Reducible means

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Abstract

A n variables mean M is said to be reducible in a certain class of means $\mathcal{N}$ when M can be represented as a composition of a finite number $M_0, \ldots, M_r$ of means belonging to $\mathcal{N}$, being less than n the number of variables of every M_i. In this paper, a basic classification of reducible means is developed and the notions of S-reducibility, a type of analytically decidible reducibility, and of complete reducibility of a mean are isolated. Several applications of these notions are presented. In particular, a continuous and scale invariant weighting procedure defined on a class $\mathcal{M}_2$ of two variables means is extended without losing its properties to the class of reducible means in $\mathcal{M}_2$.

1 Introduction

Given a real interval I and an integer number $n \in \mathbb{N}$, a function $M : I^n \to I$ defined on I is a (n variables) *mean* when it is *internal*; i.e., when the twofold inequality

$$\min\{x_1, \ldots, x_n\} \leq M(x_1, \ldots, x_n) \leq \max\{x_1, \ldots, x_n\}, \tag{1}$$

is satisfied by M for every $x_1, \ldots, x_n \in I$. x_i is said to be an *effective variable (* or *effective argument*) of M when there exists a pair $\alpha, \beta \in I$ such that $M|_{x_i=\alpha} \neq M|_{x_i=\beta}$. The number $\nu(M)$ of effective arguments of M will play a capital role along this paper and, unless otherwise agreed, in the notation $M(x_1, \ldots, x_n)$ it will assumed that $\nu(M) = n$. Exceptions which will frequently occur are the *i-th coordinate (projection) means* $X_i(x_1, \ldots, x_n) \equiv x_i,\ i = 1, \ldots, n$. These are the unique n variables means M with $\nu(M) = 1$.

A continuous mean with $\nu(M) = n$ defined on I is a mean which is a continuous function on I^n. The class $\mathcal{C}^{(0)}\mathcal{M}(I)$ of all continuous means defined on an interval I is closed under composition, so that the function defined by

$$M = M_0(M_1, \ldots, M_r)$$

is a member of $\mathcal{C}^{(0)}\mathcal{M}(I)$ provided that $M_0, \ldots, M_r \in \mathcal{C}^{(0)}\mathcal{M}(I)$ and $\nu(M_0) = r$. Borrowing a concept from Universal Algebra, it can be said that the set

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$\mathcal{C}^{(0)}\mathcal{M}(I)$ enhanced with composition has the structure corresponding to a *clone*. Note that $\nu(M) \leq \sum_{i=1}^{r} \nu(M_i)$, and that the maximum $\sum_{i=1}^{r} \nu(M_i)$ of effective arguments of M occurs when all the variables of $M_1, \ldots, M_r$ are different each other.

An informal rule can be formulated stating that the difficulty of a problem involving a mean M increases with $\nu(M)$. The theory of mean inequalities, a cornerstone in the studies on means, constitute a good example of this rule. Situated at the very beginning of this theory, the case $n = 2$ of the Arithmetic mean-Geometric mean inequality turns out to be equivalent to the nonnegativity of $(x_1 - x_2)^2$, $x_1, x_2 \in \mathbb{R}$, but no one similar reduction is possible when $n > 2$. A less simple instance of the rule is the problem of defining a *scale invariant weighting* on a class $\mathcal{M}_n(I)$ of n variables means. A function $\mathcal{W} : \mathcal{M}_n(I) \times \Delta_{n-1} \to \mathcal{N}(I)$, where Δ_m denotes the standard m-simplex and $\mathcal{N}(I) \supseteq \mathcal{M}_n(I)$ is another class of means, is said to be a *weighting procedure* (defined on $\mathcal{M}_n(I)$) when the following conditions are satisfied (cf. [6] for the case $n = 2$):

(W1) $\mathcal{W}(M, (1/n, \ldots, 1/n)) = M$,

(W2) $\mathcal{W}(M, e_i) = X_i$, where $e_i = \left(\delta_j^i\right)_{j=1}^n$ (δ_j^i is the Kronecker delta) and $X_i(x_1, \ldots, x_n) \equiv x_i$ is the i-th coordinate mean.

A weighting procedure can be understood as a generalization of the process of converting the arithmetic mean $A_n(x_1, \ldots, x_n) = (x_1 + \cdots + x_n)/n$ in the weighted arithmetic (or linear) mean $L_{n,(w_1,\ldots,w_n)}(x_1, \ldots, x_n) = w_1x_1 + \cdots + w_nx_n$, where $(w_1, \ldots, w_n) \in \Delta_{n-1}$. If the weighting process is covariant with respect to an arbitrary change of scale, then it is said scale invariant, while it is said continuous when, for every $M \in \mathcal{M}_n(I)$, $\mathcal{W}(M, \cdot)$ is continuous on Δ_{n-1}. Now, some schemes of composition like Aczél's or Ryll-Nardzewski's iterations of a two variables mean M_2 which are defined on the dyadic fractions of $[0, 1]$ can be, under mild conditions on M_2, extended to the whole interval $[0, 1]$ and therefore, continuous and scale invariant weighting procedures defined on $\mathcal{C}^{(0)}\mathcal{M}_2(I)$ (or even on more general classes of two variables means) can be based on them (cf. [6], [7], [8]). It must be added that the extension of these algorithms to $n = 3$ is not immediate (cf. [23]), while general algorithms valid for $n > 3$ are being currently studied.

In view of the difficulty of a problem generally increases with dimension, it seems natural to express a n variables mean as a composition of means with a less number of variables, and then try to solve the problem for these last ones. Accordingly, along this paper a mean $M \in \mathcal{M}(I)$ with $\nu(M) = n$ is said to be *reducible in a class* $\mathcal{N}(I) \subseteq \mathcal{M}(I)$, when M can be expressed as a composition of a finite number of means $M_0, \ldots, M_r \in \mathcal{N}(I)$ satisfying $\nu(M_i) < n$, $i = 0, \ldots, r$. Nevertheless, as reasonable as this program may seem to solve a problem involving means, a full implementation of it will require to decide whether a mean M is reducible or not in a given class $\mathcal{N}(I)$, a certainly non trivial problem.

To insert reducibility of means in a suitable context, let us remind that the problem of expressing a continuous function $F : \mathbb{R}^n \to \mathbb{R}$ as a composition of a finite number of continuous functions $F_i : \mathbb{R}^{n_i} \to \mathbb{R},\ i = 0, \dots, r$, with $n_i < n$, can be traced back at least to the year 1900, when D. Hilbert presented his collection of twenty-three problems at the International Conference of Mathematicians in Paris. In the 13th problem of the collection (cf. [4] or Chap. 11 of [20]) the conjecture was implicit that not all continuous functions of three variables can be expressed as a composition of functions of two variables. In 1957, V. I. Arnold showed the conjecture was not true: all function $f \in \mathcal{C}^{(0)}\left(I^3\right)$ can be represented in the form

$$f\left(x_1, x_2, x_3\right) = \sum_{i,j=1}^{3} h_{ij}\left(\phi_{ij}\left(x_1, x_2\right), x_3\right), \tag{2}$$

where $I = [0, 1]$ and h_{ij}, $\phi_{ij} \in \mathcal{C}^{(0)}\left(I^2\right)$ ([2], [3]). Previously, A. N. Kolmogorov had proved that, for $n \geq 3$, every continuous function $f \in \mathcal{C}^{(0)}\left(I^n\right)$ can be represented in the form

$$f\left(x_1, \dots, x_n\right) = \sum_{i=1}^{n} h_i\left(x_n, \phi_{1i}\left(x_1, \dots, x_{n-1}\right), \phi_{2i}\left(x_1, \dots, x_{n-1}\right)\right), \tag{3}$$

where $h_i \in \mathcal{C}^{(0)}\left(I^3\right)$ and ϕ_{1i}, $\phi_{2i} \in \mathcal{C}^{(0)}\left(I^{n-1}\right)$ ([17], see also [18]). An iteration of (2) and (3), together with the observation that

$$\sum_{i=1}^{n} a_i = \left(a_1 + \left(a_2 + \dots + \left(a_{n-1} + a_n\right)\right)\right),$$

show that every continuous function $f \in \mathcal{C}^{(0)}\left(I^n\right)$ can be represented as a composition of functions of two variables. The reader interested in these developments and its many ramifications is referred to [10], [14], [24] and to the references therein. It should be observed that, in view of the functions entering into (2) or (3) are not generally means, the above general results turn out to be barely useful in connection with the problem of reducibility of means.

The purpose of this paper is to show that the strategy of reduction of dimension can be successfully implemented to solve some problems. Even if the general problem of reducibility of a mean M in a class $\mathcal{N}(I)$ may be undecidable, a decidable type of reducibility, the S-reducibility, is identified. Strategies of reduction of dimension can be fully implemented for S-reducible means. For instance, if M is S-reducible and $M_i,\ i = 0, \dots, r$ are its reduced means, then the bijective M-affine functions are easily expressed in terms of the bijective M_i-affine functions. The topic of inequalities between mean, another case in which the reduction of dimension may lead to a simplification, it is also considered in this paper. Continuous and scale invariant weighting procedures can be constructed on the class of means which can be expressed as a composition of a finite number of 2 variables means. The paper contains a detailed presentation of this construction. Other related developments, like the identification of certain classes of reducible means and the presentation of the concept of tree of a formula, will hopefully exhibit some intrinsic interest.

The paper is organized as follows. A notation for simple forms of substitution of variables and the concept of lower means are both introduced in Section 2 along with other preliminary materials. In Section 3, the problem of reducibility of a mean is generally discussed and some important classes of reducible means are identified. A scheme of classification of reducible means constructed by recurrence on a first step reduction formula ("first layer representation") is presented in Section 4. There, the concept of S-reducibility arises as a especially simple case in which the first layer representation is a reduced representation. A reducible mean turns out to be a composition of means all which are the result of a specialization of variables in a S-reducible mean, a result whose proof is given in Section 5, after the introduction of the labeled tree of a formula representing a composition of functions. Besides of providing a support for concepts like that of "longest sequence of compositions of functions in a formula", these trees are used to prove some combinatorial relationships involving the numbers of variables and functional symbols in a formula. The idea of "structure" of a formula is also easily materialized in terms of its associated tree. Some applications of reducibility are developed in the last three sections of the paper. The concept of complete reducibility is applied in Section 7, where the scale invariant weighting problem is considered. Inequalities between reducible means whose reduced representations share the same structure are addressed in Section 6, while the family of M-affine functions of a S-reducible mean is studied in Section 8. The final Appendix contains a table of the notations employed in the paper.

2 Preliminaries

Throughout the paper, the symbol $\mathcal{M}_n(I)$ will denote a given class of n variables means. The exact extension of the class will depend on the context, but the symbol $\mathcal{M}(I)$ will stand always for $\bigcup_{n=2}^{+\infty} \mathcal{M}_n(I)$, a class containing means of every dimension $n > 2$. Different notations for other classes of means will be introduced here an there along the paper. For instance, $\mathcal{C}^{(k)}\mathcal{M}_n(I)$, $k = 0, 1, \ldots$, will denote the class of n variables $\mathcal{C}^{(k)}$ means defined on I.

Means are *reflexive* functions: the equality $M(x, \ldots, x) \equiv x$ is derived by equating all variables in (1). If the inequalities (1) are strict provided that the variables x_i are not all equal, then the mean M is said to be *strict*. Classical means (arithmetic) A, (geometric) G and (harmonic) H are all strict, but the functions at the leftmost and rightmost members of the inequalities (1), named the *extremal means* $\min_n$ and $\max_n$ are not. The same is true of the coordinate means X_i. Now suppose that, once increasingly ordered, the n-tuple $(x_1, \ldots, x_n) \in I^n$ takes the form $(x_{i_1}, \ldots, x_{i_n})$; i.e., $x_{i_1} \leq \cdots \leq x_{i_n}$. For every $k = 1, \ldots, n$, the k-th *order mean* $X_n^{(k)}$ is then defined by

$$X_n^{(k)}(x_1, \ldots, x_n) = x_{i_k}.$$

Clearly $\min_n = X_n^{(1)} \leq \cdots \leq X_n^{(k)} \cdots \leq X_n^{(n)} = \max_n$ and $\nu\left(X_n^{(k)}\right) = n$ for every $k = 1, \ldots, n$.

Recall that the *product order* "$\preceq$" in I^n is defined by

$$(x_1,\ldots,x_n) \preceq (y_1,\ldots,y_n) \text{ if and only if } x_i \le y_i,\ i=1,2,\ldots,n.$$

It is written $(x_1,\ldots,x_n) \prec (y_1,\ldots,y_n)$ when $(x_1,\ldots,x_n) \preceq (y_1,\ldots,y_n)$ but $(x_1,\ldots,x_n) \neq (y_1,\ldots,y_n)$. A mean M is said to be *isotone* when preserves the product order in I^n; i.e., when $M(x_1,\ldots,x_n) \le M(y_1,\ldots,y_n)$ provided that $(x_1,\ldots,x_n) \preceq (y_1,\ldots,y_n)$. M is said to be *strictly isotone* when $M(x_1,\ldots,x_n) < M(y_1,\ldots,y_n)$ provided that $(x_1,\ldots,x_n) \prec (y_1,\ldots,y_n)$.

When applied on (continuous) means, a series of operations besides of composition return new (continuous) means. For example, the symmetric group S_n acts on a class $\mathcal{M}_n(I)$ of n variables means by returning, for a given n variables mean $M \in \mathcal{M}_n(I)$ and $\sigma \in S_n$, a new mean M_σ with permuted variables. Namely, if $M \in \mathcal{M}_n(I)$ and $\sigma = (\sigma_1,\ldots,\sigma_n) \in S_n$, then

$$M_\sigma(x_1,\ldots,x_n) = M(x_{\sigma_1},\ldots,x_{\sigma_n}),\ x_1,\ldots,x_n \in I. \tag{4}$$

M is said to be symmetric when $M_\sigma = M$ for every $\sigma \in S_n$ (i.e., when $\{M\}$ is a set invariant under S_n).

The bold type $\mathbf{n}$ will be often used to denote the set of indices $\{1,\ldots,n\}$. Given a subset $J = \{i_1,\ldots,i_k\}$ of $\mathbf{n}$, the symbol $[J]$ stands for the increasingly ordered k-tuple $(i_{j_1},\ldots,i_{j_k})$ obtained by ordering the indices in J. The compact notation $(x_j)_{[J]}$ will be used instead of $\left(x_{i_{j_1}},\ldots,x_{i_{j_k}}\right)$. For instance, $[J] = [2,3,5]$ and $(x_j)_{[J]} = (x_2,x_3,x_5)$ when $J = \{5,3,2\}$)

A generalization of the action of S_n on $\mathcal{M}_n(I)$ named *specialization of variables* is obtained by considering in (4) $\sigma \in \mathbf{n}^{\mathbf{n}} (= \{\rho \mid \rho : \mathbf{n} \to \mathbf{n}\})$ instead of $\sigma \in S_n$. Indeed, when $\sigma \in \mathbf{n}^{\mathbf{n}}$ and $k \in \mathbf{n}$, the variables whose indices belong to the preimage $\sigma^{-1}(k)$ turn out to be all identified with x_k in the equality (4). In this regard, two different notations will be introduced, each one corresponding to a function σ of simple type. On one hand, for a n variables mean M defined on I and an (increasingly) ordered set of indices $[i_1,\ldots,i_k]$, $i_j \in \mathbf{n}$, $j = 1,\ldots,k$, $(1 \le k < n)$, let us denote by $M_{[i_1,\ldots,i_k]}$ to the $(k+1)$ variables mean defined on I which is obtained by identifying in M the variables x_j with $j \neq i_l$, $l = 1,\ldots,k$; i.e.,

$$M_{[i_1,\ldots,i_k]}(x_{i_1},\ldots,x_{i_k};u) = M(u,\ldots,u,x_{i_1},u,\ldots,u,x_{i_2},u,\ldots,u,x_{i_k},u,\ldots,u) \tag{5}$$

for every $x_{i_1},\ldots,x_{i_k},u \in I$. On the other hand, given a n variables mean M and a subset $J \subseteq \boldsymbol{n}$, M_J will denote the specialization of M obtained by identifying the variables in J; i.e.,

$$M_J\left((x_j)_{[\boldsymbol{n}\setminus J]};u\right) = M(x_1,\ldots,x_{i_1-1},u,x_{i_1+1},\ldots,x_{i_k-1},u,x_{i_k+1},\ldots,x_n),$$

provided that $[J] = (i_1,\ldots,i_k)$. M_J turns out to a be a $(n-k+1)$ variables mean when $\operatorname{card}(J) = k$. For example, if $M = A_5$ and $J = \{5,3,2\}$ then

$$M_{[2,3,5]}(x_2,x_3,x_5;u) = A_5(u,x_2,x_3,u,x_5) = \frac{x_1+2u+x_4+x_5}{5},$$

while

$$M_{\{5,3,2\}}\left(x_1,x_4;u\right)=A_5\left(x_1,u,u,x_4,u\right)=\frac{x_1+3u+x_4}{5}.$$

Observe that $M_{[i_1,\ldots,i_k]}=M_\sigma$, where $\sigma\in\mathbf{n}^{\mathbf{n}}$ is given by

$$\sigma_j=\begin{cases} j, & j\in\{i_1,\ldots,i_k\}\\ i_{0,} & \text{in other case}\end{cases}\,, \tag{6}$$

and that $M_J=M_\sigma$ for a $\sigma\in\mathbf{n}^{\mathbf{n}}$ defined by

$$\sigma_j=\begin{cases} j, & j\in\mathbf{n}\backslash\{i_1,\ldots,i_k\}\\ i_{0,} & \text{in other case}\end{cases}\,. \tag{7}$$

In (6) i_0 can be arbitrarily chosen in $\mathbf{n}\backslash\{i_1,\ldots,i_k\}$, while in (7) i_0 must belong to $\{i_1,\ldots,i_k\}$.

Both types of the specializations of variables just introduced can be iterated. For instance, if $J_1,\ldots,J_r\subseteq\boldsymbol{n}$ are mutually disjoint subsets of $\boldsymbol{n}$ (i.e., $J_i\cap J_j=\varnothing$ provided that $i\neq j$), the symbol $M_{J_1\cdots J_r}$ will denote the mean obtained from M by identifying the variables in every $J_i=J_i$; i.e., the mean produced by setting $x_j=u_i$ for every $j\in J_i$, $(i=1,\ldots,r)$. In this way, $M_{J_1\cdots J_r}=\left(\left(M_{J_1}\right)_{J_2}\cdots\right)_{J_k}$ depends on the $n-\sum_{i=1}^r\operatorname{card}\left(J_i\right)+r$ variables $\left(\left(x_j\right)_{[n\backslash\cup_i J_i]};u_1,\ldots,u_r\right)$ and it will be written $M_{J_1\cdots J_r}\left(\left(x_j\right)_{[n\backslash\cup_i J_i]};u_1,\ldots,u_r\right)$ when necessary. The particular instance in which $\{J_i:i=1,\ldots,r\}$ is a partition of n will arise in Section 4: since $n\backslash\cup_i J_i=\varnothing$, $M_{J_1\cdots J_r}$ depends only on the variables $u_1,\ldots,u_r$ in this case.

The means derived from M in the previous paragraphs are all said to be *specializations of* M or, sometimes, that $M_{[i_1,\ldots,i_k]}$, M_J and $M_{J_1\cdots J_r}$ are *specialized* means of M.

Now, if $f:I\rightarrow\mathbb{R}$ is an injective and continuous function and $M\in\mathcal{C}^{(0)}\mathcal{M}_n\left(I\right)$, a mean $\left(M\right)_f$ is defined on $f\left(I\right)$ by

$$\left(M\right)_f\left(x_1,\ldots,x_n\right)=f\left(M\left(f^{-1}\left(x_1\right),\ldots,f^{-1}\left(x_n\right)\right)\right),\ x_1,\ldots,x_n\in f\left(I\right). \tag{8}$$

The mean $\left(M\right)_f\in\mathcal{C}^{(0)}\mathcal{M}_n\left(I\right)$ and it is named the (mean) *conjugate of* M *by* f. Note that $(M,f)\mapsto\left(M\right)_f$ is a group action when $f\in\operatorname{Hom}\left(I\right)$, the group of homeomorphism of the interval I onto itself. The class $\mathcal{QLM}_n\left(f\left(I\right)\right)$ of n variables *quasilinear means* defined on $f\left(I\right)$ is derived by conjugacy of the class $\mathcal{LM}_n\left(\mathbb{R}\right)$ of linear means: a generic member QL_n of $\mathcal{QLM}_n\left(f\left(I\right)\right)$ has the form

$$QL_n\left(x_1,\ldots,x_n\right)=f\left(w_1f^{-1}\left(x_1\right)+\cdots+w_nf^{-1}\left(x_n\right)\right),\ x_1,\ldots,x_n\in f\left(I\right), \tag{9}$$

where $f:I\rightarrow\mathbb{R}$ is an injective and continuous function and the n-tuple of numbers $\left(w_1,\ldots,w_n\right)$ satisfies $w_i>0$, $i=1,\ldots,n$, $\sum_{i=1}^{n}w_i=1$ (so that $\left(w_1,\ldots,w_n\right)\in\Delta_{n-1}$, the standard $(n-1)$-simplex). The function f is called the *generator function* of QL_n, while the numbers w_i are said to be its *weights*. It may be sometimes useful to specify the generator function or the weights of QL_n or both ones. For example, $L_{n,(w_1,\ldots,w_n)}$ and $G_{n,(w_1,\ldots,w_n)}$ will denote respectively the n variables *linear* and *geometric* means with weights $(w_1, \ldots, w_n)$. $G_{n,(w_1,\ldots,w_n)}$ is the instance of QL_n with generator $f = \ln : \mathbb{R}^+ \to \mathbb{R}$.

Unlike what occurs with the permutation of variables and the conjugacy, which are invertible operations, a loss of information takes place when certain variables of a mean M_n are specialized: the mean M can not be reconstructed from the knowledge of $M_{[i_1,\ldots,i_k]}$ or M_J. This fact is reflected by the equalities $\nu\left(M_\sigma\right) = \nu\left(M\right) = \nu\left(\left(M\right)_f\right)$, which are in contrast with $\nu\left(M_{[i_1,\ldots,i_k]}\right) = \nu\left(M\right) - k + 1$ and $\nu\left(M_J\right) = \nu\left(M\right) - \mathrm{card}\left(J\right) + 1$.

In [1], the *lower mean (untermittel)* of a n variables analytic mean M was defined as a solution w of the equation

$$M(x_1, \ldots, x_{n-1}, w) = w \tag{10}$$

which turns out, under suitable hypotheses on M, to be a unique $(n-1)$ variables mean M_{n-1}. This concept reappears much later in [13] for the case of means defined on $\mathbb{R}^+$. There, the mean M_n is said to be "type 2 invariant" with respect to the mean M_{n-1} provided that

$$M_n\left(x_1, \ldots, x_{n-1}, M_{n-1}\left(x_1, \ldots, x_{n-1}\right)\right) = M_{n-1}\left(x_1, \ldots, x_{n-1}\right)$$

for every $x_1, \ldots, x_{n-1} \in \mathbb{R}^+$. A presentation of similar concepts for means defined on linear spaces has recently arisen in [15] . In this paper, let us consider the solutions u to (or, in other terms, functions implicitly defined by) equations of the form

$$M_{[i_1,\ldots,i_k]}\left(x_{i_1}, \ldots, x_{i_k}; u\right) = u, \tag{11}$$

where $M_{[i_1,\ldots,i_k]}$ is defined by (5). The example furnished by $M\left(x_1, \ldots, x_n\right) = \max\left\{x_1, \ldots, x_n\right\}$ and any given ordered set of indices $[i_1, \ldots, i_k]$ (every function $u = \mu\left(x_{i_1}, \ldots, x_{i_k}\right) > \max_j x_{i_j}$ solves (11) in this case), shows that a solution μ to (11) may not be a mean. Now, for a 2 variables mean M, the equation $M\left(x, u\right) = u$ has the unique solution $u = x$ provided that M is strict. In this case, the mean $\mu\left(x\right) = x$, a *coordinate mean*, is not strict. In general, it can be shown the following:

Proposition 1 *If M is a n variables strict mean, $[i_1, \ldots, i_k]$ is an ordered set of indices with $1 < k < n$, and $u = \mu\left(x_{i_1}, \ldots, x_{i_k}\right)$ is a solution to equation (11), then μ is a k variables strict mean. In the case $k = 1$, $u = \mu\left(x_i\right) = x_i$, the i-th coordinate mean.*

Proof. The case $k = 1$ is trivial, so that let us suppose that $1 < k < n$. In this case, the specialization $M_{[i_1,\ldots,i_k]}$ of the strict mean M turns out to be a strict mean. Now, suppose that μ is a solution to equation (11). If $\mu =$

$\mu(x_{i_1},\ldots,x_{i_k}) > \max_j x_{i_j}$, then the strict internality of $M_{[i_1,\ldots,i_k]}$ yields

$$\begin{aligned}\mu(x_{i_1},\ldots,x_{i_k}) &= M_{[i_1,\ldots,i_k]}(x_{i_1},\ldots,x_{i_k};\mu(x_{i_1},\ldots,x_{i_k})) \\ &< \max\left\{\max_j x_{i_j},\mu(x_{i_1},\ldots,x_{i_k})\right\} = \mu(x_{i_1},\ldots,x_{i_k}),\end{aligned}$$

a contradiction which shows that $\mu = \mu(x_{i_1},\ldots,x_{i_k}) \le \max_j x_{i_j}$. Furthermore, if $\mu(x_{i_1},\ldots,x_{i_k}) = \max_j x_{i_j}$, then

$$\begin{aligned}M_{[i_1,\ldots,i_k]}(x_{i_1},\ldots,x_{i_k};\mu(x_{i_1},\ldots,x_{i_k})) &= \mu(x_{i_1},\ldots,x_{i_k}) \\ &= \max_j x_{i_j} \\ &= \max\{x_{i_1},\ldots,x_{i_k},\mu(x_{i_1},\ldots,x_{i_k})\}\end{aligned}$$

whence $x_{i_1} = \cdots = x_{i_k}$ $(= \mu(x_{i_1},\ldots,x_{i_k}))$.

It is similarly proved that $\mu = \mu(x_{i_1},\ldots,x_{i_k}) \ge \min_j x_j$ with equality only in the case that the variables $x_{i_1},\ldots,x_{i_k}$, are identical. Thus, μ is a k variables strict mean. ■

A notation for the set of fixed points of a function will be useful in the sequel. Given a set E, a function $\phi : E \to E$ and a subset $\varnothing \ne A \subseteq E$, the set of fixed points of ϕ in A will be denoted by $\mathrm{Fix}(\phi; A)$; i.e., $\mathrm{Fix}(\phi; A) = \{t \in A : \phi(t) = t\}$.

A *mean* μ solving equation (11) will be called a $[i_1,\ldots,i_k]$-*lower mean* of M and usually denoted by $\mu_{[i_1,\ldots,i_k]}$. Note that the $[i_1,\ldots,i_k]$-lower means of a symmetric mean M, depends only on the number k of indices in the set $\{i_1,\ldots,i_k\}$.

In connection with the existence of lower means, let us pay attention to the inequalities

$$\min\left\{\min_j x_{i_j}, u\right\} \le M_{[i_1,\ldots,i_k]}(x_{i_1},\ldots,x_{i_k};u) \le \max\left\{\max_j x_{i_j}, u\right\},$$

where $[i_1,\ldots,i_k]$ is a given ordered set of indices. It follows from these that

$$\min_j x_{i_j} \le M_{[i_1,\ldots,i_k]}(x_{i_1},\ldots,x_{i_k};u) \le \max_j x_{i_j}$$

provided that

$$\min_j x_{i_j} \le u \le \max_j x_{i_j};$$

or, in other terms, that the interval $E = \left[\min_j x_{i_j}, \max_j x_{i_j}\right]$ is mapped into itself by the map $u \mapsto M_{[i_1,\ldots,i_k]}(x_{i_1},\ldots,x_{i_k};u)$. Now, for every $(x_{i_1},\ldots,x_{i_k}) \in I^k$, the map $u \mapsto M_{[i_1,\ldots,i_k]}(x_{i_1},\ldots,x_{i_k};u)$ turns out to be continuous (on $E \subseteq I$) provided that M is continuous and, in consequence, it admits a fixed point $u_0 = u_0(x_{i_1},\ldots,x_{i_k}) \in E$. Suppose that this fixed point was *unique* whichever be $(x_{i_1},\ldots,x_{i_k}) \in I^k$. Under this assumption, the function $\mu_0 = u_0(x_{i_1},\ldots,x_{i_k})$ turns out to be a well defined k variables mean. Let us prove that μ_0 is continuous on I^k. In fact, if $x^{(0)} \in I^k$ and $\left(x^{(l)}\right)_{l=1}^{+\infty} \subseteq I^k$ is a convergent sequence with $x^{(0)}$ as limit point, then $\left(\mu_0\left(x^{(l)}\right)\right)_{l=1}^{+\infty}$ turns out to be a bounded sequence contained in I, a fact that quickly follows from the twofold inequality

$$\min_j x_j^{(l)} \leq \mu_0\left(x^{(l)}\right) \leq \max_j x_j^{(l)}.$$

Let us show that $\left(\mu_0\left(x^{(l)}\right)\right)_{l=1}^{+\infty}$ really converges to $\mu_0\left(x^{(0)}\right)$. Indeed, if u_1 and u_2 were two cluster points of $\left(\mu_0\left(x^{(l)}\right)\right)_{l=1}^{+\infty}$, then there would exist two subsequences $\left(\mu_0\left(x^{(l_i)}\right)\right)_{i=1}^{+\infty}$ and $\left(\mu_0\left(x^{(l_i')}\right)\right)_{i=1}^{+\infty}$ respectively converging to u_1 and u_2, while the continuity of M enables to write

$$\begin{aligned} M_{[i_1,\ldots,i_k]}\left(x^{(0)}; u_1\right) &= \lim_{i\uparrow+\infty} M_{[i_1,\ldots,i_k]}\left(x^{(l_i)}, \mu_0\left(x^{(l_i)}\right)\right) \\ &= \lim_{i\uparrow+\infty} \mu_0\left(x^{(l_i)}\right) = u_1 \end{aligned}$$

and, similarly,

$$M_{[i_1,\ldots,i_k]}\left(x^{(0)}; u_2\right) = u_2.$$

In this way, $u_1, u_2 \in \operatorname{Fix}\left(M_{[i_1,\ldots,i_k]}, \left[\min_i x_i^{(0)}, \max_i x_i^{(0)}\right]\right)$, whence $u_1 = u_2$ by the uniqueness hypothesis. It has been proved that $\mu_0\left(x^{(0)}\right)$, the unique point in $\operatorname{Fix}\left(M_{[i_1,\ldots,i_k]}, \left[\min_i x_i^{(0)}, \max_i x_i^{(0)}\right]\right)$, is the unique cluster point of $\left(\mu_0\left(x^{(l)}\right)\right)_{l=1}^{+\infty}$ or, in an equivalent way, that $\mu_0\left(x^{(l)}\right) \to \mu_0\left(x^{(0)}\right)$ when $l \uparrow +\infty$. Since the sequence $\left(x^{(l)}\right)_{l=1}^{+\infty}$ was arbitrarily chosen, the continuity at $x^{(0)}$ follows. The continuity on I^k of μ_0 is a consequence of the arbitrariness of $x^{(0)} \in I^k$.

Summarizing the above discussion, it can be stated the following:

Proposition 2 *Let M be a n variables continuous mean defined on I and, for a given $1 \leq k < n$, consider an ordered set of indices $[i_1, \ldots, i_k]$. Assume that, for every $(x_{i_1}, \ldots, x_{i_k}) \in I^k$, the condition*

$$\operatorname{Fix}\left(M_{[i_1,\ldots,i_k]}, \left[\min_j x_{i_j}, \max_j x_{i_j}\right]\right) \quad \textit{is an unitary set} \tag{12}$$

is satisfied by the map $u \mapsto M_{[i_1,\ldots,i_k]}(x_{i_1}, \ldots, x_{i_k}; u)$. Then, there exists a unique $[i_1, \ldots, i_k]$-lower mean $\mu_{[i_1,\ldots,i_k]}$ of M. μ turns out to be a k variables continuous mean defined on I.

A mean satisfying condition (12) is said to have the *FUS property.* For continuously differentiable means, the proposition is a consequence of the Implicit Function Theorem ([19]): the hypothesis of uniqueness of the fixed point of $u \mapsto M_{[i_1,\ldots,i_k]}(x_{i_1}, \ldots, x_{i_k}; u)$ implies the global existence (and uniqueness) of the solution to equation (11).

Proof. See the previous discussion. ■

Example 3 *The function*

$$FH_n(x_1,\dots,x_n)=\frac{\sum_{i=1}^{n}f_i(x_i)\,x_i}{\sum_{i=1}^{n}f_i(x_i)}, \tag{13}$$

where $f_i: I\to\mathbb{R}^+,\ i=1,\dots,n,$ *are continuous, turns to be a strict continuous mean defined on* I. *Note that the weighted arithmetic mean is obtained from (13) by taking positive constants* $f_i=W_i\in\mathbb{R}^+,\ i=1,\dots,n$; *while the* r-th weighted counter-harmonic mean *(cf. [11], pg. 245)*

$$CH_n^{(r)}(x_1,\dots,x_n)=\frac{\sum_{i=1}^{n}W_i x_i^{r+1}}{\sum_{i=1}^{n}W_i x_i^{r}},\ x_1,\dots,x_n\in\mathbb{R}^+, \tag{14}$$

is the particular instance of (13) in which $I=\mathbb{R}^+$ *and* $f_i(u)=W_i u^r,\ u>0,(W_i>0),\ i=1,\dots,n$. *The mean* FH_n *has the FUS property. In fact, given an ordered set of indices* $[i_1,\dots,i_k]$, *it can be written*

$$(FH_n)_{[i_1,\dots,i_k]}(u)=\frac{\sum_{j=1}^{k}f_{i_j}\left(x_{i_j}\right)x_{i_j}+u\sum_{i\in\boldsymbol{n}\backslash\{i_1,\dots,i_k\}}f_i(u)}{\sum_{j=1}^{k}f_{i_j}\left(x_{i_j}\right)+\sum_{i\in\boldsymbol{n}\backslash\{i_1,\dots,i_k\}}f_i(u)}, \tag{15}$$

and then, the equation $(FH_n)_{[i_1,\dots,i_k]}(x_{i_1},\dots,x_{i_k};u)=u$ *has a unique solution given by*

$$\mu(x_{i_1},\dots,x_{i_k})=\frac{\sum_{j=1}^{k}f_{i_j}\left(x_{i_j}\right)x_{i_j}}{\sum_{j=1}^{k}f_{i_j}\left(x_{i_j}\right)},$$

which is new mean of the form (13).

Remark 4 *The mean* $M\in\mathcal{C}^{(0)}\mathcal{M}_3(\mathbb{R}^+)$ *defined in terms of the order means* $X_3^{(1)}$, $X_3^{(2)}$, $X_3^{(3)}$ *by*

$$M(x_1,x_2,x_3)=\begin{cases}\frac{1}{2}\left(\sqrt{X_3^{(1)}X_3^{(3)}}+X_3^{(2)}\right), & X_3^{(1)}\le X_3^{(2)}\le\sqrt{X_3^{(1)}X_3^{(3)}}\\ X_3^{(2)}, & \sqrt{X_3^{(1)}X_3^{(3)}}\le X_3^{(2)}\le\left(X_3^{(1)}+X_3^{(3)}\right)/2\\ \frac{1}{2}\left(\frac{X_3^{(1)}+X_3^{(3)}}{2}+X_3^{(2)}\right), & \left(X_3^{(1)}+X_3^{(3)}\right)/2\le X_3^{(2)}\le X_3^{(3)}\end{cases},$$

is a strict, continuous, symmetric and isotone mean defined on $\mathbb{R}^+$. The equation $M_{[1,2]}(x_1,x_2;u)=u$ is solved by every mean $\mu\in\mathcal{M}_3(\mathbb{R}^+)$ satisfying the inequalities

$$\sqrt{x_1x_2}\le\mu(x_1,x_2)\le\frac{x_1+x_2}{2}.$$

Indeed $X_3^{(1)}(x_1,x_2,\mu(x_1,x_2))=\min\{x_1,x_2\}$, $X_3^{(2)}(x_1,x_2,\mu(x_1,x_2))=\mu(x_1,x_2)$ and $X_3^{(3)}(x_1,x_2,\mu(x_1,x_2))=\max\{x_1,x_2\}$, and therefore the point $(x_1,x_2,\mu(x_1,x_2))$ satisfies $\sqrt{X_3^{(1)}X_3^{(3)}}\le X_3^{(2)}\le\left(X_3^{(1)}+X_3^{(3)}\right)/2$. Hence

$$M_{[1,2]}(x_1,x_2;\mu(x_1,x_2))=X_3^{(2)}=\mu(x_1,x_2)\,,\ x_1,x_2\in\mathbb{R}^+.$$

This shows that Prop. 2 is not generally true when the FUS property does not hold: for the mean M, the set $\mathrm{Fix}\left(M_{[1,2]},[\min\{x_1,x_2\},\max\{x_1,x_2\}]\right)=\left[\sqrt{x_1x_2},(x_1+x_2)/2\right]$ is not an unitary set when $x_1\neq x_2$. This example also shows that Theorem 11 in [13] is false (unless a hypothesis implying condition (12) was added to its statement).

Remark 5 *Under the conditions of Prop. 2, it is not difficult to see that a mean M has the property FUS provided that, for every $x\in I^k$, any of the following conditions is fulfilled:* ***i)*** $u\mapsto M_{[i_1,\dots,i_k]}(x;u)$ *satisfies the inequality*

$$\left|M_{[i_1,\dots,i_k]}(x;v)-M_{[i_1,\dots,i_k]}(x;u)\right|<|v-u|\,,\ u,v\in E; \tag{16}$$

where $E=\left[\min_j x_{i_j},\max_j x_{i_j}\right]$; or, ***ii)*** *$M$ is strict and $u\mapsto M_{[i_1,\dots,i_k]}(x;u)$ is strictly convex (or strictly concave) in E. In fact, if* ***i)*** *is satisfied and u_1, $u_2\in\mathrm{Fix}\left(M_{[i_1,\dots,i_k]},E\right)$, then the replacement $v=u_2$, $u=u_1$ in the inequality (16) yields*

$$|u_2-u_1|=\left|M_{[i_1,\dots,i_k]}(x;u_2)-M_{[i_1,\dots,i_k]}(x;u_1)\right|<|u_2-u_1|\,.$$

This contradiction shows that $\mathrm{Fix}\left(M_{[i_1,\dots,i_k]},E\right)$ $(\neq\varnothing)$ contains at most one point. In regards to ***ii)****, suppose that M is strict and $u\mapsto M_{[i_1,\dots,i_k]}(x;u)$ is strictly convex. Clearly, the graph of $u\mapsto M_{[i_1,\dots,i_k]}(x;u)$ can intercept the diagonal of the square E^2 at two points (u_1,u_1) and (u_2,u_2) at most. Furthermore, if $u_1<u_2$, then $u_2=\max_j x_{i_j}$, so that*

$$\begin{aligned}M_{[i_1,\dots,i_k]}\left(x;\max_j x_{i_j}\right)&=\max_j x_{i_j}\\&=\max\left\{\max_j x_{i_j},\max_j x_{i_j}\right\},\end{aligned}$$

whence, in view of the strictness of M, all x_{i_j}, $j=1,\dots,k$, must be equal each other, and then $u_1=u_2$. This contradiction proves that the set $\mathrm{Fix}\left(M_{[i_1,\dots,i_k]},E\right)$ is unitary.

3 Reducibility and irreducibility

Let $\mathcal{M}(I), \mathcal{N}(I)$ be two classes of means defined on an interval I satisfying $\mathcal{M}(I) \supseteq \mathcal{N}(I)$. A mean $M \in \mathcal{M}(I)$ with $\nu(M) > 2$ is said to be *reducible in* $\mathcal{N}(I)$ when it can be represented as a composition of a finite number of means $M_0, \ldots, M_r$ belonging to $\mathcal{N}(I)$ with $\nu(M_i) < \nu(M)$, $i = 0, \ldots, r$. The representation itself will be said to be a *reduced representation* of M, while the means $M_0, \ldots, M_r \in \mathcal{N}(I)$ appearing in it will be named *reduced means of the representation*. The discussion contained in the forthcoming paragraphs attempts to clarify these concepts.

For an injective and continuous function $f : I \to \mathbb{R}$, $M \in \mathcal{M}(I)$ is reducible in $\mathcal{N}(I)$ if and only if $(M)_f \in (\mathcal{M})_f(f(I))$ is reducible in $(\mathcal{N})_f(f(I))$. In this way, reducibility in a class $\mathcal{N}(I)$ turns out to be a notion invariant under conjugacy (provided that $\mathcal{N}(I)$ is invariant under conjugacy). When all members with $\nu > 2$ of a class of means $\mathcal{M}(I)$ turn out to be reducible in the class $\mathcal{N}(I)$, the class $\mathcal{M}(I)$ itself is said to be reducible (in $\mathcal{N}(I)$). The class $\mathcal{QLM}(I)$ of quasilinear means on an interval I is a relevant example of a reducible class into itself. To see this, first write a generic linear mean $L_n \in \mathcal{LM}(\mathbb{R})$ $(n > 2)$ in the form

$$\begin{aligned} & L_n(x_1, \ldots, x_n) \\ = \; & w_1 x_1 + \cdots + w_n x_n \\ = \; & \left(\sum_{i=1}^{k} w_i\right)\left(\sum_{i=1}^{k} \frac{w_i}{\sum_{i=1}^{k} w_i} x_i\right) + \left(\sum_{i=k+1}^{n} w_i\right)\left(\sum_{i=k+1}^{k} \frac{w_i}{\sum_{i=k+1}^{n} w_i}\right) x_i \\ = \; & L_2\left(L_k(x_1, \ldots, x_k), L_{n-k}(x_{k+1}, \ldots, x_n)\right), \end{aligned} \tag{17}$$

where $k \in \mathbf{n}$, $k < n$, and

$$\begin{cases} L_2(x_1, x_2) = \left(\sum_{i=1}^{k} w_i\right) x_1 + \left(\sum_{i=k+1}^{n} w_i\right) x_2 \\ L_k(x_1, \ldots, x_n) = \sum_{i=1}^{k} \frac{w_i}{\sum_{i=1}^{k} w_i} x_i \\ L_{n-k}(x_1, \ldots, x_{n-k}) = \sum_{i=1}^{n-k} \frac{w_{k+i}}{\sum_{i=k+1}^{n} w_i} x_i \end{cases};$$

which shows that every linear mean $M \in \mathcal{LM}(\mathbb{R})$ with $\nu(M) > 2$ is reducible in $\mathcal{LM}(\mathbb{R})$ or, in other terms, that $\mathcal{LM}(\mathbb{R})$ is a reducible class into itself. Due to the invariance under conjugacy, the generic quasilinear mean QL_n given by (9) turns out to be reducible in $(\mathcal{LM})_f(f(\mathbb{R}))$. Now, if a mean $M \in \mathcal{M}(I)$ is reducible in $\mathcal{N}_1(I)$, then it is clearly reducible in every class of means $\mathcal{N}_2(I)$ satisfying $\mathcal{N}_2(I) \supseteq \mathcal{N}_1(I)$. In this way, QL_n turns out to be reducible in $\mathcal{QLM}(f(\mathbb{R}))$, the class of all quasilinear means defined in $f(\mathbb{R})$.

It should be noted that the reduction (17) is not unique in the sense that L_n can be expressed as the composition of several different linear reduced means. Indeed, nonlinear or even discontinuous means may be reduced terms of certain representations of a linear mean L_n as, for instance, in the representation

$$A_4(x_1, x_2, x_3, x_4) = A_2\left(A_2(x_1, N(x_2, x_3)), A_2\left(\overline{N}(x_2, x_3), x_4\right)\right), \tag{18}$$

where N and $\overline{N}$ are two arbitrarily chosen *complementary means* defined on $\mathbb{R}$; i.e.,

$$N(x_1, x_2) + \overline{N}(x_1, x_2) = x_1 + x_2, \ \ x_1, x_2 \in \mathbb{R}.$$

Correspondingly, neither are unique the reductions of the quasilinear mean QL_n.

Symmetric polynomial means provides another important class of reducible means. Recall (cf. [11], Chap. V) that the *r-th symmetric polynomial functions* $e_n^{[r]}(x_1, \ldots, x_n)$ is given by

$$e_n^{[r]}(x_1, \ldots, x_n) = \frac{1}{r!} \sum ! \prod_{j=1}^{r} x_{i_j},$$

where $r \in \mathbf{n}$ and $\sum ! \prod_{j=1}^{r} x_{i_j}$ stands for the sum of all terms of the form $\prod_{j=1}^{r} x_{i_j}$ with $i_j \in \mathbf{n}$, $j = 1, \ldots, r$. The *r-th symmetric polynomial mean* $\mathfrak{S}_n^{[r]}(x_1, \ldots, x_n)$ is then defined by

$$\mathfrak{S}_n^{[r]}(x_1, \ldots, x_n) = \left(\frac{e_n^{[r]}(x_1, \ldots, x_n)}{\binom{n}{r}}\right)^{1/r}.$$

Usually, these means are defined on $\mathbb{R}^+$ or $\mathbb{R}_0^+$ (even if they are naturally defined on the whole $\mathbb{R}$ when $r \in \mathbf{n}$ is an odd number). Since

$$\mathfrak{S}_n^{[1]}(x_1, \ldots, x_n) = A_n(x_1, \ldots, x_n) \ \text{ and } \mathfrak{S}_n^{[n]}(x_1, \ldots, x_n) = G_n(x_1, \ldots, x_n), \tag{19}$$

$\mathfrak{S}_n^{[r]}(x)$ turns out to be reducible when $r = 1$ or $r = n$. In the remaining cases, the simple equality (cf. [11], Lemma 2, pg. 324)

$$e_n^{[r]}(x_1, \ldots, x_n) = e_{n-1}^{[r]}(x_1, \ldots, x_{n-1}) + x_n e_{n-1}^{[r-1]}(x_1, \ldots, x_{n-1}), \tag{20}$$

enable us to write

$$
\begin{aligned}
&\mathfrak{S}_n^{[r]}(x_1,\ldots,x_n)\\
=\;&\left(\frac{e_n^{[r]}(x_1,\ldots,x_n)}{\binom{n}{r}}\right)^{1/r}\\
=\;&\left(\frac{e_{n-1}^{[r]}(x_1,\ldots,x_{n-1})+x_n e_{n-1}^{[r-1]}(x_1,\ldots,x_{n-1})}{\binom{n}{r}}\right)^{1/r}\\
=\;&\left(\frac{\binom{n-1}{r}\left(\left(\frac{e_{n-1}^{[r]}(x_1,\ldots,x_{n-1})}{\binom{n-1}{r}}\right)^{1/r}\right)^{r}+\binom{n-1}{r-1}\left(\left(\frac{e_{n-1}^{[r-1]}(x_1,\ldots,x_{n-1})}{\binom{n-1}{r-1}}\right)^{1/r}x_n^{1/r}\right)^{r}}{\binom{n}{r}}\right)^{1/r}\\
=\;&\left(\frac{\binom{n-1}{r}\left(\mathfrak{S}_{n-1}^{[r]}(x_1,\ldots,x_{n-1})\right)^{r}+\binom{n-1}{r-1}\left(\left(\mathfrak{S}_{n-1}^{[r-1]}(x_1,\ldots,x_{n-1})\right)^{(r-1)/r}x_n^{1/r}\right)^{r}}{\binom{n}{r}}\right)^{1/r} \quad (21)
\end{aligned}
$$

Now, the function

$$P_{r/n}^{(r)}(x,y)=\left(\frac{\binom{n-1}{r}x^r+\binom{n-1}{r-1}y^r}{\binom{n}{r}}\right)^{1/r}=\left(\left(1-\frac{r}{n}\right)x^r+\frac{r}{n}y^r\right)^{1/r}$$

is the (two variables) weighted power mean with exponent r and weight $r/n=\binom{n-1}{r-1}/\binom{n}{r}$, while

$$G_{1/r}(x,y)=x^{(r-1)/r}y^{1/r},$$

is the (two variables) weighted geometric mean with weigh $1/r$. Replacing these means in the last member of the equalities (21) produces

$$\mathfrak{S}_n^{[r]}(x_1,\ldots,x_n)=P_{r/n}^{(r)}\left(\mathfrak{S}_{n-1}^{[r]}(x_1,\ldots,x_{n-1}),G_{1/r}\left(\mathfrak{S}_{n-1}^{[r-1]}(x_1,\ldots,x_{n-1}),x_n\right)\right). \quad (22)$$

Summarizing the above discussion, it can be stated the following:

Proposition 6 *The class $\mathcal{QLM}(I)$ of quasilinear means on an interval I is a reducible class into itself. $\mathfrak{S}\mathcal{M}(\mathbb{R}^+)$, the class of polynomial symmetric means, is reducible in the class $\mathfrak{S}\mathcal{M}(\mathbb{R}^+)\cup\mathcal{QLM}_{2,\mathbb{Q}}(\mathbb{R}^+)$, where $\mathcal{QLM}_{2,\mathbb{Q}}(\mathbb{R}^+)$ stands for the class of two variables quasilinear means defined on $\mathbb{R}^+$ whose weights are all rational numbers.*

Proof. After the discussion preceding the statement of the proposition, it is deduced that $\mathfrak{S}\mathcal{M}(\mathbb{R}^+)$ is reducible in the class $\mathcal{C}^{(0)}\mathcal{M}(\mathbb{R}^+)$ of continuous means. In view of (19) and (22), it turns out to be that $\mathfrak{S}\mathcal{M}(\mathbb{R}^+)$ is reducible in $\mathfrak{S}\mathcal{M}(\mathbb{R}^+)\cup\mathcal{QLM}_{2,\mathbb{Q}}(\mathbb{R}^+)$. ■

Now consider the class $\mathcal{LM}_{\mathbb{Q}}(\mathbb{R})$ consisting of all linear means with rational weights. A linear mean $L_n \in \mathcal{LM}(\mathbb{R})$ *with at least one irrational weight* turns out to be reducible in $\mathcal{LM}(\mathbb{R})$ but irreducible in $\mathcal{LM}_{\mathbb{Q}}(\mathbb{R})$. This simple example shows that the concept of reducibility crucially depends on the class $\mathcal{N}(I)$. As affirmed in the Introduction, deciding whether a mean M belonging to a class $\mathcal{M}(I)$ is reducible or not in another class $\mathcal{N}(I)$ constitutes, in general, a highly non trivial problem. To illustrate this fact, let us discuss briefly the case presented by the continuous mean

$$M(x_1, x_2, x_3) = \frac{x_1 x_2 + x_2 x_3 + x_3 x_1}{x_1 + x_2 + x_3}, \ x_1, x_2, x_3 \in \mathbb{R}^+. \tag{23}$$

Suppose that M can be represented in the form

$$M(x_1, x_2, x_3) = M_0(M_1(x_1, x_2), x_3) \tag{24}$$

with $M_0, M_1 \in \mathcal{C}^{(0)}\mathcal{M}(\mathbb{R}^+)$. Setting $x_1 = x_2 = u$ in this equality yields

$$M(u, u, x_3) = M_0(u, x_3), \ u, x_3 \in \mathbb{R}^+,$$

which shows that $M_0 = M_{\{1,2\}}$, a particular specialization of variables in M. On the other side, the replacement $x_3 = M_1(x_1, x_2)$ gives, by the reflexivity of M_0,

$$M(x_1, x_2, M_1(x_1, x_2)) = M_1(x_1, x_2),$$

which shows that M_1 must be a $[1,2]$-lower mean of M. Now, $u = \sqrt{x_1 x_2} = G(x_1, x_2)$ is the unique solution to the equation

$$\frac{x_1 x_2 + (x_1 + x_2) u}{x_1 + x_2 + u} = u,$$

and therefore, $M_1 = G$. Since

$$M_0(x_1, x_2) = M(x_1, x_1, x_2) = \frac{x_1^2 + 2x_1 x_2}{2x_1 + x_2}, \ u, x_3 \in \mathbb{R}^+,$$

the equality (24) holds if and only if

$$\frac{x_1 x_2 + 2\sqrt{x_1 x_2} x_3}{2\sqrt{x_1 x_2} + x_3} = \frac{x_1 x_2 + x_2 x_3 + x_3 x_1}{x_1 + x_2 + x_3}, \ x_1, x_2, x_3 \in \mathbb{R}^+,$$

whence $\sqrt{x_1 x_2}$ could be expressed as a rational function of the variables x_1, x_2, x_3, an absurdity. It has been proved that M can not be represented as a composition of two means $M_0, M_1 \in \mathcal{C}^{(0)}\mathcal{M}(\mathbb{R}^+)$ *in the form given by* (24). As a consequence of this fact and the symmetry of M, no one representation of M as a composition of *two* (two variables) means $M_0, M_1 \in \mathcal{C}^{(0)}\mathcal{M}(\mathbb{R}^+)$ is possible.

Now suppose that M can be represented as a composition of three (two variables) means, say, in the form

$$M(x_1, x_2, x_3) = M_0(M_1(x_1, x_2), M(x_2, x_3)), \ x_1, x_2, x_3 \in \mathbb{R}^+, \tag{25}$$

with $M_0, M_1, M_2 \in \mathcal{C}^{(0)}\mathcal{M}(\mathbb{R}^+)$; $\nu(M_i) = 2$, $i = 0,1,2$. Unlike the preceding case, the reduced means of the representation (25) can not be computed by simple substitutions of the variables. To overcome this difficulty, let assume that $M_0, M_1, M_2 \in \mathcal{C}^{(1)}\mathcal{M}(\mathbb{R}^+)$ (cf. [22], Vol. **I**, Pt. **II**, Problem 119 a)) and partially differentiate (25) to obtain

$$\left\{\begin{array}{l} M_{x_1} = M_{0x}\left(M_1(x_1,x_2), M_2(x_2,x_3)\right) M_{1x}(x_1,x_2) \\ M_{x_2} = M_{0x}\left(M_1(x_1,x_2), M_2(x_2,x_3)\right) M_{1y}(x_1,x_2) \\ \qquad\quad + M_{0y}\left(M_1(x_1,x_2), M_2(x_2,x_3)\right) M_{2x}(x_2,x_3) \\ M_{x_3} = M_{0y}\left(M_1(x_1,x_2), M_2(x_2,x_3)\right) M_{2y}(x_2,x_3) \end{array}\right. . \tag{26}$$

Since a representation of M as a composition of two means was shown to be impossible, it can be assumed that $M_{1x}(x_1,x_2) \neq 0 \neq M_{2y}(x_2,x_3)$ and then, from (26) it is derived

$$M_{x_2} = u(x_1,x_2)\,M_{x_1} + v(x_2,x_3)\,M_{x_3}, \tag{27}$$

where

$$u(x_1,x_2) = \frac{M_{1y}(x_1,x_2)}{M_{1x}(x_1,x_2)},\ v(x_2,x_3) = \frac{M_{2x}(x_2,x_3)}{M_{2y}(x_2,x_3)}.$$

Let us show that the partial derivatives of M can not satisfy a relationship like (27). In fact, taking into account that

$$M_{x_1} = \frac{x_2^2 + x_2x_3 + x_3^2}{(x_1+x_2+x_3)^2},\ M_{x_2} = \frac{x_1^2 + x_1x_3 + x_3^2}{(x_1+x_2+x_3)^2},\ M_{x_3} = \frac{x_1^2 + x_1x_2 + x_2^2}{(x_1+x_2+x_3)^2},$$

it is seen that (27) is satisfied if and only there exists a pair of functions u, v such that

$$x_1^2 + x_1x_3 + x_3^2 = u(x_1,x_2)\left(x_2^2 + x_2x_3 + x_3^2\right) + v(x_2,x_3)\left(x_1^2 + x_1x_2 + x_2^2\right), \tag{28}$$

for every $x_1, x_2, x_3 \in \mathbb{R}^+$. Setting $x_1 = x_3$ in this equality yields

$$3x_3^2 = \left(u(x_3,x_2) + v(x_2,x_3)\right)\left(x_2^2 + x_2x_3 + x_3^2\right),$$

whence

$$v(x_2,x_3) = \frac{3x_3^2}{x_2^2 + x_2x_3 + x_3^2} - u(x_3,x_2).$$

Substituting this expression for $v(x_2,x_3)$ in (28) produces, after reordering terms, the equality

$$\begin{aligned} x_1^2 + x_1x_3 + x_3^2 - 3x_3^2\frac{x_1^2 + x_1x_2 + x_2^2}{x_2^2 + x_2x_3 + x_3^2} &= u(x_1,x_2)\left(x_2^2 + x_2x_3 + x_3^2\right) \\ &\quad - u(x_3,x_2)\left(x_1^2 + x_1x_2 + x_2^2\right) \end{aligned} \tag{29}$$

Now, setting $x_1 = x_2$ in this last equality yields

$$x_2^2 + x_2x_3 + x_3^2 - \frac{9x_2^2x_3^2}{x_2^2 + x_2x_3 + x_3^2} = u(x_2,x_2)\left(x_2^2 + x_2x_3 + x_3^2\right) - u(x_3,x_2)\,3x_2^2,$$

whence

$$u(x_3,x_2)=\frac{x_2^2+x_2x_3+x_3^2}{3x_2^2}\left(u(x_2,x_2)-1\right)+\frac{3x_3^2}{x_2^2+x_2x_3+x_3^2}. \tag{30}$$

Replacing this expression for $u(x_3,x_2)$ in (29) yields

$$\begin{aligned} x_1^2+x_1x_3+x_3^2 &= u(x_1,x_2)\left(x_2^2+x_2x_3+x_3^2\right) \\ &\quad -\left(x_1^2+x_1x_2+x_2^2\right)\left(\frac{x_2^2+x_2x_3+x_3^2}{3x_2^2}\left(u(x_2,x_2)-1\right)\right), \end{aligned}$$

whence

$$u(x_1,x_2)=\frac{x_1^2+x_1x_3+x_3^2}{x_2^2+x_2x_3+x_3^2}+\frac{x_1^2+x_1x_2+x_2^2}{3x_2^2}\left(u(x_2,x_2)-1\right). \tag{31}$$

In this way, from (30) with $x_3=x_1$ and (31) it is obtained

$$\begin{aligned} & \frac{x_1^2+x_1x_2+x_2^2}{3x_2^2}\left(u(x_2,x_2)-1\right)+\frac{3x_1^2}{x_1^2+x_1x_2+x_2^2} \\ = \ & u(x_1,x_2) \\ = \ & \frac{x_1^2+x_1x_3+x_3^2}{x_2^2+x_2x_3+x_3^2}+\frac{x_1^2+x_1x_2+x_2^2}{3x_2^2}\left(u(x_2,x_2)-1\right), \end{aligned}$$

and hence

$$\frac{3x_1^2}{x_1^2+x_1x_2+x_2^2}=\frac{x_1^2+x_1x_3+x_3^2}{x_2^2+x_2x_3+x_3^2},\ \ x_1,x_2,x_3\in\mathbb{R}^+.$$

In view of this equality can not hold identically in $(\mathbb{R}^+)^3$, the partial derivatives of M can not satisfy a relationship like (27), as affirmed. Thus, it has been proved that the mean M can not be represented in the form (25) with $M_0,M_1,M_2\in\mathcal{C}^{(1)}\mathcal{M}(\mathbb{R}^+)$. Once again, the symmetry of M enables us to conclude that M can not be represented in any form obtained from (25) by a rearrangement of the variables.

Other possibilities of representing the mean M by a composition of k (≥ 3) two variables means can, in principle, be discarded by deriving from the representation formula a relationship among the partial derivatives of M of a sufficiently high order and then show that this relationship is not really fulfilled by M. However, the complexity of the procedure increases speedily with k and its usefulness is circumscribed to sufficiently regular means. Furthermore, reducibility of M is, at best, established in a narrower class of means.

4 A classification of reducible means

A classification of reducible means based on a *reduction process* will be described along the following paragraphs. First of all, note that a generic n variables mean $M\in\mathcal{M}(I)$ reducible in a class $\mathcal{N}(I)$ can be written in the form

$$M(x_1,\ldots,x_n)=M_0\left(M_1\left(x_{b_1},\ldots,x_{e_1}\right),\ldots,M_r\left(x_{b_r},\ldots,x_{e_r}\right)\right), \tag{32}$$

where $2 \leq r = \nu(M_0) < n$ and $1 \leq b_i < e_i \leq n$, $1 \leq \nu(M_i) \leq n$ for every $i = 1, \ldots, r$. (32) will be named the *first layer representation* of the reducible mean M. M_i is a coordinate mean when $\nu(M_i) = 1$, and it should be observed that the fact that the equality $\nu(M_i) = n$ may hold for any $i = 1, \ldots, r$, is not in contradiction with the notion of reducibility, but simply implies that the mean M_i can be reduced further (and finally expressed as a composition of a finite number of ν variables means with $\nu < n$). Furthermore note, on one hand, that all variables are effective in (32) and, on the other, that the class $\mathcal{N}(I)$ contains all means M_i. The mean M_0 will be named *outer mean* while the means M_i, $i = 1, \ldots, r$, will be named *inner means* of the first layer representation (32).

Depending on the nature of the inner means, let us distinguish three mutually exclusive possibilities as follows:

i) $\nu(M_i) = n$ for any $i = 1, \ldots, r$;

ii) $\nu(M_i) < n$ for every $i = 1, \ldots, r$, and there exists a pair of overlapping sets of indices $J_i = \{b_i, \ldots, e_i\}$ and $J_k = \{b_k, \ldots, e_k\}$; i.e., $J_i \cap J_k \neq \varnothing$;

iii) $\nu(M_i) < n$ for every $i = 1, \ldots, r$, and the sets of indices $J_i = \{b_i, \ldots, e_i\}$, $i = 1, \ldots, r$, are mutually disjoint.

Clearly, the above possibilities are also exhaustive.

In the case **i)**, every inner mean M_i with $\nu(M_i) = n$ must be, in its turn, a reducible mean. Suppose, for example, that $\nu(M_1) = n$; then, the first layer representation of M_1 reads as follows:

$$M_1(x_1, \ldots, x_n) = M_{10}\left(M_{11}\left(x_{b_{11}}, \ldots, x_{e_{11}}\right), \ldots, M_{1s}\left(x_{b_{1s}}, \ldots, x_{e_{1s}}\right)\right),$$

where $2 \leq s < n$ and $1 \leq b_{1i} < e_{1i} \leq n$, $1 \leq \nu(M_{1i}) \leq n$ for every $i = 1, \ldots, s$, and therefore, the three possibilities **i)**, **ii)** and **iii)** reappear. Since M is reducible, this process can be continued up to the point in which the inequality $\nu(M_{ij}) < n$ is satisfied by every inner mean M_{ij}.

In the case **iii)**, the equality $\bigcup_{i=1}^{r} J_i = \mathbf{n}$ must hold, so that the family $\{J_i : i = 1, \ldots, r\}$ constitutes a partition of $\mathbf{n}$ and therefore, there exists a permutation $\sigma \in S_n$ such that

$$\begin{aligned} & M_\sigma(x_1, \ldots, x_n) \\ = \; & M_0\left(M_1\left(x_1, \ldots, x_{e_1}\right), M_2\left(x_{e_1+1}, \ldots, x_{e_2}\right), \ldots, M_r\left(x_{e_{r-1}+1}, \ldots, x_{e_r}\right)\right). \end{aligned}$$

In this case, let us say that M is a *simply reducible* or *S-reducible* mean.

The first layer representation of a S-reducible mean M is already a reduced representation of M. This is a characteristic shared with those means falling into case **ii)** above. Our first result shows that this case corresponds to reducible means resulting from a specialization of variables in a S-reducible mean.

Proposition 7 *If a reducible mean $M \in \mathcal{M}(I)$ with $\nu(M) = n$ has a first layer representation given by (32) with (at least) a pair of sets of indices $J_i =$*

$\{b_i, \ldots, e_i\}$ *and* $J_k = \{b_k, \ldots, e_k\}$ *satisfying* $J_i \cap J_k \neq \varnothing$*, then* M *is obtained as a specialization of variables in a* S*-reducible mean* M^* *with* $n+1 \leq \nu(M^*) \leq n(n-1)$ *whose outer mean* M_0^* *satisfies* $\nu(M_0^*) \leq n-1$*.*

Proof. Assume that $M \in \mathcal{M}(I)$ has (32) as its first layer representation and that a pair (at least) of sets of indices $J_i = \{b_i, \ldots, e_i\}, J_k = \{b_k, \ldots, e_k\}$ satisfies $J_i \cap J_k \neq \varnothing$. Let us make a substitution of the variables in the expression (32) by applying the following algorithm: for every $i = 1, \ldots, n$ and every $j = 1, \ldots, r$, replace the variable x_i in M_j (whenever it appears) by the new variable $x_{(i,j)}$. Denote the resulting mean by M^*. Every new variable in M^* appears no more that one time in no more than one M_j, so that M^* is S-reducible. Now, in order to count the number of variables in M^*, define $\rho : \mathbf{n} \times \mathbf{r} \to \{0,1\}$ as follows:

$$\rho(i,j) = \begin{cases} 1, & \text{if } x_i \text{ is a variable of } M_j \\ 0, & \text{otherwise} \end{cases} .$$

Thus $\nu(M^*) = \sum_{i,j=1}^{n,r} \rho(i,j)$. Since $\rho(i,j) \leq 1,\ i = 1, \ldots, n,\ j = 1, \ldots, r$, and $r \leq n-1$, it can be written $\nu(M^*) = \sum_{i,j=1}^{n,r} \rho(i,j) \leq nr \leq n(n-1)$. Now, in view of every variable x_i do appear in any M_j, it is clear that $\sum_{j=1}^{r} \rho(i,j) \geq 1$ for any $i = 1, \ldots, n$. Furthermore, in view of the fact that $J_i \cap J_k \neq \varnothing$ for at least a pair i, k of indices, there exists $i \in \mathbf{n}$ such that $\sum_{j=1}^{r} \rho(i,j) \geq 2$. In this way, $\nu(M^*) = \sum_{i,j=1}^{n,r} \rho(i,j) = \sum_{i=1}^{n} \sum_{j=1}^{r} \rho(i,j) \geq n+1$. On the other hand, $\nu(M_0^*) = \nu(M_0) = r \leq n-1$. The proof is completed by observing that

$$M = M^*_{J_1^* \cdots J_n^*},$$

where, for every $i \in \boldsymbol{n}$, $J_i^* = \{(i,j) : i \in J_j\}$. ∎

Unlike what occurs with a general reducible mean, an analytical determination of the reduced means is always possible when M is S-reducible in a certain class $\mathcal{N}(I)$. Using a notation introduced in Section 2, the first layer representation (32) of M assumes the compact form

$$M(x_1, \ldots, x_n) = M_0\left(M_1(x_j)_{[J_1]}, \ldots, M_r(x_j)_{[J_r]}\right), \tag{33}$$

where $\{J_k : k = 1, \ldots, r\}$ is a partition of $\mathbf{n}$ provided that M is S-reducible. Now, fix $k \in \mathbf{r}$. If, for every $i \in \mathbf{n} \setminus J_k$, the variable x_i in both members of (33) is substituted by $M_k(x_j)_{[J_k]}$, then it is obtained

$$M_{[J_k]}\left((x_j)_{[J_k]}; M_k(x_j)_{[J_k]}\right) = M_0\left(M_k(x_j)_{[J_k]}, \ldots, M_k(x_j)_{[J_k]}\right) = M_k(x_j)_{[J_k]},$$

so that $M_k\left(x_j\right)_{[J_k]}$ turns out to be a $[J_k]$-lower mean of M. On the other hand, the specialization of the variables specified by $\left(x_j\right)_{[J_i]}=\left(x_i,\ldots,x_i\right),\ i=1,\ldots,r$, (applied again in both members of (33)) produces

$$M_{J_1\cdots J_r}\left(x_1,\ldots,x_r\right)=M_0\left(M_1\left(x_1,\ldots,x_1\right),\ldots,M_r\left(x_r,\ldots,x_r\right)\right)=M_0\left(x_1,\ldots,x_r\right);$$

whence $M_0=M_{J_1\cdots J_r}$. Based on these observations, let us now prove the following:

Theorem 8 *A mean $M\in\mathcal{M}(I)$ with $\nu(M)=n$ is S-reducible in a class $\mathcal{N}(I)$ if and only if the following conditions are satisfied:*

SR1) *there exists a partition $\{J_k : k=1,\ldots,r\}$ of $\mathbf{n}$ such that*

$$M_{J_1\cdots J_r}\left(\mu_{[J_1]},\ldots,\mu_{[J_r]}\right)=M, \tag{34}$$

where $M_{J_1\cdots J_r}$ is a specialized of M and, for every $k=1,\ldots,r$, $\mu_{[J_k]}$ is a $[J_k]$-lower mean of M;

SR2) *$M_{J_1\cdots J_r}$ and $\mu_{[J_k]},\ k=1,\ldots,r$, are members of $\mathcal{N}(I)$.*

Proof. The necessity of conditions **SR1)** and **SR2)** was shown in the discussion above. Now, if the equality (34) and condition **SR2)** hold, then M is clearly reducible: $M_0=M_{J_1\cdots J_r}$ and $M_i=\mu_{[J_k]},\ k=1,\ldots,r$, are, respectively, the outer and the inner reduced means of M. Since $\{J_k : k=1,\ldots,r\}$ is a partition of $\boldsymbol{n}$, M is really S-reducible. ■

Example 9 *The equalities (17) express the fact that linear means are S-reducible in the class $\mathcal{LM}(\mathbb{R})$. Correspondingly, a quasilinear mean defined in I is S-reducible in the class $\mathcal{QLM}(I)$. At the end of Section 3, the* 3 *variables symmetric and continuous mean (23) was shown to be not S-reducible in the class $\mathcal{C}^{(0)}\mathcal{M}(\mathbb{R}^+)$. The non S-reducibility in $\mathcal{C}^{(0)}\mathcal{M}(\mathbb{R}^+)$ of the counter-harmonic mean (with equal weights) CH_3 can be proved in a similar way.*

Remark 10 *After Theor. 8, in order to establish the S-reducibility of a mean M with $\nu(M)=n$, the validity of the equality (34) must be inspected, in the worst case, for every (non trivial) partition $\{J_k : k=1,\ldots,r\}$ of $\mathbf{n}$. As it is well known, the number of partitions of $\mathbf{n}$ is B_n, the n-th Bell number, so that B_n-2 is the number of equalities to be inspected. This number decreases abruptly when M is symmetric, in whose case only $p(n)-2$ inspections of the validity of the equality (34) must be performed (in the worst case), being $p(n)$ the partition function of the integer n. Of course, these combinatorial essays are possible once the lower means of M have been determined.*

The statement of a theorem of classification of reducible means is postponed until the next section, where the concept of height of a formula is introduced.

5 The tree of a formula

In the previous sections, the information contained in a formula representing a certain composition of means has been presented in the linear form which is the main characteristic of a *list* (cf. Chap. 2 of [16]). However, the nonlinear structure of a *tree* turns out to be more apt for a number of purposes, among them, to introduce some parameters useful in describing the complexity of a representation and then prove some related combinatorial results. In order to define this tree, let be given a finite number of functions $F_i : I^{n_i} \to I,\ i = 0, \ldots, r$, with $n_i \in \mathbb{N}$ for every $i = 0, \ldots, r$. It is assumed that all arguments of every F_i are effective, so that $\nu(F_i) = n_i,\ i = 0, \ldots, r$ (being defined ν and the effectiveness of an argument in the same way as it was made for means in the Introduction). When structured as a list, a composition of the functions F_i is expressed by a formula $\mathfrak{F}$ consisting in a finite sequence of variables and the functional symbols $F_i,\ i = 0, \ldots, r$, separated by parentheses which is written in observance of the standard conventions. F_0 will denote the outermost function of the formula $\mathfrak{F}$. The set of functional symbols in a formula $\mathfrak{F}$ will be denoted by $\mathcal{FS}(\mathfrak{F})$, while $\mathcal{VAR}(\mathfrak{F})$ will denote the set of variables in $\mathfrak{F}$. Functional symbols and variables may appear repeatedly in a formula and it will be useful to define a related formula in which repetitions are eliminated. Concretely, a new formula $\mathfrak{F}_R$ is derived from $\mathfrak{F}$ by replacing the j-th occurrence of the symbol F_i by F_{ij} and the j-th occurrence of the variable x_i by the new variable x_{ij}. For the terminology and basic results on Graph Theory employed in this section, the reader is referred to [5], [9] and [16].

Let us define a graph $T(\mathfrak{F})$ with labeled vertices and arcs, named the *tree of the formula* $\mathfrak{F}$, by a pair $(V(T(\mathfrak{F})), \Gamma)$, where $V(T(\mathfrak{F}))$ is a set and $\Gamma : V(T(\mathfrak{F})) \to V(T(\mathfrak{F}))$ is a set valued function, together a rule of labeling, as follows:

a) the set of vertices $V(T(\mathfrak{F}))$ is constituted by all variables and functional symbols in the formula without repetitions $\mathfrak{F}_R$; i.e.,

$$V(T(\mathfrak{F})) = \mathcal{FS}(\mathfrak{F}_R) \cup \mathcal{VAR}(\mathfrak{F}_R);$$

b) Γ is defined for every $v \in V(T(\mathfrak{F}))$ by

$$\Gamma v = \begin{cases} \varnothing, & v \in \mathcal{VAR}(\mathfrak{F}_R) \\ \{v' \in V(T(\mathfrak{F})) : v' \text{ is an argument of } v\}, & v \in \mathcal{FS}(\mathfrak{F}_R) \end{cases} ;$$

c) labeling of vertices: for every i, the vertices F_{ij} and x_{ij} receive respectively the labels F_i and x_i. Labeling of arcs: if $v' \in \Gamma v$, then the arc (v, v') is labeled with the integer $i - 1$ provided that v' is the i-th argument of v.

It is easy to see that $T(\mathfrak{F})$ is an acyclic and connected graph; i.e., it is a tree with *root vertex* $\operatorname{root}(T(\mathfrak{F}))$ labeled F_0, the outermost function of the formula $\mathfrak{F}$. The variables x_i of $\mathfrak{F}$ are the labels corresponding to the *terminal vertices* (*leaves*) of the tree $T(\mathfrak{F})$ while the functional symbols of $\mathfrak{F}$ serve to label the *branch vertices* of $T(\mathfrak{F})$. The arcs of $T(\mathfrak{F})$ are labeled with the integers $0, 1, 2, \ldots$.

The tree $T(\mathfrak{F})$ of a simple formula $\mathfrak{F}$ is illustrated by the labeled tree on the left of Fig. 1. The rightmost tree in the figure is a variation of $T(\mathfrak{F})$ in which the labeling of the arcs has been replaced by ordering: the first argument in F_i is joined to F_i by the leftmost arc, the second one is joined by an arc placed at the right of the first and so on. Both representations make use of the planarity of trees and of their natural imbedding in a plane, but an orientation must be given to the plane in order that the second representation may be implemented. Of course, labeling of arcs is at all necessary when all functions F_i are symmetric.

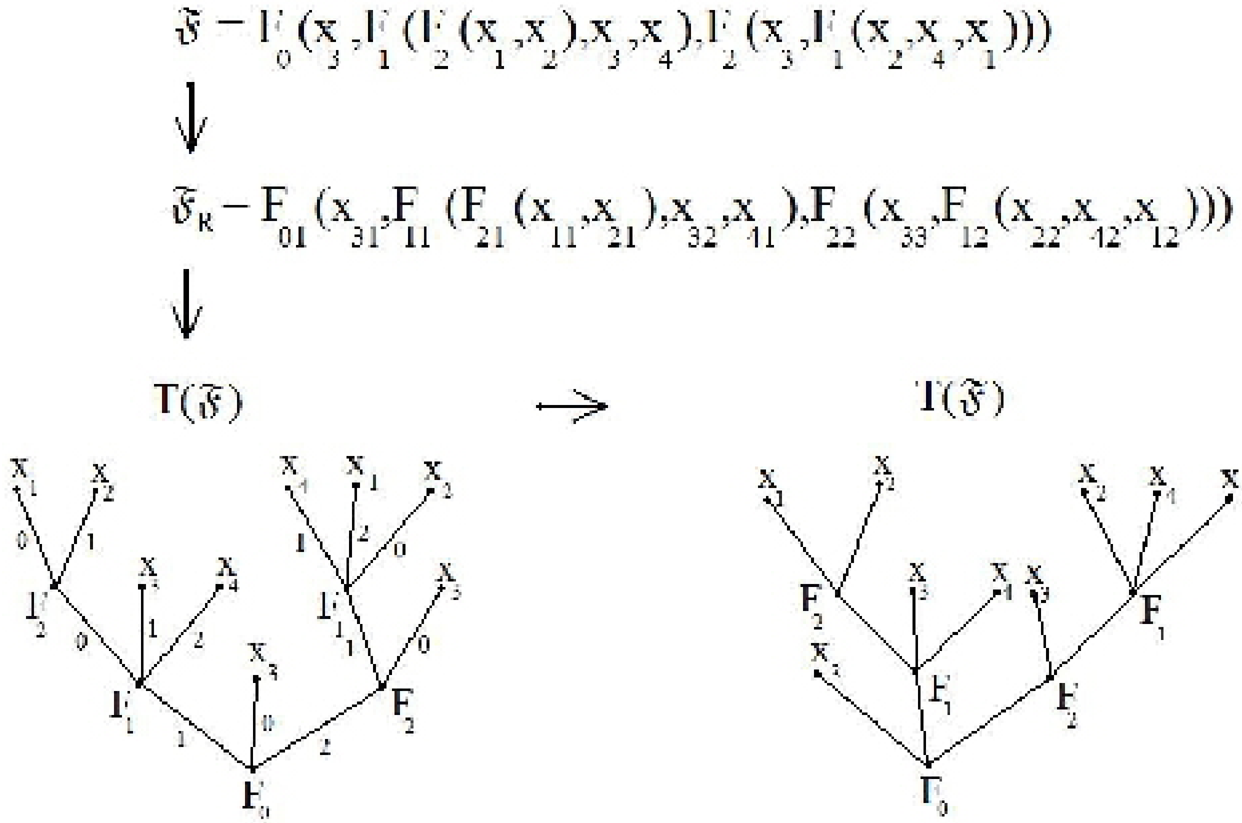

Fig. 1: the tree of a simple formula

It should be observed that the assignment of the tree $T(\mathfrak{F})$ to a formula $\mathfrak{F}$ is univocal and that the formula $\mathfrak{F}(T)$ corresponding to a given labeled tree T can be promptly written. In particular, if T_i is the subtree of $T(\mathfrak{F})$ rooted at the vertex labeled F_i, then $\mathfrak{F}(T_i)$ is a subformula of $\mathfrak{F}$.

A series of integer valued functions related to a tree $T = (V, \Gamma)$ is now presented. By definition, $\mathrm{nl}(T)$ is the the *number of leaves* of the tree T, while its height $\mathrm{h}(T)$ is the length of the longest path joining the root ($\mathrm{root}(T)$) with a leaf. For example, a formula $\mathfrak{F}$ with $\mathrm{h}(T(\mathfrak{F}))=1$ and $\mathrm{nl}(T(\mathfrak{F}))=n$ simply represents a function depending on no more than n variables. The *descent* $\mathrm{des}(v)$ of a vertex $v\in V(T)$ is given by $\mathrm{card}(\Gamma v)$, i.e., by the number of subtrees of v. Note that the relationship

$$\deg(v)=\begin{cases}\mathrm{des}(v)+1, & v\neq \mathrm{root}(T)\\ \mathrm{des}(v), & v=\mathrm{root}(T)\end{cases}$$

holds amongst the standard notion of degree $\deg(v)$ of a vertex in a graph and the descent $\mathrm{des}(v)$.

Proposition 11 *Let $T(\mathfrak{F})$ the tree of a formula $\mathfrak{F}$; then*

i) *the number of leaves $\mathrm{nl}(T(\mathfrak{F}))$ of the tree $T(\mathfrak{F})$ coincides with $\mathrm{var}(\mathfrak{F})$, the number of variables counted with repetitions in the formula $\mathfrak{F}$ (so that $\mathrm{var}(\mathfrak{F})=\mathrm{card}(\mathcal{VAR}(\mathfrak{F}_R))$);*

ii) *the height $\mathrm{h}(T(\mathfrak{F}))$ coincides with the length of the longest sequence of compositions of functions in $\mathfrak{F}_R$;*

iii) *if n_i denotes the number of arguments of F_i, $i=0,\dots,r$, then the equality*

$$\mathrm{des}(v)=\begin{cases}0, & v\in\mathcal{VAR}(\mathfrak{F}_R)\\ n_i, & v\in\mathcal{FS}(\mathfrak{F}_R)\end{cases}, \tag{35}$$

holds for the descent of the vertices v of $T(\mathfrak{F})$.

In the formula of Fig. 1, $\mathrm{var}(\mathfrak{F})=9=\mathrm{nl}(T(\mathfrak{F}))$ while $\mathrm{h}(T(\mathfrak{F}))=3$, which coincides with the length of the (longest) sequence of compositions F_{01},F_{11},F_{21} (or F_{01},F_{22},F_{12}).

Proof. The simple proof of this result is omitted. ■

Denote respectively by $\mathrm{v}(T)$ and $\mathrm{a}(T)$ the *order* (number of vertices) and *size* (number of arcs) of a tree T. These numbers are related by the equality $\mathrm{a}(T)=\mathrm{v}(T)-1$ ([9], Theor. 4.3, pg. 100).

Proposition 12 *Let F_i, $i=0,\dots,r$, be the functional symbols appearing in a formula $\mathfrak{F}$. If, for every $i=0,\dots,r$, n_i denotes the number of arguments of F_i, then*

$$\mathrm{v}(T(\mathfrak{F}))=\sum_{i=0}^{r}n_i+1, \tag{36}$$

and therefore

$$\mathrm{var}(\mathfrak{F})=\sum_{i=0}^{r}n_i-r. \tag{37}$$

Proof. Every arc of $T(\mathfrak{F})$ is counted twice in the sum $\operatorname{des}(\operatorname{root}(T(\mathfrak{F}))) + \sum_{v\in V(T(\mathfrak{F})),\, v\neq \operatorname{root}(T(\mathfrak{F}))} (1+\operatorname{des}(v))$, so that

$$\sum_{v\in V(T(\mathfrak{F}))} (1+\operatorname{des}(v)) = 2\,\mathrm{a}(T(\mathfrak{F})) + 1,$$

and taking into account the equality (35) and the fact that $\mathrm{a}(T) = \mathrm{v}(T) - 1$, it is obtained

$$\sum_{i=0}^{r} n_i + \mathrm{v}(T(\mathfrak{F})) = 2(\mathrm{v}(T(\mathfrak{F})) - 1) + 1,$$

whence the equality (36) follows. Since $\mathrm{v}(T(\mathfrak{F})) = \mathrm{nl}(T(\mathfrak{F})) + r + 1$, (37) is quickly obtained from (36) and Prop. 11-**i)**. ■

The tree $T(\mathfrak{F})$ of a reduced representation $\mathfrak{F}$ of a reducible mean $M \in \mathcal{M}_n(I)$ has the following two particular properties:

C1) if M_i, $i = 0, \ldots, r$, are the reduced means of $\mathfrak{F}$ and $\nu(M_i) = n_i$, then $r \geq 1$ and $2 \leq n_i \leq n-1$ for every $i = 0, \ldots, r$;

C2) for every $i = 1, \ldots, n$, x_i is a leaf of $T(\mathfrak{F})$.

Observe that these properties really characterize the trees T which come from a reduced representation formula of a reducible mean $M \in \mathcal{M}_n(I)$. Thus, for example, the tree of the formula $\mathfrak{F}$ in Fig. 1 may actually correspond to the reduced representation of a reducible mean $M \in \mathcal{M}_4(I)$.

Proposition 13 *Let $\mathfrak{F}$ be a representation formula of a reducible mean $M \in \mathcal{M}_n(I)$ and suppose that $r+1$ is the number of reduced means in $\mathfrak{F}$; then,*

$$\max\{r+2, n\} \leq \operatorname{var}(\mathfrak{F}) \leq (n-1)(r+1) - r. \tag{38}$$

Proof. From **C1)** it is obtained

$$2(r+1) \leq \sum_{i=0}^{r} n_i \leq (n-1)(r+1),$$

and these inequalities combined with (37) yield

$$r+2 \leq \mathrm{nl}(T(\mathfrak{F})) = \operatorname{var}(\mathfrak{F}) \leq (n-1)(r+1) - r. \tag{39}$$

Moreover, the inequality

$$\mathrm{nl}(T(\mathfrak{F})) \geq n \tag{40}$$

follows from **C2),** and thus (38) turns out to be a straightforward consequence of (39), (40). ■

The height $\mathrm{h}(T(\mathfrak{F}))$ of the reduced representation formula $\mathfrak{F}$ of a reducible mean $M \in \mathcal{M}_n(I)$ clearly satisfies $\mathrm{h}(T(\mathfrak{F})) \geq 2$, and the theorem of classification postponed in the preceding section is now stated in the following terms:

Theorem 14 *Let $M \in \mathcal{M}(I)$ be a reducible mean in a class $\mathcal{N}(I)$ and $\mathfrak{F}$ be its reduced representation formula. If $\mathrm{h}(T(\mathfrak{F})) = 2$, then one (and only one) of the following alternatives hold:*

L1) *M is S-reducible; or*

L2) *M is a specialized mean of a S-reducible mean $M^* \in \mathcal{M}(I)$ with $n+1 \leq \nu(M^*) \leq n(n-1)$.*

A reducible mean $M \in \mathcal{M}(I)$ with $\mathrm{h}(T(\mathfrak{F})) > 2$ is a composition of reducible means $N \in \mathcal{M}(I)$ with a reduced representation formula $\mathfrak{F}_N$ satisfying $\mathrm{h}(T(\mathfrak{F}_N)) \leq 2$.

Proof. Assume that $\mathfrak{F}$ is the reduced representation formula of a reducible mean M with $\nu(M) = n$. After Prop. 11-**ii)**, the equality $\mathrm{h}(T(\mathfrak{F})) = 2$ amounts to the same that the first layer representation of M corresponds to any one of the cases **ii)** or **iii)** (described in the preceding section). Thus, in view of Prop. 7, one of the alternatives **L1)** or **L2)** must occur. Now, the case **i)** must occur when $\mathrm{h}(T(\mathfrak{F})) > 2$ and an inductive reasoning on $h = \mathrm{h}(T(\mathfrak{F}))$ enable us to show the assertion that M is, in this case, a composition of reducible means $N \in \mathcal{M}(I)$ with $\mathrm{h}(T(\mathfrak{F}_N)) \leq 2$. First consider the case $h = 3$. If $\mathrm{h}(T(\mathfrak{F}')) \leq 2$ for any other reduced representation $\mathfrak{F}'$ of M, then the assertion holds trivially and thus, it can be assumed that the first layer representation of M contains a certain number of inner means M_i with $\nu(M_i) = n$. Let $M_{i_1}, \ldots, M_{i_k}$ be these means, so that

$$M = M_0(M_1, \ldots, M_{i_1}, \ldots, M_{i_k}, \ldots, M_r), \tag{41}$$

and define M_0^* to be the mean which is obtained from M after replacing every occurrence of M_{i_j} in (41) by a new variable u_j; i.e.,

$$M_0^* = M_0(M_1, \ldots, u_{i_1}, \ldots, u_{i_k}, \ldots, M_r). \tag{42}$$

Observe that $k+1 \leq \nu(M_0^*) \leq n+k$. Denoting by $\mathfrak{F}_{i_k}$ to the formula corresponding to the subtree of $T(\mathfrak{F})$ rooted at M_{i_k}, it is observed that $\mathrm{h}(T(\mathfrak{F}_{i_k})) = 2$. Now, in the case in which $\{i_1, \ldots, i_k\} = \mathbf{r}$, M_0^* is a mean satisfying $\nu(M_0^*) = r < n$ and $\mathrm{h}\left(T\left(\mathfrak{F}_{M_0^*}\right)\right) = 1$. In other case, the inequalities $1 \leq \nu(M_j) < n$ are satisfied for every $j \in \mathbf{r} \backslash \{i_1, \ldots, i_k\}$, so that M_0^* turns out to be a reducible mean (in at most $n+k$ variables) satisfying $\mathrm{h}\left(T\left(\mathfrak{F}_{M_0^*}\right)\right) = 2$. This proves that M is a composition of reducible means with a reduced representation formula $\mathfrak{F}$ satisfying $\mathrm{h}(T(\mathfrak{F})) \leq 2$, so that the assertion holds when $\mathrm{h}(T(\mathfrak{F})) = 3$. Now, suppose that the assertion was true for all reducible means whose reduced representation formula $\mathfrak{F}$ satisfies $\mathrm{h}(T(\mathfrak{F})) = h \geq 3$, and consider a reducible mean M with $\mathrm{h}(T(\mathfrak{F}_M)) = h+1$. It can be assumed that there is no reduced representation formula $\mathfrak{F}'_M$ of M with $\mathrm{h}(T(\mathfrak{F}'_M)) \leq h$ so that, as before, the first layer representation of M must contain a certain number of inner means $M_{i_1}, \ldots, M_{i_k}$ with $\nu\left(M_{i_j}\right) = n$ and $\mathrm{h}\left(T\left(\mathfrak{F}_{M_{i_j}}\right)\right) = h,\ j = 1, \ldots, k$. By the inductive hypothesis, for every $j = 1, \ldots, k$, M_{i_j} turns out to be a composition of reducible means with a reduced representation formula $\mathfrak{F}$ satisfying $\mathrm{h}(T(\mathfrak{F})) \leq 2$, and in view of $\mathrm{h}\left(T\left(\mathfrak{F}_{M_0^*}\right)\right) \leq 2$ for M_0^* defined by (42), the mean M is also a composition of reducible means $N \in \mathcal{M}(I)$ with a reduced representation formula $\mathfrak{F}$ satisfying $\mathrm{h}(T(\mathfrak{F}_N)) \leq 2$. This completes the proof. ■

The *completely reducible* means in a class $\mathcal{N}(I)$; i.e., those reducible means $M \in \mathcal{M}(I)$ whose reduced means are all *two variables* means belonging to $\mathcal{N}(I)$, deserve a special consideration. An iteration of (17) proves that linear means are completely reducible in $\mathcal{LM}(\mathbb{R})$ and, as a consequence, a quasilinear mean $M \in \mathcal{QLM}(I)$ turn out to be completely reducible in $\mathcal{QLM}(I)$. As an iteration of (22) shows, polynomial symmetric means provide an example of complete reducibility in $\mathcal{C}^{(0)}\mathcal{M}(\mathbb{R}^+)$.

If $\mathfrak{F}$ is the reduced representation formula of a completely reducible mean M, then $T(\mathfrak{F})$ turns out to be a binary tree, so that $n_i = 2$ for every $i = 0, \ldots, r$. As a consequence $\sum_{i=0}^{r} n_i = 2(r+1)$, and Props. 12 and 11-**i)** yield

$$\mathrm{v}(T(\mathfrak{F})) = 2r + 3 \text{ and } \mathrm{var}(\mathfrak{F}) = r + 2.$$

Proposition 15 *Let $M \in \mathcal{M}_n(I)$ be a mean reducible (in a class $\mathcal{N}(I)$) and assume that $M_i,\ i = 0, \ldots, r$ are the reduced means appearing in a reduced representation $\mathfrak{F}$ of M. Then M is completely reducible if and only if*

$$\mathrm{var}(\mathfrak{F}) = r + 2. \tag{43}$$

Setting $n = 3$ in the inequalities (38) of Prop. 13 yields $\mathrm{nl}(T(\mathfrak{F})) = r + 2$, which is consistent with the trivial complete reducibility of a reducible three variables mean.

Proof. The necessity of the equality (43) was proved in the paragraph preceding the statement of the proposition. In order to prove the sufficiency, assume that (43) holds for the reduced representation formula $\mathfrak{F}$ of M. Then, Props. 12 and 11-**i)** yield

$$r + 2 = \mathrm{var}(\mathfrak{F}) = \mathrm{nl}(T(\mathfrak{F})) = \sum_{i=0}^{r} n_i - r,$$

and hence

$$\sum_{i=0}^{r} n_i = 2(r+1).$$

In view of **C1)**, $n_i \geq 2$ for every $i = 0, \ldots, r$, and therefore, the last equality implies $n_i = 2$ for every $i = 0, \ldots, r$; i.e., M is completely reducible. ■

Now consider a S-reducible mean $M \in \mathcal{M}_n(I)$ whose reduced representation is given by a formula $\mathfrak{F}$. Clearly, the number of leaves $\mathrm{nl}(T(\mathfrak{F}))$ of $T(\mathfrak{F})$ is exactly n. The converse is also true. In fact, if $\mathrm{nl}\, T(\mathfrak{F}) = n$, then the leaves of $T(\mathfrak{F})$ are exactly $x_1, \ldots, x_n$ and a similar property is enjoyed by every labeled subtree of $T(\mathfrak{F})$ rooted in any child vertex of M_0: there is no pair of equally labeled leaves. Let $T_1, \ldots, T_r$ denote a complete list of these labeled subtrees and define $n_i = \mathrm{nl}(T_i),\ i = 1, \ldots, k$. Then the subformula $\mathfrak{F}(T_i)$ must reduce to a single variable when $n_i = 1$ or else represent a certain n_i variables mean M_i. In any case, M can be written in the form

$$M = M_0 \left(M_1, \ldots, M_r \right),$$

where $\nu \left(M_i \right) = n_i$. Since the family constituted by the leaves of $T_1, \ldots, T_r$ is a partition of $\{x_1, \ldots, x_n\}$, this equality shows that M is S-reducible.

Proposition 16 *Let $M \in \mathcal{M}_n \left(I \right)$ be a reducible mean M with a reduced representation given by the formula $\mathfrak{F}$. Then, M is S-reducible if and only if* $\mathrm{nl} \left(T \left(\mathfrak{F} \right) \right) = n$ *or, equivalently,*

$$\sum_{i=0}^{r} n_i = n + r. \tag{44}$$

Proof. See the previous discussion. After equality (37), $\mathrm{nl} \left(T \left(\mathfrak{F} \right) \right) = n$ is equivalent to (44). ■

Corollary 17 *Let $M \in \mathcal{M}_n \left(I \right)$ be a reducible mean M with a reduced representation given by a formula $\mathfrak{F}$. Then M is, at the same time, simply and completely reducible if and only if*

$$r = n - 2.$$

Proof. It is a direct consequence of Props. 15 and 16. ■

6 Inequalities

The *structure* of a formula $\mathfrak{F}$ is defined as a modification of $T \left(\mathfrak{F} \right)$ obtained by suppressing the labels corresponding to the branch vertices. Fig. 2 below shows the structure of formula $\mathfrak{F}$ in Fig.1.

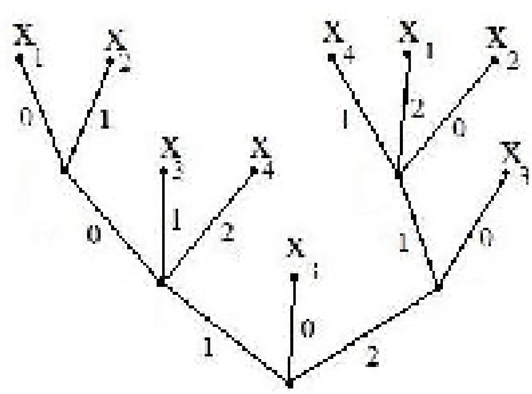

Fig. 2: structure of the formula $\mathfrak{F}$ in Fig. 1

Inequalities for reducible means possessing an identical structure can be expectably transferred to inequalities between the corresponding reduced means. Consider the case in which $M,\, N \in \mathcal{C}^{(0)}\mathcal{M}_n(I)$ are reducible means whose respective formulae $\mathfrak{F}_1, \mathfrak{F}_2$ have identical structure and $\mathrm{h}(T(\mathfrak{F}_1)) = 2\ (= \mathrm{h}(T(\mathfrak{F}_2)))$. Despite of its simplicity, this case turns out to be representative of the various results that can be reached. After Theor. 14, if $M_i,\, N_i,\ i = 0, \ldots, r$, denote the corresponding reduced means, it can be written

$$M = M_0(M_1, \ldots, M_r)$$

and

$$N = N_0(N_1, \ldots, N_r).$$

Theorem 18 *If the inequalities*

$$M_i \le N_i,$$

hold for $i = 0, \ldots, r$, and one at least of the outer reduced means M_0, N_0 is isotone, then

$$M \le N.$$

Proof. Assuming, for instance, that M_0 is isotone, it can be written

$$M = M_0(M_1, \ldots, M_r) \le M_0(N_1, \ldots, N_r) \le N_0(N_1, \ldots, N_r) = N.$$

The proof is similar when N_0 is isotone. ■

Remark 19 *In general, Theor. 18 ceases to be true when no one of M_0, N_0 is an isotone mean. For example, if $M_0 = CH_2 = N_0$ where CH_2 is the 2-nd (unweighted) counter-harmonic mean, $M_1 = A_2,\ N_1 = G_2$, and $M_2 = X_3 = N_2$, then*

$$M(x_1, x_2, x_3) = \frac{x_1 x_2 + x_3^2}{\sqrt{x_1 x_2} + x_3} \text{ and } N(x_1, x_2, x_3) = \frac{\left(\frac{x_1+x_2}{2}\right)^2 + x_3^2}{\frac{x_1+x_2}{2} + x_3}.$$

Both M and N are defined on $\mathbb{R}^+$. By homogeneity,it can be written

$$M(x_1, x_2, x_3) - N(x_1, x_2, x_3) = x_3\left(M\left(\frac{x_1}{x_3}, \frac{x_2}{x_3}, 1\right) - N\left(\frac{x_1}{x_3}, \frac{x_2}{x_3}, 1\right)\right),$$

so that, setting $x_i/x_3 = u_i^2,\ i = 1, 2$, it is obtained

$$\begin{aligned}
& M(x_1, x_2, x_3) - N(x_1, x_2, x_3) \\
= \ & x_3\left(M\left(u_1^2, u_2^2, 1\right) - N\left(u_1^2, u_2^2, 1\right)\right) \\
= \ & x_3\left(\frac{\left(u_1^2 u_2^2 + 1\right)\left(\frac{u_1^2+u_2^2}{2} + 1\right) - (u_1 u_2 + 1)\left(\left(\frac{u_1^2+u_2^2}{2}\right)^2 + 1\right)}{(u_1 u_2 + 1)\left(\frac{u_1^2+u_2^2}{2} + 1\right)}\right) \\
= \ & -\frac{x_3(u_1 - u_2)^2}{4(u_1 u_2 + 1)\left(\frac{u_1^2+u_2^2}{2} + 1\right)}\left(u_1^3 u_2 + u_1^2 + u_1 u_2^3 + 2u_1 u_2 + u_2^2 - 2\right).
\end{aligned}$$

Since the last factor in the last member of these equalities changes of sign when u_1 and u_2 vary on $\mathbb{R}^+$, M and N turn out to be non comparable means.

When $M,\ N \in \mathcal{M}_n(I)$ are both completely reducible in a class $\mathcal{N}(I)$ and share the same structure, the comparison of M and N reduces to comparisons among two variables means. In this regard, let $M_2,\ N_2 \in \mathcal{M}_2(I)$ and $\left\{M_n^{(0)} : n \geq 3\right\},\ \left\{N_n^{(0)} : n \geq 3\right\} \subseteq \mathcal{M}_2(I)$ be given and consider $M_n,\ N_n \in \mathcal{M}_n(I)$ defined for every $n \geq 3$ by

$$M_n(x_1, \ldots, x_n) = M_n^{(0)}(M_{n-1}(x_1, \ldots, x_{n-1}), x_n),\ x_1, \ldots, x_n \in I, \qquad (45)$$

and

$$N_n(x_1, \ldots, x_n) = N_n^{(0)}(N_{n-1}(x_1, \ldots, x_{n-1}), x_n),\ x_1, \ldots, x_n \in I, \qquad (46)$$

respectively.

Theorem 20 *Assume that the following conditions are satisfied:*

i) $M_2 \leq N_2$,

ii) $M_n^{(0)} \leq N_n^{(0)},\ n \geq 3$, *and*

iii) *for every* $n \geq 3$, $M_n^{(0)}$ *or* $N_n^{(0)}$ *is an isotone mean;*

then the inequalities

$$M_n \leq N_n,\ n \geq 2, \qquad (47)$$

hold among the means M_n, N_n.

Proof. The inequality (47) for $n = 2$ holds by condition **i)**. Assuming that it holds for a certain $k \geq 2$, by virtue of **ii)** and **iii)** it can be written, in the case in which $M_{k+1}^{(0)}$ is isotone:

$$\begin{aligned} M_{k+1}(x_1, \ldots, x_n) &= M_{k+1}^{(0)}(M_k(x_1, \ldots, x_{n-1}), x_n) \\ &\leq M_{k+1}^{(0)}(N_k(x_1, \ldots, x_{n-1}), x_n) \\ &\leq N_{k+1}^{(0)}(N_k(x_1, \ldots, x_{n-1}), x_n) \\ &= N_{k+1}(x_1, \ldots, x_n), \end{aligned}$$

while a similar chain of inequalities holds when $N_{k+1}^{(0)}$ is isotone. This completes the inductive proof of the proposition. ■

Example 21 *The conditions of Theor. 20 are satisfied by the means* $M_2 = A_2$, $N_2 = G_2$ *and* $M_n^{(0)} = G_{2,1/n}$, $N_n^{(0)} = L_{2,1/n},\ n \geq 3$, *where*

$$G_{2,1/n}(x_1, x_2) = x_1^{1-1/n} x_2^{1/n}$$

and

$$L_{2,1/n}(x_1,x_2)=\left(1-\frac{1}{n}\right)x_1+\frac{1}{n}x_2.$$

Now, it can be inductively proved that $M_n=A_n$ while $N_n=G_n$, so that the n variables AGM inequality follows from the inequalities

$$x_1^{1-1/n}x_2^{1/n}\le\left(1-\frac{1}{n}\right)x_1+\frac{1}{n}x_2,\ n\ge 2.$$

Observe that, conversely, these inequalities follows from a suitable specialization of the variables in the n variables AGM inequality.

7 Completely reducible means and weighting

In the case in which both classes $\mathcal{M}_n(I)$ and $\mathcal{N}_n(I)$ are closed under conjugacy, a weighting procedure $\mathcal{W}$ is said to be *scale invariant* when the equality $\mathcal{W}\left((M)_f,\cdot\right)=(\mathcal{W}(M,\cdot))_f$ holds for every homeomorphism (change of scale) $f:I\to I$. A short notation for a weighting procedure $\mathcal{W}_2$ defined on a class of two variables means $\mathcal{M}_2(I)$ will be useful in the following paragraphs: since $(1-w,w)$ with $w\in[0,1]$ is a generic point of the standard 1-simplex Δ_1, let us write $M^{(w)}$ instead of $\mathcal{W}_2(M,(1-w,w))$.

Along this section, a certain continuous and scale invariant weighting procedure $\mathcal{W}_2$ defined on a suitable subclass $\mathcal{M}_2(I)$ of $\mathcal{C}^{(0)}\mathcal{M}_2(I)$ will be extended to a continuous and scale invariant weighting procedure $\mathcal{W}$ defined on a class $\mathcal{M}(I)$ of n variables means which are completely reducible in $\mathcal{M}_2(I)$. To this purpose, let us consider a representation formula $\mathfrak{F}$ of a completely reducible mean $M\in\mathcal{M}_n(I)$ with reduced means given by $M_0,\ldots,M_r\in\mathcal{M}_2(I)$. The notation $\mathfrak{F}(M_0,\ldots,M_r)$ specifying the reduced means will be useful in the next paragraphs. Thus, if for every $i=0,\ldots,r$, $M_i^{(w_i)}$ is the weighting of M_i, a mean $M^{(w_0,\ldots,w_r)}$ depending on $(w_0,\ldots,w_r)\in[0,1]^{r+1}$ is defined by

$$M^{(w_0,\ldots,w_r)}=\mathfrak{F}\left(M_0^{(w_0)},\ldots,M_r^{(w_r)}\right). \tag{48}$$

In view of Prop. 15, $\mathrm{var}(\mathfrak{F})=r+2$, so that

$$r+1=\mathrm{var}(\mathfrak{F})-1\ge n-1; \tag{49}$$

i.e., the number $r+1$ of reduced means (counted with repetitions) of M is greater than $n-1$, the dimension of the $(n-1)$-simplex Δ_{n-1}. Now, if a continuous function $\Phi:\Delta_{n-1}\to[0,1]^{r+1}$ can be constructed so that conditions **(W1)** and **(W2)** from the Introduction are fulfilled by

$$\mathcal{W}(M,\delta)=M^{\Phi(\delta)}, \tag{50}$$

then $\mathcal{W}$ will turn out to be a continuous and scale invariant weighting (defined on a suitable subclass of completely reducible means and taking values on another subclass). As a matter of fact, let us prove the following:

Theorem 22 *A continuous and scale invariant weighting procedure $\mathcal{W}_2$ defined on class $\mathcal{M}_2(I)$ of two variables means can be extended to a weighting procedure defined on the class $\mathcal{M}(I)$ of n variables means which are completely reducible in $\mathcal{M}_2(I)$. The extension is made through (48) and (50). In this last, $\Phi : \Delta_{n-1} \to [0,1]^{r+1}$ is a suitable continuous function which depends only on the structure of M.*

Some auxiliary results will be proven before this theorem. A crucial observation is contained in the first of them.

Lemma 23 *For every $i = 1, \ldots, n$, there exists $w_{(i)} = \left(w_0^{(i)}, \ldots, w_r^{(i)}\right) \in [0,1]^{r+1}$ such that*

$$M^{\left(w_0^{(i)}, \ldots, w_r^{(i)}\right)} = X_i.$$

Furthermore, $w_{(i)}$ can be chosen as a vertex of the cube $[0,1]^{r+1}$.

Proof. In the binary tree $T(\mathfrak{F})$, a path $\mathcal{P}$ joining the root vertex M_0 and a leaf labeled with x_i has the (graphic) form

$$M_0 \xrightarrow{w_{j_0}^{(i)}} M_{j_1} \xrightarrow{w_{j_1}^{(i)}} \cdots M_{j_k} \xrightarrow{w_{j_k}^{(i)}} x_i, \tag{51}$$

where $w_{j_l}^{(i)} \in \{0,1\}$ stands for the label of the arc connecting a pair of adjacent vertices in the sequence. Note that, by Prop. 11-**ii)**, the inequality $\mathrm{h}(T(\mathfrak{F})) \leq r+1$ is satisfied by the height of $T(\mathfrak{F})$, so that the number $k+1$ of arrows in (51) does not exceed the dimension of $[0,1]^{r+1}$; i.e. $k \leq r$. After this observation, the $(k+1)$-tuple $\left(w_{j_l}^{(i)}\right)_{l=0}^{k}$ can be expanded up to obtain a $(r+1)$-tuple $\left(w_j^{(i)}\right)_{j=0}^{r}$ as follows:

$$w_j^{(i)} = \begin{cases} w_{j_l}^{(i)}, & j = j_l,\ l = 0, \ldots, k \\ \alpha_j, & \text{in other case} \end{cases}, \quad j = 0, \ldots, r, \tag{52}$$

where $\alpha_j \in [0,1]$ is arbitrarily fixed. Thus, for the mean $M^{\left(w_0^{(i)}, \ldots, w_r^{(i)}\right)}$ it is clear that

$$M^{\left(w_0^{(i)}, \ldots, w_r^{(i)}\right)}(x_1, \ldots, x_n) \equiv \mathfrak{F}\left(M_0^{\left(w_0^{(i)}\right)}, \ldots, M_r^{\left(w_r^{(i)}\right)}\right)(x_1, \ldots, x_n) \equiv x_i,$$

and therefore $M^{\left(w_0^{(i)}, \ldots, w_r^{(i)}\right)} = X_i$. Finally, observe that $\left(w_0^{(i)}, \ldots, w_r^{(i)}\right)$ is a vertex of the cube $[0,1]^{r+1}$ when all the numbers α_j in (52) are chosen to be 0 or 1. ■

Lemma 24 *If P is an interior point of the cube $[0,1]^n$, then there exists a continuous mapping $B : [0,1]^{r+1} \to [0,1]^{r+1}$ such that*

i) $B(v) = v$ *for every vertex* v *of the cube* $[0,1]^{r+1}$*; and*

ii) $B(P) = (1/2, \ldots, 1/2)$.

Proof. If $r = 0$ and $b \in (0,1)$, every continuous function $\phi_b : [0,1] \to [0,1]$ passing through the points $(0,0)$, $(b, 1/2)$ and $(1,1)$ can be taken as the function B of the statement. The existence for $r \geq 1$ is easily derived from this: if $P = (b_i)_{i=1}^n$ is an interior point of $[0,1]^n$, then $0 < b_i < 1,\ i = 1, \ldots, r+1$, and the mapping $B : [0,1]^{r+1} \to [0,1]^{r+1}$ defined by

$$B(x_1, \ldots, x_n) = \left(\phi_{b_1}(x_1), \ldots, \phi_{b_n}(x_n)\right),\ x_1, \ldots, x_n \in [0,1],$$

satisfies conditions **i)** and **ii)**. ■

Lemma 25 *Let* $\Delta(v_1, \ldots, v_n)$ *be the* $(n-1)$*-simplex with vertices* $v_1, \ldots, v_n \in \mathbb{R}^{r+1}$ $(n-1 \leq r+1)$*; then, there exists a continuous mapping* $\lambda : \Delta(v_1, \ldots, v_n) \to [0,1]$ *such that* $\lambda(v_i) = 0,\ i = 1, \ldots, n,$ *and* $\lambda\left(\left(\sum_{i=1}^n v_i\right)/n\right) = 1$.

Proof. Every Urysohn function $\lambda : \Delta(v_1, \ldots, v_n) \to [0,1]$ for the sets $\{v_1, \ldots, v_n\}$ and $\left\{\left(\sum_{i=1}^n v_i\right)/n\right\}$ satisfies the properties required. ■

Proof of Theorem 22. After (48) and (50), the proof reduces to define a continuous function $\Phi : \Delta_{n-1} \to [0,1]^{r+1}$ satisfying: **A)** $M^{\Phi(e_i^n)} = X_i$, and **B)** $M^{\Phi(1/n, \ldots, 1/n)} = M$. With this aim in mind, for every $i = 1, \ldots, n$, consider a vertex $w_{(i)}$ of the cube $[0,1]^{r+1}$ such that $M^{w_{(i)}} = X_i$. The existence of $w_{(i)}$ is guaranteed by Lemma 23. The linear map $\mathcal{L} : \mathbb{R}^n \to \mathbb{R}^{r+1}$ whose matrix in the canonical bases of $\mathbb{R}^n$ and $\mathbb{R}^{r+1}$ is given by

$$\left[w_{(1)}^T, \ldots, w_{(n)}^T\right],$$

(where w^T denotes the transpose of w) has the property

$$\mathcal{L}\left(e_i^{(n)}\right) = \left[w_{(1)}^T, \ldots, w_{(n)}^T\right]\left(e_i^{(n)}\right)^T = w_{(i)}^T,\ i = 1, \ldots, n,$$

so that, by linearity,

$$\mathcal{L}(\Delta_{n-1}) = \mathcal{L}\left(\left\{\left(e_1^{(n)}\right)^T, \ldots, \left(e_n^{(n)}\right)^T\right\}^\wedge\right) = \left\{w_{(1)}^T, \ldots, w_{(n)}^T\right\}^\wedge \subseteq [0,1]^{r+1}.$$

Clearly, the lineal function $\mathcal{L}$ depends only on the structure of M. Let us distinguish two cases according to $r+1 = n-1$ or $r+1 > n-1$. In the first of them, $\mathcal{L}(\Delta_{n-1})$ is a $(n-1)$-simplex whose vertices coincide with n vertices of $[0,1]^{n-1}$, so that the point

$$P = \mathcal{L}\left(\frac{1}{n}, \ldots, \frac{1}{n}\right) = \left[w_{(1)}^T, \ldots, w_{(n)}^T\right]\left(\frac{1}{n}\sum_{i=1}^n \left(e_i^{(n)}\right)^T\right) = \frac{1}{n}\sum_{i=1}^n w_{(i)}^T \qquad (53)$$

turns out to be an interior point of $[0,1]^{r+1}$. Define $\Phi = B \circ \mathcal{L}$, where B is the mapping given by Lemma 24 (with $n = r+1$ and P defined by (53)); then $\Phi\left(e_i^{(n)}\right) = w_{(i)},\ i = 1, \ldots, n$, and $\Phi(1/n, \ldots, 1/n) = B(\mathcal{L}(1/n, \ldots, 1/n)) = B(P) = (1/2, \ldots, 1/2)$, and therefore

$$M^{\Phi(e_i^n)} = M^{w_{(i)}} = X_i$$

and

$$M^{\Phi(1/n,\ldots,1/n)} = M^{(1/2,\ldots,1/2)} = M,$$

so that conditions **A)** and **B)** are fulfilled by Φ. Now, let us deal with the case $r+1 > n-1$. Note that in this case, the point P given by (53) is not necessarily an interior point of $[0,1]^{r+1}$. If $\lambda : \mathcal{L}(\Delta_{n-1}) \to [0,1]$ is the continuous mapping given by Lemma 25 for the simplex $\mathcal{L}(\Delta_{n-1})$ and $w_G = (1/2, \ldots, 1/2)$ is the barycenter of $[0,1]^{r+1}$, define $U : \mathcal{L}(\Delta_{n-1}) \to [0,1]^{r+1}$ by

$$U(w) = (1 - \lambda(w))\, w + \lambda(w)\, w_G,\ \ w \in \mathcal{L}(\Delta_{n-1}).$$

Thus, the function $\Phi = U \circ \mathcal{L} : \Delta_{n-1} \to [0,1]^{r+1}$ turns out to be continuous and satisfies

$$\Phi\left(e_i^{(n)}\right) = \left(1 - \lambda\left(w_{(i)}\right)\right) w_{(i)} + \lambda\left(w_{(i)}\right) w_G = w_{(i)},\ \ i = 1, \ldots, n,$$

and

$$\Phi\left(\frac{1}{n}, \ldots, \frac{1}{n}\right) = \left(1 - \lambda\left(\frac{1}{n}\sum_{i=1}^{n} w_{(i)}\right)\right)\left(\frac{1}{n}\sum_{i=1}^{n} w_{(i)}\right) + \lambda\left(\frac{1}{n}\sum_{i=1}^{n} w_{(i)}\right) w_G = w_G.$$

Like in the case $n-1 = r+1$, these equalities prove that conditions **A)** and **B)** are satisfied by $\Phi = G \circ \mathcal{L}$. To finish the proof, observe that both functions $B \circ \mathcal{L}$ and $G \circ \mathcal{L}$ depend only on the structure of the mean M. ∎

Remark 26 *It can be shown that all the* $(r+1)$*-tuples* $\left(w_0^{(i)}, \ldots, w_r^{(i)}\right)$ *for which the equality* $M^{\left(w_0^{(i)},\ldots,w_r^{(i)}\right)} = X_i$ *is true can be obtained from (52) in Lemma 23 provided that the path* $\mathcal{P}$ *is made vary on the paths joining the root vertex* M_0 *and every leaf labeled with* x_i. *On the other hand, note that the mappings* B *of Lemma 24 and* λ *of Lemma 25 can be easily constructed and thus, Theor. 22 is a result of constructive nature. The fact that* B *and* λ *can be taken not only continuous but also very regular maps and, consequently, that a regular function* Φ *could be constructed, does not represent a real improvement of Theor. 22. Indeed, for the continuous and scale invariant weighting procedures for two variables means which are known up to now, the function* $w \mapsto M^{(w)}$ *is not even differentiable.*

Example 27 *Let* $M \in \mathcal{M}_4(I)$ *be a completely reducible mean with formula* $\mathfrak{F}$ *given by*

$$M(x_1, x_2, x_3, x_4) = M_0(x_3, M_1(M_2(x_1, x_4), M_3(x_3, M_4(x_1, x_2)))).$$

The vertices of the cube $[0,1]^5$ *(whose existence is assured by Lemma 23) which are obtained by completing with* 0*'s the labels of the corresponding arcs are listed below:*

$$\left| \begin{array}{lll} X_1 & \leftarrow & (1,0,0,0,0)\,,\,(1,1,0,1,0) \\ X_2 & \leftarrow & (1,1,0,1,1) \\ X_3 & \leftarrow & (0,0,0,0,0)\,,\,(1,1,0,0,0) \\ X_4 & \leftarrow & (1,0,1,0,0) \end{array} \right. .$$

In association with these vectors, four linear functions $\mathcal{L}$ *can be constructed as in the proof of Theor. 22. Their corresponding matrices are given by*

$$\begin{pmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$

Clearly, even others matrices could be constructed by choosing a different expansion with 0*'s or* 1*'s of the vectors of labels. Take, for example, the linear function* $\mathcal{L}$ *whose matrix is the first of the list and consider the* 3*-simplex* $\Delta(v_1,v_2,v_3,v_4)$ *with vertices* $v_1=(1,0,0,0,0)\,,\, v_2=(1,1,0,1,1)\,,\, v_3=(0,0,0,0,0)$ *and* $v_4=(1,0,1,0,0)$*. A Urysohn function* $\lambda:\Delta(v_1,v_2,v_3,v_4)\to[0,1]$ *for the sets* $\{v_1,\dots,v_n\}$ *and* $\{(\sum_{i=1}^n v_i)/n\}$ *is given by* $\lambda(v)=4G_4(w_1,w_2,w_3,w_4)$*, where* (w_1,w_2,w_3,w_4) $(\in\Delta_3)$ *are the barycentric coordinates of the point* $v\in\Delta(v_1,v_2,v_3,v_4)$*; i.e.,* $v=\sum_{i=1}^4 w_i v_i$*. Indeed, the Arithmetic mean-Geometric mean inequality yields*

$$\lambda(v)=4G_4(w_1,w_2,w_3,w_4)\le 4A_4(w_1,w_2,w_3,w_4)=1,\ v\in\Delta(v_1,v_2,v_3,v_4)\,,$$

so that $0\le\lambda(v)\le 1,\ v\in\Delta(v_1,v_2,v_3,v_4)$*, while* $\lambda(v_i)=0,\ i=1,2,3,4,$ *and* $\lambda\left(\left(\sum_{i=1}^4 v_i\right)/4\right)=4G_4(1/4,1/4,1/4,1/4)=1$*. In this way, for every* $(w_1,w_2,w_3,w_4)\in\Delta_3$*, the function* $\Phi=U\circ\mathcal{L}$ *of the proof of Theor. 22 is given, for every* $(w_1,w_2,w_3,w_4)\in\Delta_3$*, by*

$$\begin{aligned} & \Phi(w_1,w_2,w_3,w_4) \\ = \ & (1-4G(w_1,w_2,w_3,w_4))\,\mathcal{L}(w_1,w_2,w_3,w_4)+4G(w_1,w_2,w_3,w_4)\,w_G, \end{aligned}$$

where $\mathcal{L}(w_1,w_2,w_3,w_4)=(w_1+w_2+w_4,w_2,w_4,w_2,w_2)$ *and*

$$w_G=(1/2,1/2,1/2,1/2,1/2)\,.$$

Observe that the first and third matrices in the preceding list both have rank 3 *while* 4 *is the rank of the remaining ones. Presumably, a selection of matrices with lower rank yields more simple functions* Φ*.*

8 Affine functions and S-reducibility

Let $M\in\mathcal{M}_n(I)$ be a mean defined on an interval I. A M-*affine* function f is ([12], [21]) a function $f:I\to I$ satisfying the functional equation

$$f(M(x_1,...,x_n))=M(f(x_1),...,f(x_n))\,,\ x_1,...,x_n\in I. \tag{54}$$

The terminology comes from the fact that the continuous A-affine functions coincide with the standard affine functions $f(x) = mx+h$, $m, h \in \mathbb{R}$ (equivalently, the standard affine functions constitute the general continuous solution to the Jensen functional equation). The family of M-affine functions is denoted by $\mathcal{A}(M;I)$, while the notation $\mathcal{BA}(M;I)$ is reserved for the family of bijective M-affine functions. Note that, whichever be the mean M, the constant functions belong to $\mathcal{A}(M;I)$ while id_I, the identity function on I, is a member of $\mathcal{BA}(M;I)$. In this way, given a mean M, $\mathcal{A}(M;I)$ and $\mathcal{BA}(M;I)$ are always non void and different each other families.

Proposition 28 *Let $M \in \mathcal{M}_n(I)$ be a mean and $[i_1,\ldots,i_k]$ an ordered set of indices with $i_j \in \mathbf{n}$, $j = 1,\ldots,k$. Then, the inclusion*

$$\mathcal{BA}(M;I) \subseteq \mathcal{BA}\left(\mu_{[i_1,\ldots,i_k]};I\right), \tag{55}$$

holds provided that there exists a unique $[i_1,\ldots,i_k]$-lower mean $\mu_{[i_1,\ldots,i_k]}$ of M.

When $M \in \mathcal{C}^{(0)}\mathcal{M}_3(\mathbb{R}^+)$ is the mean given by (23) in Section 3, it can be shown that $\mathcal{BA}(M;I) = \{f : \mathbb{R}^+ \to \mathbb{R}^+ : f(t) = at,\ t \in \mathbb{R}^+;\ a > 0\}$ but $\mathcal{BA}\left(\mu_{[1,2]};I\right) = \mathcal{BA}(G_2;I)$ contains, among many others, the function $g(t) = t^2,\ t \in \mathbb{R}^+$. This shows that the inclusion (55) is generally strict.
Proof. Setting $J = \{i_1,\ldots,i_k\}$, it can be written

$$M\left((x_j)_{[J]}, \mu_{[i_1,\ldots,i_k]}(x_j)_{[J]}\right) = \mu_{[i_1,\ldots,i_k]}(x_j)_{[J]},\ x_j \in I,\ j \in J; \tag{56}$$

whence, for every $f \in \mathcal{BA}(M;I)$,

$$\begin{aligned} M\left((f(x_j))_J, f\left(\mu_{[i_1,\ldots,i_k]}(x_j)_{[J]}\right)\right) &= f\left(M\left((x_j)_{[J]}, \mu_{[i_1,\ldots,i_k]}(x_j)_{[J]}\right)\right) \\ &= f\left(\mu_{[i_1,\ldots,i_k]}(x_j)_{[J]}\right),\ x_j \in I,\ j \in J, \end{aligned}$$

or, taking into account the bijectivity of f,

$$M\left((x_j)_{[J]}, f\left(\mu_{[i_1,\ldots,i_k]}\left(f^{-1}(x_j)\right)_J\right)\right) = f\left(\mu_{[i_1,\ldots,i_k]}\left(f^{-1}(x_j)\right)_J\right),\ x_j \in I,\ j \in J.$$

In this way, $f\left(\mu_{[i_1,\ldots,i_k]}\left(f^{-1}(x_j)\right)_J\right)$ turns out to be a $[i_1,\ldots,i_k]$-lower mean of M, and therefore

$$f\left(\mu_{[i_1,\ldots,i_k]}\left(f^{-1}(x_j)\right)_J\right) = \mu_{[i_1,\ldots,i_k]}(x_j)_{[J]},\ x_j \in I,\ j \in J,$$

by uniqueness. The last equality proves that $f \in \mathcal{BA}\left(\mu_{[i_1,\ldots,i_k]};I\right)$. ■

Now suppose that a mean M is reducible in a class $\mathcal{N}(I)$ and that $M_i \in \mathcal{N}(I)$, $i = 0,\ldots,r$, are its reduced means. If $f \in \mathcal{A}(M_i;I)$ $[f \in \mathcal{BA}(M_i;I)]$, for every $i = 0, \dots, r$, then it is clear that $f \in \mathcal{A}(M; I)$ $[f \in \mathcal{BA}(M; I)]$, so that the inclusions

$$\bigcap_{i=0}^{r} \mathcal{A}(M_i; I) \subseteq \mathcal{A}(M; I) \text{ and } \bigcap_{i=0}^{r} \mathcal{BA}(M_i; I) \subseteq \mathcal{BA}(M; I) \tag{57}$$

hold for every reducible mean M.

Remark 29 *A suitable selection of the complementary means N and $\overline{N}$ in the representation (18) of A_4 shows that (57) are, in general, strict inequalities. For example, taking $N = \max$, for the mean $M_1(x_1, x_2, x_3) = A_2(x_1, \max\{x_2, x_3\})$ it is can be see that*

$$\mathcal{A}(M_1; I) = \{f : \mathbb{R} \to \mathbb{R} : f(t) = mt + h,\ t \in \mathbb{R};\ m,\ h \in \mathbb{R},\ m \geq 0\}.$$

In fact, if the equality

$$f(A_2(x_1, \max\{x_2, x_3\})) = A_2(f(x_1), \max\{f(x_2), f(x_3)\}),\ x_1, x_2, x_3 \in \mathbb{R}, \tag{58}$$

holds for a function $f : \mathbb{R} \to \mathbb{R}$, then, the replacement $x_1 = \max\{x_2, x_3\}$ yields

$$f(\max\{x_2, x_3\}) = \max\{f(x_2), f(x_3)\},\ x_2, x_3 \in \mathbb{R},$$

which shows that f must be an increasing function. On the other hand, setting $x_3 = x_2$ in (58) produces

$$f(A_2(x_1, x_2)) = A_2(f(x_1), f(x_2)),\ x_1, x_2 \in \mathbb{R},$$

which shows that f must be a solution of the Jensen functional equation. As it is well know, an increasing solution f of the Jensen equation in $\mathbb{R}$ has the form $f(t) = mt + h,\ t \in \mathbb{R}$, with $m,\ h \in \mathbb{R}, m \geq 0$. As a consequence, for the representation (18) of A_4, it turns out to be

$$\begin{aligned} & \bigcap_{i=0}^{r} \mathcal{A}(M_i; I) \\ \subseteq\ & \mathcal{A}(M_1; I) = \{f : \mathbb{R} \to \mathbb{R} : f(t) = mt + h,\ t \in \mathbb{R};\ m,\ h \in \mathbb{R},\ m \geq 0\}, \end{aligned}$$

while

$$\mathcal{A}(M; I) = \mathcal{A}(A_4; I) = \{f : \mathbb{R} \to \mathbb{R} : f(t) = \alpha(t) + h,\ t \in \mathbb{R};\ \alpha \text{ additive},\ h \in \mathbb{R}\}.$$

A remarkable property of S-reducible means is stated by the following:

Theorem 30 *Let $M \in \mathcal{C}^{(0)}\mathcal{M}_n(I)$ be a continuous mean possessing the FUS property. Furthermore, assume that M is S-reducible in a class $\mathcal{N}(I)$ and that $M_i,\ i = 0, \dots, r$, are its reduced means; then*

$$\bigcap_{i=0}^{r} \mathcal{BA}(M_i; I) = \mathcal{BA}(M; I).$$

Proof. After the preceding discussion, it will suffice to prove the inclusion

$$\bigcap_{i=0}^{r} \mathcal{BA}(M_i; I) \supseteq \mathcal{BA}(M; I).$$

Now, since M is S-reducible in $\mathcal{N}(I)$, for a certain partition $\{J_k : k = 1, \ldots, r\}$ of $\mathbf{n}$, Theor. 8 enables to write the reduced means M_i in the form $M_0 = M_{J_1 \cdots J_r}$ and $M_i = \mu_{[J_i]}$, $i = 1, \ldots, r$, where $M_{J_1 \cdots J_r} \in \mathcal{N}(I)$ is a specialized of M and $\mu_{[J_i]} \in \mathcal{N}(I)$ is a $[J_i]$-lower mean of M. Now, if $f \in \mathcal{BA}(M; I)$, then it is clear that $f \in \mathcal{BA}(M_{J_1 \cdots J_r}; I) = \mathcal{BA}(M_0; I)$. Let us see that $f \in \mathcal{BA}\left(\mu_{[J_i]}; I\right)$ for every $i = 1, \ldots, r$. In fact, since $M \in \mathcal{C}^{(0)}\mathcal{M}_n(I)$ has the FUS property, Prop. 12 shows that lower means are unique and then, $f \in \mathcal{BA}\left(\mu_{[J_i]}; I\right)$ by Prop. 28. This finishes the proof. ■

Remark 31 *Prop. 28 and Theor. 30 are valid without changes when* $\mathcal{BA}(M_i; I)$ *denote the family of bijective and* continuous *M-affine functions.*

Example 32 *Given a strictly monotonic and continuous function $f : \mathbb{R} \to \mathbb{R}$, the hypotheses of Theor. 30 are fulfilled by the mean*

$$M(x_1, x_2, x_3) = A_2\left((A_2)_f(x_1, x_2), x_3\right), \ x_1, x_2, x_3 \in \mathbb{R}.$$

It is easy to see that a bijective and continuous $(A_2)_f$-affine function ϕ has the form

$$\phi(t) = f^{-1}(\alpha f(t) + \beta), \ t \in \mathbb{R},$$

where α, $\beta \in \mathbb{R}$, $\alpha \neq 0$. When $f = \mathrm{id}_{\mathbb{R}}$, this expression gives all bijective and continuous A_2-affine functions in the form $t \mapsto at + b$, with a, $b \in \mathbb{R}$, $a \neq 0$. Thus, Theor. 30 and Remark 31 show that an affine function $t \mapsto at + b$ (with a, $b \in \mathbb{R}$, $a \neq 0$) is a bijective and continuous M-affine function if and only if there exist α, $\beta \in \mathbb{R}$, $\alpha \neq 0$, such that the linear iterative functional equation

$$\psi(at + b) = \alpha\psi(t) + \beta, \ t \in \mathbb{R},$$

is solved by f.

9 Appendix

To facilitate the reading of the paper, a table of the main notations employed is given below.

$\mathbb{R}^+$	set of positive real numbers
I	real interval
w^T	transpose of w
$\preceq$	product order in I^n

$\nu(F)$ number of effective arguments of the function F
S_n symmetric group of order n

$X_k(x_1,\ldots,x_n)\equiv x_i,\ k=1,\ldots,n,$ coordinate means
$X_n^{(k)},\ k=1,\ldots,n,$ order means
$\min_n,\ \max_n$ n variables extremal means
A_n n variables arithmetic mean
G_n n variables geometric mean
$L_n\ [L_{n,(w_1,\ldots,w_n)}]$ n variables linear mean, [with weights $(w_1,\ldots,w_n)$]
$G_n\ [G_{n,(w_1,\ldots,w_n)}]$ n variables geometric mean [with weights $(w_1,\ldots,w_n)$]
$QL_n\ [QL_{n.(w_1,\ldots,w_n)}]$ n variables quasilinear mean [with weights $(w_1,\ldots,w_n)$]
$FH_n(x_1,\ldots,x_n)$ the n variables mean defined by $\left(\sum_{i=1}^{n} f_i(x_i)\,x_i\right)/\sum_{i=1}^{n} f_i(x_i)$
$CH_n^{(r)}$ r-th weighted n variables counter-harmonic mean
$e_n^{[r]}(x_1,\ldots,x_n)$ the r-th symmetric polynomial function
$\mathfrak{S}_n^{[r]}(x_1,\ldots,x_n)$ the r-th symmetric polynomial mean
$P_\alpha^{(r)}$ the (two variables) weighted power mean with exponent r and weight α
$\mathcal{M}(I),\mathcal{N}(I)$ general classes of means defined on I
$(\mathcal{M})_f(f(I))$ the class of means $\{(M_f):M\in\mathcal{M}(I)\}$
$\mathcal{C}^{(k)}\mathcal{M}_n(I)$ the class of n variables $\mathcal{C}^{(k)}$ means defined on I
$\mathcal{LM}(I)$ the class of linear means defined on I
$\mathcal{LM}_{\mathbb{Q}}(\mathbb{R})$ the class of linear means with a rational weight vector
$\mathcal{QLM}(I)$ the class of quasilinear means defined on I
$\mathcal{QLM}_{2,\mathbb{Q}}(\mathbb{R}^+)$ the class of two variables quasilinear means with rational weights defined on $\mathbb{R}^+$
$\mathfrak{S}\mathcal{M}(\mathbb{R}^+)$ the class of polynomial symmetric means
$(M)_f$ the mean conjugate of M by the homeomorphism f
$\overline{M}$ two variables mean complementary of M
$\mathbf{n}$ the set (of indices) $\{1,\ldots,n\}$
J a subset of $\mathbf{n}$
$[J]$ the increasingly ordered set of indices corresponding to $J\subseteq\mathbf{n}$
$(x_j)_{[J]}$ the k-tuple $(x_{i_1},\ldots,x_{i_k})$ with $[i_1,\ldots,i_k]=[J]$ $(J=\{i_1,\ldots,i_k\}\subseteq\mathbf{n})$
$M_{[i_1,\ldots,i_k]}(x_{i_1},\ldots,x_{i_k};u)$ the specialization of variables of M obtained by setting $x_j=u,\ j\notin\{i_1,\ldots,i_k\}$
$M_J\left((x_j)_{[\boldsymbol{n}\setminus J]};u\right)$ the specialization of variables of M obtained by setting $x_j=u,\ j\in J$
$M_{J_1\cdots J_r}\left((x_j)_{[n\setminus\cup_i J_i]};u_1,\ldots,u_r\right)$ the specialization of variables of M obtained by setting $x_j=u_i,\ j\in J_i,\ i=1,\ldots,r$
$\mu_{[i_1,\ldots,i_k]}$ $[i_1,\ldots,i_k]$-lower mean of a mean M
$\mathcal{FS}(\mathfrak{F})$ the set of functional symbols in a formula $\mathfrak{F}$
$\mathcal{VAR}(\mathfrak{F})$ the set of variables in a formula $\mathfrak{F}$

$T(\mathfrak{F})$ the tree of the formula F
$V(G)$ set of vertices of a graph G
$\mathrm{v}(T)$ order (number of vertices) of a tree T
$\mathrm{a}(T)$ size (number of arcs) of a tree T
$\mathrm{root}(T)$ root vertex of a tree T
$\mathrm{nl}(T)$ the number of leaves of a tree T
$\mathrm{h}(T)$ the height of a tree T (the length of the longest path joining the root with a leaf)
$\mathrm{des}(v)$ the descent of a vertex $v \in V(T)$ (the number of subtrees of v)
$\mathcal{W} : \mathcal{M}_n(I) \times \Delta_{n-1} \to \mathcal{N}(I)$ weighting procedure defined on the class of n variables means $\mathcal{M}_n(I)$
$\mathcal{A}(M; I)$ the family of M-affine functions
$\mathcal{BA}(M; I)$ the family of bijective M-affine functions

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References

[1] G. Aumann, *Aufbau von Mittelwerten mehrerer Argumente II (Analytische Mittelwerte)*, Math. Ann. **111**, (1935), 713-730.

[2] V. I. Arnold, *On functions of three variables*, Dokl. Akad. Nauk SSSR 114 (1957), 679-681; English transl., Amer. Math. Soc. Transl. (2) 38 (1963), 51-54.

[3] V. I. Arnold, *On the representation of continuous functions of three variables by superpositions of continuous functions of two variables*, Mat. Sb. (n.S.) 48 (90):1 (1959), 3-74; Corrections in Mat. Sb. (n.S.) 56 (98):3 (1962), 392; English transl., Amer. Math. Soc. Transl. (2) 28 (1963), 61-147.

[4] V. I. Arnold, *From Hilbert's superposition problem to dynamical systems*, in B. A. Khesin, S. L. Tabachnikov (Eds.), *Arnold: Swimming Against the Tide*, AMS, Providence, 2014, 11-29.

[5] C. Berge, *The Theory of Graphs*, Dover, Mineola, New York, 2001.

[6] L. R. Berrone, *The Aumann functional equation for general weighting procedures*, Aequationes Math., Volume 89, Issue 4, (2015), 1051-1073, DOI 10.1007/s00010-015-0344-4

[7] L. R. Berrone, *Weighting by iteration: the case of Ryll-Nardzewski's iterations*, Results in Math., Volume 71, Issue 3, (2017), 535–567, DOI:10.1007/s00025-016-0586-z

[8] L. R. Berrone, G. E. Sbérgamo, *Aczéls iterations for discontinuous or non strict means*, J. of Analysis **25**, Issue 1, 1-27, https://doi.org/10.1007/s41478-016-0008-3

[9] J. A. Bondy, U. S. R. Murty, *Graph Theory*, Springer, 2008.

[10] V. Brattka, *From Hilbert's 13th Problem to the theory of neural networks: constructive aspects of Kolmogorov's Superposition Theorem*, in E. Charpentier, A. Lesne, N. Nikolski (Eds.), *Kolmogorov's Heritage in Mathematics*, Springer, Berlin-Heildelberg, 2007, 253-280.

[11] P. S. Bullen, *Handbook of Means and Their Inequalities*, Series Mathematics and Its Applications, Kluwer Academic Publisher, 2nd. Ed., 2003.

[12] J. G. Dhombres, *Some recent applications of functional equations*, in "Functional Equations: History, Applications and Theory", J. Aczél (Ed.), D. Reidel, Dordrecht, (1984), 67-91.

[13] A. Horwitz, *Invariant means*, J. Math. Anal. Appl. **270**, (2002), 499-518.

[14] S. Ya. Khavinson, Best Approximations by Linear Superpositions (Approximate Nomographie), Transl. of Math. Monographs Vol. **139**, AMS, Providence, 1997.

[15] T. Kiss, Z. Páles, *Reducible means and reducible inequalities*, Aequationes Math., Volume 91, Issue 3, (2017), 505–525, DOI: 10.1007/s00010-016-0459-2

[16] D. E. Knuth, *The Art of Computer Programming*, Vol. 1, *Fundamental Algorithms*, Addison Wesley, Reading, Massachussets, 3rd. Ed., 1997.

[17] A. Kolmogorov, *The representation of continuous functions of several variables by superpositions of continuous functions of a smaller number of variables*, Doklady Akademii Nauk SSSR, 108: (2) (1956), 179-182.

[18] A. Kolmogorov, *The representation of continuous functions of many variables by superposition of continuous functions of one variable and addition*, Doklady Akademii Nauk SSSR, 114 (1957), 953–956.

[19] S. G. Krantz, H. R. Parks, *The Implicit Function Theorem. History, Theory and Applications*, Birkhäuser, Boston, 2003.

[20] G. G. Lorentz, *Approximation of Functions*, Holt, Rinehart and Winston, New York,1966.

[21] J. Matkowski, *Convex functions with respect to a mean and a characterization of quasi-arithmetic means*, Real Analysis Exchange, Vol. **29** (**1**), 2003/2004, 229-246.

[22] G. Pólya, G. Szegö, *Problems and Theorems in Analysis I*, Springer, Berlin-Heildelberg, 1972. Revised and enlarged translation of "Aufgaben und Lehrsatze aus der Analysis I", 4th ed., Heidelberger Taschenbticher, Band 73,1970.

[23] G. E. Sbérgamo, *Ponderaciones de medias de* 3 *variables*, (unpublished manuscript).

[24] A. G. Vitushkin, *On representation of functions by means of superpositions and related topics: lectures,* Monographie de l'Enseignement Mathématique, Université, L'Enseignement Mathématique, (1978).