that the improvement is only in D.

The basic scheme is that in [3], and the central tool is then the estimate contained in the next statement.

Lemma 5 *Let D be a domain of $\mathbb{R}^n$. Let $0 < \lambda < 1$ and $1 < p < \infty$. For $f \in B_p^\lambda(\mathbb{R}^n)$ with $(-\triangle)^s f = 0$ on D, we have*

$$\left(\int_D \left|\delta(x)^{1-\lambda}\nabla f(x)\right|^p dx\right)^{\frac{1}{p}} \leq C\,\|f\|_{B_p^\lambda(\mathbb{R}^n)},$$

where $\delta(x)$ is the distance from x to the boundary of D, ∇f is the gradient of f, and C is a constant depending only on n, λ, and φ.

The main difference between the local case in [3] and our nonlocal setting is precisely provided by the fact that since our mean value kernel is not localized in D, the Calderón maximal operator needs to be taken on the whole $\mathbb{R}^n$, not only on D.

The result is itself a consequence of a pointwise estimate of the gradient of f in terms of the sharp Calderón maximal operator and [4]. The result is contained in the next statement and follows from the mean value formula in Theorem 2, and the basic properties of the mean value kernel Φ_r and its first-order partial derivatives contained in Proposition 3.

Lemma 6 *Let D and λ be as in Lemma 5, and let $x \in D$ and $0 < r < \delta(x)$. Then*

$$|\nabla f(x)| \leq C r^{\lambda-1} M^{\sharp,\lambda} f(x),$$

with

$$M^{\sharp,\lambda} f(x) = \sup \frac{1}{|B|^{1+\frac{\lambda}{n}}} \int_B |f(y) - f(x)|\,dy,$$

where the supremum is taken on the family of all balls of $\mathbb{R}^n$ containing x.

Proof From the definition of Φ it is clear that $\frac{\partial}{\partial x_i}\Phi_r(x) = \frac{1}{r}\Psi_r^i(x)$ with $\Psi^i(x) = 2\int_0^\infty \int_{z\in\mathbb{R}^n} \frac{\partial\varphi}{\partial z_i}(z, y) P_y^a(x - z) y^a dz dy$, $i = 1, \ldots, n$. Since from *(e)* in Proposition 3 we have that $\Psi^i(0) = 0$, then

$$|\Psi_r^i(x)| = \left|\Psi_r^i(x) - \Psi_r^i(0)\right| \leq |x| \sup_{\xi \in \mathbb{R}^n} |\nabla \Psi_r^i(\xi)| \leq \frac{C}{r^{n+1}}|x|, \tag{4.1}$$

from *(g)* in Proposition 3. This is a good estimate in a neighborhood of 0. Applying the mean value formula for f, we get the result after the following estimates:

$$\begin{aligned}
\left|\frac{\partial f(x)}{\partial x_i}\right| &= \left|\frac{\partial}{\partial x_i}(\Phi_r * f)(x)\right| \\
&= \left|\frac{1}{r}\int_{\mathbb{R}^n} f(x-z)\Psi_r^i(z)dz\right| \\
&= \left|\frac{1}{r}\int_{\mathbb{R}^n} (f(x-z) - f(x))\,\Psi_r^i(z)dz\right| \\
&= \left|\frac{1}{r}\int_{\mathbb{R}^n} (f(z) - f(x))\,\Psi_r^i(x-z)dz\right| \\
&\leq \frac{1}{r}\int\limits_{B(x,2r)} |f(z) - f(x)||\Psi_r^i(x-z)|dz \\
&\quad + \frac{1}{r}\int\limits_{B^c(x,2r)} |f(z) - f(x)||\Psi_r^i(x-z)|dz = I + II.
\end{aligned}$$

We shall bound I using (4.1):

$$\begin{aligned}
I &= \frac{1}{r}\int_{B(x,2r)} |f(z) - f(x)||\Psi_r^i(x-z)|dz \\
&\leq \frac{C}{r^{n+2}}\int_{B(x,2r)} |f(z) - f(x)||x-z|dz \\
&= \frac{C}{r^{n+2}}\sum_{j=0}^{\infty}\int_{\left\{z:\, 2^{-j-1} \leq \frac{|x-z|}{2r} < 2^{-j}\right\}} |f(z) - f(x)||x-z|dz \\
&\leq \frac{C}{r^{n+2}}\sum_{j=0}^{\infty}\int_{B(x,2^{-j+1}r)} |f(z) - f(x)|2^{-j+1}r\,dz \\
&= \frac{C}{r^{n+1}}\sum_{j=0}^{\infty} 2^{-j+1}\left(2^{-j+1}r\right)^{n+\lambda}\frac{1}{\left(2^{-j+1}r\right)^{n+\lambda}}\int_{B(x,2^{-j+1}r)} |f(z) - f(x)|dz \\
&\leq Cr^{\lambda-1}\sum_{j=0}^{\infty}\left(2^{-j+1}\right)^{n+\lambda+1} M^{\sharp,\lambda}f(x) \\
&= Cr^{\lambda-1}M^{\sharp,\lambda}f(x).
\end{aligned}$$

Now from *(f)* in Proposition 3,

$$
\begin{aligned}
II &= \frac{1}{r}\int_{B^c(x,2r)} |f(z)-f(x)||\Psi_r^i(x-z)|dz \\
&\le \frac{C}{r}\sum_{j=0}^{\infty}\int_{\left\{z:\, 2^j\le \frac{|x-z|}{2r}<2^{j+1}\right\}} |f(z)-f(x)|\frac{r^{2-a}}{|x-z|^{n+2-a}}dz \\
&\le Cr^{1-a}\sum_{j=0}^{\infty}\int_{\left\{z:\, 2^j\le \frac{|x-z|}{2r}<2^{j+1}\right\}} |f(z)-f(x)|\frac{1}{(2^{j+1}r)^{n+2-a}}dz \\
&\le \frac{C}{r^{n+1}}\sum_{j=0}^{\infty}\left(2^{j+1}\right)^{-n-2+a}\frac{(r2^{j+2})^{n+\lambda}}{(r2^{j+2})^{n+\lambda}}\int_{B(x,2^{j+2}r)}|f(z)-f(x)|dz \\
&\le Cr^{\lambda-1}\left(\sum_{j=0}^{\infty}\left(2^{j+2}\right)^{\lambda-2+a}\right)M^{\sharp,\lambda}f(x) \\
&= Cr^{\lambda-1}M^{\sharp,\lambda}f(x),
\end{aligned}
$$

and the lemma is proved. □

Proof of Theorem 4 The proof follows closely the lines of the proof of Theorem 3 in [3]. The only point in which the nonlocal character of our situation becomes relevant is contained in the first estimates on page 11 in [3]. On the other hand, our upper restriction on λ is only a consequence of the fact that we are using only estimates for the first-order derivatives (after a fine tuning of the function φ, larger values of λ can be achieved). Our restriction guarantees the convergence of the series involved in the estimates in [3] mentioned above. □

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