Positive decompositions of selfadjoint operators.

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Abstract

Given a linear bounded selfadjoint operator a on a complex separable Hilbert space $\mathcal{H}$, we study the decompositions of a as a difference of two positive operators whose ranges satisfy an angle condition. These decompositions are related to the canonical decompositions of the indefinite metric space $(\mathcal{H}, \langle\, , \,\rangle_a)$, associated to a. As an application, we characterize the orbit of congruence of a in terms of its positive decompositions.

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1 Introduction

Consider $L(\mathcal{H})$ the algebra of linear bounded operators of a complex separable Hilbert space $\mathcal{H}$. Denote by $GL(\mathcal{H})$ the group of invertible operators of $L(\mathcal{H})$. It is well known that a selfadjoint operator $a \in L(\mathcal{H})$ can be written as a difference of two positive operators with orthogonal ranges and these operators are uniquely determined by a. The main purpose of this article is to study alternative decompositions of a selfadjoint operator a as a difference of two positive operators whose ranges are not necessarily orthogonal, but satisfy an angle condition.
On the other hand, each selfadjoint operator a defines an indefinite inner product on $\mathcal{H}$, given by

$$\langle x, y\rangle_a = \langle ax, y\rangle, \quad \text{for } x, y \in \mathcal{H}.$$

If a is also invertible, then $(\mathcal{H}, \langle\, , \,\rangle_a)$ is a Krein space and the canonical decompositions of this space, as a direct orthogonal sum of an a-positive and an a-negative subspaces are described, for example, in the classical books by J. Bognar [3] and T. Ya. Azizov and I.S. Iokhvidov [2]. More generally, for any selfadjoint operator $a \in L(\mathcal{H})$, every canonical decomposition of $(\mathcal{H}, \langle\, , \,\rangle_a)$ (in this case, as the sum of three subspaces, an a-positive, an a-negative and the nullspace of a), determines an a-selfadjoint oblique projection with a-nonnegative range and a-nonpositive nullspace, or equivalently, an a-positive reflection. We study the relationship between the positive decompositions of a and the canonical decompositions of the indefinite inner product space $(\mathcal{H}, \langle\, , \,\rangle_a)$. We prove that there is a one to one correspondence between the positive decompositions of a and the a-positive reflections, when a is injective.
As an application, we study the orbit of congruence of a selfadjoint operator a, i.e., the set $\mathcal{O}_a = \{gag^* : g \in GL(\mathcal{H})\}$. If a has closed range, it is possible

to provide $\mathcal{O}_a$ with a structure of differentiable manifold; see [7], [5] and [11]. Moreover, it holds that $(GL(\mathcal{H}), \mathcal{O}_a, \pi_a)$ is a fibre bundle, where $\pi_a(g) = gag^*$, for $g \in GL(\mathcal{H})$. In this paper we characterize the orbit of a in terms of the positive decompositions of a. When a has closed range, we show that the set of positive decompositions of a is parametrized by the elements of the isotropy group of a, i.e., the set $\mathcal{I}_a = \{g \in GL(\mathcal{H}) : gag^* = a\}$.

The article is organized as follows: Section 2 contains some basic results about angles between closed subspaces and a brief survey about equivalence and congruence of operators. In Section 3, we collect some definitions and properties of the indefinite metric space $(\mathcal{H}, \langle\ ,\ \rangle_a)$, for a selfadjoint operator a. Section 4 is devoted to study decompositions of a selfadjoint operator a as a difference of two positive operator such that the minimal angle of their ranges is positive. Any of these decompositions will be called a *positive decomposition*. We relate the positive decompositions of a to the canonical decompositions of the indefinite metric space $(\mathcal{H}, \langle, \rangle_a)$. More precisely, we prove that given a positive decomposition of a, there is an associated a-positive reflection. Conversely, an a-positive reflection determines a positive decomposition of a. If a is injective, we show that there is a bijection between the positive decompositions of a and the set of a-selfadjoint projections with a-positive range and a-negative nullspace. We prove that every positive decomposition of a induces a "pseudo polar decomposition" of a: i.e. a factorization of a as $a = \alpha w$, where α is positive and w is an (a-positive) reflection. If $a = u_a|a|$ is the polar decomposition of a, the a-positive reflections w are those of the form $w = u_a d$, where d is $|a|$-positive (in fact, this turns out to be the polar decomposition of w in the space $(\mathcal{H}, \langle, \rangle_{|a|})$). Finally, if a is injective, given two canonical decompositions of $(\mathcal{H}, \langle, \rangle_a)$, we prove that the a-positive subspaces and the a-negative subspaces have the same dimension, respectively. In the last section, we characterize the set of congruence of a fixed selfadjoint operator a. It is known that two positive operators are congruent if and only if their ranges are unitarily equivalent. We generalize this fact for selfadjoint operators, by means of their positive decompositions. Also, if $a = a_1 - a_2$ is the positive orthogonal decomposition of a and $g \in \mathcal{I}_a$, it holds that $a = ga_1g^* - ga_2g^*$ is a positive decomposition of a. When a has closed range, we show that all the positive decompositions of a can be written as $a = ga_1g^* - ga_2g^*$, for some $g \in \mathcal{I}_a$.

2 Preliminaries

Let $L(\mathcal{H})$ be the algebra of linear bounded operators of a complex separable Hilbert space $\mathcal{H}$, $GL(\mathcal{H})$ the group of invertible operators of $L(\mathcal{H})$ and $\mathcal{U}$ the subgroup of $GL(\mathcal{H})$ of unitary operators. Denote by $L(\mathcal{H})^s$ the set of selfadjoint operators and $L(\mathcal{H})^+$ the cone of positive operators. An operator $v \in L(\mathcal{H})$ is a *reflection* if $v = v^{-1}$ and v is *symmetry* if it is a selfadjoint reflection.

Given $\mathcal{M}$ and $\mathcal{N}$ two closed subspaces of $\mathcal{H}$, then $\mathcal{M} \dot{+} \mathcal{N}$ denotes the direct sum of $\mathcal{M}$ and $\mathcal{N}$, and $\mathcal{M} \oplus \mathcal{N}$ the orthogonal sum. If $\mathcal{M} \dot{+} \mathcal{N} = \mathcal{H}$, we denote by $p_{\mathcal{M}//\mathcal{N}}$ the oblique projection with range $\mathcal{M}$ and nullspace $\mathcal{N}$ and $p_{\mathcal{M}} = p_{\mathcal{M}//\mathcal{M}^\perp}$. Let $\mathcal{Q} = \{q \in L(\mathcal{H}), q^2 = q\}$ be the set of oblique projections. For every $a \in L(\mathcal{H})$, $R(a)$ denotes the range of a, $N(a)$ its nullspace and $p_a = p_{\overline{R(a)}}$.

If $a \in L(\mathcal{H})$, we fix the following *polar decomposition* of a: $a = v_a|a|$ where $|a| = (a^*a)^{1/2}$ is positive and v_a is a partial isometry from $N(a)^\perp$ onto $\overline{R(a)}$ with nullspace $N(v_a) = N(a)$. If a is selfadjoint, the isometric part of the polar decomposition can be defined to obtain a symmetry: in this case $R(a)^\perp = N(a)$ so that $u_a = v_a + p_{N(a)}$ is a symmetry and $a = u_a|a| = |a|u_a$.

The following result due to R. G. Douglas [9], characterizes the operator ranges inclusion:

Theorem 2.1 *Consider $a, b \in L(\mathcal{H})$. Then $R(a) \subseteq R(b)$ if and only if $a = bc$, for some $c \in L(\mathcal{H})$.*

Given $\mathcal{M}$ and $\mathcal{N}$ two closed subspaces of $\mathcal{H}$, the *angle* or *Friedrichs angle* between $\mathcal{M}$ and $\mathcal{N}$ is the angle $\alpha(\mathcal{M}, \mathcal{N}) \in [0, \pi/2]$ whose cosine is given by

$$c(\mathcal{M}, \mathcal{N}) = \sup\{|\langle x, y\rangle| : x \in \mathcal{M} \cap (\mathcal{M} \cap \mathcal{N})^\perp, \|x\| \leq 1, y \in \mathcal{N} \cap (\mathcal{M} \cap \mathcal{N})^\perp, \|y\| \leq 1\}.$$

The *minimal angle* or *Dixmier angle* between $\mathcal{M}$ and $\mathcal{N}$ is the angle $\alpha_0(\mathcal{M}, \mathcal{N}) \in [0, \pi/2]$ whose cosine is given by

$$c_0(\mathcal{M}, \mathcal{N}) = \sup\{|\langle x, y\rangle| : x \in \mathcal{M}, \|x\| \leq 1, x \in \mathcal{N}, \ \|y\| \leq 1\}.$$

Observe that $0 \leq c(\mathcal{M}, \mathcal{N}) \leq c_0(\mathcal{M}, \mathcal{N}) \leq 1$.
The next results about angles can be found in [8]:

Theorem 2.2 *The following statements are equivalent:*

1. $c_0(\mathcal{M}, \mathcal{N}) < 1$,
2. $\mathcal{M} \cap \mathcal{N} = \{0\}$ *and* $\mathcal{M} + \mathcal{N}$ *is closed.*

Theorem 2.3 *The following statements are equivalent:*

1. $c(\mathcal{M}, \mathcal{N}) < 1$,
2. $\mathcal{M} + \mathcal{N}$ *is closed,*
3. $\mathcal{M}^\perp + \mathcal{N}^\perp$ *is closed.*

Two operator ranges $\mathcal{R}$ and $\mathcal{S}$ are *similar* if there exists $g \in GL(\mathcal{H})$ such that $\mathcal{R} = g(\mathcal{S})$ and *unitarily equivalent* if g can be taken to be unitary. Operator ranges are similar if and only if they are unitarily equivalent.
The operators $a, b \in L(\mathcal{H})$ are *equivalent* if there exist $g, f \in GL(\mathcal{H})$ such that $b = gaf$; the operators a and b are *congruent* if there exists $g \in GL(\mathcal{H})$ such that $b = gag^*$.

Proposition 2.4 *Normal operators are equivalent if and only if their ranges are unitarily equivalent.*

Proposition 2.5 *Let $a, b \in L(\mathcal{H})^+$, then a and b are equivalent if and only if a and b are congruent.*

See [10] for the proofs of these facts.

3 The indefinite metric associated to a selfadjoint operator

Along this paper, we consider a fixed $a \in L(\mathcal{H})^s$ and the indefinite metric on $\mathcal{H}$ induced by a, given by

$$\langle x, y\rangle_a = \langle ax, y\rangle, \qquad x, y \in \mathcal{H}.$$

In the following paragraphs, we recall some notions of indefinite inner product spaces. We refer to the classical books of J. Bognar [3] and T.Ya. Azizov and I. S. Iokhvidov [2], for all basic facts of indefinite inner product spaces.

An element $x \in \mathcal{H}$ is *a-positive*, *a-negative* or *a-neutral*, respect to the indefinite metric $\langle\ ,\ \rangle_a$, if $\langle x, x\rangle_a > 0$, $\langle x, x\rangle_a < 0$ or $\langle x, x\rangle_a = 0$ respectively. The element x is called *a-nonnegative* (respectively *a-nonpositive*), if x is a-positive or a-neutral (respectively a-negative or a-neutral). A subspace $\mathcal{S}$ of $\mathcal{H}$ is called *a-positive*, *a-negative* or *a-neutral* if each non zero element of $\mathcal{S}$ is a-positive, a-negative or a-neutral, respectively.

Given $c \in L(\mathcal{H})$, an operator $d \in L(\mathcal{H})$ is an *a-adjoint* of c if $\langle cx, y\rangle_a = \langle x, dy\rangle_a$ for all $x, y \in \mathcal{H}$; or equivalently $ac = d^*a$. Observe that an operator c may admit many, only one or no a-adjoint, depending on whether the equation $c^*a = ah$ has many, only one or no solution, respectively. By Douglas' Theorem, this equation has a solution if and only if $R(c^*a) \subseteq R(a)$. The operator $c \in L(\mathcal{H})$ is *a-selfadjoint* if $ac = c^*a$ and it is *a-positive* if $\langle cx, x\rangle_a \geq 0$ for all $x \in \mathcal{H}$, or equivalently, $ac \in L(\mathcal{H})^+$.
The operator c is an *a-expansion* (respectively, *a-contraction*) if $\langle cx, cx\rangle_a \geq \langle x, x\rangle_a$ (respectively, $\langle cx, cx\rangle_a \leq \langle x, x\rangle_a$); or equivalently $c^*ac \geq a$ (respectively, $c^*ac \leq a$).

Given $x, y \in \mathcal{H}$, then x and y are *a-orthogonal* if $\langle x, y\rangle_a = 0$. In this case, we write $x \perp_a y$. Given a subspace $\mathcal{S}$ of $\mathcal{H}$, the *a-orthogonal subspace* of $\mathcal{S}$ respect to the indefinite metric is the set

$$\mathcal{S}^{\perp_a} = \{x \in \mathcal{H} : \langle x, y\rangle_a = 0,\ \forall y \in \mathcal{S}\}.$$

It is easy to see that $\mathcal{S}^{\perp_a} = a^{-1}(\mathcal{S}^{\perp}) = a(\mathcal{S})^{\perp}$. Observe that $\mathcal{S} \cap \mathcal{S}^{\perp_a}$ is not necessarily zero. If $\mathcal{S}_1, \mathcal{S}_2 \subseteq \mathcal{S}$, then $\mathcal{S}_1 \oplus_a \mathcal{S}_2 = \mathcal{S}$ denotes $\mathcal{S}_1 + \mathcal{S}_2 = \mathcal{S}$, $\mathcal{S}_1 \cap \mathcal{S}_2 = \{0\}$ and $\langle x, y\rangle_a = 0$ for all $x \in \mathcal{S}_1$, $y \in \mathcal{S}_2$.
Observe that if $q \in \mathcal{Q}$, then q is a-selfadjoint if and only if $R(q)$ and $N(q)$ are a-orthogonal.

A *canonical decomposition of* $(\mathcal{H}, \langle\ ,\ \rangle_a)$ is a decomposition of $\mathcal{H}$ as a direct sum

$$\mathcal{H} = N(a) \oplus_a \mathcal{S}^+ \oplus_a \mathcal{S}^-, \tag{1}$$

where $\mathcal{S}^+$ is an a-positive closed subspace of $\mathcal{H}$ and $\mathcal{S}^-$ is an a-negative closed subspace of $\mathcal{H}$. In particular, if $a \in L(\mathcal{H})^s$ is injective, a canonical decomposition of $\mathcal{H}$ is a decomposition of $\mathcal{H}$ as a direct sum $\mathcal{H} = \mathcal{S} \oplus_a \mathcal{S}^{\perp_a}$, where $\mathcal{S}$ is a closed subspace of $\mathcal{H}$ such that $\mathcal{S}$ is a-positive and $\mathcal{S}^{\perp_a}$ is a-negative. In this case, each canonical decomposition defines the projection $q = P_{\mathcal{S}//\mathcal{S}^{\perp_a}}$, or equivalently, the reflection $w = 2q - 1$. Observe that q is a-selfadjoint, $R(q) = \mathcal{S}$ is a-positive

and $N(q) = \mathcal{S}_a^\perp$ is a-negative. Conversely, every $q \in \mathcal{Q}$, a-selfadjoint, such that $R(q)$ is a-positive and $N(q)$ is a-negative, defines a canonical decomposition of $(\mathcal{H}, \langle\, ,\, \rangle_a)$.
S. Hassi and K. Nörstrom proved that given $q \in \mathcal{Q}$, q is an *a-expansion* (*a-contraction*) if and only if q is a-selfadjoint and $N(q)$ is a-nonpositive (a-nonnegative); [13, Proposition 5].

The following lemma characterizes those reflections associated to canonical decompositions; [15, Lemma 5.6]:

Lemma 3.1 *Let $q \in \mathcal{Q}$. Then, q is a-selfadjoint, $R(q)$ is a-nonnegative and $N(q)$ is a-nonpositive if and only if the reflection $w = 2q - 1$ is a-positive.*

Proof. If q is an a-selfadjoint projection then $aq = q^*aq$, so that if $w = 2q - 1$ and $x \in \mathcal{H}$,

$$\langle\, wx, x \,\rangle_a = \langle\, qx, qx \,\rangle_a - \langle\, (1-q)x, (1-q)x \,\rangle_a \geq 0, \tag{2}$$

because $R(q)$ is a-nonnegative and $N(q)$ is a-nonpositive. Thus, w is a-positive. Conversely, if w is a-positive then w is a-selfadjoint. Therefore, $q = \frac{w+1}{2}$ is a-selfadjoint. By (2), if $x \in R(q)$, then $\langle\, qx, qx \,\rangle_a = \langle\, wx, x \,\rangle_a \geq 0$; so that $R(q)$ is a-nonnegative. In a similar way, $N(q)$ is a-nonpositive. ∎

When a is positive, the indefinite form $\langle\, ,\, \rangle_a$ defines a semi-inner product on $\mathcal{H}$, and the associated semi-norm, $\|\ \|_a$, is given by

$$\|x\|_a = \langle x, x\rangle_a^{1/2} = \|a^{1/2}x\|, \qquad x \in \mathcal{H}.$$

The quotient space $(\mathcal{H}/N(a), |||\ |||_a)$ is a normed space, where $|||\ |||_a$ is the associated quotient norm and $|||\overline{x}|||_a = \|x\|_a$, where $\overline{x} = x + N(a)$, $x \in \mathcal{H}$. Since $(\mathcal{H}/N(a), |||\ |||_a)$ is not necessarily complete, denote by $\mathcal{H}_a$ the completion of $(\mathcal{H}/N(a), |||\ |||_a)$. Denote by $\Pi : \mathcal{H} \to \mathcal{H}_a$ the quotient map. In the context of [4], the pair $(\mathcal{H}_a, \Pi)$ is called a *Hilbert space induced* by a.
In particular, if a is injective, the indefinite form $\langle\, ,\, \rangle_a$ defines an inner product on $\mathcal{H}$. In particular, if $a \in GL(\mathcal{H})^+$, then $(\mathcal{H}, \langle\, ,\, \rangle_a)$ is a Hilbert space and the norms $\|\ \|$ and $\|\ \|_a$ are equivalent.

Remark 3.2 The operator $c \in L(\mathcal{H})$ admits an $|a|$-adjoint if and only if c admits an a-adjoint. In fact, $|a|c = f|a|$ if and only if $ac = u_a f|a| = u_a f u_a a$, where $f \in L(\mathcal{H})$.

For $c \in L(\mathcal{H})$, such that c admits an a-adjoint, define $\overline{c}$ the associated operator to c on $\mathcal{H}/N(a)$ by $\overline{c}\overline{x} = \overline{cx}$. Observe that $\overline{c}$ is well defined: if $x, y \in \overline{x}$, then $\overline{cx} = \overline{cy}$, or equivalently, if $x - y \in N(a)$ then $c(x-y) \in N(a)$. In fact, since $ac = d^*a$ for some $d \in L(\mathcal{H})$, then $c(N(a)) \subseteq N(a)$.

Proposition 3.3 *If $c \in L(\mathcal{H})$ admits an a-adjoint, then $\overline{c}$, the associated operator on $\mathcal{H}/N(a)$, is well defined and admits a unique bounded extension to $\mathcal{H}_{|a|}$.*

Proof. Since c admits an a-adjoint, then by Remark 3.2, c admits an $|a|$-adjoint. As we prove above, in this case, $\overline{c}$ is well defined on $\mathcal{H}/N(a)$. From $|a|c = u_a d^* u_a |a|$ and [14, Theorem 5.1], there exists $h \in L(\mathcal{H})$ such that $|a|^{1/2}c =$

$h|a|^{1/2}$. Since $\||\overline{c}\overline{x}\||_{|a|} = \|cx\|_{|a|} = \||a|^{1/2}cx\| = \|h|a|^{1/2}x\| \leq \|h\|\||\overline{x}\||_{|a|}$ and $\mathcal{H}/N(a)$ is dense in $\mathcal{H}_{|a|}$, then $\overline{c}$ admits a unique bounded extension to $\mathcal{H}_{|a|}$. ∎

The above result is similar to [4, Theorem 3.1]. See also [1, Proposition 1.2].

Notice that the a-adjoint of $c \in L(\mathcal{H})$ is unique on $\mathcal{H}_{|a|}$, as expected, since $\mathcal{H}_{|a|}$ is a Hilbert space. In fact, given $d, h \in L(\mathcal{H})$ such that $ac = d^*a = h^*a$, then $ad = ah$. Therefore, $R(d-h) \subseteq N(a)$, so that $\overline{h} = \overline{d}$.

4 Positive decompositions

In this section we study the decompositions of a selfadjoint operator a as a suitable (in some sense we will establish) difference of two positive operators and the relation of these decompositions with the canonical decompositions of the space $(\mathcal{H}, \langle\,,\,\rangle_a)$, defined in (1).

Definition 4.1 *Given $c_1, c_2 \in L(\mathcal{H})^+$, then $a = c_1 - c_2$ is a positive decomposition of a if $c_0(\overline{R(c_1)}, \overline{R(c_2)}) < 1$.*

It is well known that every $a \in L(\mathcal{H})^s$ admits a unique positive decomposition $a = a_1 - a_2$ such that the ranges of a_1 and a_2 are orthogonal, or equivalently, such that $c_0(\overline{R(a_1)}, \overline{R(a_2)}) = 0$. In fact, consider $a_1 = \frac{|a|+a}{2}$ and $a_2 = \frac{|a|-a}{2}$. This decomposition will be called the *positive orthogonal decomposition* (p.o.d.). In this case, the operator a_1 is the *positive part* of a, and $-a_2$ its *negative part.*

By theorems 2.2 and 2.3, the condition $c_0(\overline{R(c_1)}, \overline{R(c_2)}) < 1$ is equivalent to $c(N(c_1), N(c_2)) < 1$ and $\overline{R(c_1)} \cap \overline{R(c_2)} = \{0\}$. If $a = c_1 - c_2$ is a positive decomposition of a, then $N(c_1) + N(c_2)$ is closed and $N(c_1) \cap N(c_2) = N(a)$. Also $(N(c_1) + N(c_2))^\perp = \overline{R(c_1)} \cap \overline{R(c_2)} = \{0\}$, then $N(c_1) + N(c_2) = \mathcal{H}$.

Lemma 4.2 *Consider $a = c_1 - c_2$ with $c_1, c_2 \in L(\mathcal{H})^+$. Then $a = c_1 - c_2$ is a positive decomposition of a if and only if $\overline{R(c_1)} \dot{+} \overline{R(c_2)} = \overline{R(a)}$. In this case, $R(a) = R(c_1) \dot{+} R(c_2)$. In particular, $R(a)$ is closed if and only if $R(c_i)$ is closed, for $i = 1, 2$.*

Proof. By definition, $a = c_1 - c_2$ is a positive decomposition of a if and only if $c_0(\overline{R(c_1)}, \overline{R(c_2)}) < 1$. Then, by Theorem 2.2, $\overline{R(c_1)} \cap \overline{R(c_2)} = \{0\}$ and $\overline{R(c_1)} \dot{+} \overline{R(c_2)}$ is closed. Observe that $(\overline{R(c_1)} \dot{+} \overline{R(c_2)})^\perp = N(c_1) \cap N(c_2) = N(a)$. Therefore, $\overline{R(c_1)} \dot{+} \overline{R(c_2)} = \overline{R(a)}$. The converse follows by Theorem 2.2.
Suppose that $c_0(\overline{R(c_1)}, \overline{R(c_2)}) < 1$, then $\overline{R(c_2)} \oplus N(a)$ is closed and $\mathcal{H} = \overline{R(c_1)} \dot{+} \overline{R(c_2)} \dot{+} N(a)$. Consider the oblique projection $p_1 = p_{\overline{R(c_1)}//\overline{R(c_2)} \oplus N(a)} \in L(\mathcal{H})$, then $p_1 a = c_1 = a p_1^*$ so that $R(c_1) \subseteq R(a)$. In the same way, $R(c_2) \subseteq R(a)$. Hence $R(c_1) \dot{+} R(c_2) \subseteq R(a)$. But from $a = c_1 - c_2$, it follows that $R(a) \subseteq R(c_1) + R(c_2)$. ∎

Remark 4.3 If $a = c_1 - c_2$ is a positive decomposition of a, denote by $p_1 = p_{\overline{R(c_1)}//\overline{R(c_2)} \oplus N(a)}$ and $p_2 = p_{\overline{R(c_2)}//\overline{R(c_1)} \oplus N(a)}$. Since $p_1 p_2 = p_2 p_1 = 0$ then $p_1 + p_2 \in \mathcal{Q}$ and $R(p_1 + p_2) = \overline{R(c_1)} \dot{+} \overline{R(c_2)} = \overline{R(a)}$. Also $N(p_1 + p_2) = N(p_1) \cap N(p_2) = N(a)$. Therefore $p_1 + p_2 = p_a$.

From now on, given $a = c_1 - c_2$ a positive decomposition of a, we consider

$$q = p_1^* \text{ with } p_1 = p_{\overline{R(c_1)}//\overline{R(c_2)}\oplus N(a)} \text{ and } w = 2q - 1. \tag{3}$$

Note that $q = p_{N(c_2)\cap\overline{R(a)}//N(c_1)} \in \mathcal{Q}$ and w is a symmetry. The next theorem shows the relation between positive decompositions and a-positive reflections.

Theorem 4.4 *If $a = c_1 - c_2$ is a positive decomposition of a, then $w = 2q-1$ is a-positive. Conversely, given an a-positive reflection w, consider $q = \frac{w+1}{2} \in \mathcal{Q}$, $c_1 = aq$ and $c_2 = a(q-1)$, then $a = c_1 - c_2$ is a positive decomposition of a.*

Proof. Suppose that $a = c_1 - c_2$ is a positive decomposition of a and let $q = p_1^*$ as in (3). Then $aq = c_1 = q^*a$, so that q is a-selfadjoint. If $x \in R(q)$, then $\langle x, x\rangle_a = \langle aqx, x\rangle = \langle c_1 x, x\rangle \geq 0$, because c_1 is positive; so that $R(q)$ is a-nonnegative. In a similar way, $N(q)$ is a-nonpositive, because $a(1-q) = -c_2$. By Lemma 3.1, $w = 2q - 1$ is a-positive.
Conversely, let w be an a-positive reflection and consider $q = \frac{w+1}{2}$, $c_1 = aq$ and $c_2 = a(q-1)$, then $a = c_1 - c_2$. Note that $c_1, c_2 \in L(\mathcal{H})^+$: in fact, if $x \in \mathcal{H}$, $\langle c_1 x, x\rangle = \langle aqx, qx + (1-q)x\rangle = \langle aqx, qx\rangle \geq 0$, because, by Lemma 3.1, $R(q)$ is a-nonnegative. Similarly for c_2. Observe that $\overline{R(c_1)} \subseteq \overline{R(q^*)}$ and $\overline{R(c_2)} \subseteq N(q^*)$, since $c_1 = q^*a$ and $c_2 = (q^*-1)a$. Therefore, $c_0(\overline{R(c_1)}, \overline{R(c_2)}) \leq c_0(R(q^*), N(q^*)) < 1$; so that $a = c_1 - c_2$ is a positive decomposition of a. ∎

In particular, if a is injective, we obtain the following correspondence. In case a is not injective, we can consider the operator $\tilde{a} = a|_{\overline{R(a)}} \in L(\overline{R(a)})$.

Corollary 4.5 *Consider a injective. For $a = c_1 - c_2$ a positive decomposition of a, define $\phi(c_1, c_2) = 2p_{N(c_2)//N(c_1)} - 1$. Then ϕ is a bijection from the set of positive decompositions of a onto the set of a-positive reflections.*

Proof. Let $a = c_1 - c_2$ be a positive decomposition of a and let $q = \phi(c_1, c_2) = 2p_{N(c_2)//N(c_1)} - 1$. By Theorem 4.4, $\phi(c_1, c_2)$ is an a-positive reflection. To see that ϕ is a bijection, consider w an a-positive reflection. Define $\varphi(w) = (a(\frac{w+1}{2}), a(\frac{w-1}{2}))$. By Theorem 4.4, if $c_1 = a(\frac{w+1}{2})$ and $c_2 = a(\frac{w-1}{2})$, then $a = c_1 - c_2$ is a positive decomposition of a. Let $q = \frac{w+1}{2}$, then $\phi(\varphi(w)) = \phi(aq, a(q-1)) = w$, since $N(a(q-1)) = N(q-1) = R(q)$ and $N(aq) = N(q)$ because a is injective. Moreover, if $a = c_1 - c_2$ is a positive decomposition of a, then $\varphi(\phi(c_1, c_2)) = (ap_{N(c_2)//N(c_1)}, a(p_{N(c_2)//N(c_1)} - 1)) = (c_1, c_2)$. Therefore $\varphi = \phi^{-1}$. ∎

Under the hypothesis of the above corollary, let $q = p_{N(c_2)//N(c_1)}$. By Theorem 4.4 and Lemma 3.1, $R(q)$ is a-nonnegative and $N(q)$ is a-nonpositive. Moreover, if $x \in R(q) = N(c_2)$ is such that $\langle x, x\rangle_a = 0$ then $\|c_1^{1/2}x\| = \langle c_1 x, x\rangle = \langle x, x\rangle_a = 0$. Therefore $x \in N(c_1)$. Since $N(c_1) \cap N(c_2) = N(a) = \{0\}$, it follows that $x = 0$. Hence $R(q)$ is a-positive. Similarly, $N(q)$ is a-negative.
In this case,

$$\mathcal{H} = N(c_2) \oplus_a N(c_1)$$

is the canonical decomposition of $(\mathcal{H}, \langle\ ,\ \rangle_a)$ determined by the positive decomposition $a = c_1 - c_2$ of a.

If $a = a_1 - a_2$ is the p.o.d. of a, then $|a| = a_1 + a_2$ and $a = u_a|a|$, where u_a is a symmetry. In a similar way, each positive decomposition of a induces a decomposition of a as a product of a reflection and a positive operator, as shows the following corollary:

Corollary 4.6 *Suppose that $a = c_1 - c_2$ is a positive decomposition of a. If $\alpha = c_1 + c_2$ and $w = 2q - 1$ as in (3), then $a = \alpha w$ where $\alpha \in L(\mathcal{H})^+$, $w^2 = 1$ and w is a-positive. Conversely, if $a = \alpha w$, with $\alpha \in L(\mathcal{H})^+$, $w^2 = 1$ and w a-positive, consider $c_1 = a(\frac{w+1}{2})$ and $c_2 = c_1 - a$, then $a = c_1 - c_2$ is a positive decomposition of a.*

Proof. If $w = 2q - 1$ and $\alpha = c_1 + c_2 \in L(\mathcal{H})^+$, then $w^2 = 1$ and $w^*\alpha = (2p_1 - 1)(c_1 + c_2) = c_1 - c_2 = a = \alpha w$. Also $aw = \alpha$ so that w is a-positive.
Conversely, consider $a = \alpha w$ where $\alpha \in L(\mathcal{H})^+$, $w^2 = 1$ and w is a-positive. Let $q = \frac{w+1}{2}$. By Lemma 3.1, q is a-selfadjoint, $R(q)$ is a-nonnegative and $N(q)$ is a-nonpositive. If $c_1 = aq$ and $c_2 = a(q-1)$, by Theorem 4.4, it follows that $a = c_1 - c_2$ is a positive decomposition of a. ∎

By the above corollary, if $a = c_1 - c_2$ is a positive decomposition of a, then $a = \alpha w$ where $\alpha = c_1 + c_2 \in L(\mathcal{H})^+$, $w^2 = 1$ and w is a-positive. In this case, $\alpha = (w^*|a|^2 w)^{1/2}$. In fact, $\alpha = aw = w^*a$ so that $\alpha^2 = w^*a^2w = w^*|a|^2w$.

If a is injective, then, by the previous results, a positive decomposition of a is uniquely determined either by a cannonical decomposition of $(\mathcal{H}, \langle\,,\rangle_a)$, or an a-positive reflection; or equivalently, an a-selfadjoint projection with a-positive range and a-negative nullspace.

The next proposition characterizes the a-positive reflections.

Proposition 4.7 *Let $a = u_a|a|$ be the polar decomposition of a. A reflection w is a-positive if and only if w admits a polar decomposition in $(\mathcal{H}, \langle\,,\rangle_{|a|})$ given by $w = u_a d$, where $d \in GL(\mathcal{H})$ is $|a|$-positive.*

Proof. Suppose that $w = 2q - 1$ is an a-positive reflection, then $aw = |a|u_a w \in L(\mathcal{H})^+$. Therefore, $d = u_a w \in GL(\mathcal{H})$ is $|a|$-positive. Then $w = u_a d$, where d is $|a|$-positive. Since $|a|u_a = u_a|a| = u_a^*|a|$, then u_a is $|a|$-unitary and $w = u_a d$ is the polar decomposition of w in $(\mathcal{H}, \langle, \rangle_{|a|})$.
Conversely, suppose that $w = u_a d$, where d is $|a|$-positive. Then $aw = |a|d$ is positive. ∎

Consider a injective and suppose that $\mathcal{H} = \mathcal{S} \oplus_a \mathcal{S}^{\perp_a} = \mathcal{S}' \oplus_a \mathcal{S}'^{\perp_a}$ are two canonical decompositions of $(\mathcal{H}, \langle\,,\rangle_a)$. Denote by $\overline{\mathcal{S}}^{|a|}$ the completion of $\mathcal{S}$ respect to $\langle\,,\rangle_{|a|}$. As a consequence of Proposition 4.7, the next theorem shows that the completion of the a-positive subspaces of the two canonical decompositions have the same Hilbert space dimension; i.e., $\dim_{|a|} \overline{\mathcal{S}}^{|a|} = \dim_{|a|} \overline{\mathcal{S}'}^{|a|}$. Moreover, $\dim \mathcal{S} = \dim \mathcal{S}'$, when $\mathcal{H}$ is separable. The same holds for the a-negative subspaces. Compare this result with [3, Corollary 7.4, Chapter IV].

Theorem 4.8 *Consider a injective. Let $\mathcal{H} = \mathcal{S} \oplus_a \mathcal{S}^{\perp_a} = \mathcal{S}' \oplus_a \mathcal{S}'^{\perp_a}$ be canonical decompositions of $\mathcal{H}$, then* $\dim \mathcal{S} = \dim \mathcal{S}'$ *and* $\dim \mathcal{S}^{\perp_a} = \dim \mathcal{S}'^{\perp_a}$.

Proof. Suppose that $\mathcal{H} = \mathcal{S}' \oplus_a \mathcal{S}'^{\perp_a}$ is the decomposition of $\mathcal{H}$ given by the p.o.d. $a = a_1 - a_2$. In this case, the associated projection is $p_1 = p_{a_1}$. Consider the oblique projection $q = p_{\mathcal{S}//\mathcal{S}^{\perp_a}}$ and the reflection $w = 2q-1$. By Proposition 4.7, $w = u_a d$, where $d \in GL(\mathcal{H})$ is $|a|$-positive. Since w is a reflection, $u_a d = d^{-1}u_a$, so that $u_a d u_a = d^{-1}$. Observe that u_a, d, d^{-1} are $|a|$-selfadjoint. Then, by Proposition 3.3 and Remark 3.2, it holds that u_a, d, d^{-1} admit bounded extensions to $\mathcal{H}_{|a|}$, denoted by $\overline{u_a}, \overline{d}, \overline{d^{-1}}$. Note that $\overline{d^{-1}} = (\overline{d})^{-1}$. Since $\overline{d^{-1}}$ is positive in $\mathcal{H}_{|a|}$, then $\overline{d^{-1}}$ admits a unique (positive) square root in $\mathcal{H}_{|a|}$, $(\overline{d^{-1}})^{1/2} \in GL(\mathcal{H}_{|a|})^+$. Note that $\overline{u_a}(\overline{d})^{1/2}\overline{u_a}$ is positive in $\mathcal{H}_{|a|}$, because $\overline{u_a}$ is selfadjoint in $\mathcal{H}_{|a|}$ and $(\overline{d})^{1/2}$ is positive in $\mathcal{H}_{|a|}$. Also, $(\overline{u_a}(\overline{d})^{1/2}\overline{u_a})^2 = \overline{u_a d u_a}$, so that $(\overline{u_a d u_a})^{1/2} = \overline{u_a}(\overline{d})^{1/2}\overline{u_a}$. Then $(\overline{d^{-1}})^{1/2} = (\overline{u_a d u_a})^{1/2} = \overline{u_a}(\overline{d})^{1/2}\overline{u_a}$ and $(\overline{d^{-1}})^{1/2}\overline{u_a} = \overline{u_a}(\overline{d})^{1/2}$. Hence, $2\overline{q} - 1 = \overline{w} = \overline{u_a}\overline{d} = (\overline{d^{-1}})^{1/2}\overline{u_a}(\overline{d})^{1/2} = (\overline{d^{-1}})^{1/2}(2\overline{p_1} - 1)(\overline{d})^{1/2} = 2(\overline{d^{-1}})^{1/2}\overline{p_1}(\overline{d})^{1/2} - 1$, since $(\overline{d^{-1}})^{1/2} = [(\overline{d})^{-1})]^{1/2} = (\overline{d})^{-1/2}$. Therefore, $\overline{q} = (\overline{d^{-1}})^{1/2}\overline{p_1}(\overline{d})^{1/2}$. Then, $R(\overline{p_1}) = (\overline{d^{-1}})^{1/2}R(\overline{q})$ and $N(\overline{p_1}) = (\overline{d^{-1}})^{1/2}N(\overline{q})$, so that $\dim_{|a|} R(\overline{p_1}) = \dim_{|a|} R(\overline{q})$ and $\dim_{|a|} N(\overline{p_1}) = \dim_{|a|} N(\overline{q})$, where $\dim_{|a|}\mathcal{U}$ is the dimension of a subspace $\mathcal{U}$ of $\mathcal{H}_{|a|}$.
Then $\dim_{|a|} \overline{R(p_1)}^{|a|} = \dim_{|a|} \overline{R(q)}^{|a|}$, because $R(\overline{q}) = \overline{R(q)}^{|a|}$. Since $\mathcal{H}$ is separable, it is easy to see that $(\mathcal{H}, \langle\, , \rangle_{|a|})$ is separable. Hence, $\mathcal{H}_{|a|}$ is separable, because $(\mathcal{H}, \langle\, , \rangle_{|a|})$ is dense in $\mathcal{H}_{|a|}$. In this case, if $\mathcal{S}$ is a subspace of $\mathcal{H}$, then $\dim_{|a|} \overline{\mathcal{S}}^{|a|} = \dim_{|a|} \mathcal{S} = \dim \mathcal{S}$, with $\dim_{|a|} \mathcal{S}$ the cardinal of any maximal orthonormal subset of $\mathcal{S}$ in $\mathcal{H}_{|a|}$ and $\dim \mathcal{S}$ is the dimension of $\mathcal{S}$ as a subspace of $\mathcal{H}$. Therefore, $\dim R(p_1) = \dim R(q)$. Similarly, $\dim N(p_1) = \dim N(q^*)$. ∎

In the proof of the previous proposition, we concluded that $\dim_{|a|} \overline{\mathcal{S}}^{|a|} = \dim \mathcal{S}$ for any closed subspace $\mathcal{S}$ of $\mathcal{H}$. This holds because the Hilbert space $\mathcal{H}$ is separable, so that $(\mathcal{H}, \langle\, , \rangle_{|a|})$ and (therefore) $\mathcal{H}_{|a|}$ are separable. There are examples of inner product spaces $\mathcal{E}$ with completions $\overline{\mathcal{E}}$ such that $\dim \mathcal{E} < \dim \overline{\mathcal{E}}$, where $\dim \mathcal{E}$ is the cardinal of any maximal orthonormal set, see [12].

Corollary 4.9 *Consider a with closed range such that $a = a_1 - a_2$ is the p.o.d. of a and $a = c_1 - c_2$ is a positive decomposition of a. Then* $\dim R(a_i) = \dim R(c_i)$ *and* $\dim N(a_i) = \dim N(c_i)$ *for $i = 1, 2$.*

Proof. Suppose first that a is invertible and consider p_1 and q as in the proof of the previous proposition; i.e. $p_1 = p_{a_1}$ and $q = p_{N(c_2)//N(c_1)}$. By the above proof, there exists $g \in GL(\mathcal{H})^+$ such that $q = g^{-1}p_1 g$ so that $q^* = g^* p_1 g^{*-1}$. Then, since $q^* = p_{R(c_1)//R(c_2)}$, we get that $\dim R(a_i) = \dim R(c_i)$ and $\dim N(a_i) = \dim N(c_i)$ for $i = 1, 2$.
More generally, if a has closed range and $a = c_1 - c_2$ is a positive decomposition of a, notice that $c_i p_a = c_i$, because $N(a) \subseteq N(c_i)$, for $i = 1, 2$. Then $R(c_i|_{R(a)}) = R(c_i)$ and $a|_{R(a)} = c_1|_{R(a)} - c_2|_{R(a)}$ is a positive decomposition of $a|_{R(a)}$. Since $a|_{R(a)} \in GL(R(a))^s$, it follows that $\dim R(c_i|_{R(a_i)}) = \dim R(a_i|_{R(a_i)})$, where $a = a_1 - a_2$ is the p.o.d of a. Then $\dim R(c_i) = \dim R(a_i)$, for $i = 1, 2$. Also, $\dim N(c_i|_{R(a)}) = N(a_i|_{R(a)})$. But, $N(c_i) = N(c_i) \cap R(a) \oplus N(a)$. Then, $\dim N(a_i) = \dim N(c_i)$ for $i = 1, 2$. ∎

5 Congruence of a selfadjoint operator

Two operators $a, b \in L(\mathcal{H})$ are *congruent* if there exists $g \in GL(\mathcal{H})$ such that $b = gag^*$. In this section we study the set of operators in $L(\mathcal{H})^s$ which are congruent to a fixed selfadjoint operator a. We characterize this set in terms of the positive decompositions of a.

The congruence between selfadjoint operators defines the following natural action of the group $GL(\mathcal{H})$ over the set $L(\mathcal{H})^s$:

$$L : GL(\mathcal{H}) \times L(\mathcal{H})^s \to L(\mathcal{H})^s,\ L_g a = gag^*, \quad a \in L(\mathcal{H})^s,\ g \in GL(\mathcal{H}).$$

Given $a \in L(\mathcal{H})^s$, the *orbit* of a corresponding to the action L is the set $\mathcal{O}_a$ of operators which are congruent to a, i.e.

$$\mathcal{O}_a = \{gag^* : g \in GL(\mathcal{H})\}.$$

Denote by $\mathcal{I}_a$ the *isotropy group* of a, i.e. $\mathcal{I}_a = \{g \in GL(\mathcal{H}) : gag^* = a\}$.

The following result is a consequence of Proposition 2.4 and Theorem 2.5, and provides a characterization of $\mathcal{O}_a$, when a is positive.

Proposition 5.1 *Let $a, b \in L(\mathcal{H})^+$; then $b \in \mathcal{O}_a$ if and only if $R(a)$ and $R(b)$ are unitarily equivalent.*

Remark 5.2 If $a, b \in L(\mathcal{H})^+$, it also holds that b is in the orbit of a if and only if the ranges of their square roots are unitarily equivalent, see [10, Theorem 3.5].

If a, b are positive closed range operators, then b is congruent to a if and only if $\dim R(b) = \dim R(a)$ and $\dim N(b) = \dim N(a)$, see [5, Theorem 3.4]. The next result generalizes this fact, for selfadjoint operators.

Proposition 5.3 *Consider $a = a_1 - a_2$ the p.o.d of a. Then $b \in \mathcal{O}_a$ if and only if there exists a positive decomposition $b = b_1 - b_2$ such that $R(b_i)$ is (unitarily) equivalent to $R(a_i)$ for $i = 1, 2$ and $\dim N(b) = \dim N(a)$.*

Proof. If $b \in \mathcal{O}_a$, then $b = gag^*$ for some $g \in GL(\mathcal{H})$, so that $\dim N(b) = \dim N(a)$. Consider $b_i = ga_ig^* \in L(\mathcal{H})^+$, for $i = 1, 2$, then it is easy to see that $b = b_1 - b_2$ is a positive decomposition of b. Also, $b_i \in \mathcal{O}_{a_i}$, for $i = 1, 2$, and by Proposition 5.1, $R(b_i)$ is unitarily equivalent to $R(a_i)$ for $i = 1, 2$.
Conversely, since $\dim N(b) = \dim N(a)$, there exists a partial isometry v such that $v(N(a)) = N(b)$. By Proposition 5.1 and Remark 5.2, there exist $u_1, u_2 \in \mathcal{U}$ such that $R(b_i^{1/2}) = u_i R(a_i^{1/2})$, for $i = 1, 2$. Then $b_i^{1/2}$ and $u_i a_i^{1/2} u_i^*$ have the same range and nullspace, so that (see [10, Corollary 1]) there exists $g_i \in GL(\mathcal{H})$ such that $b_i^{1/2} = g_i u_i a_i^{1/2} u_i^*$, or $b_i^{1/2} u_i = g_i u_i a_i^{1/2}$, $i = 1, 2$. Consider $p_1 = p_{\overline{R(b_1)}//\overline{R(b_2)}\oplus N(b)}$, $p_2 = p_{\overline{R(b_2)}//\overline{R(b_1)}\oplus N(b)}$ and $w = p_1 g_1 u_1 p_{a_1} + p_2 g_2 u_2 p_{a_2} + v(1 - p_a)$. Then $w \in GL(\mathcal{H})$. In fact, by Remark 4.3, it is easy to see that $w^{-1} = p_{a_1} u_1^* g_1^{-1} p_1 + p_{a_2} u_2^* g_2^{-1} p_2 + v^*(1 - p_b)$. On the other hand, $waw^* = w(a_1 - a_2)w^* = b_1^{1/2} u_1 u_1^* b_1^{1/2} - b_2^{1/2} u_2 u_2^* b_2^{1/2} = b$. Hence $b \in \mathcal{O}_a$. ∎

The above result also holds if $a = a_1 - a_2$ is any positive decomposition of a; in fact, in the proof of the proposition, it is sufficient to consider the oblique projections $p_{\overline{R(a_1)}//\overline{R(a_2)}\oplus N(a)}$ and $p_{\overline{R(a_2)}//\overline{R(a_1)}\oplus N(a)}$ instead of p_{a_1} and p_{a_2}.

Corollary 5.4 *Let $a = a_1 - a_2$ be any positive decomposition of a. Then $\mathcal{O}_a = \{b_1 - b_2 : b_i \in \mathcal{O}_{a_i}, i = 1,2;$ and $\overline{R(b_1)}\dot{+}\overline{R(b_2)}$ is unitarily equivalent to $\overline{R(a)}\}$.*

Proof. Observe that $\overline{R(b_1)} + \overline{R(b_2)}$ is an operator range by [10, Theorem 2.2]. If $b \in \mathcal{O}_a$ then by Proposition 5.3, there exists a positive decomposition $b = b_1 - b_2$ such that $R(b_i)$ is unitarily equivalent to $R(a_i)$ for $i = 1,2$ and $N(b)$ is unitarily equivalent to $N(a)$; or equivalently, $b_i \in \mathcal{O}_{a_i}, i = 1,2$ and $\overline{R(b)}$ is unitarily equivalent to $\overline{R(a)}$. But, by Lemma 4.2, $\overline{R(b)} = \overline{R(b_1)}\dot{+}\overline{R(b_2)}$. Conversely, let $b = b_1 - b_2$ with $b_i \in \mathcal{O}_{a_i}, i = 1,2$ and $\overline{R(b_1)} + \overline{R(b_2)} = u\overline{R(a)}$, for $u \in \mathcal{U}$. Then $\overline{R(b_1)} + \overline{R(b_2)}$ is closed and $\overline{R(b_1)} \cap \overline{R(b_2)} = \{0\}$, so by Theorem 2.2, $c_0(\overline{R(b_1)}, \overline{R(b_2)}) < 1$. Note that $b_i \in L(\mathcal{H})^+$ because $b_i \in \mathcal{O}_{a_i}$, for $i = 1,2$. Hence $b = b_1 - b_2$ is a positive decomposition of b, so that $\overline{R(b)} = \overline{R(b_1)}\dot{+}\overline{R(b_2)} = u\overline{R(a)}$ and then $N(b)$ is unitarily equivalent to $N(a)$. Therefore, by Proposition 5.3, $b \in \mathcal{O}_a$. ∎

If $g \in \mathcal{I}_a$, then $a = ga_1g^* - ga_2g^*$, where $a = a_1 - a_2$ is the p.o.d. of a. It is not difficult to see that $a = ga_1g^* - ga_2g^*$ is a positive decomposition of a. Therefore, it is natural to ask if all the positive decomposition of a can be written as $a = ga_1g^* - ga_2g^*$ for some $g \in \mathcal{I}_a$. The following proposition shows that this holds if a has closed range.

Proposition 5.5 *Consider a with closed range and p.o.d. $a = a_1 - a_2$. Then $\{ga_1g^* - ga_2g^* : g \in \mathcal{I}_a\}$ is the set of positive decomposition of a.*

Proof. Given $g \in \mathcal{I}_a$, then $a = gag^*$. It follows easily that $a = ga_1g^* - ga_2g^*$ is a positive decomposition of a. Conversely, let $a = c_1 - c_2$ be a positive decomposition of a. By Corollary 4.9, it holds that $\dim R(a_i) = \dim R(c_i)$ and $\dim N(a_i) = \dim N(c_i)$ for $i = 1,2$. Therefore, by Remark 5.2, $c_i \in \mathcal{O}_{a_i}$, for $i = 1,2$. Then, there exist $g_1, g_2 \in GL(\mathcal{H})$ such that $c_1 = g_1a_1g_1^*$ and $c_2 = g_2a_2g_2^*$. Consider $g = g_1p_{a_1} + g_2p_{a_2} + p_{N(a)}$. By Remark 4.3, it is not difficult to see that $g \in GL(\mathcal{H})$ and $g^{-1} = g_1^{-1}p_1 + g_2^{-1}p_2 + p_{N(a)}$, where $p_1 = p_{R(c_1)//R(c_2)\oplus N(a)}$ and $p_2 = p_{R(c_2)//R(c_1)\oplus N(a)}$. Also, $ga_1g^* = g_1a_1g_1^*p_1^* = c_1p_1^* = c_1$. Similarly, $ga_2g^* = c_2$. Finally, $gag^* = ga_1g^* - ga_2g^* = c_1 - c_2 = a$, so that $g \in \mathcal{I}_a$. ∎

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