M-affine functions composing Sturm-Liouville families

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Abstract

Given a n variables mean M defined on a real interval I, an M-affine function is a solution to the functional equation

$$f\left(M\left(x_1,\ldots,x_n\right)\right)=M\left(f\left(x_1\right),\ldots,f\left(x_n\right)\right),\ x_1,\ldots,x_n\in I. \quad (1)$$

When M is a quasilinear mean, the set of continuous M-affine functions is a Sturm-Lioville family on every compact interval $[a,b]\subseteq I$; i.e., for every $\alpha,\beta\in[a,b]$, there exists an M-affine function f such that $f\left(a\right)=\alpha$ and $f\left(b\right)=\beta$. The validity of the converse statement is explored in this paper and several consequences are derived from this study. New characterizations of quasilinear means and the solution to equation (1) under suitable conditions are among the more important of them.

1 Introduction and preliminaries

Let $I\neq\emptyset$ be a real interval. A *n variables mean M defined on I* is a function $M:I^n\to I$ which is *internal*; i.e., it satisfies the property

$$\min\{x_1,\ldots,x_n\}\leq M(x_1,\ldots,x_n)\leq\max\{x_1,\ldots,x_n\},\ x_1,\ldots,x_n\in I. \quad (2)$$

M is said to be *strict* when the inequalities (2) turn out to be strict provided that the variables x_i are not all equal. Immediate consequences of the (2) are both the equality

$$M\left(x,...,x\right)=x,\quad x\in I,$$

(which show that means are *reflexive* functions) and the fact that a mean M is continuous at every point of the diagonal $\{(x,...,x):x\in I\}$ of I^n. A mean invariant under rearrangements of their arguments is said to be a *symmetric mean, so that* a n variables mean M is symmetric when $M\left(x_{\sigma_1},\ldots,x_{\sigma_n}\right)=$

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$M(x_1, \ldots, x_n)$ for every $\sigma = (\sigma_1, , \ldots, \sigma_n) \in S_n$, the symmetric group of order n. The restriction to a subinterval $J \subseteq I$ of a n variables mean M defined on I is a n mean on J which will be denoted by $M|_J$.

The set of all [continuous] n variables means defined on an interval I will be denoted by $\mathcal{M}_n(I)$ $[\mathcal{CM}_n(I)]$. When a change of variable $f : I \to J$ is performed, a given mean $M \in \mathcal{CM}_n(I)$ becomes another mean $N \in \mathcal{CM}_n(J)$ and, by identifying the so related means M and N, an equivalence relationship is introduced on $\mathcal{CM}_n$. Namely, given $M \in \mathcal{CM}_n(I)$ and $N \in \mathcal{CM}_n(J)$, it is said that M and N are *conjugated means* when there exists a homeomorphism $f : I \to J$ such that the equality

$$f(M(x_1, x_2, ..., x_n)) = N(f(x_1), f(x_2), ..., f(x_n)), \tag{3}$$

holds for every $x_1, x_2, ..., x_n \in I$. This relationship decomposes $\mathcal{CM}_n(I)$ in classes named *conjugacy classes*. For instance, the conjugacy class of the *linear mean*

$$L(x_1, ..., x_n; w) = \sum_{j=1}^{n} w_j x_j,$$

(where the coefficients w_i, the *weights* of the mean, satisfy $w_i > 0$, $i = 1, \ldots, n$; $\sum_{j=1}^{n} w_j = 1$) is given by the class of *quasilinear means*; i.e., the means of the form

$$L_f(x_1, , ..., x_n; w) = f^{-1}\left(\sum_{j=1}^{n} w_j f(x_j)\right), \quad x_1, ..., x_n \in I, \tag{4}$$

where $f : I \to \mathbb{R}$ varies on the set of strictly monotonic and continuous functions. The function f is called the *generator* of the quasilinear mean L_f. In the literature (v.g. [8], pg. 266; [9], pg. 215; [14], pg. 208), nonnegative weights are often admitted in definition (4) but, throughout this paper, quasilinear means are ever strict means. (Note that the annulation of some weights in (4) simply produces a quasilinear mean in fewer variables). Particularly relevant is the equal weights (or symmetric) case: the conjugacy class of the arithmetic mean $A(x_1, x_2, ..., x_n) = \left(\sum_{j=1}^{n} x_j\right)/n$, $x_1, x_2, ..., x_n \in \mathbb{R}$, is given by the means of the form

$$A_f(x_1, ..., x_n) = f^{-1}\left(\frac{1}{n}\sum_{j=1}^{n} f(x_j)\right), \quad x_1, ..., x_n \in I, \tag{5}$$

where, as before, $f : I \to \mathbb{R}$ denotes a generic continuous and strictly monotonic function. These means are named *quasiarithmetic means*. It must be added that, in reference to the means defined by (4), a non uniform terminology was employed. In the recent literature, they are frequently named *weighted quasiarithmetic means,* but in Chap. III of [12], the explicit denomination *mean values with an arbitrary function* was preferred.

A well known result (cf. [12], Sect. 3.2; [1], Theor. 2, pg. 67; [2], Cor. 5, pg. 246; [14], pg. 382 and ff.) establishes that the generator f of a quasilinear mean M defined on I is determined only up to an affine homeomorphism by M: the equality $L_f = L_g$ holds if and only if $g = mf + h$ for certain real constants m and h, $m \neq 0$. On the other hand, the differentiability of the quasilinear mean L_f is strictly related to the differentiability of its generator, as established by the following:

Proposition 1 *A quasilinear mean L_f defined on I is differentiable if and only if its generator f is differentiable in I.*

Proof. The "if" part easily follows from the chain rule. To prove the converse it is enough to consider two variables means $M(x, y)$, for which (4) can be rewritten in the form

$$f(x) = \frac{1}{w_1} \left(f(M(x, y)) - w_2 f(y) \right), \; x, y \in I. \tag{6}$$

Since f is differentiable almost everywhere in I by the Lebesgue´s Theorem, for a given $x \in I$, there exists $y \in I$ such that f is differentiable at the point $M(x, y)$. This fact and the chain rule applied to the right hand side of (6) show that f is differentiable at x. The proposition follows from the arbitrariness of $x \in I$. ■

Given two means $M \in \mathcal{CM}_n(I)$ and $N \in \mathcal{CM}_n(J)$, one can look for functions f satisfying the equality (3). This type of functional equations (or even a more general one in which M and N are continuous functions) have been studied since the first decades of the past century (for $n = 2$ see [1], pgs. 62, 79, 145, and the corresponding references; [7], pg. 239 and ff.; [10]; [4], [11]), but the problem of finding conditions on the means M and N in order that functional equation (3) admits nontrivial (non constant) solutions has not been fully solved. When $M = N$, (3) takes the form

$$f(M(x_1, ..., x_n)) = M(f(x_1), ..., f(x_n)), \quad x_1, ..., x_n \in I, \tag{7}$$

a functional equation which can be seen as a generalization of the Jensen equation

$$f\left(\frac{x_1 + \cdots + x_n}{n}\right) = \frac{f(x_1) + \cdots + f(x_n)}{n}, \; x_1, ..., x_n \in I, \tag{8}$$

and whose solutions are, by this reason, named *M-affine functions* ([10], [17]). Indeed, for every $n \geq 2$ and every real interval I, the A-affine functions have the form $f(x) = \alpha(x) + h$, where $\alpha : \mathbb{R} \to \mathbb{R}$ is an additive function and b is a real constant, but the continuous A-affine functions reduce to the set of affine functions $f(x) = mx + h$, $m, h \in \mathbb{R}$. Along this paper, the general solution to (7); i.e., the set of M-affine functions, will be denoted by $\mathcal{A}(M; I)$, while the notation $\mathcal{AC}(M; I)$ is reserved for the continuous M-affine functions. Clearly, $\mathcal{A}(M; I)$ and $\mathcal{AC}(M; I)$ are semigroups under "$\circ$", the usual composition of functions.

The set constituted by the affine functions on an interval $I \subseteq \mathbb{R}$ (i.e., the set $\mathcal{AC}(A; I)$,) will be denoted by $\mathrm{Aff}(I)$; i.e.,

$$\mathrm{Aff}(I) = \{f(t) = mt + h : m,\ h \in \mathbb{R},\ mI + h \subseteq I\}.$$

For instance, $\mathrm{Aff}(\mathbb{R}^+) = \left\{f(t) = mt + h : m \in \mathbb{R}_0^+,\ h \in \mathbb{R}^+\right\}$ and $\mathrm{Aff}([0,1]) = \{f(t) = mt + h : m \in [-1,1],\ h,\ m + h \in [0,1]\}$. The set of parameters $(m,h) \in \mathbb{R}^2$ such that the affine function $t \mapsto mt+h$ is a member of $\mathrm{Aff}(I)$ will be denoted by $AFF(I)$; i.e.,

$$AFF(I) = \left\{(m,h) \in \mathbb{R}^2 : mI + h \subseteq I\right\}.$$

In this way, $AFF(\mathbb{R}^+) = \mathbb{R}_0^+ \times \mathbb{R}^+$ and $AFF([0,1]) = \{(0,0),(1,0),(0,1),(-1,1)\}^\wedge$, where $E^\wedge$ denotes the convex hull of the set E. Clearly, $\mathrm{Aff}(I)$ and $AFF(I)$ are convex sets whichever be the interval I. Further properties of these sets are to be considered in Section 5, where it will be appretiated that the visual representation $AFF(I)$ of $\mathrm{Aff}(I)$ can clarify some developments.

This paper deals with a sort of inverse problem: to deduce properties of the means M from the knowledge of some properties of $\mathcal{A}(M; I)$ or $\mathcal{AC}(M; I)$. For example, if for a strict mean M defined on $\mathbb{R}$, the functions $f(x) = mx + h$, $m, h \in \mathbb{R}$, were solutions to the equation

$$f(M(x,y)) = M(f(x), f(y)),\ x, y \in \mathbb{R}, \tag{9}$$

or, in other terms, if the inclusion $Aff(\mathbb{R}) \subseteq \mathcal{A}(M; I)$ holds, then necessarily $M(x,y) = M(0(y-x)+x, 1(y-x)+x) = M(0,1)(y-x)+x$ is a linear mean (cf. [1], Theor. 1, pg. 234). Note that no hypothesis was made on the regularity of M; furthermore, note that the same result is true whenever M is strict mean defined on an interval I provided that the inclusion $Aff(I) \subseteq \mathcal{A}(M; I)$ holds, a fact that quickly follows from the equality

$$M(x,y) = \frac{M(x_0,y_0) - x_0}{y_0 - x_0}(y-x) + x,\ x, y \in I,$$

where $x_0,\ y_0 \in I,\ x_0 < y_0$. Unfortunately, this is no longer true when the number of variables is greater than 2 (cf. [1], pg. 237): given the three variables linear means L_1, L_2 and L_3, with $L_i \neq L_j$ at least for a pair $i,\ j,\ i \neq j$, the (continuous) strict mean $M(x,y,z)$ defined on $\mathbb{R}$ by $M(x,x,x) = x,\ x \in \mathbb{R}$, by

$$M(x,x,y) = L_3(x,x,y),\ M(x,y,x) = L_2(x,y,x),\ M(y,x,x) = L_1(y,x,x), \tag{10}$$

when $x,\ y \in \mathbb{R}$, $x \neq y$, and by

$$\begin{aligned} &M(x,y,z) \\ = \ &\frac{(x-y)^2 e^{-\left(\frac{x-y}{x-z}\right)^2} L_1(x,y,z) + (y-z)^2 e^{-\left(\frac{y-z}{y-x}\right)^2} L_2(x,y,z) + (z-x)^2 e^{-\left(\frac{z-x}{z-y}\right)^2} L_3(x,y,z)}{(x-y)^2 e^{-\left(\frac{x-y}{x-z}\right)^2} + (y-z)^2 e^{-\left(\frac{y-z}{y-x}\right)^2} + (z-x)^2 e^{-\left(\frac{z-x}{z-y}\right)^2}} \end{aligned} \tag{11}$$

when $x, y, z \in \mathbb{R},\ x \neq y,\ y \neq z,\ z \neq x$, serves as a counterexample. However, it can be proved the following:

Proposition 2 *Let I be a real interval with* $\operatorname{int}(I) \neq \emptyset$ *and* $M \in \mathcal{CM}_n(I)$ *be a strict mean such that* $Aff(I) \subseteq \mathcal{A}(M; I)$. *If M is differentiable at a point of the diagonal of* $(\operatorname{int}(I))^n$, *then M is a linear mean.*

Note that a mean fulfilling the hypotheses of the proposition is not only continuous but also differentiable at every point of the diagonal of I^n.

Proof. First of all observe that, whichever be the interval I, $\{0\} \times I \subseteq AFF(I)$ and $(m, h_0) \in AFF(I)$ provided that $h_0 \in \operatorname{int}(I)$ and $m > 0$ is small enough. Thus, if $(t_0, ..., t_0) \in (\operatorname{int}(I))^n$ is the point at which M is differentiable and $x_1, ..., x_n \in I$ are fixed, the map

$$u \mapsto M(ux_1 + t_0, ..., ux_n + t_0),$$

(which is, by the former observation, defined and continuous on an interval of the form $[0, \delta)$ $(\delta > 0)$,) has a right-hand derivative D^+M at $u = 0$ given by

$$D^+M(ux_1 + t_0, ..., ux_n + t_0)\big|_{u=0} = \sum_{i=1}^{n} \frac{\partial M}{\partial x_i}(t_0, ..., t_0)\, x_i.$$

Now, by the assumptions it can be written

$$M(ux_1 + t_0, ..., ux_n + t_0) = uM(x_1, ..., x_n) + t_0, \ u \in [0, \delta),$$

and taking (right-hand) derivatives at $u = 0$ in this equality,

$$\sum_{i=1}^{n} \frac{\partial M}{\partial x_i}(t_0, ..., t_0)\, x_i = M(x_1, ..., x_n). \tag{12}$$

Differentiating the identity $M(x, ..., x) \equiv x$ at $x = t_0$ yields

$$\sum_{i=1}^{n} \frac{\partial M}{\partial x_i}(t_0, ..., t_0) = 1; \tag{13}$$

and, from (12) and the assumed strictness of M, it is derived

$$0 < M(0, ..., 1, ..., 0) = \frac{\partial M}{\partial x_i}(t_0, ..., t_0)_i < 1. \tag{14}$$

The equality (12) together (13) and (14) show that M is a linear mean. ■

In a noteworthy result by J. Matkowski (cf. Theorem 3 in [17]), the two variables quasilinear means are characterized as those strict means M such that their corresponding semigroup $\mathcal{A}(M; I)$ are, in a certain sense, extense. Let us quote this theorem as follows:

Theorem 3 (J. Matkowski, 2003) *Let $I \subseteq \mathbb{R}$ be an* open *interval and M be a* two variables strict mean *defined on I. Suppose that $\mathcal{A}(M; I)$ contains a continuous (multiplicative) iteration group $\{f^t : t > 0\}$ with generator γ. Furthermore, suppose that the function $h : \mathbb{R}^+ \to \mathbb{R}^+$ defined by* $h(u) = \gamma\left(M\left(\gamma^{-1}(u), \gamma^{-1}(1)\right)\right)$,

$u>0$ *is twice differentiable, and that* $0\neq h'(1)\neq 1$*. If there exists an* M*-affine function,* continuous at a point*, that is neither constant nor an element of the iteration group* $\{f^t : t>0\}$*, then*

$$M(x,y)=\phi^{-1}(w\phi(x)+(1-w)\phi(y)), \quad x,y\in I,$$

for some continuous and strictly monotonic function $\phi : I\to\mathbb{R}^+$ *and* $w=h'(1)$*.*

In this paper, a class of functions $\mathcal{F}$ will be considered an extense one whenever there exists a function $f\in\mathcal{F}$ passing through every pair of points. More precisely, if $I=[a,b]$, $(a,b\in\mathbb{R},\ a<b)$ and $J\neq\emptyset$ are two real intervals and $\mathcal{F}\subseteq J^I=\{f|\, f: I\to J\}$, $\mathcal{F}\neq\emptyset$, is a family of functions, then let us say that $\mathcal{F}$ is a *Sturm-Liouville family* when, for every $\alpha,\beta\in J$, there exists $f\in\mathcal{F}$ such that

$$f(a)=\alpha \text{ and } \ f(b)=\beta.$$

The terminology comes from the denomination of the boundary conditions in the theory of boundary value problems for second order linear differential equations. When $I=[a,b]$ and $J=\mathbb{R}$, the family constituted by all monotonic functions $f:[a,b]\to\mathbb{R}$ is a Sturm-Liouville family. If J reduces to the single point $\{c\}$, then the unitary set $\mathcal{F}=\{f\equiv c\}$ is also a Sturm-Liouville family (whichever be the interval I). It should be observed that the property of being a Sturm-Liouville family is invariant under congugacy: given a homeomorphism $\phi:[a,b]\to\mathbb{R}$, a family $\mathcal{F}$ is a Sturm-Liouville family of functions defined on $[a,b]$ if and only $\phi\circ\mathcal{F}\circ\phi^{-1}=\left\{\phi\circ f\circ\phi^{-1} : f\in\mathcal{F}\right\}$ is a Sturm-Liouville family of functions defined on $\phi([a,b])$.

A remarkable example of a Sturm-Liouville family is furnished by the semigroup $\mathcal{AC}(L_\phi;J)$ corresponding to the n variables quasilinear mean L_ϕ with generator $\phi:J\to\mathbb{R}$ and weights w_i, $i=1,\ldots,n$. In fact, observe that equation (7) takes, in this case, the form

$$f\left(\phi^{-1}\left(\sum_{i=1}^{n} w_i\phi(x_i)\right)\right)=\phi^{-1}\left(\sum_{i=1}^{n} w_i\phi(f(x_i))\right),\ x_i\in J,\ i=1,\ldots,n, \quad (15)$$

for a unknown function $f:J\to J$. Replacing $g=\phi\circ f\circ\phi^{-1}:\phi(J)\to\phi(J)$ in (**??**), reduce it to the equation

$$g\left(\sum_{i=1}^{n} w_i t_i\right)=\sum_{i=1}^{n} w_i g(t_i),\ t_i\in\phi(J),\ i=1,\ldots,n,$$

whose general continuous solution is given by

$$g(t)=mt+h,\ t\in\phi(J), \quad (16)$$

where m, h are real constants such that $g(t)\in\phi(J)$, $t\in\phi(J)$, (this is a simple consequence of [1], Theor. 2, pg. 67 or also [14], pg. 382 and ff.) and hence, a solution f to equation (15) must have the form

$$f(x)=\phi^{-1}(m\phi(x)+h),\ x\in J, \quad (17)$$

where $(m,h) \in AFF(\phi(J))$. A replacement of (17) in (**??**) shows that (17) really solves this equation, so that (cf. [2], Chap. 15, Prop. 6, for the case $n=2$ and M symmetric)

$$\mathcal{AC}(L_\phi; J) = \left\{f : f(x) = \phi^{-1}(m\phi(x) + h),\ x \in J,\ (m,h) \in AFF(\phi(J))\right\}.$$

A straightforward consequence of this characterization of $\mathcal{AC}(L_\phi; J)$ is the following:

Proposition 4 *Let J be a real interval and L_ϕ be a quasilinear mean with generator $\phi : J \to \mathbb{R}$; then, for every compact interval $[a,b] \subseteq J$, the family of restrictions $\left\{ f|_{[a,b]} : f \in \mathcal{AC}(L_\phi; J)\right\}$ ($= \mathcal{AC}(L_\phi; [a,b])$) is a Sturm-Liouville family.*

Proof. It is sufficient to observe that, for any pair of numbers $\alpha, \beta \in J$, the system of equations

$$\begin{cases} \phi^{-1}(m\phi(a) + h) = \alpha \\ \phi^{-1}(m\phi(b) + h) = \beta \end{cases},$$

has the solution

$$m = \frac{\phi(\beta) - \phi(\alpha)}{\phi(b) - \phi(a)},\ h = \phi(\alpha) - \frac{\phi(\beta) - \phi(\alpha)}{\phi(b) - \phi(a)}\phi(a), \tag{18}$$

and that the pair (m,h) given by (18) really is a member of $AFF(\phi([a,b]))$. ■

Remark 5 *Defining the increasing homeomorphism $\psi : [a,b] \to [0,1]$ by*

$$\psi(x) = \frac{\phi(x) - \phi(a)}{\phi(b) - \phi(a)},\ x \in [a,b], \tag{19}$$

it turns out to be

$$\begin{aligned} & \phi^{-1}\left(\frac{\phi(\beta) - \phi(\alpha)}{\phi(b) - \phi(a)}(\phi(x) - \phi(a)) + \phi(\alpha)\right) \\ = \quad & \psi^{-1}\left(\frac{\frac{\phi(\beta)-\phi(\alpha)}{\phi(b)-\phi(a)}(\phi(x) - \phi(a)) + \phi(\alpha) - \phi(a)}{\phi(b) - \phi(a)}\right) \\ = \quad & \psi^{-1}\left(\frac{\phi(\beta) - \phi(\alpha)}{\phi(b) - \phi(a)}\frac{\phi(x) - \phi(a)}{\phi(b) - \phi(a)} + \frac{\phi(\alpha) - \phi(a)}{\phi(b) - \phi(a)}\right) \\ = \quad & \psi^{-1}((\psi(\beta) - \psi(\alpha))\psi(x) + \psi(\alpha)), \end{aligned}$$

so that, from the proof of Prop. 4 it is seen that

$$\mathcal{AC}(L_\phi; [a,b]) = \left\{\psi^{-1}((\psi(\beta) - \psi(\alpha))\psi(\cdot) + \psi(\alpha)) : \alpha, \beta \in [a,b]\right\},$$

where $\psi : [a,b] \to [0,1]$ is given by (19).

Now, assume that M is a strict and continuous mean defined on J such that, for every compact interval $[a,b] \subseteq J$, the family $\mathcal{AC}\left(M|_{[a,b]};[a,b]\right)$ is a Sturm-Liouville family. ¿Must be M a quasilinear mean? This paper is addressed to answer to this question. Concretely, along Sections 2 and 3, a proof of the following result will be developed.

Theorem 6 *Let $M \in \mathcal{CM}_n([a,b])$ be a strict and continuous mean defined on a* compact *interval $[a,b]$. If $\mathcal{AC}(M;[a,b])$ is a Sturm-Liouville family, then there exist a unique increasing homeomorphism $\psi:[a,b] \to [0,1]$ from $[a,b]$ onto $[0,1]$ such that, for the conjugated mean M_ψ defined on $[0,1]$ by*

$$M_\psi(x_1,\ldots,x_n) = \psi\left(M\left(\psi^{-1}(x_1),\ldots,\psi^{-1}(x_n)\right)\right), \tag{20}$$

the semigroup $\mathcal{AC}(M_\psi;[0,1])$ coincides with $\mathrm{Aff}([0,1])$.

In other words, when for a certain strict continuous mean M, $\mathcal{AC}(M;[a,b])$ is a Sturm-Liouville family, then, there exists a unique increasing homeomorphism $\psi:[a,b] \to [0,1]$ from $[a,b]$ onto $[0,1]$ such that every $f \in \mathcal{AC}(M;[a,b])$ can be represented in the form

$$f(t) = \psi^{-1}(m\psi(t)+h),\ t \in [a,b],$$

where $(m,h) \in AFF([0,1])$. Remarkably, when $n=2$, Theor. 6 implies that M_ψ is a linear mean, so that M turns out to be a quasilinear mean, so that the converse of Prop. 4 turns out to be true in this case. Now, ¿what if the interval I is not compact? In Section 4, the following result will be shown.

Theorem 7 *Let $M \in \mathcal{CM}_2(I)$ be a two variables mean defined on a real interval I. If $\{[a_k,b_k] : k \in \mathbb{N}\}$ is a sequence of nested ($[a_k,b_k] \subseteq [a_{k+1},b_{k+1}]$, $k \in \mathbb{N}$) and exhaustive ($\bigcup_k [a_k,b_k] = I$) compact subintervals of I such that $\mathcal{AC}\left(M|_{[a_k,b_k]};[a_k,b_k]\right)$ is a Sturm-Liouville family for every $k \in \mathbb{N}$, then M is a quasilinear mean.*

Other consequences of Theor. 6 for two variables means are explained in Section 4. Among them, a special mention deserves the characterization of two variables quasilinear means through the theory of bases, which is now presented by setting aside the technical differentiability hypotheses made in [3]. Section 5 is devoted to study the case of n variables means. The following result, which can be considered as an ample generalization of Prop. 2, will be shown there.

Theorem 8 *Let $M \in \mathcal{CM}_n(I)$ be a n variables, strict and continuous mean defined on a real interval I. If $\{[a_k,b_k] : k \in \mathbb{N}\}$ is a sequence of nested and exhaustive compact subintervals of I such that $\mathcal{AC}(M;[a_k,b_k])$ is a Sturm-Liouville family for every $k \in \mathbb{N}$, then there exists a strictly increasing and continuous function $\phi : I \to \mathbb{R}$ such that $\mathcal{AC}(M_\phi;\phi(I)) = Aff(\phi(I))$. Furthermore, M is a quasilinear mean in I provided that M is differentiable.*

The final Section 6 serves to gather together some examples and remarks. Particularly, the use of the above results in solving the functional equation (7) will be illustrated there.

2 Continuous M-affine functions constituting a Sturm-Liouville family

A useful tool in the study of continuous M-affine functions are the *Aczel dyadic iterations* of a two variables mean M. Concretely, denoting the set of dyadic numbers of the interval $[0,1]$ by $\mathbf{D}([0,1])$, a family of means $\left\{M^{(d)} : d \in \mathbf{D}([0,1])\right\}$ is defined on I^2 as follows: fix $x, y \in I$ and set

$$M^{(0)}(x,y) = x, \quad M^{(1)}(x,y) = y;$$

then, assuming that $M^{\frac{j}{2^n}}(x,y)$ is known for $n \geq 0$ and every $0 \leq j \leq 2^n$, define

$$M^{\left(\frac{k}{2^{n+1}}\right)}(x,y) = \begin{cases} M^{\left(\frac{h}{2^n}\right)}(x,y), & k = 2h,\ 0 \leq h \leq 2^n \\ M\left(M^{\left(\frac{h}{2^n}\right)}(x,y), M^{\left(\frac{h+1}{2^n}\right)}(x,y)\right), & k = 2h+1,\ 0 \leq h \leq 2^n - 1 \end{cases}.$$

In the following result, whose proof can be found in [3] (see also [4] and [5]), the main properties of the Aczel dyadic iterations are established.

Theorem 9 *a) Let I and J two real intervals and $M \in \mathcal{M}_2(I)$ and $N \in \mathcal{M}_2(J)$. If the equality*

$$f(M(x,y)) = N(f(x), f(y)),\ x, y \in I,$$

holds for any $x, y \in I$, then, for every $d \in \mathbf{D}([0,1])$,

$$f\left(M^{(d)}(x,y)\right) = N^{(d)}(f(x), f(y)).$$

b) If M is a strict continuous *mean, the map $\mathbf{D}([0,1]) \ni d \mapsto M^{(d)}(x,y)$ can be continuously extended to the interval $[0,1]$. Moreover, the extension $\phi_{(x,y)}(\delta) = M^{(\delta)}(x,y)$ is a homeomorphism from $[0,1]$ onto $[\min(x,y), \max(x,y)]$ which turns out to be increasing when $x < y$ and decreasing when $x > y$.*

c) For each $\delta \in (0,1)$, $M^{(\delta)}$ is a continuous strict mean defined on I (while $M^{(0)}(x,y) \equiv x$ and $M^{(1)}(x,y) \equiv y$).

For example, if $M(x,y) = A_f(x,y)$ is a quasiarithmetic mean with generator f, then it is inductively shown that

$$A_f^{(d)}(x,y) = f^{-1}((1-d)f(x) + df(y)),\ x, y \in I,$$

for every $d \in \mathbf{D}([0,1])$, whence it is easily deduced that

$$A_f^{(\delta)}(x,y) = f^{-1}((1-\delta)f(x) + \delta f(y)),\ x, y \in I, \tag{21}$$

for every $\delta \in [0,1]$.

As a first application of the Aczel dyadic iterations, let us prove the following:

Proposition 10 *Let $M \in \mathcal{CM}_n(J)$ be a continuous and strict mean defined on a real interval J and consider a compact subinterval $[a,b] \subseteq J$. Furthermore, for given $\alpha, \beta \in [a,b]$, let f be a continuous M-affine function such that $f(a) = \alpha$ and $f(b) = \beta$. Then $f|_{[a,b]}$ is a homeomorphism from $[a,b]$ onto $[\min(\alpha,\beta), \max(\alpha,\beta)]$.*

Proof. In the first place, let us consider the case $n = 2$. From part a) of Theor. 9 and the continuity of f, it can be written in this case

$$f\left(M^{(\delta)}(x,y)\right) = M^{(\delta)}(f(x), f(y)), \ x, y \in [a,b], \ \delta \in [0,1],$$

whence, setting $x = a$, $y = b$, and using the notation introduced in Theor. 9-b), it is derived

$$f\left(\phi_{(a,b)}(\delta)\right) = \phi_{(\alpha,\beta)}(\delta), \ \delta \in [0,1],$$

or, by substituting $u = \phi_{(a,b)}(\delta)$,

$$f(u) = \left(\phi_{(\alpha,\beta)} \circ \phi_{(a,b)}^{-1}\right)(u), \ u \in [a,b]. \tag{22}$$

This expression and Theor. 9- b) shows that $f|_{[a,b]}$ is a homeomorphism from $[a,b]$ onto $[\min(\alpha,\beta), \max(\alpha,\beta)]$, as stated. Now, if $M \in \mathcal{CM}_n(J)$, let define a two variables mean N by

$$N(x,y) = M(x, y, ..., y), \ x, y \in J. \tag{23}$$

Clearly N is a strict and continuous mean, and if f is M-affine, then it is also N-affine. Hence, the general case follows from the case $n = 2$. This completes the proof. ∎

Like in the previous proposition, consider a continuous and strict mean $M \in \mathcal{CM}_n(J)$ and suppose, for a given compact subinterval $[a,b] \subseteq J$, that $\mathcal{AC}\left(M|_{[a,b]}; [a,b]\right)$ is a Sturm-Liouville family. By definition, for every $\alpha, \beta \in [a,b]$ there exists $f \in \mathcal{AC}\left(M|_{[a,b]}; [a,b]\right)$ such that $f(a) = \alpha$ and $f(b) = \beta$. It is asserted that this f is unique. In fact, if $g \in \mathcal{AC}(M; [a,b])$ was another M-affine function satisfying $g(a) = \alpha$ and $g(b) = \beta$, then there would exist $t_0 \in (a,b)$ such that $f(t_0) \neq g(t_0)$ and, by continuity, $f(t) \neq f_{\alpha,\beta}(t)$ for every t in a maximal open neighborhood (c,d) of t_0. Since the equalities $f(c) = g(c)$ and $f(d) = g(d)$ hold by the maximality of (c,d), it can be written

$$\begin{aligned} f(M(c,c,...,c,d)) &= M(f(c), f(c), ..., f(c), f(d)) \\ &= M(g(c), g(c), ..., g(c), g(d)) \\ &= g(M(c,c,...,c,d)), \end{aligned}$$

which, taking into account that $M(c,c,...,c,d) \in (c,d)$ by the strict internality of M, is a contradiction. This proves the above assertion and justifies the use of the notation $f_{\alpha,\beta}$ for the unique $f \in \mathcal{AC}(M; [a,b])$ satisfying $f(a) = \alpha$ and

$f(b) = \beta$. Now, let us see that $\alpha \mapsto f_{\alpha,\beta}(t)$ is a continuous and monotonic function on $[a,b]$. Indeed, in view of (22) in the proof of Prop. 10, it can be written

$$f_{\alpha,\beta}(t) = \phi_{(\alpha,\beta)}\left(\phi_{(a,b)}^{-1}(t)\right) = M^{\phi_{(a,b)}^{-1}(t)}(\alpha,\beta),$$

so that the continuity of $\alpha \mapsto f_{\alpha,\beta}(t)$ follows from Theor. 9, c). In order to prove the monotonicity, let us consider $\alpha, \alpha' \in [a,b]$, $\alpha < \alpha'$, and suppose that there exists $t_0 \in (a,b)$ such that $f_{\alpha,\beta}(t_0) > f_{\alpha',\beta}(t_0)$. In view of $f_{\alpha,\beta}(a) = \alpha < \alpha' = f_{\alpha',\beta}(a)$ and of $f_{\alpha,\beta}$ and $f_{\alpha',\beta}$ are both continuous functions, there exists $c \in (a,t_0)$ such that $f_{\alpha,\beta}(c) = f_{\alpha',\beta}(c)$ and therefore, an argument like that used above to prove the uniqueness of $f_{\alpha,\beta}$ shows that $f_{\alpha,\beta}(t) = f_{\alpha',\beta}(t)$, $t \in [c,b]$. Since $t_0 > c$, this is in contradiction with the former assumption and thus $f_{\alpha,\beta}(t) \le f_{\alpha',\beta}(t)$, $t \in [a,b)$. Since $f_{\alpha,\beta}(b) = \beta = f_{\alpha',\beta}(b)$, $\alpha \mapsto f_{\alpha,\beta}(t)$ is monotonic for every $t \in [a,b]$.

Summarizing the above discussion, the following result can be established.

Proposition 11 *Let $M \in \mathcal{CM}_n(J)$ be a continuous and strict mean and suppose, for a given compact subinterval $[a,b] \subseteq J$, that $\mathcal{AC}\left(M|_{[a,b]};[a,b]\right)$ is a Sturm-Liouville family. Then, there exists a unique function $f_{\alpha,\beta} \in \mathcal{AC}\left(M|_{[a,b]};[a,b]\right)$ such that $f_{\alpha,\beta}(a) = \alpha$ and $f_{\alpha,\beta}(b) = \beta$. Furthermore, the functions $\alpha \mapsto f_{\alpha,\beta}(t)$ and $\beta \mapsto f_{\alpha,\beta}(t)$ turn out to be monotonic and continuous on $[a,b]$.*

Proof. After the previous discussion, it remains prove only that $\beta \mapsto f_{\alpha,\beta}(t)$ is monotonic and continuous on $[a,b]$. This is a immediate consequence of the representation

$$f_{\alpha,\beta}(t) = f_{\beta,\alpha}(f_{b,a}(t))$$

and the corresponding properties of $\alpha \mapsto f_{\alpha,\beta}(t)$. ■

Under the hypotheses of Prop. 11 and remembering that $\mathcal{AC}\left(M|_{[a,b]};[a,b]\right)$ is a semigroup, it turns out to be that, for every α_i, $\beta_i \in [a,b]$, $i = 1,2$, there exist a unique pair α, $\beta \in [a,b]$ such that

$$f_{\alpha_1,\beta_1} \circ f_{\alpha_2,\beta_2} = f_{\alpha,\beta}.$$

Since

$$\alpha = f_{\alpha,\beta}(a) = \left(f_{\alpha_1,\beta_1} \circ f_{\alpha_2,\beta_2}\right)(a) = f_{\alpha_1,\beta_1}\left(f_{\alpha_2,\beta_2}(a)\right) = f_{\alpha_1,\beta_1}(\alpha_2)$$

and, similarly,

$$\beta = f_{\alpha,\beta}(b) = f_{\alpha_1,\beta_1}(\beta_2),$$

for every α_i, $\beta_i \in [a,b]$, $i = 1,2$, it can be written

$$f_{\alpha_1,\beta_1} \circ f_{\alpha_2,\beta_2} = f_{f_{\alpha_1,\beta_1}(\alpha_2), f_{\alpha_1,\beta_1}(\beta_2)}. \tag{24}$$

Now, consider the function $F : [a,b]^3 \to [a,b]$ defined by

$$F(t,\alpha,\beta) = f_{\alpha,\beta}(t). \tag{25}$$

Proposition 12 *Let $M \in \mathcal{CM}_n(J)$ be a strict and continuous mean and suppose, for a given compact subinterval $[a,b] \subseteq J$, that $\mathcal{AC}\left(M|_{[a,b]}; [a,b]\right)$ is a Sturm-Liouville family. If $f_{\alpha,\beta} \in \mathcal{AC}\left(M|_{[a,b]}; [a,b]\right)$ is the unique M-affine function satisfying $f_{\alpha,\beta}(a) = \alpha$ and $f_{\alpha,\beta}(b) = \beta$, then the function F defined by (25) is a solution to the composite functional equation*

$$F\left(F\left(t,\alpha_1,\beta_1\right),\alpha_2,\beta_2\right) = F\left(t, F\left(\alpha_1,\alpha_2,\beta_2\right), F\left(\beta_1,\alpha_2,\beta_2\right)\right), \ t,\alpha_i,\beta_i \in [a,b], \ (i=1,2), \tag{26}$$

in the class constituted by the functions with the following properties:

i) *F is continuous;*

ii) *$F(t,\alpha,\beta)$ is monotonic with respect to each variable, and strictly monotonic with respect to the variable t provided that $\alpha \neq \beta$ ($t \mapsto F(t,\alpha,\beta)$ is strictly increasing when $\alpha < \beta$ and strictly decreasing when $\alpha > \beta$);*

iii) *$F(a,\alpha,\beta) = \alpha$ and $F(b,\alpha,\beta) = \beta$.*

Proof. The functional equation (26) is a rewriting of (24) using (25). **ii)** and **iii)** are an immediate consequence of (25) and Props. 10 and 11 (the strict monotonicity of $t \mapsto F(t,\alpha,\beta)$ follows from the representation (22) and Theor. 9). In regards to **i),** observe that the function $F : [a,b]^3 \to [a,b]$ is separately continuous and monotonic in each variable and therefore, F is continuous. In fact, the argument employed by R. L. Kruse and J. J. Deely in [13], Prop. 2, to prove the joint continuity on an given open set can be easily extended to prove joint continuity on the cube $[a,b]^3$. ■

After this result, Prop. 4 and Remark 5 imply that

$$F(t,\alpha,\beta) = \psi^{-1}\left(\left(\psi(\beta) - \psi(\alpha)\right)\psi(t) + \psi(\alpha)\right),$$

where $\psi : [a,b] \to [0,1]$ is an increasing homeomorphism, must be a solution to the functional equation (26) in the class of functions satisfying the properties **i), ii)** and **iii)**. A direct checking of this fact is an easy task. As it will be seen in the next section, this expression really provides the general solution to (26).

3 The functional equation (26)

The purpose of this section is to prove the following:

Theorem 13 *The general solution to the functional equation (26) in the class of functions fulfilling the conditions **i)**, **ii)** and **iii)** is given by*

$$F(t,\alpha,\beta) = \psi^{-1}\left(\left(\psi(\beta) - \psi(\alpha)\right)\psi(t) + \psi(\alpha)\right), \tag{27}$$

where $\psi : [a,b] \to [0,1]$ is an increasing homeomorphism. ψ is the unique increasing homeomorphism satisfying (27).

A proof of Theor. 6 will easily follow from this result.

As a first observation note that, in view of $f_{a,b}$ must be an increasing homeomorphism onto $[a,b]$ and $f_{a,b}^2 = f_{a,b} \circ f_{a,b} = f_{a,b}$ (idempotency), it turns out to be $f_{a,b} = \mathrm{id}$, the identity on $[a,b]$. On the other hand, $f_{b,a}^2 = f_{b,a} \circ f_{b,a} = f_{a,b} = \mathrm{id}$, so that $f_{b,a}$ is a decreasing *involutory* homeomorphism onto $[a,b]$. Furthermore, after (24), a generic $f_{\alpha,\beta} \in \mathcal{AC}(M;[a,b])$ can be written as a product $f_{\alpha,\beta} = f_{\alpha,b} \circ f_{a,\beta}$ of the "boundary elements" $f_{\alpha,b}$, $f_{a,\beta}$, and thus, the whole semigroup $\mathcal{AC}(M;[a,b])$ can be reconstructed when $f_{\alpha,b}$ and $f_{a,\beta}$ are known for every $\alpha,\beta \in [a,b]$. This fact is the basis of the following:

Proposition 14 *If $F : [a,b]^3 \to [a,b]$ is a solution to the functional equation (26) satisfying the conditions **i), ii)** and **iii)**, then F can be written in the form*

$$F(t,\alpha,\beta) = \begin{cases} H(H(t,\psi(\beta,\alpha)),\alpha), & \alpha \le \beta \\ H(G(t,\psi(\alpha,\beta)),\beta), & \alpha > \beta \end{cases}, \quad t,\alpha,\beta \in [a,b], \tag{28}$$

where $H,G : [a,b]^2 \to [a,b]$ are solutions to the system of functional equations

$$\begin{cases} G(G(t,\alpha),\beta) = G(t,G(\alpha,\beta)) \\ G(H(t,\alpha),\beta) = H(t,G(\alpha,\beta)) \end{cases}, \quad t,\alpha,\beta \in [a,b], \tag{29a}$$

which are continuous, monotonic in both variables and strictly monotonic in the first variable when $\alpha \neq b$, while Ψ is a continuous function implicitly defined by

$$H(\Psi(t,\alpha),\alpha) = t, \quad \alpha \le t \le b, \; a \le \alpha \le b. \tag{30}$$

Proof. If $F : [a,b]^3 \to [a,b]$ is a solution to (26) satisfying the conditions **i)**, **ii)** and **iii)**, and the functions G and H are respectively defined by

$$G(t,\alpha) = F(t,\alpha,b), \; t,\alpha \in [a,b],$$

and

$$H(t,\alpha) = F(t,b,\alpha), \; t,\alpha \in [a,b],$$

then both G and H turn out to be continuous on $[a,b]^2$ by condition **i)**, while condition **ii)** shows that G and H must be monotonic functions in both variables. Moreover, $t \mapsto G(t,\alpha)$ is strictly monotonic for every $\alpha \neq b$ and the same is true for $t \mapsto H(t,\alpha)$ (but $G(t,b) \equiv b \equiv H(t,b)$). By this reason, the (continuous) function Ψ defined by

$$\Psi(t,\alpha) = \begin{cases} f_{b,\alpha}^{-1}(t), & a \le \alpha < b \\ b, & \alpha = b \end{cases}, \quad \alpha \le t \le b,$$

is the unique solution to equation (30). Now, let us prove that system (29a) is solved by the above defined functions G and H. In fact, from (26) and condition **iii**) it is deduced

$$\begin{aligned} G(G(t,\alpha),\beta) &= F(F(t,\alpha,b),\beta,b) \\ &= F(t,F(\alpha,\beta,b),F(b,\beta,b)) \\ &= F(t,F(\alpha,\beta,b),b) \\ &= G(t,G(\alpha,\beta)), \; t,\alpha,\beta \in [a,b]. \end{aligned}$$

Analogously, it can be written

$$\begin{aligned}
G\left(H\left(t,\alpha\right),\beta\right) &= F\left(F\left(t,b,\alpha\right),\beta,b\right) \\
&= F\left(t,F\left(b,\beta,b\right),F\left(\alpha,\beta,b\right)\right) \\
&= F\left(t,b,F\left(\alpha,\beta,b\right)\right) \\
&= H\left(t,G\left(\alpha,\beta\right)\right),\ t,\alpha,\beta\in\left[a,b\right].
\end{aligned}$$

It remains to prove that, in terms of G and H, F is expressed by (28). To this end, first consider the case $\alpha \leq \beta$; thus, from (26) and condition **iii**), it is derived

$$\begin{aligned}
H\left(H\left(t,\beta_1\right),\alpha\right) &= F\left(F\left(t,b,\beta_1\right),b,\alpha\right) \\
&= F\left(t,F\left(b,b,\alpha\right),F\left(\beta_1,b,\alpha\right)\right) \\
&= F\left(t,\alpha,F\left(\beta_1,b,\alpha\right)\right) \\
&= F\left(t,\alpha,H\left(\beta_1,\alpha\right)\right),\ t,\alpha,\beta_1\in\left[a,b\right];
\end{aligned}$$

whence, introducing $\beta_1 = \Psi\left(\beta,\alpha\right)$ and taking into account (30), it is obtained

$$H\left(H\left(t,\psi\left(\beta,\alpha\right)\right),\alpha\right) = F\left(t,\alpha,H\left(\psi\left(\beta,\alpha\right),\alpha\right)\right) = F\left(t,\alpha,\beta\right).$$

Similarly, when $\alpha > \beta$, it can be written

$$\begin{aligned}
H\left(G\left(t,\alpha_1\right),\beta\right) &= F\left(F\left(t,\alpha_1,b\right),b,\beta\right) \\
&= F\left(t,F\left(\alpha_1,b,\beta\right),F\left(b,b,\beta\right)\right) \\
&= F\left(t,F\left(\alpha_1,b,\beta\right),\beta\right) \\
&= F\left(t,H\left(\alpha_1,\beta\right),\beta\right),\ t,\alpha_1,\beta\in\left[a,b\right];
\end{aligned}$$

and the substitution $\alpha_1 = \psi\left(\alpha,\beta\right)$ gives

$$\begin{aligned}
H\left(G\left(t,\psi\left(\alpha,\beta\right)\right),\beta\right) &= F\left(t,H\left(\psi\left(\alpha,\beta\right),\beta\right),\beta\right) \\
&= F\left(t,\alpha,\beta\right).
\end{aligned}$$

This completes the proof. ■

Remark 15 *Note that the function G is really increasing in both variables and strictly increasing in the first variable when $\alpha \neq b$. In its turn, H is strictly decreasing in the first variable when $\alpha \neq b$, while it is increasing in the second variable.*

In the next paragraph, the system of composite equations (29a) is to be solved. The first equation in this system is no other than the associativity equation. Fortunately, its solution in our setting is furnished by a result due to C. H. Ling (see [15], Main Theorem, or also [16]Theor. 3.2 in). In the next paragraphs, $\overline{\mathbb{R}}$ and $[0,+\infty]$ will stand respectively for the extended real numbers and the nonnegative extended real numbers.

Theorem 16 (C. H. Ling, 1965) *Let $I = [a,b] \subseteq \overline{\mathbb{R}}$ be a closed interval. A function $\Gamma : I \times I \to I$ is an associative function satisfying the following conditions: Γ is continuous, increasing in both variables, the endpoint a is a left unit (i.e., $\Gamma(a,\alpha) = \alpha$ for all α in I) and, for every $\alpha \in (a,b)$, $\Gamma(\alpha,\alpha) > \alpha$, if and only if there exists a continuous and strictly increasing function $f : I \to [0,+\infty]$ with $f(a) = 0$, such that*

$$\Gamma(t,\alpha) = f^{-1}(\min(f(t) + f(\alpha), f(b))), \ t,\alpha \in I.$$

The function G in Prop. 14 is easily cheeked to satisfy the hypotheses of Ling's theorem. Moreover, from the strict monotonicity of G in the first variable it follows that $f(b) = +\infty$. Indeed, in view of $a < f^{-1}(f(b) - f(\alpha)) < b$ for every $\alpha \in (a,b)$, the assumption $f(b) < +\infty$ would imply $\Gamma(t,\alpha) = b$ for every $f^{-1}(f(b) - f(\alpha)) < t \leq b$, a contradiction. Then, for a continuous and strictly increasing function $f : I \to [0,+\infty]$ with $f(a) = 0$, it can be written

$$G(t,\alpha) = f^{-1}(f(t) + f(\alpha)), \ t,\alpha \in [a,b]. \tag{31}$$

In order to solve the second equation in (29a), let us substitute the expression (31) for G in it to obtain

$$f^{-1}(f(H(t,\alpha)) + f(\beta)) = H\left(t, f^{-1}(f(\alpha) + f(\beta))\right), \ t,\alpha,\beta\,[a,b].$$

Setting $\xi = f(t)$, $\eta = f(\alpha)$, $\zeta = f(\beta)$ and introducing the function $K : \mathbb{R}_0^+ \times \mathbb{R}_0^+ \to \mathbb{R}_0^+$ defined by $K(\xi,\eta) = f\left(H\left(f^{-1}(\xi), f^{-1}(\eta)\right)\right)$, this equation can be written as

$$K(\xi,\eta) + \zeta = K(\xi,\eta+\zeta), \quad \xi,\eta,\zeta \in \mathbb{R}_0^+,$$

and then

$$K(\xi,\eta) + \zeta = K(\xi,\eta+\zeta) = K(\xi,\zeta+\eta) = K(\xi,\zeta) + \eta,$$

or, equivalently,

$$K(\xi,\eta) - \eta = K(\xi,\zeta) - \zeta, \quad \xi,\eta,\zeta \in \mathbb{R}_0^+.$$

In other words, the function $(\xi,\eta) \to K(\xi,\eta) - \eta$ depends only on ξ; i.e., there exists $p : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ such that

$$K(\xi,\eta) = p(\xi) + \eta,$$

and therefore

$$H(t,\alpha) = f^{-1}(p(f(t)) + f(\alpha)). \tag{32}$$

The replacement $\alpha = a$ in the last expression produces

$$f_{b,a}(t) = H(t,a) = f^{-1}(p(f(t)) + f(a)) = f^{-1}(p(f(t))),$$

whence it is deduced that p is a strictly decreasing involutory function satisfying $p(0^+) = +\infty$ and $p(+\infty) = 0$. An expression for the function Ψ of Prop. 14 is promptly derived from (32) in the form

$$\Psi(t,\alpha) = f^{-1}\left(p\left(f(t) - f(\alpha)\right)\right), \ \alpha \le t \le b, \ a \le \alpha \le b.$$

From the above discussion and Prop. 14, it follows that any solution to equation (26) satisfying conditions **i)**, **ii)** and **iii)** can be written as

$$F(t,\alpha,\beta) = \begin{cases} f^{-1}\left(p\left(p\left(f(t)\right) + p\left(f(\beta) - f(\alpha)\right)\right) + f(\alpha)\right), & \alpha \le \beta \\ f^{-1}\left(p\left(f(t) + p\left(f(\alpha) - f(\beta)\right)\right) + f(\beta)\right), & \alpha > \beta \end{cases}, \quad (33)$$

where $f : I \to [0,+\infty]$ is a continuous and strictly increasing function with $f(a) = 0$ and p is a strictly decreasing involutory function satisfying $p(0^+) = +\infty$ and $p(+\infty) = 0$. Now, assume that the function F represented by (33) is a solution to equation (26); then, taking $\alpha_1, \beta_1, \beta_2 \in [a,b]$ with $\alpha_1 \le \beta_1$, it can be written

$$F(t,\alpha_1,\beta_1) = f^{-1}\left(p\left(p\left(f(t)\right) + p\left(f(\beta_1) - f(\alpha_1)\right)\right) + f(\alpha_1)\right),$$

and, in view of $f(a) = 0$, it follows that

$$F\left(F(t,\alpha_1,\beta_1),\alpha_2,\beta_2\right) = f^{-1}\left(p\left(p\left(p\left(p(s) + p(b_1 - a_1)\right) + a_1\right) + p(b_2)\right)\right), \quad (34)$$

where $s = f(t)$, $a_1 = f(\alpha_1)$ and $b_i = f(\beta_i)$, $i = 1,2$. On the other hand,

$$F(\alpha_1,\alpha_2,\beta_2) = f^{-1}\left(p\left(p\left(f(\alpha_1)\right) + p\left(f(\beta_2) - f(\alpha_2)\right)\right) + f(\alpha_2)\right),$$

and

$$F(\beta_1,\alpha_2,\beta_2) = f^{-1}\left(p\left(p\left(f(\beta_1)\right) + p\left(f(\beta_2) - f(\alpha_2)\right)\right) + f(\alpha_2)\right),$$

whence, since $F(\alpha_1,\alpha_2,\beta_2) \le F(\beta_1,\alpha_2,\beta_2)$, the following equality is deduced

$$\begin{aligned} & F\left(t, F(\alpha_1,\alpha_2,\beta_2), F(\beta_1,\alpha_2,\beta_2)\right) \\ = \ & f^{-1}\left(p\left(p(s) + p\left(p\left(p(b_1) + p(b_2)\right) - p\left(p(a_1) + p(b_2)\right)\right)\right)\right. \\ & \left.+ p\left(p(a_1) + p(b_2)\right)\right), \end{aligned} \quad (35)$$

where again $s = f(t)$, $a_1 = f(\alpha_1)$ and $b_i = f(\beta_i)$, $i = 1,2$. Since the left hand sides of (34) and (35) are equal, their corresponding right hand sides must be equal as well and therefore, the equality

$$\begin{aligned} & p\left(p\left(p\left(p(s) + p(b_1 - a_1)\right) + a_1\right) + p(b_2)\right) \\ = \ & p\left(p(s) + p\left(p\left(p(b_1) + p(b_2)\right) - p\left(p(a_1) + p(b_2)\right)\right)\right) + p\left(p(a_1) + p(b_2)\right), \end{aligned}$$

must hold for every $s, a_1, b_1, b_2 \in [a,b]$ or, after the substitutions $x = b_1 - a_1$, $y = a_1$, $z = p(b_2)$ and $s = p(s)$,

$$p\left(p\left(p\left(s + p(x)\right) + y\right) + z\right) = p\left(s + p\left(p\left(p(x+y) + z\right) - p\left(p(y) + z\right)\right)\right) + p\left(p(y) + z\right) \quad (36)$$

where $x, y, z, s \in \mathbb{R}_0^+$.

Summarizing the above developments, the following result can be established.

Proposition 17 *Let $F : [a,b]^3 \to [a,b]$ be a solution to functional equation (26) satisfying the conditions **i)**, **ii)** and **iii)**. Then, F can be represented in the form (33), where $f : I \to [0,+\infty]$ is a continuous and strictly increasing function with $f(a) = 0$ (and $f(+\infty) = +\infty$) and $p : [0,+\infty] \to [0,+\infty]$ is a strictly decreasing involutory function with $p(0^+) = +\infty$ and $p(+\infty) = 0$ which solves the functional equation (36).*

Proof. The proof follows from Prop. 14 and the preceding discussion. ■

Now, let us pay attention to the functional equation (36). In the first place, observe that in view of the continuity of p and the the fact that $p(+\infty) = 0$, taking limits when $x \uparrow +\infty$ in (36) produces

$$p(p(p(s)+y)+z) = p(s + p(p(z) - p(p(y)+z))) + p(p(y)+z), \qquad (37)$$

where $x, y, z, s \in \mathbb{R}_0^+$. In order to simplify the expressions, let us define a commutative operation (quasisum) $\Delta : [0,+\infty]^2 \to [0,+\infty]$ by

$$x \Delta y = p(p(x) + p(y)). \qquad (38)$$

In this way, the substitutions $s = p(s)$ and $z = p(z)$ in (37) enables to write it in the form

$$(s+y)\Delta z = s\Delta(z - y\Delta z) + y\Delta z. \qquad (39)$$

Note that $0 \leq z - y\Delta z \leq z$, being the inequalities strict provided that y, $z > 0$.

As it is shown by the following result, the function $s \mapsto s\Delta z$ has nice properties.

Lemma 18 *Let $p : [0,+\infty] \to [0,+\infty]$ be a continuous and strictly decreasing involutory function solving the functional equation (37) and $\Delta : [0,+\infty]^2 \to [0,+\infty]$ be the quasisum defined by (38). Then, for every $z \in \mathbb{R}^+$, the function $s \mapsto s\Delta z$ is strictly subadditive., strictly increasing, strictly concave and continuously differentiable in $\mathbb{R}^+$.*

By commutativity, the function $z \mapsto s\Delta z$ has the same properties as $s \mapsto s\Delta z$.

Proof. Fix $z \in \mathbb{R}^+$ and consider the function $s \mapsto s\Delta z$. Since p is a strictly decreasing function, $s \mapsto s\Delta z$ turns out to be strictly increasing. As a consequence, (39) and the inequality $z - t\Delta z < z$ yields

$$(s+t)\Delta z = s\Delta(z - t\Delta z) + t\Delta z < s\Delta z + t\Delta z, \quad s, t \in \mathbb{R}^+;$$

i.e., $s \mapsto s\Delta z$ is subadditive. To prove the strict concavity of $s \mapsto s\Delta z$, choose a pair $s, t \in \mathbb{R}^+$ with $s \neq t$, say $s < t$; then, a repeated use of (39) produces

$$\begin{aligned}
t\Delta z &= \left(\frac{t-s}{2} + \left(\frac{t-s}{2} + s\right)\right)\Delta z \\
&= \frac{t-s}{2}\Delta\left(z - \left(\frac{t-s}{2} + s\right)\Delta z\right) + \left(\frac{t-s}{2} + s\right)\Delta z \\
&= \frac{t-s}{2}\Delta\left(z - \frac{t-s}{2}\Delta\left(z - s\Delta z\right) - s\Delta z\right) + \frac{t-s}{2}\Delta\left(z - s\Delta z\right) + s\Delta z \\
&< \frac{t-s}{2}\Delta\left(z - s\Delta z\right) + \frac{t-s}{2}\Delta\left(z - s\Delta z\right) + s\Delta z, \qquad (40)
\end{aligned}$$

where the last inequality holds by the strict monotonicity of $z \mapsto s\Delta z$. On the other hand,

$$\begin{aligned}
\left(\frac{s+t}{2}\right)\Delta z &= \left(\frac{s-t}{2} + t\right)\Delta z \\
&= \frac{s-t}{2}\Delta\left(z - t\Delta z\right) + t\Delta z; \qquad (41)
\end{aligned}$$

so that, combining (40) and (41) it is deduced

$$t\Delta z < 2\left(\left(\frac{s+t}{2}\right)\Delta z\right) - s\Delta z,$$

or, equivalently,

$$\left(\frac{s+t}{2}\right)\Delta z > \frac{s\Delta z + t\Delta z}{2}. \qquad (42)$$

By symmetry, inequality (42) holds also when $s > t$ and, due to the continuity of $s \mapsto s\Delta z$, it implies the strict concavity of this function.

Now, for every $s \in \mathbb{R}^+$, the existence of the lateral derivatives $D_s^+\left(s\Delta z\right)$ and $D_s^-\left(s\Delta z\right)$ is ensured by the concavity of $s \mapsto s\Delta z$. In particular, in view of (39), for the right derivative $D_s^+\left(s\Delta z\right)$ it can be written

$$\begin{aligned}
D_s^+\left(s\Delta z\right) &= \lim_{t\downarrow 0}\frac{\left(s+t\right)\triangle z - s\triangle z}{t} \\
&= \lim_{t\downarrow 0}\frac{t\triangle\left(z - s\triangle z\right)}{t} \\
&= \lim_{t\downarrow 0}\frac{p\left(p\left(t\right) + p\left(z - s\triangle z\right)\right)}{t} \\
&= \lim_{u\uparrow +\infty}\frac{p\left(u + p\left(z - s\triangle z\right)\right)}{p\left(u\right)}, \ \ s \geq 0.
\end{aligned}$$

The last of these equalities was obtained by replacing $t = p\left(u\right)$. Since $0 \leq z - s\triangle z \leq z$ and $z \in \mathbb{R}^+$ was arbitrarily chosen, it is concluded that the function

$$\Phi\left(\lambda\right) = \lim_{u\uparrow +\infty}\frac{p\left(u+\lambda\right)}{p\left(u\right)}$$

is defined for every $\lambda \geq 0$. Clearly, Φ is decreasing and the equalities

$$\begin{aligned} \Phi(\lambda+\mu) &= \lim_{u\uparrow+\infty} \frac{p(u+\lambda+\mu)}{p(u)} \\ &= \lim_{u\uparrow+\infty} \frac{p(u+\lambda+\mu)}{p(u+\lambda)} \frac{p(u+\lambda)}{p(u)} \\ &= \lim_{u\uparrow+\infty} \frac{p(u+\lambda+\mu)}{p(u+\lambda)} \lim_{u\uparrow+\infty} \frac{p(u+\lambda)}{p(u)} \\ &= \Phi(\lambda)\,\Phi(\mu), \end{aligned}$$

hold for every $\lambda, \mu \geq 0$. In other words, Φ is a decreasing solution to the exponential Cauchy equation and, in consequence, $\Phi(\lambda) \equiv 0$ or $\Phi(\lambda) \equiv e^{-k\lambda}$ for any $k \geq 0$. Indeed, the instances $\Phi = 0$ or $\Phi = 1$ must be excluded since, in these cases, it would be $D_s^+(s\Delta z) \equiv 0$ or $D_s^+(s\Delta z) \equiv 1$ and therefore, $D_s(s\Delta z) \equiv 0$ or $D_s(s\Delta z) \equiv 1$, two identities contradicting the strict concavity of $s \mapsto s\Delta z$. In this way, there exists $k > 0$ such that

$$D_s^+(s\Delta z) = \lim_{u\uparrow+\infty} \frac{p(u+p(z-s\triangle z))}{p(u)} = e^{-kp(z-s\triangle z)}.$$

This equality shows that $s \mapsto D_s^+(s\Delta z)$ is continuous on $\mathbb{R}^+$ and hence, there exists the standard derivative $D_s(s\Delta z)$ and

$$D_s(s\Delta z) = e^{-kp(z-s\triangle z)},\ \ s, z \in \mathbb{R}^+. \tag{43}$$

This completes the proof. ■

A result on regularity of the solutions to the functional equation (36) is now proved.

Proposition 19 *Let $p : [0, +\infty] \to [0, +\infty]$ be a strictly decreasing involutory function with $p(0^+) = +\infty$ and $p(+\infty) = 0$ which solves the functional equation (36); then p is continuously differentiable in $\mathbb{R}^+$. Moreover, $p'(0^+) = -\infty$ and $p'(+\infty) = 0$.*

Proof. Let us denote by $\mathrm{Diff}(p)$ the set of points where the derivative p' exists. By the Lebesgues´s Theorem, $\mathrm{Diff}(p)$ contains almost every point of $\mathbb{R}^+$ so that, for a given $s \in \mathbb{R}^+$, one can chose t, $z_0 > 0$ such that $p(s) + t$ and $p(p(s)+t) + p(z_0)$ are both in $\mathrm{Diff}(p)$. Thus, the chain rule applied to $s \mapsto s\Delta z_0 = p(p(s) + p(z_0))$ at $p(s) + t$ yields

$$D_s(s\Delta z_0)\big|_{s=p(s)+t} = p'(p(p(s)+t) + p(z_0))\, p'(p(s)+t).$$

Now, in view of (43),

$$D_s(s\Delta z_0)\big|_{s=p(s)+t} = e^{-kp(z_0-(p(s)+t)\triangle z_0)} > 0,$$

so that it must be $p'(p(s)+t) \neq 0$, and therefore

$$\lim_{h\to 0} \frac{p(s+h)-p(s)}{h} = \lim_{h\to 0} \frac{\frac{p(p(s+h)+t)-p(p(s)+t)}{h}}{\frac{p(p(s+h)+t)-p(p(s)+t)}{p(s+h)-p(s)}}$$
$$= \frac{D_s\left(s\Delta p(t)\right)\big|_s}{p'(p(s)+t)}.$$

This shows that $s \in \mathrm{Diff}(p)$, and thus $\mathrm{Diff}(p) = \mathbb{R}^+$.

Now, from (38) and (43) it is obtained

$$p'(p(s)+p(z))\, p'(s) = e^{-kp(z-p(p(s)+p(z)))},\ \ s,z \in \mathbb{R}^+;$$

or, replacing $s = p(s)$ and $z = p(z)$,

$$p'(s+z) = \frac{e^{-kp(p(z)-p(s+z))}}{p'(p(s))}, \tag{44}$$

whence the continuity of p' on $(s,+\infty)$ is easily derived. Since s can be arbitrarily chosen in $\mathbb{R}^+$, p' turns out to be continuous on $\mathbb{R}^+$. Moreover, making $z \uparrow +\infty$ in (44) yields

$$p'(+\infty) = \lim_{z\uparrow+\infty} \frac{e^{-kp(p(z)-p(s+z))}}{p'(p(s))} = 0. \tag{45}$$

Finally, being p an involutory function, it turns out to be

$$p'(p(s))\, p'(s) = 1,\ s \in \mathbb{R}^+; \tag{46}$$

whence, in view of (45) and the fact that p is strictly decreasing, it is deduced

$$p'\left(0^+\right) = \lim_{s\downarrow 0} p'(s) = \lim_{s\downarrow 0} \frac{1}{p'(p(s))} = \lim_{z\uparrow+\infty} \frac{1}{p'(z)} = -\infty.$$

This completes the proof. ■

It should be noted that an inductive reasoning based on (44) shows that p is really a $\mathcal{C}^\infty$ function in $\mathbb{R}^+$. At this point, the solutions to equation (36) can be determined.

Proposition 20 *Let $p : [0,+\infty] \to [0,+\infty]$ be a strictly decreasing involutory function with $p(0^+) = +\infty$ and $p(+\infty) = 0$ which solves the functional equation (36), then there exists $k > 0$ such that*

$$p(t) = -\frac{1}{k} \ln\left(1-e^{-kt}\right),\ \ t > 0. \tag{47}$$

Proof. By Prop. 19, p is continuously differentiable in $\mathbb{R}^+$. Thus, deriving both members of (36) with respect to z and then taking limits when $z \downarrow 0$, it is obtained

$$p'(p(p(s+p(x))+y)) = p'(s+p(x))\, p'(x)\left(p'(p(x+y)) - p'(p(y))\right) + p'(p(y)). \tag{48}$$

Observe that $p'(p(x+y)) - p'(p(y)) \neq 0$ for every $x, y \in \mathbb{R}^+$. In fact, if $p'(p(x+y)) = p'(p(y))$ for any pair $x, y \in \mathbb{R}^+$; then, $p'(x+y) = p'(y)$ by (46), an equality which, together (44) with $s = y$ and $z = x$, would imply

$$1 = p'(y)\, p'(p(y)) = p'(x+y)\, p'(p(y)) = e^{-kp(p(x)-p(x+y))},$$

whence

$$p(p(x) - p(x+y)) = 0.$$

Since $p(x) - p(x+y) \in \mathbb{R}^+$, the last equality is an absurdity. In this way, (48) can be rewritten in the form

$$p'(x) = \frac{p'(p(p(s+p(x))+y)) - p'(p(y))}{p'(s+p(x))\,(p'(p(x+y)) - p'(p(y)))}.$$

and then, using (46), it is deduced

$$\begin{aligned}
1 &= \lim_{x\uparrow+\infty} p'(x)\, p'(p(x)) \\
&= \lim_{x\uparrow+\infty} \left(\frac{p'(p(p(s+p(x))+y)) - p'(p(y))}{p'(s+p(x))\,(p'(p(x+y)) - p'(p(y)))} p'(p(x)) \right) \\
&= \lim_{x\uparrow+\infty} \frac{p'(p(p(s+p(x))+y)) - p'(p(y))}{p'(s+p(x))} \lim_{x\uparrow+\infty} \frac{p'(p(x))}{p'(p(x+y)) - p'(p(y))} \\
&= \frac{p'(p(p(s)+y)) - p'(p(y))}{p'(s)} \lim_{x\uparrow+\infty} \frac{p'(p(x))}{p'(p(x+y)) - p'(p(y))},
\end{aligned}$$

whence, for every $s, y \in \mathbb{R}^+$,

$$\lim_{x\uparrow+\infty} \frac{p'(p(x))}{p'(p(x+y)) - p'(p(y))} = \frac{p'(s)}{p'(p(p(s)+y)) - p'(p(y))}. \tag{49}$$

Now, a new application of (46) produces

$$\begin{aligned}
\lim_{x\uparrow+\infty} \frac{p'(p(x))}{p'(p(x+y)) - p'(p(y))} &= \lim_{x\uparrow+\infty} \frac{1}{\frac{p'(p(x+y))}{p'(p(x))} - \frac{p'(p(y))}{p'(p(x))}} \\
&= \lim_{x\uparrow+\infty} \frac{1}{\frac{p'(x)}{p'(x+y)} - \frac{p'(x)}{p'(y)}} \\
&= \lim_{x\uparrow+\infty} \frac{p'(x+y)}{p'(x)},
\end{aligned} \tag{50}$$

where the last equality follows from the fact that $p'(+\infty) = 0$. Moreover, from (46) and (44) with $s = x$ and $z = y$, it is obtained

$$\frac{p'(x+y)}{p'(x)} = p'(x+y)\, p'(p(x)) = e^{-kp(p(y)-p(x+y))}. \tag{51}$$

Now, from (49), (50) and (51) it is deduced

$$e^{-ky} = \lim_{x\uparrow+\infty} e^{-kp(p(y)-p(x+y))} = \lim_{x\uparrow+\infty} \frac{p'(x+y)}{p'(x)} = \frac{p'(s)}{p'(p(p(s)+y)) - p'(p(y))},$$

where $s,\, y \in \mathbb{R}^+$ and $k > 0$ is a constant. Substituting $s = p(s)$ in the last member of these equalities, gives

$$e^{-ky} = \frac{p'(p(s))}{p'(p(s+y)) - p'(p(y))},$$

whence, for every $s,\, y \in \mathbb{R}^+$,

$$\frac{1}{p'(s+y)} = p'(p(s+y)) = p'(p(y)) + p'(p(s))\, e^{ky} = \frac{1}{p'(y)} + \frac{1}{p'(s)} e^{ky}.$$

The first member of these equalities is symmetric in its arguments, which shows that

$$\frac{1}{p'(y)} + \frac{1}{p'(s)} e^{ky} = \frac{1}{p'(s)} + \frac{1}{p'(y)} e^{ks}$$

or, equivalently,

$$p'(s)\left(1 - e^{ks}\right) = p'(y)\left(1 - e^{ky}\right),\ \ s,\, y \in \mathbb{R}^+.$$

In other terms, there exist a positive constant A such that

$$p'(s) = \frac{A}{1 - e^{ks}},\ \ s \in \mathbb{R}^+. \tag{52}$$

An integration of the equality (52) yields

$$p(s) = -\frac{A}{k} \ln\left(1 - e^{-ks}\right),\ \ s \in \mathbb{R}^+. \tag{53}$$

Let us see that a function p expressed by (53) is involutory if and only if $A = 1$. In fact, this occurs if and only if, for every $s \in \mathbb{R}^+$,

$$1 = p'(p(s))\, p'(s) = \frac{A}{\left(1 - \left(1 - e^{-ks}\right)^{-A}\right)} \frac{A}{1 - e^{ks}},$$

or, setting $x = \left(1 - e^{-ks}\right)^{-1}$,

$$1 - x^A = A^2 (1 - x),\ \ x > 1,$$

an equality which holds if and only if $A = 1$. This proves that a solution to equation (36) which satisfies the hypotheses of the proposition must be of the form (47). A simple substitution shows that (47) is really a solution to equation (36). This completes the proof. ■

Remark 21 *Note that, for p given by (47), the quasisum Δ is expressed by*

$$\begin{aligned} x \Delta y &= -\frac{1}{k} \ln\left(1 - e^{-k\left(-\frac{1}{k}\ln\left(1-e^{-kx}\right) - \frac{1}{k}\ln\left(1-e^{-ky}\right)\right)}\right) \\ &= -\frac{1}{k} \ln\left(1 - \left(1 - e^{-kx}\right)\left(1 - e^{-ky}\right)\right),\ \ x, y \in \mathbb{R}^+, \end{aligned}$$

while $x \Delta 0 = 0$ and $x \Delta (+\infty) = x$ for every $x \in [0, +\infty]$.

Theors. 13 and 6 are now proved.

Proof of Theor. 13. Suppose that $F : [a,b]^3 \to [a,b]$ is a solution to the functional equation (26) in the class of functions fulfilling the conditions **i)**, **ii)** and **iii).** Then, from propositions 17 and 20, it turns out to be that F can be written in the form

$$F(t,\alpha,\beta) = \begin{cases} f^{-1}\left(-\frac{1}{k}\ln\left(1-\left(1-e^{-kf(t)}\right)\left(1-e^{-k(f(\beta)-f(\alpha))}\right)\right)+f(\alpha)\right), & \alpha\leq\beta \\ f^{-1}\left(-\frac{1}{k}\ln\left(1-e^{-kf(t)}\left(1-e^{-k(f(\alpha)-f(\beta))}\right)\right)+f(\beta)\right), & \alpha>\beta \end{cases}, \ t\in[a,b] \tag{54}$$

where $f : [a,b] \to [0,+\infty]$ is a continuous and strictly increasing function with $f(a) = 0$ ($f(+\infty) = +\infty$) and $k > 0$. In this way, the function $\psi(t) = 1-e^{-kf(t)}$, $t \in [a,b]$, is an increasing homeomorphism from $[a,b]$ onto $[0,1]$ and, in view of $f(t) = -k^{-1}\ln(1-\psi(t))$ and $f^{-1}(t) = \psi^{-1}\left(1-e^{-kt}\right)$, (54) is, in terms of ψ, expressed by

$$F(t,\alpha,\beta) = \begin{cases} \psi^{-1}\left((\psi(\beta)-\psi(\alpha))\,\psi(t)+\psi(\alpha)\right), & \alpha\leq\beta \\ \psi^{-1}\left((\psi(\beta)-\psi(\alpha))\,\phi(t)+\psi(\alpha)\right), & \alpha>\beta \end{cases}, \ t\in[a,b],$$

which is no other than (27).

Now, if ψ_* was another increasing homeomorphism satisfying (27), then, the equality

$$\psi_*^{-1}\left((\psi_*(\beta)-\psi_*(a))\,\phi(t)+\psi_*(\alpha)\right) = \psi^{-1}\left((\psi(\beta)-\psi(a))\,\phi(t)+\psi(\alpha)\right)$$

would hold for every $t,\alpha,\beta \in [a,b]$. Setting $\phi = \psi\circ\psi_*^{-1} : [0,1] \to [0,1]$, $\alpha = \psi_*^{-1}(p)$ and $\beta = \psi_*^{-1}(q)$ in this equality, it turns out to be that ϕ is a continuous solution to the functional equation

$$\phi((1-t)\,p+tq) = (1-\phi(t))\,\phi(p)+\phi(t)\,\phi(q),\ t,p,q\in[0,1],$$

whence ([1], Theor. 2, pg. 67, or [14], pg. 382 and ff.) $\phi(t) = t$, $t \in [0,1]$. In this way, $\psi = \psi_*$ and the proof is complete. ■

Proof of Theor. 6. Let M be a mean fulfilling the hypotheses of the theorem and, for $\alpha,\beta \in [a,b]$, consider $f_{\alpha,\beta} \in \mathcal{AC}(M;[a,b])$ such that $f_{\alpha,\beta}(a) = \alpha$ and $f_{\alpha,\beta}(b) = \beta$. After Theor. 13, it can be written

$$f(t) = \psi^{-1}\left((\psi(\beta)-\psi(\alpha)\,\psi(t)+\psi(\alpha))\right),\ t\in[a,b], \tag{55}$$

where ψ is a uniquely determined increasing homeomorphism from $[a,b]$ onto $[0,1]$. In this way, the equality

$$\begin{aligned} &\psi^{-1}\left((\psi(\beta)-\psi(\alpha)\,\psi(M(x_1,\ldots,x_n))+\psi(\alpha))\right) \\ = \ &M\left(\psi^{-1}\left((\psi(\beta)-\psi(\alpha)\,\psi(x_1)+\psi(\alpha))\right),\ldots,\psi^{-1}\left((\psi(\beta)-\psi(\alpha)\,\psi(x_n)+\psi(\alpha))\right)\right), \end{aligned}$$

which holds for every $x_1,\ldots,x_n,\alpha,\beta \in [a,b]$ M_ψ, turns out to be equivalent to

$$mM_\psi(t_1,\ldots,t_n)+k = M_\psi(mt_1+h,\ldots,mt_n+h),$$

where $t_1,\ldots,t_n \in [0,1]$, $m \in [-1,1]$, h, $m+h \in [0,1]$ and M_ψ is the mean conjugated of M by ψ (defined by (20)). This shows that $\mathrm{Aff}([0,1]) \subseteq \mathcal{AC}(M_\psi;[0,1])$.

To prove the opposite inclusion observe that, if $g \in \mathcal{AC}(M_\psi;[0,1])$, then $\psi^{-1} \circ g \circ \psi \in \mathcal{AC}(M;[a,b])$, and therefore, there exists $(m,h) \in AFF([0,1])$ such that

$$\left(\psi^{-1} \circ g \circ \psi\right)(t) = \psi^{-1}\left((m\psi(t)+h)\right),\ t \in [a,b],$$

whence $g(t) = mt+h,\ t \in [0,1]$; i.e., $g \in \mathrm{Aff}([0,1])$. ■

4 Two variables means

The following result, which is not devoid of intrinsic interest, will be the key to derive Theor. 7 from Theor. 6.

Proposition 22 *Let M be a two variables mean defined on an interval I and $\{[a_k,b_k] : k \in \mathbb{N}\}$ be a nested and exhaustive sequence of subintervals of I. If , for every $k \in \mathbb{N}$, $M|_{[a_k,b_k]}$ is a quasilinear mean on $[a_k,b_k]$, then M is a quasilinear mean on I.*

Proof. The hypotheses ensure the existence, for every $k \in \mathbb{N}$, of a strictly monotonic and continuous function $\psi_k : [a_k,b_k] \to \mathbb{R}$ and a real number $w_k \in (0,1)$ such that

$$M|_{[a_k,b_k]}(x,y) = \psi_k^{-1}\left((1-w_k)\psi_k(x)+w_k\psi_k(y)\right),\ x,y \in [a_k,b_k]. \tag{56}$$

Since the second member of (56) is not altered by taking $-\psi_k$ instead of ψ_k, it can be assumed that ψ_k is strictly increasing. Now, in view of$M|_{[a_k,b_k]}(x,y) = M|_{[a_j,b_j]}(x,y)$ for every $x,y \in [a_l,b_l]$, $l = \min\{k,j\}$, (56) yields

$$\psi_k^{-1}\left((1-w_k)\psi_k(x)+w_k\psi_k(y)\right) = \psi_j^{-1}\left((1-w_j)\psi_j(x)+w_j\psi_j(y)\right),\ x,y \in [a_l,b_l].$$

and thus, the function $\psi_{k,j} = \psi_k \circ \psi_j^{-1}$ is a continuous solution to the equation

$$\psi_{k,j}\left((1-w_j)s+w_jt\right) = (1-w_k)\psi_{k,j}(s)+w_k\psi_{k,j}(t),\ s,t \in [0,1].$$

In this way ([1], Theor. 2, pg. 67 or also [14], pg. 382 and ff.), $w_k = w_j,\ k,\ j \in \mathbb{N}$, and, for certain $p_{k,j},\ q_{k,j} \in \mathbb{R}$, $p_{k,j} \neq 0$, $\psi_{k,j}(t) = p_{k,j}t+q_{k,j},\ t \in [a_l,b_l]$; hence

$$\psi_k(t) = p_{k,j}\psi_j(t)+q_{k,j},\ t \in [a_l,b_l]. \tag{57}$$

Note on one hand that, in view of ψ_k and ψ_j are both strictly increasing functions, it must really occur that $p_{k,j} > 0$ for every $k,j \in \mathbb{N}$, and, on the other, that the equality (56) can be written in the form

$$M(x,y) = \psi_k^{-1}\left((1-w_1)\psi_k(x)+w_1\psi_k(y)\right),\ x,y \in [a_k,b_k]. \tag{58}$$

In what follows, a particular instance of (57) will be used; namely, setting $p_k = p_{k+1,k}$ and $q_k = q_{k+1,k}$ for every $k \geq 1$, (57) takes the form

$$\psi_{k+1}(t) = p_k\psi_k(t)+q_k,\ t \in [a_k,b_k]. \tag{59}$$

Now, define a sequence of strictly increasing and continuous functions $\phi_k : [a_k, b_k] \to \mathbb{R},\ k \in \mathbb{N}$, by $\phi_1(t) = \psi_1(t),\ t \in [a_1, b_1]$, and for $k \geq 1$,

$$\phi_{k+1}(t) = \left(\prod_{j=1}^{k} p_j^{-1}\right) \psi_{k+1}(t) - \sum_{i=1}^{k} \left(q_i \prod_{j=1}^{i} p_j^{-1}\right),\ t \in [a_{k+1}, b_{k+1}].$$

From (59) it is deduced that, for every $t \in [a_k, b_k]$,

$$\begin{aligned}
\phi_{k+1}(t) &= \left(\prod_{j=1}^{k} p_j^{-1}\right) (p_k \psi_k(t) + q_k) - \sum_{i=1}^{k} \left(q_i \prod_{j=1}^{i} p_j^{-1}\right) \\
&= \left(\prod_{j=1}^{k-1} p_j^{-1}\right)^{-1} \psi_k(t) - \sum_{i=1}^{k-1} \left(q_i \prod_{j=1}^{i} p_j^{-1}\right) \\
&= \phi_k(t),
\end{aligned}$$

so that the expression

$$\phi(t) = \phi_k(t),\ t \in [a_k, b_k],$$

defines a function $\phi : I \to \mathbb{R}$ which turns out to be strictly increasing and continuous on I. Since a quasilinear mean L_f does not change when its generator f is replaced by $mf+h$ with $m, h \in \mathbb{R},\ m \neq 0$, the equality (58) can be rewritten in the form

$$M(x,y) = \phi_k^{-1}((1-w_1)\phi_k(x) + w_1 \phi_k(y)),\ x, y \in [a_k, b_k],$$

whence

$$M(x,y) = \phi^{-1}((1-w_1)\phi(x) + w_1 \phi(y)),\ x, y \in I,$$

which shows that M is quasilinear on I, as affirmed. ■

Proof of Theor. 7. Since $M|_{[a_1,b_1]}$ satisfies the hypotheses of Theor. 6, it follows that there exist a homeomorphism $\psi_1 : [a_1, b_1] \to [0,1]$ such that $\mathcal{AC}\left(M_{\psi_1}; [0,1]\right) = \mathrm{Aff}([0,1])$. After what was said in Section 1, M_{ψ_1} is a linear mean, and therefore, there exist $w_1 \in (0,1)$ such that

$$M(x,y) = \psi_1^{-1}((1-w_1)\psi_1(x) + w_1 \psi_1(y)),\ x, y \in [a_1, b_1].$$

The same reasoning applied to the interval $[a_k, b_k]$ yields, for every $k \in \mathbb{N}$,

$$M(x,y) = \psi_k^{-1}((1-w_k)\psi_k(x) + w_k \psi_k(y)),\ x, y \in [a_k, b_k], \tag{60}$$

where $\psi_k : [a_k, b_k] \to [0,1]$ is an increasing homeomorphism and $w_k \in (0,1)$. This proves that, for every $k \in \mathbb{N}$, the restriction $M|_{[a_k,b_k]}$ to the subinterval $[a_k, b_k]$ is a quasilinear mean. By Prop. 22, this implies that M is quasilinear on I. ■

Remark 23 *Under the hypotheses of Theor. 7, consider the mean M_ϕ conjugated of M by ϕ, being ϕ the function defined in the proof of Prop. 22. For the M_ϕ-affine functions, the equality*

$$\mathcal{AC}\left(M_\phi;\phi\left(I\right)\right)=\mathrm{Aff}\left(\phi\left(I\right)\right)$$

holds.

A simple consequence of Theor. 7 is the following:

Proposition 24 *Let I, J be non void real intervals and $M\in\mathcal{CM}_2\left(I\right)$, $N\in\mathcal{CM}_2\left(J\right)$ be a pair of strict continuous means. Suppose that for every a, $b\in I$, $a\neq b$, and for every c, $d\in J$, $c\neq d$, there exists a strictly monotonic and continuous solution $f:I\to J$ to the equation*

$$f\left(M\left(x,y\right)\right)=N\left(f\left(x\right),f\left(y\right)\right),\ x,y\in I,$$

with $f\left(a\right)=c$ and $f\left(b\right)=d$; then M and N are both quasilinear means.

Proof. Fix a, $b\in I$, $a<b$, and choose α, $\beta\in\left[a,b\right]$, $\alpha\neq\beta$. By the assumptions, given c, $d\in J$, $c\neq d$, there exist continuous and strictly monotonic functions f, $g:I\to J$ respectively solving the equations

$$f\left(M\left(x,y\right)\right)=N\left(f\left(x\right),f\left(y\right)\right),\ x,y\in I,$$

and

$$g\left(M\left(x,y\right)\right)=N\left(g\left(x\right),g\left(y\right)\right),\ x,y\in I,$$

while $f\left(a\right)=c=g\left(\alpha\right)$ and $f\left(b\right)=d=g\left(\beta\right)$. Then, the function $f_{\alpha,\beta}:I\to I$ defined by $f_{\alpha,\beta}=g^{-1}\circ f$ turns out to be a homeomorphism which solves the equation

$$h\left(M\left(x,y\right)\right)=M\left(h\left(x\right),h\left(y\right)\right),\ x,y\in I,$$

and satisfies $f_{\alpha,\beta}\left(a\right)=\alpha$ and $f_{\alpha,\beta}\left(b\right)=\beta$. In this way, after defining $f_{\alpha,\alpha}\equiv\alpha$, it is seen that $\mathcal{AC}\left(\left.M\right|_{[a,b]};[a,b]\right)$ is a Sturm-Liouville family on every compact subinterval $[a,b]$ of I and therefore, Theor. 7 implies that M is a quasilinear mean. The quasilinearity of N follows from a similar argument. ■

The remaining of this section is engaged with bases of two variables means, a concept introduced in [3]. Before stating a result on characterization of quasilinear means in terms of bases, an abridged recall will be presented of the involved ideas.

If $M\in\mathcal{CM}_2\left(I\right)$ is a strict and continuous mean defined on an interval I and, for a pair u, $v\in[0,1]$, $M^{(u)},M^{(v)}$ are Aczel dyadic iterations of M, then, in view of the monotonicity and continuity of $\delta\mapsto M^{(\delta)}(x,y)$ ensured by Theor. 9, there exists a unique real number $\mu\in[0,1]$ such that

$$M(M^u(x,y),M^v(x,y))=M^\mu(x,y).$$

Denoting by P to the point $(x, y) \in I^2$, this equality can be written in the form

$$M(M^u(x, y), M^v(x, y)) = M^{\mu_P(u,v)}(x, y), \tag{61}$$

in which the dependence of the number μ on the pair u, v as well as on $P = (x, y)$ has been emphasized. As shown in [3], $\mathcal{B}(M) = \{\mu_P : P \in I^2\}$ is a family of strict and continuous means defined on the unit interval $[0, 1]$. Every member μ_P belonging to the family $\mathcal{B}(M)$ is named a *base mean* of the mean M, while the entire family $\mathcal{B}(M)$ is said to be the *base* of M.

The base of a quasiarithmetic mean A_f can be easily computed. Indeed, using the expression of $A_f^{(\delta)}$ given by (21) it is obtained

$$\begin{aligned} A_f\left(A_f^{(u)}(x, y), A_f^{(v)}(x, y)\right) &= f^{-1}\left(\frac{f\left(f^{-1}\left((1-u) f(x) + v f(y)\right)\right) + f\left(f^{-1}\left((1-v) f(x) + v f(y)\right)\right)}{2}\right) \\ &= f^{-1}\left(\left(1 - \frac{u+v}{2}\right) f(x) + \frac{u+v}{2} f(y)\right), \ x, y \in I, \ u, v \in [0, 1], \end{aligned}$$

while

$$A_f^{\mu_P(u,v)}(x, y) = f^{-1}\left((1 - \mu_P(u, v)) x + \mu_P(u, v) y\right), \ x, y \in I, \ u, v \in [0, 1],$$

and therefore, the base mean

$$\mu_P(u, v) = \frac{u+v}{2} = A(u, v), \ u, v \in [0, 1],$$

does not depend on $P \in I^2$. A slightly more involved computation shows that a quasilinear mean L_f possesses a unitary base as well. Now, ¿what can be said on a two variables, strict an continuous mean M when its base is a unitary set? In [3] the following result was established.

Theorem 25 *Let $M \in \mathcal{CM}_2(I)$ be a differentiable strict mean. Then, the base mean of M is a unitary family if and only M is a quasilinear mean.*

Let us see that, with the help of Theor. 7, the differentiability hypothesis in the above statement can be omitted.

Theorem 26 *Let $M \in \mathcal{CM}_2(I)$ be a continuous and strict mean on a real interval I. Then, the base mean of M is a unitary family if and only M is a quasilinear mean.*

Proof. The "if" part of the proof proceeds along the same lines of the particular case in which M is quasiarithmetic. The details can be seen in [3]. To prove the converse, suppose that $\mathcal{B}(M) = \{\mu\}$ is a base of M; then, given a, $b \in I$, with $a < b$, (61) yields

$$M(M^u(a, b), M^v(a, b)) = M^{\mu(u,v)}(a, b), \ u, v \in [0, 1], \tag{62}$$

or, using the notation introduced in the statement of Theor. 9,

$$M\left(\phi_{(a,b)}(u),\phi_{(a,b)}(v)\right)=\phi_{(a,b)}(\mu(u,v)),\ u,v\in[0,1].$$

Similarly, choosing a pair of numbers α, $\beta\in[a,b]$, it can be written

$$M\left(\phi_{(\alpha,\beta)}(u),\phi_{(\alpha,\beta)}(v)\right)=\phi_{(\alpha,\beta)}(\mu(u,v)),\ u,v\in[0,1]. \tag{63a}$$

Now well, from (62) and (63a) it is deduced

$$\phi^{-1}_{(a,b)}\left(M\left(\phi_{(a,b)}(u),\phi_{(a,b)}(v)\right)\right)=\phi^{-1}_{(\alpha,\beta)}\left(M\left(\phi_{(\alpha,\beta)}(u),\phi_{(\alpha,\beta)}(v)\right)\right),\ u,v\in[0,1],$$

or, equivalently

$$\left(\phi_{(\alpha,\beta)}\circ\phi^{-1}_{(a,b)}\right)(M(x,y))=M\left(\left(\phi_{(\alpha,\beta)}\circ\phi^{-1}_{(a,b)}\right)(x),\left(\phi_{(\alpha,\beta)}\circ\phi^{-1}_{(a,b)}\right)(y)\right),\ x,y\in[a,b].$$

This equality expresses the fact that the function $f_{\alpha,\beta}:[a,b]\to[a,b]$ given by $f_{\alpha,\beta}=\phi_{(\alpha,\beta)}\circ\phi^{-1}_{(a,b)}$ is a continuous M-affine function. Moreover, since $f_{\alpha,\beta}(a)=\beta$ and $f_{\alpha,\beta}(b)=\alpha$, the arbitrariness of a,b,α and β shows that $\mathcal{AC}\left(M|_{[a_k,b_k]};[a_k,b_k]\right)$ is a Sturm-Liouville family and therefore, Theor. 7 implies that M is quasilinear. ■

5 n variables means

Due to its usefulness in proving Theor. 8, the following paragraphs deep into the connections existing among $Aff(I)$ and $AFF(I)$. First of all, note that the algebraic and topological structures of $Aff(I)$ and $AFF(I)$ find a natural correspondence through the bijective map $i:Aff(I)\to AFF(I)$ given by $i(m(\cdot)+h)=(m,h)$. In this way, if $Aff(I)$ is equipped with the topology of the uniform convergence on compact subsets of I while $AFF(I)$ is given the topology induced by the usual topology on $\mathbb{R}^2$, then the map i becomes a homeomorphism. On the other hand, the law defined on $AFF(I)$ by

$$(m_1,h_1)\bullet(m_2,h_2)=(m_1m_2,m_1h_2+h_1)$$

turns out to be an associative operation, and the map $i:\langle Aff(I),\circ\rangle\to\langle AFF(I),\bullet\rangle$ becomes an isomorphism of semigroups.

Now, if $\phi\in Aff(\mathbb{R})$, it is clear that $f\in Aff(I)$ if and only if $\phi\circ f\circ\phi^{-1}\in Aff(\phi(I))$, which is compactly expressed by the equality

$$Aff(\phi(I))=\phi\circ Aff(I)\circ\phi^{-1}. \tag{64}$$

Correspondingly,

$$AFF(\phi(I))=i(\phi)\bullet AFF(I)\bullet i\left(\phi^{-1}\right).$$

In particular, taking $I=[0,1]$ and $\phi(t)=(b-a)\,t+a$, it is obtained

$$\begin{aligned} AFF\left([a,b]\right) &= (b-a,a)\bullet AFF\left([0,1]\right)\bullet\left(\frac{1}{b-a},-\frac{a}{b-a}\right) \\ &= \left\{(b-a,a)\bullet(m,h)\bullet\left(\frac{1}{b-a},-\frac{a}{b-a}\right):(m,h)\in AFF\left([0,1]\right)\right\} \\ &= \left\{(m,-am+(b-a)\,h+a):(m,h)\in AFF\left([0,1]\right)\right\} \\ &= T_{a,b}\left(AFF\left([0,1]\right)\right), \end{aligned}$$

where $T:\mathbb{R}^2\rightarrow\mathbb{R}^2$ is the transformation given by

$$T_{a,b}\left(x,y\right)=\left(\begin{array}{cc} 1 & 0 \\ -a & b-a \end{array}\right)\left(\begin{array}{c} x \\ y \end{array}\right)+\left(\begin{array}{c} 0 \\ a \end{array}\right).$$

Since $T_{a,b}$ is affine and, as noted in Section 1, $AFF\left([0,1]\right)=\left\{(0,0),(1,0),(0,1),(-1,1)\right\}^{\wedge}$, it turns out to be

$$\begin{aligned} AFF\left([a,b]\right) &= \left\{T_{a,b}\left(0,0\right),T_{a,b}\left(1,0\right),T_{a,b}\left(0,1\right),T_{a,b}\left(-1,0\right)\right\}^{\wedge} \\ &= \left\{(0,a),(1,0),(0,b),(-1,a+b)\right\}^{\wedge}. \end{aligned} \tag{65}$$

The following result will play a relevant role in the proof of Theor. 8.

Proposition 27 *Let $I\neq\emptyset$ a real interval and $\mathcal{S}\neq\emptyset$ be a closed subset of $Aff\left(I\right)$ such that, for a given nested and exhaustive sequence $\left\{[a_k,b_k]:k\in\mathbb{N}\right\}$ of compact subintervals of I, the inclusion*

$$Aff\left([a_k,b_k]\right)\cap Aff\left(I\right)\subseteq\mathcal{S},$$

holds for every $k\in\mathbb{N}$. Then, the equality

$$\mathcal{S}=Aff\left(I\right)$$

holds provided that

i) *I is bounded or,*

ii) *I is unbounded and $\mathcal{S}$ is a subsemigroup of $Aff\left(I\right)$ with the property that, if $f\in\mathcal{S}$ and $f^{-1}\in Aff\left(I\right)$, then $f^{-1}\in\mathcal{S}$.*

Proof. After the equality (64), it will be sufficient to prove the proposition for the instances $I=[0,1]$, $[0,1)$, $(0,1)$, $[0,+\infty)$, $(0,+\infty)$, $\mathbb{R}$. Now, when $I=[0,1]$ there is nothing to prove, so that it must be considered only the five remaining cases. From these, the treatment of the instances $I=[0,1)$, $(0,1)$ reveals to be very similar, and the same occurs when $I=[0,+\infty)$, $(0,+\infty)$, so that a detailed argument is to be given below only for the three cases $I=[0,1)$, $[0,+\infty)$, $\mathbb{R}$. On the other hand, the isomorphism $i:\left\langle Aff\left(I\right),\circ\right\rangle\rightarrow\left\langle AFF\left(I\right),\bullet\right\rangle$ can be applied to derive an equivalent formulation of the proposition in terms of subsets of $\mathbb{R}^2$. After all these simplifications, the statement to be proved is the following: let I be one of the real intervals $[0,1)$, $[0,+\infty)$, $\mathbb{R}$, and let $S \neq \emptyset$ be a closed subset of $AFF(I)$ such that, for a certain nested and exhaustive sequence $\{[a_k, b_k] : k \in \mathbb{N}\}$ of compact subintervals of I, the inclusion

$$AFF([a_k, b_k]) \cap AFF(I) \subseteq S, \tag{66}$$

holds for every $k \in \mathbb{N}$. Then, the equality

$$S = AFF(I)$$

holds provided that $I = [0,1)$ or $I = [0,+\infty)$, $\mathbb{R}$ and S is a subsemigroup of $AFF(I)$ with the property that, if $f \in S$ and $f^{-1} \in AFF(I)$, then $f^{-1} \in S$. A separate analysis of the three cases follows.

$I = [0,1)$: In this case, a nested and exhaustive sequence of compact subintervals of I has the form $\{[0, b_k] : k \in \mathbb{N}\}$, where (b_k) is a sequence of numbers satisfying $0 < b_k < 1$, $k \in \mathbb{N}$, and $b_k \uparrow 1$. Thus, (65) yields

$$AFF([0, b_k]) = \{(0,0), (1,0), (0, b_k), (-1, b_k)\}^{\wedge}, \tag{67}$$

and taking into account that $AFF([0,1)) = AFF([0,1]) \cap \big\{(m,h) \in \mathbb{R}^2 : h < 1\big\}$, the inclusion (66) gives

$$\begin{aligned} S &\supseteq \bigcup_{k=1}^{+\infty} \left(AFF([0,b_k]) \cap AFF([0,1))\right) \\ &= AFF([0,1)) \cap \left\{(m,h) \in \mathbb{R}^2 : m + h < 1\right\}, \end{aligned}$$

or, since S is closed in $AFF([0,1))$,

$$S \supseteq \overline{\bigcup_{k=1}^{+\infty} \left(AFF([0,b_k]) \cap AFF([0,1))\right)} = AFF([0,1)).$$

The closure operator in the above equalities is taken with respect to relative topology induced on $AFF([0,1))$ by the usual topology of $\mathbb{R}^2$.

$I = [0,+\infty)$: Here, a generic nested and exhaustive sequence of compact subintervals of I is given by $\{[0, b_k] : k \in \mathbb{N}\}$, where (b_k) is a sequence satisfying $0 < b_k$, $k \in \mathbb{N}$, and $b_k \uparrow +\infty$. Accordingly, $AFF([0, b_k])$ is also given by (67), and taking into account that $AFF([0,+\infty)) = [0,+\infty)^2$, from (66) it is deduced

$$\begin{aligned} S &\supseteq \overline{\bigcup_{k=1}^{+\infty} \left(AFF([0,b_k]) \cap AFF([0,+\infty))\right)} \\ &= [0,1] \times [0,+\infty). \end{aligned}$$

As a consequence, $(m,0) \in S$ for every $0 < m < 1$ and, in view of $(m,0)^{-1} = \left(m^{-1}, 0\right) \in AFF([0,+\infty))$, it must be $\left(m^{-1}, 0\right) \in S$, so that

$(m,0) \in S$ for every $m \geq 0$. Finally, if $(m,h) \in AFF\left([0,+\infty)\right)$, it can be written

$$(m,h) = (m,0) \bullet (1, h/m)$$

where $(m,0)\,,\,(1,h/m) \in S$, and thus $(m,h) \in S$.

$I = \mathbb{R}:$ $\{[a_k,b_k] : k \in \mathbb{N}\}$, where (a_k) and (b_k) are sequences of real numbers satisfying $a_k < b_k,\ k \in \mathbb{N}$, and $a_k \downarrow -\infty,\ b_k \uparrow +\infty$, is a generic nested and exhaustive sequence of compacts subintervals of I. For this sequence, (65) yields

$$AFF\left([a_k,b_k]\right) = \left\{(0,a_k)\,,(1,0)\,,(0,b_k)\,,(-1,a_k+b_k)\right\}^{\wedge}.$$

It can be seen that

$$\overline{\bigcup_{k=1}^{+\infty}\left(AFF\left([a_k,b_k]\right) \cap AFF\left(\mathbb{R}\right)\right)} \supseteq \left\{(-1,a_1+b_1)\right\} \cup \left([0,1]\times\mathbb{R}\right),$$

(where the closure operator is the closure on $\mathbb{R}^2$,) so that

$$S \supseteq \left\{(-1,a_1+b_1)\right\} \cup \left([0,1]\times\mathbb{R}\right).$$

It is shown like in the case $I = [0,+\infty)$ that $(m,h) \in S$ for every $m \geq 0$ and $h \in \mathbb{R}$. Now, if $m < 0$ and $h \in \mathbb{R}$, then $(-m,0)\,,\,(-1,a_1+b_1)\,,\,(1,h/m+a_1+b_1) \in S$, and the equality

$$(m,h) = (-m,0) \bullet (-1,a_1+b_1) \bullet (1,h/m+a_1+b_1)\,,$$

shows that $(m,h) \in S$. It has been thus proved that $S = \mathbb{R}^2 = AFF\left(\mathbb{R}\right)$.

■

Proof of Theor. 8. Let $M \in \mathcal{CM}_n(I)$ be a continuous and strict mean defined on I. Suppose that $\{[a_k,b_k] : k \in \mathbb{N}\}$ is a nested and exhaustive sequence of compact subintervals of I, and that $\mathcal{AC}\left(M|\,;[a_k,b_k]\right)$ is a Sturm-Liouville family for every $k \in \mathbb{N}$. Clearly, the two variables mean N defined on I by (23) satisfies the hypotheses of Teor. 7, so that there exist both a strictly monotonic and continuous function $\phi : I \to \mathbb{R}$ and a number $w \in (0,1)$ such that

$$N(x,y) = \phi^{-1}\left((1-w)\,\phi(x) + w\phi(y)\right),\ x,y \in I. \tag{68}$$

Now, consider the mean M_ϕ conjugated of M by ϕ; i.e.,

$$M_\phi(x_1,\ldots,x_n) = \phi\left(M\left(\phi^{-1}(x_1),\ldots,\phi^{-1}(x_n)\right)\right),\ x_1,\ldots,x_n \in \phi(I). \tag{69}$$

The inclusion

$$\mathcal{S} = \mathcal{AC}\left(M_\phi;\phi(I)\right) \subseteq \mathcal{AC}\left(N_\phi;\phi(I)\right) = \mathrm{Aff}\left(\phi(I)\right)$$

follows from a simple specialization of the variables in the equation (7). Let us see that the above inclusion is really an equality. With this purpose, let us note that

$$Aff\left(\left[\phi\left(a_k\right),\phi\left(b_k\right)\right]\right)\cap Aff\left(\phi\left(I\right)\right)\subseteq\mathcal{S}.$$

recall from Section 1 that,

In view of the equality

$$\mathcal{AC}\left(\left.M_\phi\right|_{\left[\phi(a_k),\phi(b_k)\right]};\left[\phi\left(a_k\right),\phi\left(b_k\right)\right]\right)=\phi\circ\mathcal{AC}\left(\left.M\right|_{\left[a_k,b_k\right]};\left[a_k,b_k\right]\right)\circ\phi^{-1},$$

$\mathcal{AC}\left(\left.M_\phi\right|_{\left[\phi(a_k),\phi(b_k)\right]};\left[\phi\left(a_k\right),\phi\left(b_k\right)\right]\right)\subseteq\mathcal{AC}\left(\left.N_\phi\right|_{\left[\phi(a_k),\phi(b_k)\right]};\left[\phi\left(a_k\right),\phi\left(b_k\right)\right]\right)=$ $Aff\left(\left[\phi\left(a_k\right),\phi\left(b_k\right)\right]\right)$ is a Sturm-Liouville family, thus $\mathcal{AC}\left(\left.M_\phi\right|_{\left[\phi(a_k),\phi(b_k)\right]};\left[\phi\left(a_k\right),\phi\left(b_k\right)\right]\right)=$ $Aff\left(\left[\phi\left(a_k\right),\phi\left(b_k\right)\right]\right)$

$$\mathcal{AC}\left(\left.M_\phi\right|_{\left[\phi(a_k),\phi(b_k)\right]};\left[\phi\left(a_k\right),\phi\left(b_k\right)\right]\right)\subseteq\mathcal{AC}\left(\left.N_\phi\right|_{\left[\phi(a_k),\phi(b_k)\right]};\left[\phi\left(a_k\right),\phi\left(b_k\right)\right]\right)=Aff\left(\left[\phi\left(a_k\right),\phi\left(b_k\right)\right]\right),$$

the set $\mathcal{S}$ satisfies the inclusion

$$Aff\left(\left[\phi\left(a_k\right),\phi\left(b_k\right)\right]\right)\cap Aff\left(\phi\left(I\right)\right)\subseteq\mathcal{S}$$

for the sequence $\left\{\left[\phi\left(a_k\right),\phi\left(b_k\right)\right]:k\in\mathbb{N}\right\}$. Since ϕ is strictly monotonic and continuous, $\left\{\left[\phi\left(a_k\right),\phi\left(b_k\right)\right]:k\in\mathbb{N}\right\}$ is a nested and exhaustive sequence of compact subintervals of $\phi\left(I\right)$. On the other hand, $\mathcal{S}$ is clearly closed subsemigroup of $\mathrm{Aff}\left(\phi\left(I\right)\right)$ and, moreover, if $(m,h)\in\mathcal{S}$ for a certain $(m,h)\in AFF\left(\phi\left(I\right)\right)$ such that $(m,h)^{-1}=(1/m,-h/m)\in AFF\left(\phi\left(I\right)\right)$, then the equality

$$M_\phi\left(mx_1+h,\ldots,mx_n+h\right)=mM_\phi\left(x_1,\ldots,x_n\right)+h,\ x_1,\ldots,x_n\in\phi\left(I\right),\quad(70)$$

holds for every $x_1,\ldots,x_n\in\phi\left(I\right)$, so that the substitutions $x_i=\left(y_i-h\right)/m,\ i=1,\ldots,n$, yield

$$\frac{1}{m}M_\phi\left(y_1,\ldots,y_n\right)-\frac{h}{m}=M_\phi\left(\left(y_i-h\right)/m,\ldots,\left(y_n-h\right)/m\right),\ y_1,\ldots,y_n\in\phi\left(I\right),$$

which shows that $(m,h)\in\mathcal{S}$. Since the constants are contained $\mathcal{S}$, all the hypotheses of Prop. 27 are fulfilled by $\mathcal{S}$, and therefore $\mathcal{S}=Aff\left(\phi\left(I\right)\right)$, as affirmed.

Finally, assuming that the mean M is differentiable, the mean N given by (68) turns out to be differentiable as well, and then, the generator ϕ is differentiable in I by Prop. 1. In consequence, the conjugated mean M_ϕ is differentiable and $\mathcal{AC}\left(M_\phi;\phi\left(I\right)\right)=Aff\left(\phi\left(I\right)\right)$. By Prop. 2, M_ϕ turns out to be a linear mean on $\phi\left(I\right)$, and therefore, M is quasilinear on I. This completes the proof. ■

Before finishing this section, a result is proved which will show its usefulness in the next one.

Proposition 28 *Let I be a real interval with* $\operatorname{int}(I) \neq \emptyset$*. The inclusion*

$$\phi \circ Aff\,(I) \subseteq Aff\,(\phi\,(I)) \circ \phi \tag{71}$$

holds for a continuous function $\phi : I \to \mathbb{R}$ if and only if ϕ is affine.

Proof. First assume that ϕ is affine; i.e., for a certain pair $p,\ q \in \mathbb{R}$, $\phi\,(t) = pt + q,\ \ t \in I$. A generic $f \in \phi \circ Aff\,(I)$ has the form $f\,(t) = p\,(mt + h) + q$, with $(m, h) \in AFF\,(I)$ and thus, setting $\mu = m$ and $\nu = ph - mq + q$, it turns out to be

$$f\,(t) = p\,(mt + h) + q = \mu\,(pt + q) + \nu.$$

Let us see that $(\mu, \nu) \in AFF\,(\phi\,(I))$. In fact, since $(m, h) \in AFF\,(I)$, it can be written

$$\begin{aligned} \mu\phi\,(I) + \nu &= m\,(pI + q) + ph - mq + q \\ &= p\,(mI + h) + q \\ &\subseteq pI + q \\ &= \phi\,(I)\,. \end{aligned}$$

In consequence, $f \in Aff\,(\phi\,(I)) \circ \phi$ and the inclusion (71) follows. Conversely, if (71) holds for a continuous function $\phi : I \to \mathbb{R}$, then for every $(m, h) \in AFF\,(I)$ there exists $(\mu\,(m, h)\,, \nu\,(m, h)) \in AFF\,(\phi\,(I))$ such that

$$\phi\,(mt + h) = \mu\,(m, h)\,\phi\,(t) + \nu\,(m, h)\,,\ \ t \in I. \tag{72}$$

Since $\operatorname{int}(I) \neq \emptyset$, (72) can be evaluated at two different points $t_0, t_1 \in I$, $t_0 < t_1$, to obtain

$$\begin{cases} \phi\,(mt_0 + h) = \mu\,(m, h)\,\phi\,(t_0) + \nu\,(m, h) \\ \phi\,(mt_1 + h) = \mu\,(m, h)\,\phi\,(t_1) + \nu\,(m, h) \end{cases},$$

whence

$$\mu\,(m, h) = \frac{\phi\,(mt_1 + h) - \phi\,(mt_0 + h)}{\phi\,(t_1) - \phi\,(t_0)},\ \nu\,(m, h) = \phi\,(mt_0 + h) - \frac{\phi\,(mt_1 + h) - \phi\,(mt_0 + h)}{\phi\,(t_1) - \phi\,(t_0)}\phi\,(t_0)\,.$$

Introducing these expressions for μ and ν in (72), it is obtained

$$\phi\,(mt + h) = \frac{\phi\,(mt_1 + h) - \phi\,(mt_0 + h)}{\phi\,(t_1) - \phi\,(t_0)}\,(\phi\,(t) - \phi\,(t_0)) + \phi\,(mt_0 + h)\,,\ \ t \in I,$$

an equality which, when expressed in terms of the function $\psi : I \to \mathbb{R}$ defined by

$$\psi\,(t) = \frac{\phi\,(t) - \phi\,(t_0)}{\phi\,(t_1) - \phi\,(t_0)}, \tag{73}$$

becomes

$$\psi\,(mt + h) = (1 - \psi\,(t))\,\psi\,(mt_0 + h) + \psi\,(t)\,\psi\,(mt_1 + h)\,,\ t \in I,\ (m, h) \in AFF\,(I)\,. \tag{74}$$

Now, in view of the identity,

$$mt+h=\left(1-\frac{t-t_0}{t_1-t_0}\right)(mt_0+h)+\frac{t-t_0}{t_1-t_0}(mt_1+h),$$

and the fact that $[t_0,t_1]\subseteq I$, the equality

$$\psi\left(\left(1-\frac{t-t_0}{t_1-t_0}\right)x_0+\frac{t-t_0}{t_1-t_0}x_1\right)=(1-\psi(t))\,\psi(x_0)+\psi(t)\,\psi(x_1)\,,\ t\in\left[\dot{t}_0,t_1\right],\ x_0,x_1\in I,$$

can be easily derived from (74), and then ([1], Theor. 2, pg. 67, or [14], pg. 382 and ff.)

$$\psi(t)=\frac{t-t_0}{t_1-t_0},\ t\in\left[\dot{t}_0,t_1\right].$$

This equality and (73) yields

$$\frac{\phi(t)-\phi(t_0)}{\phi(t_1)-\phi(t_0)}=\frac{t-t_0}{t_1-t_0},\ t\in\left[\dot{t}_0,t_1\right],$$

so that ϕ is affine in every subinterval $\left[\dot{t}_0,t_1\right]\subseteq I$, which implies that ϕ is affine.
■

6 Final remarks

The nice properties of Aczél's iterations ensured by Theor. 9 can not extended to non strict means (even though the continuity can be somewhat relaxed). Further studies on Aczél´s iterations in absence of strictness or continuity are contained in [6]. In view of the basic role played by Theor. 9, there is no hope that the main results stated in Section 1 continue to be true for non strict means. For instance, if M is a two variables continuous mean defined on the compact interval $[a,b]$ such that $M(a,b)=a$ and $\mathcal{AC}(M;[a,b])$ is a Sturm-Liouville family, then, for every α, $\beta\in I$, there exists $f\in\mathcal{AC}(M;[a,b])$ satisfying $f(a)=\alpha$ and $f(b)=\beta$, and therefore

$$M(\alpha,\beta)=M(f(a),f(b))=f(M(a,b))=f(a)=\alpha,\ \alpha,\beta\in I,$$

which shows that M is not quasilinear.

The generalization of other results in this paper is possible. Particularly, Prop. 22 can be extended without difficulty to n variables means. However, Prop. 24 is no longer valid for n variables means when $n>2$, whereas a generalization of the concept of base to n variables means is a dubbious question. The main theorems can be restated for symmetric means by simply introducing the word "quasiarithmetic" instead of the word "quasilinear" wherever this last appears in a statement.

Results like Matkowski´s theorem 3 or Theors. 6, 7 and 8 in this paper can be used to determine the whole families of affine or continuous affine functions of a given mean. This fact is illustrated by the following examples.

Example 29 *The* counterharmonic *mean* is *defined by*

$$CH(x,y) = \frac{x^2+y^2}{x+y},\ x,y>0.$$

Note that the CH-affine functions $f^t(u) = tu,\ t>0$, make up a continuous iteration group (with generator $\gamma(u) = u,\ u>0$), and that $h(u) = CH(u,1)$, $u>0$, turns out to be a rational infinite differentiable function with $0 \neq h'(1) \neq 1$. Then, the hypotheses of Matkowski´s Theor. 3 are satisfied by CH and, taking into account that CH is not a quasiarithmetic mean, it follows that $\mathcal{AC}(CH;\mathbb{R}^+) = \{f^t : t>0\} \cup \{f=c : c>0\}$.

Example 30 *Consider a strict and continuous mean M defined on $\mathbb{R}$ for which the inclusion*

$$Aff(\mathbb{R}) \subseteq \mathcal{AC}(M;\mathbb{R}) \tag{75}$$

is satisfied. The mean M defined by (10) and (10) in Section 1 serves as example of this kind of means. Let us see that (75) must be really an equality. Since $Aff([a,b]) \subseteq Aff(\mathbb{R})$ for every compact interval $[a,b]$, the hypotheses of Theor. 8 are fulfilled by M and then, there exists a strictly increasing and continuous function $\phi : \mathbb{R} \to \mathbb{R}$ such that

$$\mathcal{AC}(M_\phi;\phi(\mathbb{R})) = Aff(\phi(\mathbb{R})),$$

or, equivalently,

$$\mathcal{AC}(M;\mathbb{R}) = \phi^{-1} \circ Aff(\phi(\mathbb{R})) \circ \phi. \tag{76}$$

From (75) and (76) it is deduced

$$\phi \circ Aff(\mathbb{R}) \subseteq Aff(\phi(\mathbb{R})) \circ \phi,$$

an inclusion which, after Prop. 28, implies that ϕ is affine. Clearly, ϕ does not reduce to a constant, so that $\phi^{-1} \circ Aff(\phi(\mathbb{R})) \circ \phi = Aff(\mathbb{R})$ and then,

$$\mathcal{AC}(M;\mathbb{R}) = Aff(\mathbb{R}),$$

as affirmed.

References

[1] J. Aczél, *Lectures on Functional Equations and their Applications*, Academic Press, New York and London, (1966).

[2] J. Aczél, J. Dhombres, *Functional Equations in Several Variables*, Cambridge Univ. Press, Cambridge, 1989.

[3] L. R. Berrone, *A dynamical characterization of quasilinear means*, Aequationes Math. 84, 1, (2012), 51-70.

[4] L. R. Berrone, A. L. Lombardi, *A note on equivalence of means*, Publ. Math. Debrecen **58**, Fasc. **1-2**, (2001), 49-56.

[5] L. R. Berrone, G. E. Sbérgamo, *La familia de bases de una media continua y la representación de las medias cuasiaritméticas*, Bol. de la Soc. Venezolana de Matemática, Vol. XIX, No. 1, (2012), 3-18.

[6] L. R. Berrone, G. E. Sbérgamo, *Aczél´s iterations for discontinuous or non strict means*, J. of Analysis, (to appear).

[7] J.M. Borwein, P.B. Borwein, *Pi and the AGM,* John Wiley and Sons, New York, 1987.

[8] P. S. Bullen, *Handbook of Means and Their Inequalities*, Kluwer Academic Publishers, Dordrecht, 2010.

[9] P. S. Bullen, D. S. Mitrinović, P. M. Vasić, *Means and Their Inequalities*, D. Reidel Publishing Company, Dordrecht, 1988.

[10] J. G. Dhombres, *Some recent applications of functional equations*, in "Functional Equations: History, Applications and Theory", J. Aczél (Ed.), D. Reidel, Dordrecht, (1984), 67-91.

[11] R. Ger, T. Kochanek, *An inconsistency equation involving means*, Colloq. Math. **115** (**1**), 87-99, (2009).

[12] G. H. Hardy, J. E. Littlewood, G. Pólya, *Inequalities*, Cambridge Univ. Press., Cambridge, 1934.

[13] R. L. Kruse and J. J. Deely, *Joint continuity of monotonic functions,* Amer. Math. Monthly **76**, No. **1,** (1969), 74-76.

[14] M. Kuczma, *An Introduction to the Theory of Functional Equations and Inequalities*, Birkhäuser, Basel-Boston-Berlin, 2nd. Ed., 2009.

[15] C.H. Ling, *Representation of associative functions,* Publ. Math. Debrecen **12**, Fasc. **1-4**, (1965), 189-212.

[16] J.-L. Marichal, *On the associativity functional equation*, Fuzzy Sets and Systems, **114** (**3**), (2000), 381-389.

[17] J. Matkowski, *Convex functions with respect to a mean and a characterization of quasi-arithmetic means*, Real Analysis Exchange, Vol. **29** (**1**), 2003/2004, 229-246.