Journal of Knot Theory and Its Ramifications
Vol. 27, No. 10 (2018) 1850053 (15 pages)

DOI: 10.1142/S0218216518500530

On unrolled Hopf algebras

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Received 6 February 2017
Accepted 20 February 2018
Published 27 July 2018

ABSTRACT

We show that the definition of unrolled Hopf algebras can be naturally extended to the Nichols algebra $\mathcal{B}(V)$ of a Yetter–Drinfeld module V on which a Lie algebra $\mathfrak{g}$ acts by biderivations. As a special case, we find unrolled versions of the small quantum group.

Keywords: Unrolled quantum groups; (pre-, post-)Nichols algebras; Gelfand–Kirillov dimension.

Mathematics Subject Classification 2010: 16W30

1. Introduction

1.1.

In the recent papers [9, 11], a so-called unrolled version of quantum sl(2) was introduced, with applications to quantum topology; the definition was generalized to simple finite-dimensional Lie algebras in [10]. In this paper, we propose a generalization of this notion and embed it into the appropriate conceptual context.

Recall that the unrolled quantum sl(2) is defined as the smash product of $U_q(\mathrm{sl}(2))$ by the universal enveloping algebra of the Lie algebra of dimension 1. Our starting point is the observation in Lemma 2.6: given an action of the universal enveloping algebra $U(\mathfrak{g})$ of a Lie algebra $\mathfrak{g}$ on a Hopf algebra H, the smash product is a Hopf algebra, if and only if $\mathfrak{g}$ acts on H by biderivations. We next observe that, if V is a Yetter–Drinfeld module over a group G, then the Lie algebra $\mathfrak{bd}_V := \mathrm{End}^G_G(V)$

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of endomorphisms of the Yetter–Drinfeld module V acts by biderivations on the Nichols algebra $\mathcal{B}(V)$. Hence, we can form the Hopf algebra $(\mathcal{B}(V)\#\Bbbk G) \rtimes U(\mathfrak{bd}_V)$ which we call the *unrolled bosonization* of V. If $\dim V$ is finite, then its Gelfand–Kirillov dimension can be expressed in terms of the Gelfand–Kirillov dimension of $\mathcal{B}(V)$ and the dimension of $\mathfrak{bd}_V$.

The construction of unrolled bosonizations extends to a Lie subalgebra $\mathfrak{g}$ of $\mathfrak{bd}_V$, pre- or post-Nichols algebras (in the place of $\mathcal{B}(V)$), and to deformations thereof, provided that the action of the Lie algebra $\mathfrak{g}$ preserves the relevant defining relations. In particular, we define the unrolled version of the quantum double of a finite-dimensional Nichols algebra of diagonal type.

1.2. *Preliminaries*

Fix a field $\Bbbk$ and let H be a Hopf algebra over $\Bbbk$. We use standard notation: Δ, ε, $\mathcal{S}$, $\overline{\mathcal{S}}$ are respectively the comultiplication, the counit, the antipode (always assumed to be bijective) and the inverse of the antipode.

We denote by ${}^H_H\mathcal{YD}$ the category of Yetter–Drinfeld modules over H as in [5]. For $V, W \in {}^H_H\mathcal{YD}$, we denote by $\mathrm{Hom}^H_H(V, W)$, $\mathrm{End}^H_H(V)$, $\mathrm{Aut}^H_H(V)$ the spaces of morphisms, respectively endomorphisms, automorphisms in ${}^H_H\mathcal{YD}$. Let R be a Hopf algebra in the braided monoidal category ${}^H_H\mathcal{YD}$, with comultiplication denoted by $r \mapsto r^{(1)} \otimes r^{(2)}$. Recall that the *bosonization* $R\#H$ is the Hopf algebra over $\Bbbk$ with underlying vector space $R \otimes H$, smash product multiplication and smash coproduct comultiplication; i.e. for all $r, s \in R$, $a, b \in H$,

$$(r\#a)(s\#b) = r(a_{(1)} \cdot s)\#a_{(2)}b, \tag{1.1}$$

$$\Delta(r\#a) = r^{(1)}\#(r^{(2)})_{(-1)}a_{(1)} \otimes (r^{(2)})_{(0)}\#a_{(2)}. \tag{1.2}$$

Here we write $r\#h$ for $r \otimes h$.

We also introduce the category $\mathcal{YD}^H_H = {}^{H^{\mathrm{bop}}}_{H^{\mathrm{bop}}}\mathcal{YD}$ of *right–right* Yetter–Drinfeld modules over H. Thus $M \in \mathcal{YD}^H_H$ means that M is a right H-module and a right H-comodule (with coaction ϱ), and satisfies the compatibility axiom

$$\varrho(m \cdot h) = m_{(0)} \cdot h_{(2)} \otimes \mathcal{S}(h_{(1)})m_{(1)}h_{(3)}, \quad m \in M, \ \ h \in H. \tag{1.3}$$

The tensor category $\mathcal{YD}^H_H$ is braided, with braiding $c(m \otimes n) = n \cdot m_{(1)} \otimes m_{(0)}$, for all $m \in M$, $n \in N$, $M, N \in \mathcal{YD}^H_H$. For right–right Yetter–Drinfeld modules $V, W \in \mathcal{YD}^H_H$, we use the notions $\mathrm{Hom}^H_H(V, W)$, $\mathrm{End}^H_H(V)$, $\mathrm{Aut}^H_H(V)$ are as before.

Let T be a Hopf algebra in the braided monoidal category $\mathcal{YD}^H_H$ of right–right Yetter–Drinfeld modules, with comultiplication denoted by $t \mapsto t^{(1)} \otimes t^{(2)}$. In this case, the *bosonization* $H\#T$ is the Hopf algebra over $\Bbbk$ with underlying vector space $H \otimes T$, smash product multiplication and smash coproduct comultiplication; i.e.

$$(a\#t)(b\#u) = ab_{(1)}\#(t \cdot b_{(2)})u, \tag{1.4}$$

$$\Delta(a\#t) = a_{(1)}\#(t^{(1)})_{(0)} \otimes a_{(2)}(t^{(1)})_{(1)}\#t^{(2)}, \tag{1.5}$$

for all $t, u \in R$, $a, b \in H$. Here we write $h\#t$ for $h \otimes t$.

If Γ is an abelian group, then we denote by $\Bbbk_g^\chi$ the one-dimensional object in ${}^{\Bbbk\Gamma}_{\Bbbk\Gamma}\mathcal{YD}$ with coaction given by the group element $g \in \Gamma$ and action given by the character $\chi \in \widehat{\Gamma}$. For a Yetter–Drinfeld module $V \in {}^{\Bbbk\Gamma}_{\Bbbk\Gamma}\mathcal{YD}$, the corresponding isotypic component is denoted by V_g^χ. A Yetter–Drinfeld module has a natural structure of a braided vector space. For a braided vector space V, denote by $\mathcal{B}(V)$ its Nichols algebra and by $\mathcal{J} = \mathcal{J}(V)$ its ideal of defining relations, cf. [5]; so that $\mathcal{B}(V) \simeq T(V)/\mathcal{J}(V)$.

2. Unrolled Hopf Algebras

2.1.

Let L be a Hopf algebra. Recall that a (left) L-module algebra is an algebra A which is also an L-module with action $\cdot : L \otimes A \to A$ such that for all $\ell \in L$ and all $a, b \in A$ the compatibility conditions

$$\ell \cdot (ab) = (\ell_{(1)} \cdot a)(\ell_{(2)} \cdot b), \tag{2.1}$$

$$\ell \cdot 1 = \varepsilon(\ell)1 \tag{2.2}$$

for product and unit hold. It is well-known that (2.1) and (2.2) mean that A is an algebra in the monoidal category ${}_L\mathcal{M}$ of left L-modules.

In this paper, we are interested in the case of a Hopf algebra H that is also an L-module algebra, where L is a Hopf algebra as well. In this case, we impose the following consistency conditions:

$$\Delta(\ell \cdot a) = \ell_{(1)} \cdot a_{(1)} \otimes \ell_{(2)} \cdot a_{(2)}, \tag{2.3}$$

$$\varepsilon(\ell \cdot a) = \varepsilon(\ell)\varepsilon(a), \tag{2.4}$$

$$\ell_{(1)} \otimes \ell_{(2)} \cdot a = \ell_{(2)} \otimes \ell_{(1)} \cdot a, \tag{2.5}$$

for all $\ell \in L$ and all $a, b \in H$. Then $H \rtimes L := H \otimes L$ with the tensor product structure as a coalgebra and with the smash product (1.1) for the algebra structure is a Hopf algebra; see [15; 4, 1.2.10] (in this second paper a different notation is used). We shall say that H is a *L-module Hopf algebra*.

Remark 2.1. The following perspective shows that it is natural to impose these consistency conditions. The category ${}_L\mathcal{M}$ of left L-modules is monoidal, but not braided; thus H cannot be interpreted as a Hopf algebra in ${}_L\mathcal{M}$. Still, it can be interpreted in terms of monads. Recall that A has the structure of an algebra in the monoidal category ${}_L\mathcal{M}$ of left L-modules, if and only if the endofunctor $T : {}_L\mathcal{M} \to {}_L\mathcal{M}$, $T(X) = A \otimes X$ has the structure of a monad.

Also recall [8] that a bimonad structure on a monad T on a monoidal category consists of a comonoidal structure on the functor T, i.e. a natural transformation

$$T_2 : T(X \otimes Y) = H \otimes (X \otimes Y) \to T(X) \otimes T(Y) = (H \otimes X) \otimes (H \otimes Y),$$

and a morphism $T_0 : T(1) \to 1$. They have to obey axioms generalizing coassociativity and counitality. If H is a bialgebra in a braided monoidal category, the monad $T(-) = H \otimes -$ can be endowed via the coproduct $\Delta : H \to H \otimes H$ with the natural transformation

$$T_2(a \otimes x \otimes y) = (a_{(1)} \otimes x) \otimes (a_{(2)} \otimes y),$$

where we used Sweedler notation for Δ. The morphism T_0 is induced from the counit $\varepsilon : H \to \Bbbk$.

Now let L be another Hopf algebra and H be an L-module algebra. The fact that T_2 is a morphism in ${}_L\mathcal{M}$ is then equivalent to the consistency conditions (2.3) and (2.5), while condition (2.4) amounts to the fact that ε is a morphism in ${}_L\mathcal{M}$. Thus $T(-) = H \otimes -$ is a bimonad on the monoidal category ${}_L\mathcal{M}$, if and only if the requirements (2.3)–(2.5) hold. It is a Hopf monad, if and only if H is a Hopf algebra. The Hopf monad in $\mathrm{Vec}_\Bbbk$ (i.e. Hopf algebra) $H \rtimes L$ corresponds to the forgetful functor as described in [8, Proposition 4.3].

Remark 2.2. Here is another way to interpret $H \rtimes L$, dual to [4, 1.1.5]. Let H be a L-module Hopf algebra. Then H, endowed with the trivial coaction, is a Hopf algebra in ${}^L_L\mathcal{YD}$ and $H \rtimes L \simeq H\#L$. Indeed, (2.5) is equivalent to the compatibility in ${}^L_L\mathcal{YD}$.

2.2.

Now turn to the situation of two Hopf algebras H and U, provided with a non-degenerate bilinear form $(\,|\,) : H \otimes U \to \Bbbk$. We extend this bilinear form to a non-degenerate bilinear form $(\,|\,) : H \otimes H \otimes U \otimes U \to \Bbbk$ by

$$(a \otimes \widetilde{a} \,|\, u \otimes \widetilde{u}) := (a \,|\, \widetilde{u})(\widetilde{a} \,|\, u), \quad \text{for } a, \widetilde{a} \in H,\ \ u, \widetilde{u} \in U. \tag{2.6}$$

We assume that the pairing $(\,|\,)$ is such that for every $a, \widetilde{a} \in H$, $u, \widetilde{u} \in U$, the following identities hold

$$(a\widetilde{a} \,|\, u) = (a \otimes \widetilde{a} \,|\, \Delta(u)) = (a \,|\, u_{(2)})(\widetilde{a} \,|\, u_{(1)}), \quad (1 \,|\, u) = \epsilon(u), \tag{2.7}$$

$$(a \,|\, u\widetilde{u}) = (\Delta(a) \,|\, u \otimes \widetilde{u}) = (a_{(2)} \,|\, u)(a_{(1)} \,|\, \widetilde{u}), \quad (a \,|\, 1) = \epsilon(a), \tag{2.8}$$

$$(\mathcal{S}(a) \,|\, u) = (a \,|\, \mathcal{S}(u)). \tag{2.9}$$

Such a pairing is called a Hopf pairing on H and U.

Lemma 2.3. *Assume that the two Hopf algebras H and U are L-modules and that there is a Hopf pairing on H and U. Assume that the pairing is compatible with the L-action involving the antipode of L,*

$$(\ell \cdot a \,|\, u) = (a \,|\, \mathcal{S}(\ell) \cdot u), \quad a \in H,\ \ u \in U,\ \ \ell \in L. \tag{2.10}$$

Then the Hopf algebra H is an L-module Hopf algebra, if and only if U is so.

Proof. Let $\ell \in L$, $u, v \in U$ and $a \in H$. We compute

$$\begin{aligned}(a \,|\, \ell \cdot (uv)) &= (\overline{\mathcal{S}}(\ell) \cdot a \,|\, uv) = ((\overline{\mathcal{S}}(\ell) \cdot a)_{(2)} \,|\, u)((\overline{\mathcal{S}}(\ell) \cdot a)_{(1)} \,|\, v);\\ (a \,|\, (\ell_{(1)} \cdot u)(\ell_{(2)} \cdot v)) &= (a_{(2)} \,|\, \ell_{(1)} \cdot u)(a_{(1)} \,|\, \ell_{(2)} \cdot v)\\ &= (\overline{\mathcal{S}}(\ell_{(1)}) \cdot a_{(2)} \,|\, u)(\overline{\mathcal{S}}(\ell_{(2)}) \cdot a_{(1)} \,|\, v)\\ &= (\overline{\mathcal{S}}(\ell)_{(2)} \cdot a_{(2)} \,|\, u)(\overline{\mathcal{S}}(\ell)_{(1)} \cdot a_{(1)} \,|\, v).\end{aligned}$$

Hence (2.1) holds for U if and only if $(a \,|\, \ell \cdot (uv)) = (a \,|\, (\ell_{(1)} \cdot u)(\ell_{(2)} \cdot v))$ for all $\ell \in L$, $u, v \in U$, $a \in H$, if and only if $((\widetilde{\ell} \cdot a)_{(2)} \,|\, u)((\widetilde{\ell} \cdot a)_{(1)} \,|\, v) = (\widetilde{\ell}_{(2)} \cdot a_{(2)} \,|\, u)(\widetilde{\ell}_{(1)} \cdot a_{(1)} \,|\, v)$ for all $\widetilde{\ell} \in L$, $u, v \in U$, $a \in H$, if and only if (2.3) holds for H. Thus (2.1) holds for H if and only if (2.3) holds for U.

Similarly (2.2) holds for U if and only if (2.4) holds for H and vice versa. Finally, (2.5) holds for H if and only if it holds for U:

$$\begin{aligned}\ell_{(1)} \otimes \ell_{(2)} \cdot u = \ell_{(2)} \otimes \ell_{(1)} \cdot u, \quad \forall\, u \Leftrightarrow \overline{\mathcal{S}}(\ell_{(1)})(a \,|\, \otimes\, \ell_{(2)} \cdot u)\\ = \overline{\mathcal{S}}(\ell_{(2)})(a \,|\, \ell_{(1)} \cdot u), \quad \forall\, u, a \Leftrightarrow \overline{\mathcal{S}}(\ell_{(1)})(\overline{\mathcal{S}}(\ell_{(2)}) \cdot a \,|\, u),\\ = \overline{\mathcal{S}}(\ell_{(2)})(\overline{\mathcal{S}}(\ell_{(1)}) \cdot a \,|\, u), \quad \forall\, u, a \Leftrightarrow \overline{\mathcal{S}}(\ell)_{(2)}(\overline{\mathcal{S}}(\ell)_{(1)} \cdot a \,|\, u)\\ = \overline{\mathcal{S}}(\ell)_{(1)}(\overline{\mathcal{S}}(\ell)_{(2)} \cdot a \,|\, u,), \quad \forall\, u, a \Leftrightarrow \overline{\mathcal{S}}(\ell)_{(2)} \otimes \overline{\mathcal{S}}(\ell)_{(1)} \cdot a\\ = \overline{\mathcal{S}}(\ell)_{(1)} \otimes \overline{\mathcal{S}}(\ell)_{(2)} \cdot a, \quad \forall\, a.\end{aligned}$$ □

2.3.

We next extend our construction to Hopf algebras in braided monoidal categories. To this end, let now K be a Hopf algebra, $\mathcal{B}$ a Hopf algebra in the braided category ${}^{K}_{K}\mathcal{YD}$. Let L be another Hopf algebra as before, and assume that $\mathcal{B}$ is also an L-module algebra. We extend the action of the Hopf algebra L to the bosonization $H := \mathcal{B}\#K$ by $\ell \cdot (b\#k) := (\ell \cdot b)\#k$, for $\ell \in L$, $b \in \mathcal{B}$ and $k \in K$:

Then straightforward verifications show that:

- The bosonization H is a L-module algebra ⇔ The actions of L and K on $\mathcal{B}$ commute.
- Equation (2.4) holds for H ⇔ (2.4) holds for $\mathcal{B}$.
  From now on, we assume that this is the case.
- Equation (2.3) holds for H ⇔ (2.3) holds for $\mathcal{B}$ and the action of ℓ on $\mathcal{B}$ is a morphism of K-comodules for all $\ell \in L$.
- Equation (2.5) holds for H ⇔ (2.5) holds for $\mathcal{B}$.

In other words, the action of L on the bosonization $H = \mathcal{B}\#K$ satisfies (2.4), (2.3) and (2.5), if and only if so does the action of L on $\mathcal{B}$, and the homothety η_ℓ for $\ell \in L$ is a morphism of Yetter–Drinfeld modules, $\eta_\ell \in \operatorname{End}^K_K \mathcal{B}$ for all $\ell \in L$. This leads to the following.

Definition 2.4. An *L-module braided Hopf algebra* is a Hopf algebra $\mathcal{B}$ in the braided category ${}^K_K\mathcal{YD}$ that is also a L-module algebra, that satisfies (2.4), (2.3) and (2.5), and such that the homothety $\eta_\ell \in \operatorname{End}^K_K \mathcal{B}$ for all $\ell \in L$.

We have just seen: for an L-module braided Hopf algebra, the bosonization $H := \mathcal{B}\#K$ is an L-module Hopf algebra over $\Bbbk$ and we can form the Hopf algebra $H \rtimes L = (\mathcal{B}\#K) \rtimes L$.

As in Sec. 2.2, we consider the situation with non-degenerate pairings; this time internal to the braided monoidal category ${}^K_K\mathcal{YD}$ instead of $\operatorname{vect}\Bbbk$. Concretely, let $\mathcal{E}$ be another Hopf algebra in the category ${}^K_K\mathcal{YD}$ provided with a non-degenerate bilinear form $(\,|\,) : \mathcal{B}\otimes\mathcal{E} \to \Bbbk$, and extend it by (2.6) to a pairing $\mathcal{B}\otimes\mathcal{B}\otimes\mathcal{E}\otimes\mathcal{E} \to \Bbbk$.

- ◇ The fact that the pairing is internal to the category ${}^K_K\mathcal{YD}$ means that the bilinear form $(\,|\,)$ is a morphism in the monoidal category ${}^K_K\mathcal{YD}$, where $\Bbbk$ is endowed with the structure of a trivial Yetter–Drinfeld module.
- ◇ We assume that for every $a, \widetilde{a} \in \mathcal{B}$, $u, \widetilde{u} \in \mathcal{E}$, the conditions (2.7)–(2.9) of a Hopf pairing, relating coproduct, product, unit and counit of $\mathcal{B}$ and $\mathcal{E}$ hold.

Then we have in the braided category ${}^K_K\mathcal{YD}$ exactly the same situation we considered in Lemma 2.3 in the braided category $\operatorname{vect}\Bbbk$. The same calculations, this time in the category ${}^K_K\mathcal{YD}$, yield the following.

Lemma 2.5. *Assume that both $\mathcal{B}$ and $\mathcal{E}$ are L-modules and that condition* (2.10) *on the Hopf pairing* $(\,|\,)$ *holds. Then $\mathcal{B}$ is a L-module braided Hopf algebra, if and only if $\mathcal{E}$ is so.*

2.4.

Let $\mathfrak{g}$ be a Lie algebra over the field $\Bbbk$. We specialize to L-module braided Hopf algebras where the Hopf algebra $L = U(\mathfrak{g})$ is the universal enveloping algebra of $\mathfrak{g}$. Then the conditions (2.1) and (2.4) in the definition of an L-module Hopf algebra H just mean that $\mathfrak{g}$ acts on H by $\Bbbk$-derivations, while condition (2.5) is for free, due to the cocommutativity of $U(\mathfrak{g})$. Condition (2.3) amounts to the condition

$$\Delta(x \cdot a) = x \cdot a_{(1)} \otimes a_{(2)} + a_{(1)} \otimes x \cdot a_{(2)}, \quad \varepsilon(x \cdot a) = 0, \tag{2.11}$$

for all $x \in \mathfrak{g}$ and $a \in H$. In other words, condition (2.11) tells us that $\mathfrak{g}$ acts on H by $\Bbbk$-coderivations. We summarize all conditions by saying that $\mathfrak{g}$ acts on H by $\Bbbk$-*biderivations*: $\mathfrak{g}$ acts by endomorphisms that are simultaneously $\Bbbk$-derivations and $\Bbbk$-coderivations. Thus we have the following.

Lemma 2.6. *Let H be a Hopf algebra and let $\mathfrak{g}$ be a Lie algebra acting on H by $\Bbbk$-biderivations. Then H is a $U(\mathfrak{g})$-module Hopf algebra and we can form the Hopf algebra $H \rtimes U(\mathfrak{g})$.*

The following remarks on biderivations are useful:

- ⋄ For any Hopf algebra H, the subspace $\operatorname{Bider}_{\Bbbk}(H) := \{x \in \operatorname{Der}_{\Bbbk}(H) : x$ is a coderivation$\}$ is a Lie subalgebra of $\operatorname{Der}_{\Bbbk}(H)$.
- ⋄ If $x \in \operatorname{Der}(H)$ and if $a, b \in H$ fulfill (2.11) for x, then so does their product ab. Hence it is enough to check the biderivation property (2.11) for a given derivation x on a family of generators of H.

Remark 2.7. Let H be a Hopf algebra and let $\mathfrak{g}$ be a Lie algebra acting on H by $\Bbbk$-coderivations. Let H_0 be the coradical, and $(H_n)_{n\geq 0}$ the coradical filtration, of H. If H_0 is $\mathfrak{g}$-stable, then H_n is $\mathfrak{g}$-stable for all $n \geq 0$ by the defining condition (2.11). Hence $\mathfrak{g}$ acts on $\operatorname{gr} H$ by $\Bbbk$-coderivations.

Assume that H_0 is a Hopf subalgebra, that $\mathfrak{g}$ acts on H by $\Bbbk$-biderivations and that H_0 is $\mathfrak{g}$-stable. Then $\mathfrak{g}$ acts on the graded object $\operatorname{gr} H$ by $\Bbbk$-biderivations.

Notice that $\mathfrak{g}$ may act on H by $\Bbbk$-biderivations with H_0 not being $\mathfrak{g}$-stable. For instance, let $x \in H$ primitive. Then $D = \operatorname{ad} x$ is a $\Bbbk$-biderivation. If there exists $g \in G(H)$ such that $gx = qxg$ with $q \in \Bbbk^{\times} - \{1\}$, then $D(g) = (1-q)xg \notin H_0$.

2.5.

In this context, suppose that H is pointed and set $G := G(H)$ the group of group-like elements of H. Let $\mathfrak{g}$ act on H by derivations; assume that $\mathfrak{g}$ acts trivially on $\Bbbk G$. Let $g, t \in G$ and $\mathcal{P}_{g,t}(H) := \{a \in H : \Delta(a) = g \otimes a + a \otimes t\}$ the space of (g, t) skew-primitive elements. Then the coderivation property (2.11) implies that $\mathcal{P}_{g,t}(H)$ is a $\mathfrak{g}$-submodule for all $g, t \in G$. Summarizing, we have the following.

Lemma 2.8. *Let $\mathfrak{g}$ be a Lie algebra acting by derivations on a pointed Hopf algebra H, $G = G(H)$. Assume that*:

- *$\mathfrak{g}$ acts trivially on $\Bbbk G$.*
- *H is generated by group-like and skew-primitive elements.*

Then the following are equivalent:

(1) *$\mathfrak{g}$ acts on H by $\Bbbk$-biderivations, i.e.* (2.11) *holds.*
(2) *$\mathcal{P}_{g,t}(H)$ is a $\mathfrak{g}$-submodule for all g, $t \in G$.*
(3) *$\mathcal{P}_{g,1}(H)$ is a $\mathfrak{g}$-submodule for all $g \in G$.*

2.6.

Let K be a Hopf algebra and $V \in {}^{K}_{K}\mathcal{YD}$. It is well-known that every $d \in \operatorname{Hom}(V, T(V))$ extends uniquely to a derivation $D \in \operatorname{Der}(T(V))$ on the tensor algebra $T(V)$ by $D(1) = 0$ and

$$D_{|T^n(V)} = \sum_{1\leq j\leq n} \operatorname{id}_{T^{j-1}(V)} \otimes d \otimes \operatorname{id}_{T^{n-j}(V)}, \tag{2.12}$$

for $n > 0$. Thus every Lie algebra map $\mathfrak{g} \to \operatorname{End}(V)$ extends to a Lie algebra map $\mathfrak{g} \to \operatorname{Der}(T(V))$.

Proposition 2.9. *Let $V \in {}^K_K\mathcal{YD}$. Every morphism of Lie algebras $\mathfrak{g} \to \operatorname{End}^K_K(V)$ extends to an action of the universal enveloping algebra $U(\mathfrak{g})$ on $T(V)\#K$ and to an action on $\mathcal{B}(V)\#K$, giving rise to the Hopf algebras $(T(V)\#K) \rtimes U(\mathfrak{g})$ and $(\mathcal{B}(V)\#K) \rtimes U(\mathfrak{g})$.*

Proof. As explained, the action of $\mathfrak{g}$ on V extends uniquely to an action of $\mathfrak{g}$ on the tensor algebra $T(V)$ by derivations. Formula (2.12) and the assumptions imply that this action is by morphisms in the category ${}^K_K\mathcal{YD}$. By definition, (2.3) holds in V, hence it holds in $T(V)$. By Sec. 2.3, the action extended to $T(V)\#K$ satisfies the requirements in Sec. 2.1, hence we can form $(T(V)\#K) \rtimes U(\mathfrak{g})$. Second, the action of $\mathfrak{g}$ on $T^n(V)$ commutes with that of the braid group $\mathbb{B}_n$; since the kernel of the projection $T^n(V) \to \mathcal{B}^n(V)$ is the kernel of the quantum symmetrizer, $\mathfrak{g}$ acts on the Nichols algebra $\mathcal{B}(V)$ with the desired requirements. □

Definition 2.10. Let K be a Hopf algebra, $V \in {}^K_K\mathcal{YD}$ and $\mathfrak{g}$ a Lie subalgebra of $\mathfrak{bd}_V := \operatorname{End}^K_K(V)$. We call the Hopf algebra $(\mathcal{B}(V)\#K) \rtimes U(\mathfrak{g})$ the *unrolled bosonization* of the Nichols algebra of V by $\mathfrak{g}$.

One may define unrolled versions of bosonizations of pre-Nichols or post-Nichols algebras, see e.g. [13], or of deformations of Nichols algebras, provided that the ideals of defining relations are preserved by the action of $\mathfrak{bd}_V$, or if $\mathfrak{bd}_V$ is replaced by a suitable subalgebra.

2.7. *Finite* GK-dim

Our main reference for this subsection is [14]. Let A be an associative $\Bbbk$-algebra. We say that a finite-dimensional subspace $V \subseteq A$ is *GK-deterministic* if

$$\operatorname{GK-dim} A = \lim_{n\to\infty} \log_n \dim \sum_{0\le j\le n} V^n.$$

Lemma 2.11 ([2, Lemma 2.2]). *Let K be a Hopf algebra, R a Hopf algebra in ${}^K_K\mathcal{YD}$, A a K-module algebra and B an R-module algebra in ${}^K_K\mathcal{YD}$. Assume that the actions of K on A, of K on B, of K on R, and of R on B are locally finite.*

(a) $\operatorname{GK-dim} A\#K \le \operatorname{GK-dim} A + \operatorname{GK-dim} K$. *If either K or A has a GK-deterministic subspace, then* $\operatorname{GK-dim} A\#K = \operatorname{GK-dim} A + \operatorname{GK-dim} K$.
(b) $\operatorname{GK-dim} B\#R \le \operatorname{GK-dim} B + \operatorname{GK-dim} R$. *If either R or B has a GK-deterministic subspace, then* $\operatorname{GK-dim} B\#R = \operatorname{GK-dim} B + \operatorname{GK-dim} R$.

Clearly, a finite-dimensional Lie algebra $\mathfrak{g}$ is a GK-deterministic subspace of $U(\mathfrak{g})$. Thus we have the following.

Example 2.12. Let H be a Hopf algebra and let $\mathfrak{g}$ be a Lie subalgebra of $\mathrm{Bider}_{\Bbbk}(H)$ such that GK-dim H, $\dim \mathfrak{g} < \infty$. If the action of $\mathfrak{g}$ on H is locally finite, then

$$\text{GK-dim}(H \rtimes U(\mathfrak{g})) = \text{GK-dim}\, H + \dim \mathfrak{g} < \infty. \tag{2.13}$$

Here are some particular cases:

- If H is a finite-dimensional Hopf algebra and $\mathfrak{g}$ is a Lie subalgebra of $\mathrm{Bider}_{\Bbbk}(H)$, then

$$\text{GK-dim}(H \rtimes U(\mathfrak{g})) = \dim \mathfrak{g} < \infty.$$

- Let K be a Hopf algebra, $V \in {}^{K}_{K}\mathcal{YD}$, $\mathfrak{g}$ a Lie subalgebra of $\mathfrak{bd}_V$, $\mathcal{B} \in {}^{K}_{K}\mathcal{YD}$ a pre-Nichols algebra of V and $\mathcal{E} \in {}^{K}_{K}\mathcal{YD}$ a post-Nichols algebra of V. Assume that the action of $\mathfrak{g}$ descends to $\mathcal{B}$ and $\mathcal{E}$,

$$\text{GK-dim}\, K < \infty, \quad \dim V < \infty, \quad \text{GK-dim}\, \mathcal{B} < \infty, \quad \text{GK-dim}\, \mathcal{E} < \infty.$$

  Clearly, $\dim \mathfrak{g} < \infty$ and $\mathfrak{g}$ acts locally finitely on $\mathcal{B}\#K$ and $\mathcal{E}\#K$. If either K or $\mathcal{B}$, respectively $\mathcal{E}$, have a GK-deterministic subspace, then

$$\text{GK-dim}((\mathcal{B}\#K) \rtimes U(\mathfrak{g})) = \text{GK-dim}\, \mathcal{B} + \text{GK-dim}\, K + \dim \mathfrak{g} < \infty,$$

$$\text{GK-dim}((\mathcal{E}\#K) \rtimes U(\mathfrak{g})) = \text{GK-dim}\, \mathcal{E} + \text{GK-dim}\, K + \dim \mathfrak{g} < \infty.$$

3. The Dual Construction

3.1.

Let J be a Hopf algebra. A J-comodule coalgebra is a coalgebra C which is also a right J-comodule with coaction $\varrho : C \to C \otimes J$, $\varrho(c) = c_{[0]} \otimes c_{[1]}$, and counit ε_C such that for all $c \in C$

$$(c_{(1)})_{[0]} \otimes (c_{(2)})_{[0]} \otimes (c_{(1)})_{[1]}(c_{(2)})_{[1]} = (c_{[0]})_{(1)} \otimes (c_{[0]})_{(2)} \otimes c_{[1]}, \tag{3.1}$$

$$\varepsilon_C(c_{[0]})c_{[1]} = \varepsilon_C(c). \tag{3.2}$$

Here (3.1) and (3.2) mean that C is a coalgebra in the monoidal category $\mathcal{M}^J$ of right J-comodules. Assume that $C = H$ is a Hopf algebra and a J-comodule coalgebra that satisfies

$$(ab)_{[0]} \otimes (ab)_{[1]} = a_{[0]}b_{[0]} \otimes a_{[1]}b_{[1]}, \tag{3.3}$$

$$\varrho(1) = 1 \otimes 1, \tag{3.4}$$

$$a_{[0]} \otimes ja_{[1]} = a_{[0]} \otimes a_{[1]}j, \tag{3.5}$$

$j \in J$, $a, b \in H$; (3.3) and (3.5) say that H is a J-comodule algebra. Then $J \ltimes H := J \otimes H$ with the tensor product structure as an algebra and with the smash coproduct

(1.5) for the coalgebra structure is a Hopf algebra; see e.g. [4, 1.1.4].[a] We shall say that H is a *J-comodule Hopf algebra*.

3.2.

Let H and U be Hopf algebras, provided with a non-degenerate Hopf pairing $(\,|\,): H \otimes U \to \Bbbk$.

Lemma 3.1. *Assume that H and U are J-comodules and that the pairing is compatible with J-coaction involving the antipode of J, i.e.*

$$(a_{[0]} \,|\, u)a_{[1]} = (a \,|\, u_{[0]})\mathcal{S}(u_{[1]}), \quad a \in H, \ u \in U. \tag{3.6}$$

Then H is a J-comodule Hopf algebra if and only if U is so.

Proof. Let $u, v \in U$, $a, b \in H$. We compute

$$((ab)_{[0]} \,|\, u)(ab)_{[1]} = (ab \,|\, u_{[0]})\mathcal{S}(u_{[1]}) = (a \,|\, (u_{[0]})_{(2)})(b \,|\, (u_{[0]})_{(1)})\mathcal{S}(u_{[1]});$$

$$\begin{aligned}(a_{[0]}b_{[0]} \,|\, u)a_{[1]}b_{[1]} &= (a_{[0]} \,|\, u_{(2)})(b_{[0]} \,|\, u_{(1)})a_{[1]}b_{[1]} \\ &= (a \,|\, (u_{(2)})_{[0]})(b \,|\, (u_{(1)})_{[0]})\mathcal{S}((u_{(2)})_{[1]})\mathcal{S}((u_{(1)})_{[1]}) \\ &= (a \,|\, (u_{(2)})_{[0]})(b \,|\, (u_{(1)})_{[0]})\mathcal{S}((u_{(1)})_{[1]}(u_{(2)})_{[1]}).\end{aligned}$$

Hence (3.1) holds for U if and only if (3.3) holds for H and vice versa. Similarly (3.2) holds for U if and only if (3.4) holds for H and vice versa. Finally, (3.5) holds for H if and only if it holds for U:

$$(a_{[0]} \,|\, u)ja_{[1]} = (a \,|\, u_{[0]})j\mathcal{S}(u_{[1]}) = (a \,|\, u_{[0]})\mathcal{S}(u_{[1]}\overline{\mathcal{S}}(j));$$

$$(a_{[0]} \,|\, u)a_{[1]}j = (a \,|\, u_{[0]})\mathcal{S}(u_{[1]})j = (a \,|\, u_{[0]})\mathcal{S}(\overline{\mathcal{S}}(j)u_{[1]}). \qquad \square$$

3.3.

Let now K be a Hopf algebra, $\mathcal{B}$ a Hopf algebra in ${}^K_K\mathcal{YD}$ and also a J-comodule coalgebra. Extend the coaction of J to $H = K\#\mathcal{B}$ by $\varrho(k\#b) = k\#b_{[0]} \otimes b_{[1]}$, $b \in \mathcal{B}$ and $k \in K$. Then

- H is a J-comodule coalgebra $\Leftrightarrow$ the coactions of J and K on $\mathcal{B}$ commute, i.e. for all $b \in \mathcal{B}$

$$(b_{(0)})_{[0]} \otimes b_{(1)} \otimes (b_{(0)})_{[1]} = (b_{[0]})_{(0)} \otimes (b_{[0]})_{(1)} \otimes b_{[1]} \in \mathcal{B} \otimes K \otimes J. \tag{3.7}$$

- Equation (3.4) holds for $H \Leftrightarrow$ (3.4) holds for $\mathcal{B}$. Assume this is the case.
- Equation (3.3) holds for $H \Leftrightarrow$ (3.3) holds for $\mathcal{B}$ and the action of k on $\mathcal{B}$ is a morphism of J-comodules for all $k \in K$.
- Equation (3.5) holds for $H \Leftrightarrow$ (3.5) holds for $\mathcal{B}$.

[a] In [4, p. 10] a left version is presented, with a different notation. The proof is equally straightforward.

In other words, the coaction of J on $H = K\#\mathcal{B}$ satisfies (3.4), (3.3) and (3.5), if and only if so does the coaction of J on $\mathcal{B}$, and the coaction of J on $\mathcal{B}$ commutes both with the action and the coaction of K. This can be phrased also as: the homothety η_ℓ for $\ell \in J^*$ is a morphism of Yetter–Drinfeld modules, i.e. $\eta_\ell \in \operatorname{End}_K^K \mathcal{B}$.

Definition 3.2. A *J-comodule braided Hopf algebra* is a Hopf algebra $\mathcal{B}$ in the braided category ${}^{K}_{K}\mathcal{YD}$ that is also a J-comodule coalgebra, that satisfies (3.4), (3.3) and (3.5), and such that the coaction of J on $\mathcal{B}$ commutes both with the action and the coaction of K. In such a case, the bosonization $H = K\#\mathcal{B}$ is a J-comodule Hopf algebra and we can form the Hopf algebra $J \ltimes H = J \ltimes (K\#\mathcal{B})$.

As in Sec. 3.2, we consider the situation with non-degenerate pairings; this time internal to the braided monoidal category $\mathcal{YD}_K^K$ instead of $\operatorname{vect}\Bbbk$. Concretely, let $\mathcal{E}$ be a Hopf algebra in $\mathcal{YD}_K^K$ provided with a non-degenerate bilinear form $(\,|\,) : \mathcal{B} \otimes \mathcal{E} \to \Bbbk$, and extend it by (2.6) to a pairing $\mathcal{B} \otimes \mathcal{B} \otimes \mathcal{E} \otimes \mathcal{E} \to \Bbbk$.

- ◇ The fact that the pairing is internal to the category $\mathcal{YD}_K^K$ means that the bilinear form $(\,|\,)$ is a morphism in the monoidal category $\mathcal{YD}_K^K$, where $\Bbbk$ is endowed with the structure of a trivial Yetter–Drinfeld module.
- ◇ We assume that for every $a, \widetilde{a} \in \mathcal{B}$, $u, \widetilde{u} \in \mathcal{E}$, the conditions (2.7)–(2.9) of a Hopf pairing, relating coproduct, product, unit and counit of $\mathcal{B}$ and $\mathcal{E}$ hold.

Then we have in the braided category $\mathcal{YD}_K^K$ exactly the same situation we considered in Lemma 3.1 in the braided category $\operatorname{vect}\Bbbk$. The same calculations, this time in the category $\mathcal{YD}_K^K$, yield the following.

Lemma 3.3. *Assume that both $\mathcal{B}$ and $\mathcal{E}$ are J-comodules and that* (3.6) *holds. Then $\mathcal{B}$ is a J-comodule braided Hopf algebra, if and only if $\mathcal{E}$ is so.*

3.4.

Let G be an affine algebraic group over $\Bbbk$ and let $J = \Bbbk[G]$ be the algebra of functions on $G = \operatorname{Hom}_{\text{alg}}(J, \Bbbk)$. Here we use the convention (2.6), i.e.

$$\langle \gamma\eta, j \rangle = \langle \gamma, j_{(2)} \rangle \langle \eta, j_{(1)} \rangle, \quad \gamma,\ \eta \in G.$$

Thus, being a (right) J-comodule means being a rational (right) G-module: $m \cdot \gamma = m_{[0]} \langle \gamma, m_{[1]} \rangle$; which of course is equivalent to being rational left G-module. So, in what follows we work with left rational modules. The conditions (3.1) and (3.2), respectively (3.3) and (3.4), in the definition of J-comodule Hopf algebra just say that G acts on H by coalgebra, respectively algebra, automorphisms, while (3.5) is automatic by the commutativity of $\Bbbk[G]$. We summarize our findings.

Proposition 3.4. *Let H be a Hopf algebra and let G be an affine algebraic group acting rationally on H by Hopf algebra maps. Then H is a $\Bbbk[G]$-comodule Hopf algebra and we can form $\Bbbk[G] \ltimes H$.*

Remark 3.5. Since J is commutative, GK-dim$(\Bbbk[G] \ltimes H) = \dim G + \text{GK-dim}\, H$, see e.g. [14, 3.10].

3.5.

Let K be a Hopf algebra and $V \in \mathcal{YD}_K^K$, $\dim V < \infty$. Then $\operatorname{Aut}_K^K(V)$ is an algebraic group, whose Lie algebra is $\operatorname{End}_K^K(V)$. Every morphism of algebraic groups $G \to \operatorname{Aut}_K^K(V)$ extends to an action of G on $T(V)$ by Hopf algebra automorphisms in $\mathcal{YD}_K^K$; hence it descends to an action of G on $\mathcal{B}(V)$ by Hopf algebra automorphisms in $\mathcal{YD}_K^K$. It extends to an action of G on $K\#\mathcal{B}(V)$, trivially on K, giving rise to the Hopf algebra $\Bbbk[G] \ltimes (K\#\mathcal{B}(V))$. One may define analogous actions of these Hopf algebras from bosonizations of pre-Nichols or post-Nichols algebras, or of deformations of Nichols algebras, provided that the ideals of defining relations are preserved by the action of G.

4. Hopf Algebras Arising from Nichols Algebras of Diagonal Type

4.1.

Let $\theta \in \mathbb{N}$, $\mathbb{I} = \mathbb{I}_\theta = \{1, 2, \dots, \theta\}$. Denote by $(\alpha_i)_{i\in\mathbb{I}}$ the canonical basis of $\mathbb{Z}^\theta$.

Let (V, c) be a braided vector space of diagonal type of dimension θ; let $(x_i)_{i\in\mathbb{I}}$ be a basis of V. Since (V, c) is assumed to be of diagonal type, there is a matrix $\mathbf{q} = (q_{ij})_{i,j\in\mathbb{I}} \in (\Bbbk^\times)^{\mathbb{I}\times\mathbb{I}}$ such that $c(x_i \otimes x_j) = q_{ij} x_j \otimes x_i$ for all $i, j \in \mathbb{I}$. Then the tensor algebra $T(V)$ and the Nichols algebra $\mathcal{B}(V)$ are $\mathbb{Z}^\theta$-graded (as braided Hopf algebras), by $\deg x_i = \alpha_i$, $i \in \mathbb{I}$.

Let K be a Hopf algebra. To realize the braided vector space (V, c) as a Yetter–Drinfeld module over K we need some extra data.

- ◇ A pair $(g, \chi) \in G(K) \times \operatorname{Hom}_{\text{alg}}(K, \Bbbk)$ is called a *YD-pair* [1] if $\chi(a)\, g = \chi(a_{(2)})\, a_{(1)}\, g\, \mathcal{S}(a_{(3)})$ for all $a \in K$. This implies $g \in Z(G(K))$.
- ◇ Then $\Bbbk_g^\chi := \Bbbk$ with coaction given by g and action given by χ is a simple object in ${}_K^K\mathcal{YD}$.

A *principal realization* of the braided vector space (V, c) over the Hopf algebra K is a family $((g_i, \chi_i))_{i\in\mathbb{I}}$ of YD-pairs such that

$$\chi_j(g_i) = q_{ij}, \quad \text{for all } i,\ \ j \in \mathbb{I}. \tag{4.1}$$

A principal realization allows us to see braided vector space as a Yetter–Drinfeld module, $V \in {}_K^K\mathcal{YD}$, by declaring $x_i \in V_{g_i}^{\chi_i}$, $i \in \mathbb{I}$. Let $d_g^\chi = \dim V_g^\chi = |\{i \in \mathbb{I} : (g_i, \chi_i) = (g, \chi)\}|$. Then

$$\mathfrak{bd}_V = \operatorname{End}_K^K(V) \simeq \bigoplus_{g\in\Gamma, \chi\in\widehat{\Gamma}} \mathfrak{gl}(d_g^\chi, \Bbbk).$$

Despite the notation, the Lie algebra $\mathfrak{bd}_V$ depends on the way the braided vector space V is realized as a K-Yetter–Drinfeld module and not merely on the braided vector space V itself.

For $h = (h_i)_{i \in \mathbb{I}_\theta} \in \Bbbk^\theta$ we denote by $D_h \in \operatorname{End}(V)$ the map defined by $D_h(x_i) = h_i x_i$, $i \in \mathbb{I}_\theta$. By abuse of notation, we denote by D_h the corresponding derivation of $T(V)\#\Bbbk\Gamma$ or $\mathcal{B}(V)\#\Bbbk\Gamma$. Let

$$\mathfrak{t}_V = \{D_h : h \in \Bbbk^\theta\} \subseteq \mathfrak{bd}_V.$$

The abelian Lie algebra $\mathfrak{t}_V$ depends only on (V, c). If $(g_i, \chi_i) = (g_j, \chi_j)$ implies $i = j$, then $\mathfrak{bd}_V = \mathfrak{t}_V$.

Remark 4.1. The action of the Lie algebra $\mathfrak{t}_V$ preserves the $\mathbb{Z}^\theta$-grading. Indeed, let $h \in \Bbbk^\theta$ and let $\alpha \mapsto h_\alpha$ be the unique group homomorphism $\mathbb{Z}^\theta \to \Bbbk$ such that $h_{\alpha_i} = h_i$, $i \in \mathbb{I}$. Then D_h acts by h_β in the homogeneous component $T(V)_\beta$ for all $\beta \in \mathbb{Z}^\theta$. Hence every Hopf ideal $\mathcal{I}$ of $T(V)$ generated by $\mathbb{Z}^\theta$-homogeneous elements is stable under $\mathfrak{t}_V$ and $\mathfrak{t}_V$ acts by derivations and coderivations on $\mathcal{T}(V)/\mathcal{I}$.

Remark 4.2. In fact, the $\mathbb{Z}^\theta$-grading is tantamount to a comodule structure over the group algebra $\Bbbk\mathbb{Z}^\theta$, which is the algebra of functions on the algebraic torus $\mathbb{T}_V$; $\mathfrak{t}_V$ is its Lie algebra, and the action of $\mathfrak{t}_V$ is the derivation of the natural action of $\mathbb{T}_V$.

4.2.

From now on, we assume that $\operatorname{char}\Bbbk = 0$. We keep the notation above and assume that $\dim \mathcal{B}(V) < \infty$. The classification of the finite-dimensional Nichols algebras of diagonal type was given in [12]. An efficient set of defining relations of $\mathcal{B}(V)$, i.e. generators of the ideal $\mathcal{J}_{\mathbf{q}}$, was provided in [6]. Besides $\mathcal{B}(V)$, there are two other Hopf algebras in ${}^K_K\mathcal{YD}$ that are expected to play a role in representation theory:

(a) ([6, 7]) The *distinguished pre-Nichols algebra* of (V, c) is the quotient $\widetilde{\mathcal{B}}(V) := T(V)/\mathcal{I}_{\mathbf{q}}$ by a suitable ideal $\mathcal{I}_{\mathbf{q}}$. Thus, there are projections $T(V) \twoheadrightarrow \widetilde{\mathcal{B}}(V) \twoheadrightarrow \mathcal{B}(V)$.
(b) ([13]) The *Lusztig algebra* of (V, c) is the graded dual $\mathcal{L}(V)$ of $\widetilde{\mathcal{B}}(V)$.

Proposition 4.3. *Let K be a Hopf algebra provided with a principal realization of (V, c) and let $L = U(\mathfrak{t}_V)$. Then $\widetilde{\mathcal{B}}(V)$ and $\mathcal{L}(V)$ are L-module braided Hopf algebras in ${}^K_K\mathcal{YD}$ and we can form the unrolled bosonizations $(\widetilde{\mathcal{B}}(V)\#K) \rtimes L$ and $(\mathcal{L}(V)\#K) \rtimes L$.*

Proof. The claim for $\widetilde{\mathcal{B}}(V)$ follows from Remark 4.1 and implies the one for $\mathcal{L}(V)$ by Lemma 2.5. ∎

Example 4.4. If $\theta = 1$ and $\mathbf{q}$ is a root of 1 of even order, then we recover the construction in [9, 11].

4.3.

Let (V, c) be of diagonal type with $\dim \mathcal{B}(V) < \infty$. Fix a principal realization over the group algebra $\Bbbk\Gamma$, where Γ is abelian. Then each of the Hopf algebras $\mathcal{B}(V)$, $\widetilde{\mathcal{B}}(V)$ and $\mathcal{L}(V)$ in ${}^{\Bbbk\Gamma}_{\Bbbk\Gamma}\mathcal{YD}$ gives rise to Hopf algebras $\mathfrak{u}(V), U(V), \mathcal{U}(V)$, respectively; they are suitable Drinfeld doubles of the bosonizations $\mathcal{B}(V)\#\Bbbk\Gamma$, $\widetilde{\mathcal{B}}(V)\#\Bbbk\Gamma$ and $\mathcal{L}(V)\#\Bbbk\Gamma$. See [3, 7, 13]. If $\mathbf{q}$ is symmetric, then we may divide that Drinfeld double by a central Hopf subalgebra. If furthermore $\mathbf{q}$ is of Cartan type, then we recover the small and the De Concini–Procesi quantum group, respectively. Then we may define unrolled quantum groups

$$\mathfrak{u}(V) \rtimes U(\mathfrak{t}_V), \quad U(v) \rtimes U(\mathfrak{t}_V), \quad \mathcal{U}(V) \rtimes U(\mathfrak{t}_V).$$

Indeed, the Lie algebra $\mathfrak{t}_{V\oplus W}$ acts on $T(V \oplus W)\#\Bbbk\Gamma$, but if $\zeta \in \Bbbk^{2\theta}$, then D_ζ preserves the relations of the quantum double if and only if ζ belongs to the image of the map $\mathfrak{t}_V \to \mathfrak{t}_{V\oplus W}$, $\xi \mapsto (\xi, -\xi)$.

Acknowledgment

The work of N. A. was partially supported by CONICET, Secyt (UNC) and the MathAmSud project GR2HOPF. C. S. was partially supported by the Collaborative Research Centre 676 "Particles, Strings and the Early Universe — the Structure of Matter and Space-Time", and by the RTG 1670 "Mathematics inspired by String theory and Quantum Field Theory". Most of this work was done during a visit of N. A. to the University of Hamburg, supported by the Alexander von Humboldt Foundation, in April 2015.

N. A. thanks I. Angiono for some interesting exchanges.

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