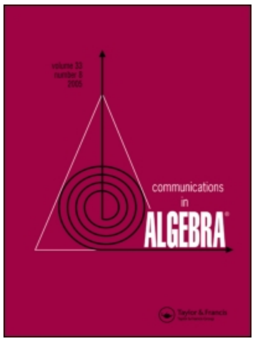


Communications in Algebra

ISSN: 0092-7872 (Print) 1532-4125 (Online) Journal homepage: http://www.tandfonline.com/loi/lagb20

Simple modules of the quantum double of the Nichols algebra of unidentified diagonal type ###(7)

Nicolás Andruskiewitsch, Iván Angiono, Adriana Mejía & Carolina Renz

To cite this article: Nicolás Andruskiewitsch, Iván Angiono, Adriana Mejía & Carolina Renz (2018) Simple modules of the quantum double of the Nichols algebra of unidentified diagonal type ###(7), Communications in Algebra, 46:4, 1770-1798, DOI: 10.1080/00927872.2017.1357726

To link to this article: https://doi.org/10.1080/00927872.2017.1357726

Accepted author version posted online: 26 Jul 2017.
Published online: 07 Sep 2017.

COMMUNICATIONS IN ALGEBRA®
2018, VOL. 46, NO. 4, 1770–1798
https://doi.org/10.1080/00927872.2017.1357726

Check for updates

Simple modules of the quantum double of the Nichols algebra of unidentified diagonal type $\mathfrak{ufo}(7)$

Nicolás Andruskiewitsch[a], Iván Angiono[a], Adriana Mejía[b], and Carolina Renz[c]

[a]FaMAF-CIEM (CONICET), Universidad Nacional de Córdoba, Ciudad Universitaria, Córdoba, República Argentina; [b]CAPES-PNPD, Departamento de matemáticas/Campus Trindade, Universidade Federal de Santa Catarina, Florianopolis, SC, Brasil; [c]Universidade do Vale do Rio dos Sinos, São Leopoldo, RS, Brasil

ABSTRACT
The finite-dimensional simple modules over the Drinfeld double of the bosonization of the Nichols algebra $\mathfrak{ufo}(7)$ are classified.

ARTICLE HISTORY
Received 23 August 2016
Revised 6 July 2017
Communicated by M. Cohen

KEYWORDS
Hopf algebras; highest weight; Nichols algebras; quantum groups; simple modules

2010 MATHEMATICS SUBJECT CLASSIFICATION
16T20; 17B37

1. Introduction

1.1. Motivations and context

The purpose of this paper is to compute explicitly all simple finite-dimensional modules of the Hopf algebra $\mathcal{U}$ introduced by generators and relations in Definition 1.1. In short, $\mathcal{U} \simeq D(H)$ arises as the Drinfeld double of $H = \mathcal{B}(V)\#\Bbbk\Lambda$, where Λ is an abelian group, V is a braided vector space of diagonal type of dimension 2 with Dynkin diagram (1) (realized as a Yetter-Drinfeld module over Λ) and $\mathcal{B}(V)$ denotes its Nichols algebra.

The general context where our results fit is the following. Let W be a braided vector space of diagonal type and assume that its Nichols algebra $\mathcal{B}(W)$ is finite-dimensional; see [2] for an introduction to Nichols algebras and [3] for a survey on Nichols algebras of diagonal type. We recall that finite-dimensional Nichols algebras of diagonal type were classified in [13]. It is useful to organize the classification in four classes:

- Standard type [8], including Cartan type [7].
- Super type [5].
- (Super) modular type [3].
- Unidentified type [9].

Let Γ be an an abelian group such that W is realized as a Yetter-Drinfeld module over it and let U be the Drinfeld double of $\mathcal{B}(W)\#\Bbbk\Gamma$. The representation theory of such Drinfeld doubles U or slight variations thereof was treated in many papers, among them [1, 6, 14, 15, 17–19]. Indeed, the first two articles deal with the representation theory of the finite quantum groups or Frobenius-Lusztig kernels (that roughly arise from W of Cartan type), while in the others some general results are established. Presently we know that the simple U-modules are parametrized by highest weights but we ignore the character formulas and the dimensions in general, except for Frobenius-Lusztig kernels under appropriate conditions.

CONTACT Iván Angiono angiono@famaf.unc.edu.ar FaMAF-CIEM (CONICET), Universidad Nacional de Córdoba, Ciudad Universitaria, Medina Allende s/n, 5000 Córdoba, República Argentina.

Back to the particular V, the goal of working out this example, establishing the dimensions of all simple $\mathcal{U}$-modules, is to gain experience for further developments. The algebra $\mathcal{U}$ is small enough to allow an approach by elementary computations. Arguing as in [6, Theorem 3.7], see also [14, Proposition 5.6], it is possible to prove that $\mathcal{U}$ is a quasi-triangular Hopf algebra, even a ribbon one by the criterion in [16, Theorem 3], what makes it susceptible of applications. If Λ is finite, then the simple $\mathcal{U}$-modules are just the simple Yetter-Drinfeld H-modules; therefore the classification here might have applications to the study of basic Hopf algebras. Also, in the organization in classes mentioned above, $\mathcal{B}(V)$ is the smallest Nichols algebra of unidentified type; in the terminology from [3], V is of type $\mathfrak{ufo}(7)$. Indeed, $\dim \mathcal{B}(V) < \infty$ by [13, Table 1, row 7]; more precisely, cf. (13),

$$\dim \mathcal{B}(V) = 2^4 3^2 = 144.$$

By [9], a consequence of [10, 11], we know that $\mathcal{B}(V)$ has a presentation by generators E_1, E_2 and relations (5) below. Thus $\mathcal{B}(V)$ is manageable yet does not arise from any Lie algebra, what makes it attractive.

There is another reason to address the representation theory of $\mathcal{U}$. A finite-dimensional Nichols algebra of diagonal type admits both a distinguished pre-Nichols algebra [12] and a distinguished post-Nichols algebra [4]; the representation theories of the corresponding Drinfeld doubles seem to be very rich. However our $\mathcal{B}(V)$ coincides with its distinguished pre-Nichols and post-Nichols algebras, being therefore of singular interest (the only other Nichols algebra with this feature has diagram $\circ^{\omega} \xrightarrow{\ -\omega\ } \circ^{-1}$, $\omega \in \mathbb{G}'_3$, which is of standard type B_2). This peculiar behaviour appeals to the consideration of V.

1.2. The algebra $\mathcal{U}$

We now introduce formally $\mathcal{U}$. Let us begin with some notation.

If $k, \ell \in \mathbb{N}_0$, then we denote $\mathbb{I}_{k,\ell} = \{n \in \mathbb{N}_0 : k \leq n \leq \ell\}$; also $\mathbb{I}_\ell := \mathbb{I}_{1,\ell}$. Let $\Bbbk$ be an algebraically closed field of characteristic zero and $\Bbbk^\times = \Bbbk - 0$. Let $\mathbb{G}_{12}$ be the group of 12-roots of unity in $\Bbbk$, and let $\mathbb{G}'_{12}$ be the subset of primitive roots of order 12.

To define $\mathcal{U}$, we need some data:

- A matrix $\mathbf{q} = (q_{ij})_{1\leq i,j\leq 2} = \begin{pmatrix} \zeta^4 & q_{12} \\ q_{21} & -1 \end{pmatrix} \in \Bbbk^{2\times 2}$ such that $q_{12}q_{21} = \zeta^{11}$; that is, its associated generalized Dynkin diagram is given by

$$\underset{1}{\circ}^{\zeta^4} \xrightarrow{\ \zeta^{11}\ } \underset{2}{\circ}^{-1}. \tag{1}$$

- An abelian group Λ whose group of characters is denoted by $\widehat{\Lambda}$. We set $\Gamma = \Lambda \times \widehat{\Lambda}$.
- $g_1, g_2 \in \Lambda, \sigma_1, \sigma_2 \in \widehat{\Lambda}$ such that $\begin{pmatrix} \sigma_1(g_1) & \sigma_2(g_1) \\ \sigma_1(g_2) & \sigma_2(g_2) \end{pmatrix} = \begin{pmatrix} \zeta^4 & q_{12} \\ q_{21} & -1 \end{pmatrix}$.

Starting from these data, we consider vector spaces V and W with bases v_i, respectively w_i, $i \in \mathbb{I}_2$ and define an action and a Γ-grading on V and W by

$$g \cdot v_i = \sigma_i(g)v_i, \qquad \sigma \cdot v_i = \sigma(g_i)v_i, \qquad g \cdot w_i = \sigma_i^{-1}(g)w_i, \qquad \sigma \cdot w_i = \sigma(g_i^{-1})w_i; \tag{2}$$

$$\deg v_i = g_i, \qquad \deg w_i = \sigma_i, \qquad g \in \Lambda,\ \sigma \in \widehat{\Lambda}, \qquad i \in \mathbb{I}_2. \tag{3}$$

Then $V \oplus W$ is a Yetter-Drinfeld module over $\Bbbk\Gamma$ and $T(V \oplus W)$ is a braided Hopf algebra in ${}^{\Bbbk\Gamma}_{\Bbbk\Gamma}\mathcal{YD}$. In particular, V is a braided vector space of diagonal type $\mathfrak{ufo}(7)$, as said.

It is convenient to start with the auxiliary Hopf algebra $\overline{\mathcal{U}} = T(V \oplus W)\#\Bbbk\Gamma$; in particular, $T(V \oplus W)$ and $\Bbbk\Gamma$ are subalgebras of $\overline{\mathcal{U}}$ and

$$gv_i = \sigma_i(g)v_i, \qquad \sigma v_i = \sigma(g_i)v_i\sigma, \qquad gw_i = \sigma_i^{-1}(g)w_ig, \qquad \sigma w_i = \sigma(g_i^{-1})w_i\sigma,$$

$g \in \Lambda, \sigma \in \widehat{\Lambda}, i \in \mathbb{I}_2$. To stress the similarity with quantum groups, we denote in $\overline{\mathcal{U}}$ or any quotient thereof, as in [6, 14, 15],

$$E_i = v_i, \qquad F_i = w_i\sigma_i^{-1}, \qquad i \in \mathbb{I}_2. \tag{4}$$

Thus

$$gE_i = \sigma_i(g)E_ig, \qquad \sigma E_i = \sigma(g_i)E_i\sigma, \qquad gF_i = \sigma_i^{-1}(g)F_ig, \qquad \sigma F_i = \sigma(g_i^{-1})F_i\sigma.$$

We also need the notation of the so-called root vectors, needed for the relations and for the PBW-basis:

$$E_{12} = E_1E_2 - q_{12}E_2E_1, \qquad E_{112} = E_1E_{12} - q_{12}\zeta^4E_{12}E_1, \qquad E_{11212} = E_{112}E_{12} - q_{12}\zeta E_{12}E_{112},$$
$$F_{12} = F_1F_2 - q_{21}F_2F_1, \qquad F_{112} = F_1F_{12} - q_{21}\zeta^4F_{12}F_1, \qquad F_{11212} = F_{112}F_{12} - q_{21}\zeta F_{12}F_{112}.$$

We are now ready to define $\mathcal{U}$.

Definition 1.1. The algebra $\mathcal{U}$ is the quotient of $\overline{\mathcal{U}}$ by the ideal generated by

$$E_1^2 = 0, \qquad E_2^2 = 0, \qquad E_{11212}E_{12} = \zeta^{10}q_{12}E_{12}E_{11212}, \tag{5}$$

$$F_1^2 = 0, \qquad F_2^2 = 0, \qquad F_{11212}F_{12} = \zeta^4q_{21}F_{12}F_{11212}, \tag{6}$$

$$E_kF_i - F_iE_k = \delta_{ki}(g_i - \sigma_i^{-1}). \tag{7}$$

The algebra $\mathcal{U}$ is a Hopf algebra with coproduct given by

$$\Delta(E_i) = E_i \otimes 1 + g_i \otimes E_i, \qquad \Delta(F_i) = F_i \otimes \sigma_i^{-1} + 1 \otimes F_i, \qquad \Delta(g) = g \otimes g, \qquad g \in \Gamma.$$

Let $\mathcal{U}^-$ (respectively $\mathcal{U}^+$) be the subalgebra of $\mathcal{U}$ generated by F_1, F_2 (respectively E_1, E_2). The following facts are not difficult to prove and can be derived from general results in the literature cited above:

- $\mathcal{U}$ has a triangular decomposition $\mathcal{U} \simeq \mathcal{U}^+ \otimes \Bbbk\Gamma \otimes \mathcal{U}^-$, given by the multiplication map.
- $\mathcal{U}^+ \simeq \mathcal{B}(V)$; in what follows we identify these two algebras.
- $\mathcal{U}, \mathcal{U}^+$ and $\mathcal{U}^-$ admit a $\mathbb{Z}^2$-graduation $\mathcal{U} = \oplus_{\beta\in\mathbb{Z}^2}\mathcal{U}_\beta$ such that $\deg E_i = \alpha_i = -\deg F_i$, $i \in \mathbb{I}_2$, and $\deg x = 0$ for $x \in \Gamma$.
  Here $(\alpha_i)_{i\in\mathbb{I}_2}$ is the canonical basis of $\mathbb{Z}^2$.

1.3. Verma modules

We recall succinctly the description of the simple modules in terms of highest weights.

Let ${}_{\mathcal{U}}\mathcal{M}$ be the category of left $\mathcal{U}$-modules and let $\operatorname{Irr}\mathcal{U}$ be the set of isomorphism classes of finite-dimensional simple $\mathcal{U}$-modules. If $M \in {}_{\mathcal{U}}\mathcal{M}$ and $\lambda \in \widehat{\Gamma}$, then

$$M^\lambda = \{m \in M : g \cdot m = \lambda(g)m \;\forall g \in \Gamma\}$$

is the space of weight vectors with weight λ; if $M = \oplus_{\lambda\in\widehat{\Gamma}}M^\lambda$, then we say that M is diagonalizable.

Let $\lambda \in \widehat{\Gamma}$. We denote by $\Bbbk_\lambda$ the $\Bbbk\Gamma \otimes \mathcal{U}^-$-module defined by $\lambda \otimes \varepsilon$ (the counit). The *Verma module* $M(\lambda)$ associated to λ is the induced module

$$M(\lambda) = \operatorname{Ind}_{\Bbbk\Gamma\otimes\mathcal{U}^-}^{\mathcal{U}}\Bbbk_\lambda \simeq \mathcal{U}/\big(\mathcal{U}F_1 + \mathcal{U}F_2 + \sum_{g\in\Gamma}\mathcal{U}(g - \lambda(g))\big). \tag{8}$$

Let v_λ be the residue class of 1 in $M(\lambda)$; then we have an isomorphism of $\mathcal{U}^+$-modules

$$\mathcal{U}^+ \simeq M(\lambda), \qquad 1 \longmapsto v_\lambda.$$

Hence $\dim M(\lambda) = \dim \mathcal{B}(V) = 144$. Thus the PBW-basis of $\mathcal{U}^+ \simeq \mathcal{B}(V)$ becomes via this isomorphism a basis of $M(\lambda)$.

The $\mathbb{Z}^2$-grading on $\mathcal{U}^+ \simeq \mathcal{B}(V)$ induces a $\mathbb{Z}^2$-grading on $M(\lambda)$ such that

$$M(\lambda)_\beta = \mathcal{U}_\beta \cdot v_\lambda, \qquad \beta \in \mathbb{Z}^2.$$

Thus

$$M(\lambda)_0 = \Bbbk v_\lambda, \qquad \mathcal{U}_\beta \cdot M(\lambda)_\gamma \subset M(\lambda)_{\beta+\gamma}, \qquad \beta, \gamma \in \mathbb{Z}^2.$$

The family of $\mathcal{U}$-submodules of $M(\lambda)$ contained in $\sum_{\beta\neq 0} M(\lambda)_\beta$ has a unique maximal element $N(\lambda)$. We set

$$L(\lambda) = M(\lambda)/N(\lambda).$$

Since $\mathcal{U}$ satisfies the conditions on [19, Section 2], [19, Corollary 2.6] implies that

$$\textit{The map} \qquad \lambda \mapsto L(\lambda) \qquad \textit{provides a bijective correspondence} \qquad \widehat{\Gamma} \simeq \operatorname{Irr}\mathcal{U}. \tag{9}$$

Alternatively we see that $L(\lambda)$ is simple arguing as in [18, Theorem 1]; then [18, Theorem 3] gives (9). Notice that $L(\lambda)$ inherits the grading from $M(\lambda)$. Also, it follows that every simple $M \in {}_{\mathcal{U}}\mathcal{M}$ is diagonalizable.

Lowest weight modules of weight λ are defined as usual; $M(\lambda)$ covers every lowest weight module of weight λ, that in turn covers $L(\lambda)$. Highest weight modules are defined similarly.

1.4. Main result

In our main theorem, we give the dimension of $L(\lambda)$ for each $\lambda \in \widehat{\Gamma}$, in terms of certain equalities arising from the Shapovalov determinant [15] satisfied by

$$\lambda_i = \lambda(g_i\sigma_i), \qquad i \in \mathbb{I}_2.$$

Indeed, the Shapovalov determinant in the context of this paper is

$$\begin{aligned} Ш = {}& (\zeta^4\lambda_1^{-1} - \zeta^4)(\zeta^4\lambda_1^{-1} - \zeta^8)(\zeta^2\lambda_1^{-2}\lambda_2^{-1} - \zeta^8)(\zeta^2\lambda_1^{-2}\lambda_2^{-1} - \zeta^4)(\lambda_1^{-3}\lambda_2^{-2} + 1) \\ & \times (\zeta^{10}\lambda_1^{-1}\lambda_2^{-1} - \zeta^9)(\zeta^{10}\lambda_1^{-1}\lambda_2^{-1} + 1)(\zeta^{10}\lambda_1^{-1}\lambda_2^{-1} - \zeta^3)(\lambda_2^{-1} - 1). \end{aligned} \tag{10}$$

Then $Ш = 0$ if and only if one of the factors in (10) vanishes. Let

$$\mathtt{S}_1 = \{1, \zeta^8\}, \qquad \mathtt{S}_2 = \{-1, \zeta^{10}\}, \qquad \mathtt{S}_3 = \{\zeta, \zeta^4, \zeta^7\}. \tag{11}$$

The equalities alluded above can be packed as the conditions:

$$\lambda_1 \overset{?}{\in} \mathtt{S}_1, \qquad \lambda_1^2\lambda_2 \overset{?}{\in} \mathtt{S}_2, \qquad \lambda_1^3\lambda_2^2 \overset{?}{=} -1, \qquad \lambda_1\lambda_2 \overset{?}{\in} \mathtt{S}_3, \qquad \lambda_2 \overset{?}{=} 1. \tag{12}$$

To organize the information, we consider 47 subsets of $\widehat{\Gamma}$, organized in classes $\mathcal{C}_j$ according to the quantity j of conditions in (12) satisfied. The class $\mathcal{C}_0$ contains just one family:

$$\mathfrak{I}_1 = \{\lambda \in \widehat{\Gamma} \mid \lambda_1 \notin \mathtt{S}_1,\ \lambda_1^2\lambda_2 \notin \mathtt{S}_2,\ \lambda_1^3\lambda_2^2 \neq -1,\ \lambda_1\lambda_2 \notin \mathtt{S}_3,\ \lambda_2 \neq 1\};$$

Here is the class $\mathcal{C}_1$:

$$\begin{aligned} \mathfrak{I}_2 &= \{\lambda \in \widehat{\Gamma} \mid \lambda_1 = 1,\ \lambda_1^2\lambda_2 \notin \mathtt{S}_2,\ \lambda_1^3\lambda_2^2 \neq -1,\ \lambda_1\lambda_2 \notin \mathtt{S}_3,\ \lambda_2 \neq 1\} \\ &= \{\lambda \in \widehat{\Gamma} \mid \lambda_1 = 1, \lambda_2 \notin \{1, \zeta, \zeta^4, \zeta^7, \zeta^3, \zeta^9, -1, \zeta^{10}\}\}; \\ \mathfrak{I}_3 &= \{\lambda \in \widehat{\Gamma} \mid \lambda_1 = \zeta^8,\ \lambda_1^2\lambda_2 \notin \mathtt{S}_2,\ \lambda_1^3\lambda_2^2 \neq -1,\ \lambda_1\lambda_2 \notin \mathtt{S}_3,\ \lambda_2 \neq 1\} \\ &= \{\lambda \in \widehat{\Gamma} \mid \lambda_1 = \zeta^8,\ \lambda_2 \notin \{\pm 1, \zeta^2, \zeta^3, \zeta^5, \zeta^8, \zeta^9, \zeta^{11}\}\}; \end{aligned}$$

$$\begin{aligned}
\mathfrak{I}_4 &= \{\lambda \in \widehat{\Gamma} \mid \lambda_1 \notin \mathbb{S}_1,\ \lambda_1^2\lambda_2 = -1,\ \lambda_1^3\lambda_2^2 \neq -1,\ \lambda_1\lambda_2 \notin \mathbb{S}_3,\ \lambda_2 \neq 1\} \\
&= \{\lambda \in \widehat{\Gamma} \mid \lambda_1^2\lambda_2 = -1,\ \lambda_1 \notin \{\pm 1, \zeta^8, \zeta^{10}, \zeta^4, \zeta^2\}\}; \\
\mathfrak{I}_5 &= \{\lambda \in \widehat{\Gamma} \mid \lambda_1 \notin \mathbb{S}_1,\ \lambda_1^2\lambda_2 = \zeta^{10},\ \lambda_1^3\lambda_2^2 \neq -1,\ \lambda_1\lambda_2 \notin \mathbb{S}_3,\ \lambda_2 \neq 1\} \\
&= \{\lambda \in \widehat{\Gamma} \mid \lambda_1^2\lambda_2 = \zeta^{10},\ \lambda_1 \notin \{\pm 1, \zeta^8, \zeta^{10}, \zeta^4, \zeta^2\}\}; \\
\mathfrak{I}_6 &= \{\lambda \in \widehat{\Gamma} \mid \lambda_1 \notin \mathbb{S}_1,\ \lambda_1^2\lambda_2 \notin \mathbb{S}_2,\ \lambda_1^3\lambda_2^2 = -1,\ \lambda_1\lambda_2 \notin \mathbb{S}_3,\ \lambda_2 \neq 1\} \\
&= \{\lambda \in \widehat{\Gamma} \mid \lambda_1^3\lambda_2^2 = -1,\ \lambda_1 \notin \{\pm 1, \zeta^8, \zeta^{10}, \zeta^4, \zeta^2\}\}; \\
\mathfrak{I}_7 &= \{\lambda \in \widehat{\Gamma} \mid \lambda_1 \notin \mathbb{S}_1,\ \lambda_1^2\lambda_2 \notin \mathbb{S}_2,\ \lambda_1^3\lambda_2^2 \neq -1,\ \lambda_1\lambda_2 = \zeta,\ \lambda_2 \neq 1\} \\
&= \{\lambda \in \widehat{\Gamma} \mid \lambda_1\lambda_2 = \zeta,\ \lambda_1 \notin \{1, \zeta^8, \zeta, \zeta^4, \zeta^9\}\}; \\
\mathfrak{I}_8 &= \{\lambda \in \widehat{\Gamma} \mid \lambda_1 \notin \mathbb{S}_1,\ \lambda_1^2\lambda_2 \notin \mathbb{S}_2,\ \lambda_1^3\lambda_2^2 \neq -1,\ \lambda_1\lambda_2 = \zeta^4,\ \lambda_2 \neq 1\} \\
&= \{\lambda \in \widehat{\Gamma} \mid \lambda_1\lambda_2 = \zeta^4,\ \lambda_1 \notin \{1, \zeta^8, \zeta^4, \zeta^2, -1, \zeta^{10}\}\}; \\
\mathfrak{I}_9 &= \{\lambda \in \widehat{\Gamma} \mid \lambda_1 \notin \mathbb{S}_1,\ \lambda_1^2\lambda_2 \notin \mathbb{S}_2,\ \lambda_1^3\lambda_2^2 \neq -1,\ \lambda_1\lambda_2 = \zeta^7,\ \lambda_2 \neq 1\} \\
&= \{\lambda \in \widehat{\Gamma} \mid \lambda_1\lambda_2 = \zeta^7,\ \lambda_1 \notin \{1, \zeta^8, \zeta^7, \zeta^4, \zeta^{11}\}\}; \\
\mathfrak{I}_{10} &= \{\lambda \in \widehat{\Gamma} \mid \lambda_1 \notin \mathbb{S}_1,\ \lambda_1^2\lambda_2 \notin \mathbb{S}_2,\ \lambda_1^3\lambda_2^2 \neq -1,\ \lambda_1\lambda_2 \notin \mathbb{S}_3,\ \lambda_2 = 1\} \\
&= \{\lambda \in \widehat{\Gamma} \mid \lambda_1 \notin \mathbb{G}_{12},\ \lambda_2 = 1\};
\end{aligned}$$

All the 37 remaining subsets belong to class $\mathcal{C}_2$:

$$\begin{aligned}
&\mathfrak{I}_{11} = \{\lambda \in \widehat{\Gamma} \mid \lambda_1 = 1, \lambda_2 = \zeta\}, && \mathfrak{I}_{12} = \{\lambda \in \widehat{\Gamma} \mid \lambda_1 = 1, \lambda_2 = \zeta^4\}, \\
&\mathfrak{I}_{13} = \{\lambda \in \widehat{\Gamma} \mid \lambda_1 = 1, \lambda_2 = \zeta^7\}, && \mathfrak{I}_{14} = \{\lambda \in \widehat{\Gamma} \mid \lambda_1 = 1, \lambda_2 = \zeta^3\}, \\
&\mathfrak{I}_{15} = \{\lambda \in \widehat{\Gamma} \mid \lambda_1 = 1, \lambda_2 = \zeta^9\}, && \mathfrak{I}_{16} = \{\lambda \in \widehat{\Gamma} \mid \lambda_1 = 1, \lambda_2 = -1\}, \\
&\mathfrak{I}_{17} = \{\lambda \in \widehat{\Gamma} \mid \lambda_1 = 1, \lambda_2 = \zeta^{10}\}, && \mathfrak{I}_{18} = \{\lambda \in \widehat{\Gamma} \mid \lambda_1 = \zeta^8, \lambda_2 = \zeta^5\}, \\
&\mathfrak{I}_{19} = \{\lambda \in \widehat{\Gamma} \mid \lambda_1 = \zeta^8, \lambda_2 = \zeta^8\}, && \mathfrak{I}_{20} = \{\lambda \in \widehat{\Gamma} \mid \lambda_1 = \zeta^8, \lambda_2 = \zeta^{11}\}, \\
&\mathfrak{I}_{21} = \{\lambda \in \widehat{\Gamma} \mid \lambda_1 = \zeta^8, \lambda_2 = \zeta^3\}, && \mathfrak{I}_{22} = \{\lambda \in \widehat{\Gamma} \mid \lambda_1 = \zeta^8, \lambda_2 = \zeta^9\}, \\
&\mathfrak{I}_{23} = \{\lambda \in \widehat{\Gamma} \mid \lambda_1 = \zeta^8, \lambda_2 = \zeta^2\}, && \mathfrak{I}_{24} = \{\lambda \in \widehat{\Gamma} \mid \lambda_1 = \zeta^8, \lambda_2 = -1\}, \\
&\mathfrak{I}_{25} = \{\lambda \in \widehat{\Gamma} \mid \lambda_1 = \zeta^{11}, \lambda_2 = \zeta^8\}, && \mathfrak{I}_{26} = \{\lambda \in \widehat{\Gamma} \mid \lambda_1 = \zeta^5, \lambda_2 = \zeta^8\}, \\
&\mathfrak{I}_{27} = \{\lambda \in \widehat{\Gamma} \mid \lambda_1 = \zeta^4, \lambda_2 = \zeta^9\}, && \mathfrak{I}_{28} = \{\lambda \in \widehat{\Gamma} \mid \lambda_1 = \zeta^9, \lambda_2 = \zeta^4\}, \\
&\mathfrak{I}_{29} = \{\lambda \in \widehat{\Gamma} \mid \lambda_1 = -1, \lambda_2 = -1\}, && \mathfrak{I}_{30} = \{\lambda \in \widehat{\Gamma} \mid \lambda_1 = \zeta^2, \lambda_2 = \zeta^2\}, \\
&\mathfrak{I}_{31} = \{\lambda \in \widehat{\Gamma} \mid \lambda_1 = -1, \lambda_2 = \zeta^{10}\}, && \mathfrak{I}_{32} = \{\lambda \in \widehat{\Gamma} \mid \lambda_1 = \zeta^{10}, \lambda_2 = -1\}, \\
&\mathfrak{I}_{33} = \{\lambda \in \widehat{\Gamma} \mid \lambda_1 = \zeta^2, \lambda_2 = -1\}, && \mathfrak{I}_{34} = \{\lambda \in \widehat{\Gamma} \mid \lambda_1 = \zeta^4, \lambda_2 = \zeta^3\}, \\
&\mathfrak{I}_{35} = \{\lambda \in \widehat{\Gamma} \mid \lambda_1 = \zeta^3, \lambda_2 = \zeta^4\}, && \\
&\mathfrak{I}_{36} = \{\lambda \in \widehat{\Gamma} \mid \lambda_1 = \zeta, \lambda_2 = 1\}, && \mathfrak{I}_{37} = \{\lambda \in \widehat{\Gamma} \mid \lambda_1 = \zeta^2, \lambda_2 = 1\}, \\
&\mathfrak{I}_{38} = \{\lambda \in \widehat{\Gamma} \mid \lambda_1 = \zeta^3, \lambda_2 = 1\}, && \mathfrak{I}_{39} = \{\lambda \in \widehat{\Gamma} \mid \lambda_1 = \zeta^4, \lambda_2 = 1\}, \\
&\mathfrak{I}_{40} = \{\lambda \in \widehat{\Gamma} \mid \lambda_1 = \zeta^5, \lambda_2 = 1\}, && \mathfrak{I}_{41} = \{\lambda \in \widehat{\Gamma} \mid \lambda_1 = -1, \lambda_2 = 1\}, \\
&\mathfrak{I}_{42} = \{\lambda \in \widehat{\Gamma} \mid \lambda_1 = \zeta^7, \lambda_2 = 1\}, && \mathfrak{I}_{43} = \{\lambda \in \widehat{\Gamma} \mid \lambda_1 = \zeta^8, \lambda_2 = 1\}, \\
&\mathfrak{I}_{44} = \{\lambda \in \widehat{\Gamma} \mid \lambda_1 = \zeta^9, \lambda_2 = 1\}, && \mathfrak{I}_{45} = \{\lambda \in \widehat{\Gamma} \mid \lambda_1 = \zeta^{10}, \lambda_2 = 1\}, \\
&\mathfrak{I}_{46} = \{\lambda \in \widehat{\Gamma} \mid \lambda_1 = \zeta^{11}, \lambda_2 = 1\}, && \mathfrak{I}_{47} = \{\lambda \in \widehat{\Gamma} \mid \lambda_1 = 1, \lambda_2 = 1\}.
\end{aligned}$$

Main Theorem. *The dimension and the maximal degree of $L(\lambda)$ depend on λ_i, $i \in \mathbb{I}_2$, and appear in Table 1.*

The paper is organized as follows. We collect some general information about $\mathcal{U}$ and the Verma modules in Section 2, where we also deal with $\mathfrak{I}_1$. The proof of the Main Theorem for the families in the class 1, resp. 2, is given in Section 3, respectively 4.

If $M \in \mathcal{U}$, then we write $N \leq M$ to express that N is a submodule of M.

Table 1. Dimensions and highest degrees of simple modules.

Family	dim $L(\lambda)$	max. degree	$L(\lambda)^{\varphi}$
$\mathfrak{I}_1$	144	(12, 8)	$\mathfrak{I}_1$
$\mathfrak{I}_2$	48	(10, 8)	$\mathfrak{I}_2$
$\mathfrak{I}_3$	96	(11, 8)	$\mathfrak{I}_3$
$\mathfrak{I}_4$	48	(8, 6)	$\mathfrak{I}_4$
$\mathfrak{I}_5$	96	(10, 7)	$\mathfrak{I}_5$
$\mathfrak{I}_6$	72	(9, 6)	$\mathfrak{I}_6$
$\mathfrak{I}_7$	36	(9, 5)	$\mathfrak{I}_7$
$\mathfrak{I}_8$	72	(10, 6)	$\mathfrak{I}_8$
$\mathfrak{I}_9$	108	(11, 7)	$\mathfrak{I}_9$
$\mathfrak{I}_{10}$	72	(12, 7)	$\mathfrak{I}_{10}$
$\mathfrak{I}_{11}$	11	(5, 4)	$\mathfrak{I}_{12}$
$\mathfrak{I}_{12}$	11	(5, 4)	$\mathfrak{I}_{11}$
$\mathfrak{I}_{13}$	23	(7, 5)	$\mathfrak{I}_{44}$
$\mathfrak{I}_{14}$	25	(7, 5)	$\mathfrak{I}_{28}$
$\mathfrak{I}_{15}$	37	(9, 6)	$\mathfrak{I}_{41}$
$\mathfrak{I}_{16}$	37	(8, 6)	$\mathfrak{I}_{30}$
$\mathfrak{I}_{17}$	47	(10, 7)	$\mathfrak{I}_{46}$
$\mathfrak{I}_{18}$	11	(5, 3)	$\mathfrak{I}_{38}$
$\mathfrak{I}_{19}$	35	(8, 5)	$\mathfrak{I}_{40}$
$\mathfrak{I}_{20}$	71	(11, 7)	$\mathfrak{I}_{42}$
$\mathfrak{I}_{21}$	61	(9, 6)	$\mathfrak{I}_{32}$
$\mathfrak{I}_{22}$	49	(9, 6)	$\mathfrak{I}_{45}$
$\mathfrak{I}_{23}$	47	(8, 6)	$\mathfrak{I}_{29}$
$\mathfrak{I}_{24}$	85	(10, 7)	$\mathfrak{I}_{35}$
$\mathfrak{I}_{25}$	37	(8, 5)	$\mathfrak{I}_{37}$
$\mathfrak{I}_{26}$	25	(8, 5)	$\mathfrak{I}_{43}$
$\mathfrak{I}_{27}$	35	(9, 5)	$\mathfrak{I}_{36}$
$\mathfrak{I}_{28}$	25	(7, 5)	$\mathfrak{I}_{14}$
$\mathfrak{I}_{29}$	47	(8, 6)	$\mathfrak{I}_{23}$
$\mathfrak{I}_{30}$	37	(8, 6)	$\mathfrak{I}_{16}$
$\mathfrak{I}_{31}$	61	(10, 6)	$\mathfrak{I}_{39}$
$\mathfrak{I}_{32}$	61	(9, 6)	$\mathfrak{I}_{21}$
$\mathfrak{I}_{33}$	71	(9, 6)	$\mathfrak{I}_{34}$
$\mathfrak{I}_{34}$	71	(9, 6)	$\mathfrak{I}_{33}$
$\mathfrak{I}_{35}$	85	(10, 7)	$\mathfrak{I}_{24}$
$\mathfrak{I}_{36}$	35	(9, 5)	$\mathfrak{I}_{27}$
$\mathfrak{I}_{37}$	37	(8, 5)	$\mathfrak{I}_{25}$
$\mathfrak{I}_{38}$	11	(5, 3)	$\mathfrak{I}_{18}$
$\mathfrak{I}_{39}$	61	(10, 6)	$\mathfrak{I}_{31}$
$\mathfrak{I}_{40}$	35	(8, 5)	$\mathfrak{I}_{19}$
$\mathfrak{I}_{41}$	37	(9, 6)	$\mathfrak{I}_{15}$
$\mathfrak{I}_{42}$	71	(11, 7)	$\mathfrak{I}_{20}$
$\mathfrak{I}_{43}$	25	(8, 5)	$\mathfrak{I}_{26}$
$\mathfrak{I}_{44}$	23	(7, 5)	$\mathfrak{I}_{13}$
$\mathfrak{I}_{45}$	49	(9, 6)	$\mathfrak{I}_{22}$
$\mathfrak{I}_{46}$	47	(10, 7)	$\mathfrak{I}_{17}$
$\mathfrak{I}_{47}$	1	(0, 0)	$\mathfrak{I}_{47}$

2. Preliminaries

2.1. The algebra $\mathcal{U}$

The Nichols algebra $\mathcal{B}(V)$ has a PBW-basis given by

$$\left\{E_2^{a_2} E_{12}^{a_{12}} E_{11212}^{a_{11212}} E_{112}^{a_{112}} E_1^{a_1} \mid \quad a_2, a_{11212} \in \mathbb{I}_{0,1}; \quad a_{12} \in \mathbb{I}_{0,3}; \quad a_{112}, a_1 \in \mathbb{I}_{0,2}\right\}. \tag{13}$$

See [9]. We obtain a new PBW-basis by reordering the PBW-generators:

$$\left\{E_1^{a_1} E_{112}^{a_{112}} E_{11212}^{a_{11212}} E_{12}^{a_{12}} E_2^{a_2} \mid \quad a_2, a_{11212} \in \mathbb{I}_{0,1}; \quad a_{12} \in \mathbb{I}_{0,3}; \quad a_{112}, a_1 \in \mathbb{I}_{0,2}\right\}. \tag{14}$$

Thus the set of positive roots of $\mathcal{B}(V)$ (the degrees of the generators of the PBW-basis) is

$$\Delta_+^V = \{\alpha_1, 2\alpha_1 + \alpha_2, 3\alpha_1 + 2\alpha_2, \alpha_1 + \alpha_2, \alpha_2\}.$$

By [11, Theorem 4.9], we have

$$E_{112}^3 = E_{11212}^2 = E_{12}^4 = 0. \tag{15}$$

From the defining relations (5), we can deduce that the following are valid in $\mathcal{B}(V)$:

$$\begin{aligned}
E_1E_{112} &= q_{12}\zeta^8E_{112}E_1,\\
E_{112}E_2 &= -q_{12}^2E_2E_{112} + q_{12}\zeta^8E_{12}^2,\\
E_1E_{11212} &= q_{12}^2E_{11212}E_1 + q_{12}\zeta^7(1+\zeta)E_{112}^2\\
E_1E_{12}^2 &= E_{11212} + q_{12}\zeta(1+\zeta^3)E_{12}E_{112} + q_{12}^2\zeta^8E_{12}^2E_1\\
E_1E_{12}^3 &= q_{12}\zeta^{10}E_{12}E_{11212} + q_{12}^2\zeta^5E_{12}^2E_{112} + q_{12}^3E_{12}^3E_1,\\
E_1^2E_2 &= E_{112} + q_{12}^2\zeta^2E_{12}E_1 + q_{12}^2E_2E_1^2,\\
E_1^2E_{12} &= -q_{12}^2E_{112}E_1 + q_{12}^2\zeta^8E_{12}E_1^2,\\
E_{112}E_{12}^2 &= -q_{12}\zeta^4(1+\zeta^3)E_{12}E_{11212} + q_{12}^2\zeta^2E_{12}^2E_{112}\\
E_{112}E_{12}^3 &= q_{12}^2\zeta^{11}E_{12}^2E_{11212} + q_{12}^3\zeta^3E_{12}^3E_{112},\\
E_{11212}E_{12} &= q_{12}\zeta^{10}E_{12}E_{11212},\\
E_{112}E_{11212} &= q_{12}\zeta^9E_{11212}E_{112},\\
E_{11212}E_2 &= q_{12}^3E_2E_{11212} + q_{12}^2\zeta^2(1+\zeta)E_{12}^3,\\
E_{12}E_2 &= -q_{12}E_2E_{12}.
\end{aligned}$$

The following equalities hold by direct computation from (5) and the previous ones:

$$\begin{aligned}
F_1E_{12} &= E_{12}F_1 + q_{12}(\zeta-1)E_2\sigma_1^{-1},\\
F_1E_{112} &= E_{112}F_1 + q_{12}\zeta^8(1+\zeta^3)E_{12}\sigma_1^{-1},\\
F_1E_{11212} &= E_{11212}F_1 + q_{12}^2(\zeta^5-1)E_{12}^2\sigma_1^{-1},\\
F_1E_{112}^2 &= E_{112}^2F_1 - q_{12}(1+\zeta^3)(E_{11212}\sigma_1^{-1} + \zeta^4E_{112}E_{12}\sigma_1^{-1}),\\
F_1E_{12}^2 &= E_{12}^2F_1 + q_{12}^2(3)_{\zeta^5}E_2E_{12}\sigma_1^{-1},\\
F_1E_{12}^3 &= E_{12}^2F_1 + q_{12}^3\zeta^3(\zeta-1)E_2E_{12}^2\sigma_1^{-1},\\
F_2E_{12} &= E_{12}F_2 + (\zeta^{11}-1)E_1g_2,\\
F_2E_{112} &= E_{112}F_2 - (3)_{\zeta^7}E_1^2g_2,\\
F_2E_{11212} &= E_{11212}F_2 - E_{112}E_1g_2,\\
F_2E_{12}^2 &= E_{12}^2F_2 + q_{21}(1+\zeta^5)E_{112}g_2 - (3)_{\zeta^7}E_{12}E_1g_2,\\
F_2E_{112}^2 &= E_{112}^2F_2 + (3)_{\zeta^7}\zeta^4E_{112}E_1^2g_2,\\
F_2E_{12}^3 &= E_{12}^3F_2 + \zeta^8(1-\zeta)(E_{12}^2E_1g_2 - q_{21}\zeta^3E_{12}E_{112}g_2 + q_{21}^2\zeta^3E_{11212}g_2),\\
F_{11212}E_{11212} &= E_{11212}F_{11212} + \sigma_1^{-3}\sigma_2^{-2} - g_{11212},\\
F_{12}E_2 &= E_2F_{12} + (1-\zeta^{11})F_1\sigma_2^{-1},\\
F_{12}E_{12} &= E_{12}F_{12} + \sigma_1^{-1}\sigma_2^{-1} - g_1g_2,\\
F_{12}E_{112} &= E_{112}F_{12} + \zeta^3(3)_{\zeta^7}E_1g_1g_2,\\
F_{12}E_{112}^2 &= E_{112}^2F_{12} + \zeta^{11}(3)_{\zeta^7}E_{112}E_1g_1g_2,\\
F_{12}E_1 &= E_1F_{12} + q_{21}(1-\zeta)F_2g_1,\\
F_{12}E_{11212} &= E_{11212}F_{12} + \zeta^{11}E_{112}g_1g_2,\\
F_{112}E_{112} &= E_{112}F_{112} + \sigma_1^{-2}\sigma_2^{-1} - g_1^2g_2,\\
F_{112}E_2 &= E_2F_{112} + (\zeta-1)F_1^2\sigma_2^{-1}.
\end{aligned}$$

2.2. Verma modules

We shall use the notation for q-factorial numbers: for each $q \in \Bbbk^\times$,

$$(n)_q = 1 + q + \ldots + q^{n-1}, \qquad (n)_q! = (1)_q(2)_q \cdots (n)_q, \qquad n \in \mathbb{N}.$$

We shall investigate the lattice of submodules of a Verma module. We record the following standard fact for future use.

Remark 2.1. Let $v \in M(\lambda)_\alpha$ be such that $F_i \cdot v = 0$ for $i \in \mathbb{I}_2$. By the triangular decomposition of $\mathcal{U}$, $\mathcal{U} \cdot v = \mathcal{U}^+ \cdot v$. In particular, if $\alpha \neq 0$, then $\mathcal{U} \cdot v \cap \Bbbk v_\lambda = 0$.

We consider two families in $M(\lambda)$, corresponding to PBW-bases (13) and (14). We set

$$\widetilde{m}_{a,b,c,d,e} := E_2^a E_{12}^b E_{11212}^c E_{112}^d E_1^e \cdot v_\lambda, \qquad \widetilde{n}_{a,b,c,d,e} := E_1^e E_{112}^d E_{11212}^c E_{12}^b E_2^a \cdot v_\lambda$$

for $a, b, c, d, e \in \mathbb{Z}$. Clearly, $v_\lambda = \widetilde{m}_{0,0,0,0,0} = \widetilde{n}_{0,0,0,0,0}$ and

$$\widetilde{m}_{a,b,c,d,e} \neq 0 \iff a, c \in \mathbb{I}_{0,1}, b \in \mathbb{I}_{0,3}, d, e \in \mathbb{I}_{0,2} \iff \widetilde{n}_{a,b,c,d,e} \neq 0.$$

We denote by $\langle S \rangle$ the subspace generated by a subset S of a vector space. Let

$$\begin{aligned} W_1(\lambda) &= \langle \widetilde{m}_{a,b,c,d,e} \mid a, c \in \mathbb{I}_{0,1}, b \in \mathbb{I}_{0,3}, d \in \mathbb{I}_{0,2}, e \in \mathbb{I}_{1,2} \rangle, \\ W_2(\lambda) &= \langle \widetilde{m}_{a,b,c,d,2} \mid a, c \in \mathbb{I}_{0,1}, b \in \mathbb{I}_{0,3}, d \in \mathbb{I}_{0,2} \rangle, \\ W(\lambda) &= \langle \widetilde{n}_{1,b,c,d,e} \mid c \in \mathbb{I}_{0,1}, b \in \mathbb{I}_{0,3}, d, e \in \mathbb{I}_{0,2} \rangle. \end{aligned}$$

By a direct computation, we can prove:

Lemma 2.2.
(a) $F_2 \cdot W_i(\lambda) \subseteq W_i(\lambda)$, $i \in \mathbb{I}_2$,
(b) $F_1 \cdot \widetilde{m}_{a,b,c,d,i} \in \lambda(\sigma_1^{-1})(i)_{\zeta^4}(\zeta^{(i-1)8} - \lambda_1)\widetilde{m}_{a,b,c,d,i-1} + W_i(\lambda)$, $i \in \mathbb{I}_2$,
(c) $F_1 \cdot W(\lambda) \subseteq W(\lambda)$,
(d) $F_2 \cdot \widetilde{n}_{1,b,c,d,e} \in \lambda(\sigma_2^{-1})(1 - \lambda_2)\widetilde{n}_{0,b,c,d,e} + W(\lambda)$.

In consequence,
- $W_1(\lambda)$ *is a* $\mathcal{U}$*-submodule if and only if* $\lambda_1 = 1$;
- $W_2(\lambda)$ *is a* $\mathcal{U}$*-submodule if and only if* $\lambda_1 = \zeta^8$;
- $W(\lambda)$ *is a* $\mathcal{U}$*-submodule if and only if* $\lambda_2 = 1$. □

We denote by $m_{a,b,c,d,e}$, $n_{a,b,c,d,e}$ the classes of $\widetilde{m}_{a,b,c,d,e}$, $\widetilde{n}_{a,b,c,d,e}$ in $L(\lambda)$. We order lexicographically the set of all $m_{a,b,c,d,e}$:

$$m_{a,b,c,d,e} < m_{a',b',c',d',e'} \iff a < a', \text{ or } a = a', b < b', \text{ or } \cdots . \tag{16}$$

2.3. Simple modules

Let $\varphi : \mathcal{U} \to \mathcal{U}$ be the algebra automorphism such that

$$\varphi(K_i) = K_i^{-1}, \qquad \varphi(L_i) = L_i^{-1}, \qquad \varphi(E_i) = F_i L_i^{-1}, \qquad \varphi(F_i) = K_i^{-1} E_i,$$

$i \in \mathbb{I}_2$, cf. [14, Proposition 4.9]; this resembles the Chevalley involution. If M is a $\mathcal{U}$-module, then we denote by M^φ the $\mathcal{U}$-module with $M^\varphi = M$ as vector space and action given by $a \rhd v = \varphi(a) \cdot v$, $v \in V$, $a \in \mathcal{U}$. If $v \in M$ has weight λ (with respect the action of Γ), then $v \in M^\varphi$ has weight λ^{-1}. The functor $M \mapsto M^\varphi$ preserves simple objects and sends lowest weight modules to highest weight modules, and vice versa. The following result is standard.

Lemma 2.3. *The subspace* $X(\lambda) := \{x \in L(\lambda) : E_i x = 0 \text{ for all } i\}$ *of* $L(\lambda)$ *is one-dimensional and there exists* $\mu \in \widehat{\Gamma}$ *such that* $X(\lambda) \overset{(1)}{=} L(\lambda)_\mu$, $L(\lambda)^\varphi \overset{(2)}{\simeq} L(\mu^{-1})$.

Proof. $X(\lambda) \neq 0$ because there exists $\beta \in \mathbb{N}_0^2$ maximal such that $L(\lambda)_\beta \neq 0$. Since $X(\lambda)$ is Γ-stable, there exists a weight vector $0 \neq x \in X(\lambda)$ with weight $\mu \in \widehat{\Gamma}$. Thus $\mathcal{U}^- x = \mathcal{U}x = L(\lambda)$ and (1) follows. Also $L(\lambda)^\varphi = (\mathcal{U}^- x)^\varphi \twoheadrightarrow L(\mu^{-1})$ implying (2). □

Lemma 2.4. *Let* $M \in {}_{\mathcal{U}}\mathcal{M}$ *a highest weight module of highest weight* μ *and* $0 \neq v \in M^\mu$. *If* $m_{a,b,c,d,e} \neq 0$ *in* $L(\mu^{-1})$ *then* $z := F_2^a F_{12}^b F_{11212}^c F_{112}^d F_1^e v \neq 0$.

There is an analogue statement for $n_{a,b,c,d,e}$.

Proof. Indeed M^φ is lowest weight of lowest weight μ^{-1}, hence $M^\varphi \twoheadrightarrow L(\mu^{-1})$; up to a non-zero scalar, $z \mapsto m_{a,b,c,d,e} \neq 0$, hence $z \neq 0$. □

2.4. A relative of $\mathfrak{u}_q(\mathfrak{sl}_2)$

We consider for a moment the algebra $\mathcal{V}$ constructed as $\mathcal{U}$ above but starting from a braided vector space of dimension 1, with braiding given by $q = \sigma(g) \in \mathbb{G}'_N$, $g \in \Lambda$, $\sigma \in \widehat{\Lambda}$. The algebra $\mathcal{V}$ is close to $\mathfrak{u}_q(\mathfrak{sl}_2)$ and has a presentation by generators $h \in \Lambda$, $\tau \in \widehat{\Lambda}$, E, F with relations

$$E^N = F^N = 0, \qquad hE = \sigma(h)Eh, \qquad \tau E = \tau(g)E\tau,$$
$$EF - FE = g - \sigma^{-1}, \qquad hF = \sigma^{-1}(h)Fh, \qquad \tau F = \tau(g^{-1})F\tau,$$

and $h\tau = \tau h$ for $h \in \Lambda$, $\tau \in \widehat{\Lambda}$, and the relations defining Λ, $\widehat{\Lambda}$. Thus

$$E^j F - FE^j = (j)_q E^{j-1}(g - q^{1-j}\sigma^{-1}), \qquad j \in \mathbb{N}. \tag{17}$$

Let $\lambda \in \widehat{\Gamma}$. Let $L(\lambda)$ be lowest weight $\mathcal{V}$-module of lowest weight λ defined in the same usual way. The same argument as for $\mathfrak{u}_q(\mathfrak{sl}_2)$ gives the following.

Lemma 2.5.
(a) *If there exists* $j \in \mathbb{I}_{N-1}$ *such that* $\lambda(g\sigma) = q^{1-j}$, *then* $\dim L(\lambda) = j$.
(b) *If* $\lambda(g\sigma) \notin \{q^h | h \in \mathbb{I}_{0,N-2}\}$, *then* $\dim L(\lambda) = N$.
(c) $L(\lambda)$ *has a basis* $v_0, \dots, v_{\dim L(\lambda)-1}$ *such that for all i,*

$$Ev_i = v_{i+1}, \qquad Fv_i = (i)_q(q^{1-i}\lambda(\sigma_1^{-1}) - \lambda(g_1))v_{i-1}, \qquad h\tau v_i = \lambda(h\tau)\sigma^i(h)\tau(g^i)v_i. \tag{18}$$

(d) *Let M be a lowest weight* $\mathcal{V}$-*module with lowest weight* $\lambda \in \widehat{\Gamma}$. *If* $0 \neq v \in M^\lambda$, *then* $v, Ev, \dots, E^{n-1}v$ *are linearly independent, where*
   (1) *either* $n = j$ *if* $\lambda(g\sigma) = q^{1-j}$ *for some (unique)* $j \in \mathbb{I}_{N-1}$,
   (2) *or else* $n = N - 1$ *if* $\lambda(g\sigma) \notin \{q^h | h \in \mathbb{I}_{0,N-2}\}$.
   Moreover $F^i E^i v = a_i v$ *for some* $a_i \in \Bbbk^\times$ *when* $i \in \mathbb{I}_{0,n-1}$. □

2.5. The class $\mathcal{C}_0$

The first family is easy to deal with.

Lemma 2.6. *If* $\lambda \in \mathfrak{I}_1$, *then* $M(\lambda)$ *is simple.*

Proof. By [15, 5.16] that says: if Ш $\neq 0$, then $M(\lambda)$ is simple. □

3. Simple $\mathcal{U}$-modules in class $\mathcal{C}_1$

Here we deal with the class of families satisfying exactly one of the conditions in (12). Recall that $\Gamma = \Lambda \times \widehat{\Lambda}$; we introduce $\chi_i \in \widehat{\Gamma}$ by

$$\chi_i(g, \sigma) = \sigma_i(g)\sigma(g_i), \qquad i \in \mathbb{I}_2.$$

For simplicity, we introduce the following notation:

$$\begin{aligned} g_{12} &= g_1g_2, & g_{112} &= g_1^2g_2, & g_{11212} &= g_1^3g_2^2, \\ \sigma_{12} &= \sigma_1\sigma_2, & \sigma_{112} &= \sigma_1^2\sigma_2, & \sigma_{11212} &= \sigma_1^3\sigma_2^2. \end{aligned}$$

We outline the method to compute $L(\lambda)$, $\lambda \in \mathfrak{I}_j$, $j \in \mathbb{I}_{2,10}$.

(a) As (exactly) one of the factors of the Shapovalov determinant III vanishes, there exists $\beta \neq 0$ and $w \in M(\lambda)_\beta - 0$, such that $F_iw = 0$, $i \in \mathbb{I}_2$, see Remarks 3.5, 3.8, 3.11, 3.14, 3.17, 3.20, or Lemma 2.2. Thus $\mathcal{U}w$ is a proper submodule.

(b) Assume we are dealing with $\mathfrak{I}_j$, $j \in \mathbb{I}_{2,6}$. Write $w = \sum \mathrm{p}_{a,b,c,d,e}\, \widetilde{m}_{a,b,c,d,e}$. Then there exist a, b, c, d, e such that $\mathrm{p}_{a,b,c,d,e} \neq 0$ and exactly four of the integers $a, \dots, e$ are zero. The same holds for $j \in \mathbb{I}_{7,10}$ exchanging $\widetilde{m}_{a,b,c,d,e}$ by $\widetilde{n}_{a,b,c,d,e}$. From here we describe a basis B_j of the quotient $L'(\lambda)$ of $M(\lambda)$ by $\mathcal{U}w$, $j \in \mathbb{I}_{2,10}$.

(c) Let v be the element of maximal degree of $L'(\lambda)$. A short computation shows that v belongs to every submodule of $L'(\lambda)$. Because of the inequalities defining $\mathfrak{I}_j$, there exists $F \in \mathcal{U}$ such that $Fv = v_\lambda$. Hence $L'(\lambda)$ is simple.

We work out the details for $\mathfrak{I}_2$, with shorter expositions for the other families in $\mathcal{C}_1$.

3.1. The family $\mathfrak{I}_2$

Recall that

$$\mathfrak{I}_2 = \{\lambda \in \widehat{\Gamma} \mid \lambda_1 = 1, \lambda_2 \notin \{1, \zeta, \zeta^4, \zeta^7, \zeta^3, \zeta^9, -1, \zeta^{10}\}\}.$$

Lemma 3.1. *If $\lambda \in \mathfrak{I}_2$, then $\dim L(\lambda) = 48$. A basis of $L(\lambda)$ is given by*

$$\mathrm{B}_2 = \{m_{a,b,c,d,0} : a, c \in \mathbb{I}_{0,1},\ b \in \mathbb{I}_{0,3},\ d \in \mathbb{I}_{0,2}\}.$$

Proof. Let $w = \widetilde{m}_{0,0,0,0,1}$; then $F_iw = 0$, $i \in \mathbb{I}_2$, hence $\mathcal{U}^+w = W_1(\lambda) \leq M(\lambda)$ is proper by Lemma 2.2. Let $L'(\lambda) = M(\lambda)/\mathcal{U}^+w$. Let $\widehat{m}_{a,b,c,d,0}$ be the class of $\widetilde{m}_{a,b,c,d,0}$ in $L'(\lambda)$. Then

$$\widehat{\mathrm{B}}_2 = \{\widehat{m}_{a,b,c,d,0} : a, c \in \mathbb{I}_{0,1},\ b \in \mathbb{I}_{0,3},\ d \in \mathbb{I}_{0,2}\}$$

is a basis of $L'(\lambda)$, ordered by (16). Thus, it is enough to show that $L'(\lambda)$ is simple. Let $0 \neq W \leq L'(\lambda)$ and pick $u \in W - 0$. Fix $\widehat{m}_{a,b,c,d,0} \in \widehat{\mathrm{B}}_2$ minimal among those whose coefficient in u is non-zero. Then

$$E_{112}^{2-d}E_{11212}^{1-c}E_{12}^{3-b}E_2^{1-a}u \in \Bbbk^\times \widehat{m}_{1,3,1,2,0} \implies \widehat{m}_{1,3,1,2,0} \in W.$$

By abuse of notation, we denote by v_λ its class in $L'(\lambda)$. We claim that

$$F_2F_{12}^3F_{11212}F_{112}^2\widehat{m}_{1,3,1,2,0} \in \Bbbk^\times v_\lambda; \tag{19}$$

this implies that $v_\lambda \in W$, so $L'(\lambda)$ is simple.

To prove (19), we first consider the subalgebra $\mathcal{V}_1 = \Bbbk\langle g, \sigma, E_{112}, F_{112}\rangle$ of $\mathcal{U}$; clearly $\mathcal{V}_1 \simeq \mathcal{V}$ from §2.4. Then

$$F_{112}\widehat{m}_{1,3,1,0,0} = 0, \qquad g_{112}\sigma_{112}\widehat{m}_{1,3,1,0,0} = -\lambda_2\widehat{m}_{1,3,1,0,0}, \qquad E_{112}^2\widehat{m}_{1,3,1,0,0} = \sigma_{112}^2(g_{12}^{-6})\widehat{m}_{1,3,1,2,0}.$$

By Lemma 2.5, we conclude that

$$F_{112}^2\widehat{m}_{1,3,1,2,0} \in \Bbbk^\times \widehat{m}_{1,3,1,0,0} \implies \widehat{m}_{1,3,1,0,0} \in W.$$

We next consider $\mathcal{V}_2 = \Bbbk\langle g, \sigma, E_{11212}, F_{11212}\rangle \hookrightarrow \mathcal{U}$; again, $\mathcal{V}_2 \simeq \mathcal{V}$. Then

$$\begin{aligned}&F_{11212}\widehat{m}_{1,3,0,0,0} = 0, \qquad g_{11212}\sigma_{11212}\widehat{m}_{1,3,0,0,0} = -\lambda_2^2\widehat{m}_{1,3,0,0,0},\\ &E_{11212}\widehat{m}_{1,3,0,0,0} = \sigma_{11212}(g_1^{-3}g_2^{-4})\widehat{m}_{1,3,1,0,0},\\ &\overset{\text{Lemma 2.5}}{\Longrightarrow} F_{11212}\widehat{m}_{1,3,1,0,0} \in \Bbbk^\times\widehat{m}_{1,3,0,0,0} \implies \widehat{m}_{1,3,0,0,0} \in W.\end{aligned}$$

Once again, we consider $\mathcal{V}_3 = \Bbbk\langle g, \sigma, E_{12}, F_{12}\rangle \hookrightarrow \mathcal{U}$; thus $\mathcal{V}_3 \simeq \mathcal{V}$ from §2.4. Then

$$\begin{aligned}&F_{12}\widehat{m}_{1,0,0,0,0} = 0, \qquad g_{12}\sigma_{12}\widehat{m}_{1,0,0,0,0} = \lambda_2\zeta^{11}\widehat{m}_{1,0,0,0,0}, \qquad E_{12}^3\widehat{m}_{1,0,0,0,0} = \sigma_{12}^3(g_2^{-1})\widehat{m}_{1,3,0,0,0}\\ &\overset{\text{Lemma 2.5}}{\Longrightarrow} F_{12}^3\widehat{m}_{1,3,0,0,0} \in \Bbbk^\times\widehat{m}_{1,0,0,0,0} \implies \widehat{m}_{1,0,0,0,0} \in W.\end{aligned}$$

Now $F_2\widehat{m}_{1,0,0,0,0} = \lambda(\sigma_2)^{-1}(\lambda_2 - 1)v_\lambda \neq 0$, and (19) follows. □

Corollary 3.2. *If $\lambda \in \mathfrak{I}_2$, then $N(\lambda) \simeq L(\chi_1\lambda)$ and $\chi_1\lambda \in \mathfrak{I}_3$.*

Proof. By the proof of the Lemma, $N(\lambda)$ is of lowest weight $\chi_1\lambda$ and $\dim N(\lambda) = 96$. It is easy to see that $\chi_1\lambda \in \mathfrak{I}_3$; hence $\dim L(\chi_1\lambda) = 96$ by Lemma 3.3 and the claim follows. □

3.2. The family $\mathfrak{I}_3$

Recall that

$$\mathfrak{I}_3 = \{\lambda \in \widehat{\Gamma} \mid \lambda_1 = \zeta^8,\ \lambda_2 \notin \{\pm 1, \zeta^2, \zeta^3, \zeta^5, \zeta^8, \zeta^9, \zeta^{11}\}\}.$$

Lemma 3.3. *If $\lambda \in \mathfrak{I}_3$, then $\dim L(\lambda) = 96$. A basis of $L(\lambda)$ is given by*

$$\mathrm{B}_3 = \{m_{a,b,c,d,e} | a, c \in \mathbb{I}_{0,1},\ b \in \mathbb{I}_{0,3},\ d \in \mathbb{I}_{0,2}, e \in \mathbb{I}_{0,1}\}.$$

Proof. Let $w = \widetilde{m}_{0,0,0,0,2}$ and $L'(\lambda) = M(\lambda)/\mathcal{U}^+w$. We identify B_3 with a basis of $L'(\lambda)$. Now $F_2F_{12}^3F_{11212}F_{112}^2F_1m_{1,3,1,2,1} \in \Bbbk^\times v_\lambda$, hence $L'(\lambda)$ is simple. □

Exactly as for Corollary 3.2, we conclude:

Corollary 3.4. *If $\lambda \in \mathfrak{I}_3$, then $N(\lambda) \simeq L(\chi_1^2\lambda)$ and $\chi_1^2\lambda \in \mathfrak{I}_2$.* □

3.3. The family $\mathfrak{I}_4$

Recall that

$$\mathfrak{I}_4 = \{\lambda \in \widehat{\Gamma} \mid \lambda_1^2\lambda_2 = -1,\ \lambda_1 \notin \{\pm 1, \zeta^8, \zeta^{10}, \zeta^4, \zeta^2\}\}.$$

We start by a Remark that will be useful elsewhere.

Remark 3.5. Let $\lambda \in \widehat{\Gamma}$. If $\lambda_1^2\lambda_2 = -1$, then $w = F_1^2E_{112}E_1^2v_\lambda \in M(\lambda)$ satisfies

$$F_1w = F_2w = 0. \tag{20}$$

Proof. By a direct computation,

$$F_{112}E_{112}E_1^2v_\lambda = \lambda(\sigma_1^{-2}\sigma_2^{-1})q_{21}^2\zeta^4(\lambda_1^2\lambda_2 + 1)E_1^2v_\lambda.$$

As $M(\lambda)_{4\alpha_1} = M(\lambda)_{3\alpha_1} = 0$, we have that $F_2E_{112}E_1^2v_\lambda = F_1E_{112}E_1^2v_\lambda = 0$, so

$$0 = F_{112}E_{112}E_1^2v_\lambda = \zeta^8q_{12}^2F_2F_1^2E_{112}E_1^2v_\lambda.$$

This shows that $F_2w = 0$; on the other hand, $F_1w = F_1^3(E_{112}E_1^2v_\lambda) = 0$, since $F_1^3 = 0$. □

Lemma 3.6. *If $\lambda \in \mathfrak{I}_4$, then* $\dim L(\lambda) = 48$. *A basis of* $L(\lambda)$ *is given by*

$$\mathrm{B}_4 = \{m_{a,b,c,0,e} : a, c \in \mathbb{I}_{0,1}, b \in \mathbb{I}_{0,3}, e \in \mathbb{I}_{0,2}\}.$$

Proof. Let $w = F_1^2 E_{112} E_1^2 v_\lambda$. By Remark 3.5, $\mathcal{U}w$ is a proper submodule. We identify B_4 with a basis of $L'(\lambda) := M(\lambda)/\mathcal{U}w$. We check that there exists $F \in \mathcal{U}$ such that $Fm_{1,3,1,0,2} = v_\lambda$. Then $L'(\lambda)$ is simple. □

Exactly as for Corollary 3.2, we conclude:

Corollary 3.7. *If $\lambda \in \mathfrak{I}_4$, then $N(\lambda) \simeq L(\chi_1^2\chi_2\lambda)$ and $\chi_1^2\chi_2\lambda \in \mathfrak{I}_5$.*

3.4. The family $\mathfrak{I}_5$

Recall that

$$\mathfrak{I}_5 = \{\lambda \in \widehat{\Gamma} \mid \lambda_1^2\lambda_2 = \zeta^{10}, \lambda_1 \notin \{\pm 1, \zeta^8, \zeta^{10}, \zeta^4, \zeta^2\}\}.$$

Here is another Remark that will be useful later, proved as Remark 3.5.

Remark 3.8. Let $\lambda \in \widehat{\Gamma}$. If $\lambda_1^2\lambda_2 = \zeta^{10}$, then $w = F_1^2 E_{112}^2 E_1^2 v_\lambda \in M(\lambda)$ satisfies (20).

Lemma 3.9. *If $\lambda \in \mathfrak{I}_5$, then* $\dim L(\lambda) = 96$. *A basis of* $L(\lambda)$ *is given by*

$$\mathrm{B}_5 = \{m_{a,b,c,d,e} | a, c, d \in \mathbb{I}_{0,1}, b \in \mathbb{I}_{0,3}, e \in \mathbb{I}_{0,2}\}.$$

Proof. Let $w = F_1^2 E_{112}^2 E_1^2 v_\lambda$. By Remark 3.8, $\mathcal{U}w$ is a proper submodule. We identify B_5 with a basis of $L'(\lambda) := M(\lambda)/\mathcal{U}w$. We check that there exists $F \in \mathcal{U}$ such that $Fm_{1,3,1,1,2} = v_\lambda$. Then $L'(\lambda)$ is simple. □

Exactly as for Corollary 3.2, we conclude:

Corollary 3.10. *If $\lambda \in \mathfrak{I}_5$, then $N(\lambda) \simeq L(\chi_1^4\chi_2^2\lambda)$ and $\chi_1^4\chi_2^2\lambda \in \mathfrak{I}_4$.*

3.5. The family $\mathfrak{I}_6$

Recall that

$$\mathfrak{I}_6 = \{\lambda \in \widehat{\Gamma} \mid \lambda_1^3\lambda_2^2 = -1, \lambda_1 \notin \{\pm 1, \zeta^8, \zeta^{10}, \zeta^4, \zeta^2\}\}.$$

Still another Remark useful elsewhere, with an analogous proof as above.

Remark 3.11. Let $\lambda \in \widehat{\Gamma}$. If $\lambda_1^3\lambda_2^2 = -1$, then $w = F_1^2 F_{112}^2 E_{11212} E_{112}^2 E_1^2 v_\lambda$ satisfies (20).

Lemma 3.12. *If $\lambda \in \mathfrak{I}_6$, then* $\dim L(\lambda) = 72$. *A basis of* $L(\lambda)$ *is given by*

$$\mathrm{B}_6 = \{m_{a,b,0,d,e} | a \in \mathbb{I}_{0,1}, b \in \mathbb{I}_{0,3}, d, e \in \mathbb{I}_{0,2}\}.$$

Proof. Let w be as in Remark 3.11; then $\mathcal{U}w$ is proper. Again B_6 is identified with a basis of $L'(\lambda) = M(\lambda)/\mathcal{U}w$; since there is $F \in \mathcal{U}$ such that $Fm_{1,3,0,2,2} = v_\lambda$, $L'(\lambda)$ is simple. □

Exactly as for Corollary 3.2, we conclude:

Corollary 3.13. *If $\lambda \in \mathfrak{I}_6$, then $N(\lambda) \simeq L(\chi_1^3\chi_2^2\lambda)$ and $\chi_1^3\chi_2^2\lambda \in \mathfrak{I}_6$.*

3.6. The family $\mathfrak{J}_7$

Recall that

$$\mathfrak{J}_7 = \{\lambda \in \widehat{\Gamma} \mid \lambda_1\lambda_2 = \zeta,\ \lambda_1 \notin \{1, \zeta^8, \zeta, \zeta^4, \zeta^9\}\}.$$

Again we start by a useful remark.

Remark 3.14. Let $\lambda \in \widehat{\Gamma}$. If $\lambda_1\lambda_2 = \zeta$, then $w = F_2E_2E_{12}v_\lambda \in M(\lambda)$ satisfies (20).

Lemma 3.15. *If $\lambda \in \mathfrak{J}_7$, then $\dim L(\lambda) = 36$. A basis of $L(\lambda)$ is given by*

$$\mathrm{B}_7 = \{n_{a,0,c,d,e} | a, c \in \mathbb{I}_{0,1}, d, e \in \mathbb{I}_{0,2}\}.$$

Proof. Let $w = F_2E_2E_{12}v_\lambda$. By Remark 3.14, $\mathcal{U}w \subsetneq M(\lambda)$. Let $L'(\lambda) = M(\lambda)/\mathcal{U}w$, so B_7 is a basis of $L'(\lambda)$. There exists $F \in \mathcal{U}$ such that $Fn_{1,0,1,2,2} = v_\lambda$. Then $L'(\lambda)$ is simple. □

Exactly as for Corollary 3.2, we conclude:

Corollary 3.16. *If $\lambda \in \mathfrak{J}_7$, then $N(\lambda) \simeq L(\chi_1\chi_2\lambda)$ and $\chi_1\chi_2\lambda \in \mathfrak{J}_9$.*

3.7. The family $\mathfrak{J}_8$

Recall that

$$\mathfrak{J}_8 = \{\lambda \in \widehat{\Gamma} \mid \lambda_1\lambda_2 = \zeta^4,\ \lambda_1 \notin \{1, \zeta^8, \zeta^4, \zeta^2, -1, \zeta^{10}\}\}.$$

Remark 3.17. Let $\lambda \in \widehat{\Gamma}$. If $\lambda_1\lambda_2 = \zeta^4$, then $w = F_2E_2E_{12}^2v_\lambda \in M(\lambda)$ satisfies (20).

Proof. Analogous to Remark 3.5. □

Lemma 3.18. *If $\lambda \in \mathfrak{J}_8$, then $\dim L(\lambda) = 72$. A basis of $L(\lambda)$ is given by*

$$\mathrm{B}_8 = \{n_{a,b,c,d,e} | a, b, c \in \mathbb{I}_{0,1}, d, e \in \mathbb{I}_{0,2}\}.$$

Proof. Let $w = F_2E_2E_{12}^2v_\lambda$. By Remark 3.17, $\mathcal{U}w \subsetneq M(\lambda)$. Now B_8 identifies with a basis of $L'(\lambda) := M(\lambda)/\mathcal{U}w$. Since there is $F \in \mathcal{U}$ such that $Fn_{1,1,1,2,2} = v_\lambda$, $L'(\lambda)$ is simple. □

Exactly as for Corollary 3.2, we conclude:

Corollary 3.19. *If $\lambda \in \mathfrak{J}_8$, then $N(\lambda) \simeq L(\chi_1^2\chi_2^2\lambda)$ and $\chi_1^2\chi_2^2\lambda \in \mathfrak{J}_8$.*

3.8. The family $\mathfrak{J}_9$

Recall that

$$\mathfrak{J}_9 = \{\lambda \in \widehat{\Gamma} \mid \lambda_1\lambda_2 = \zeta^7,\ \lambda_1 \notin \{1, \zeta^8, \zeta^7, \zeta^4, \zeta^{11}\}\}.$$

Remark 3.20. Let $\lambda \in \widehat{\Gamma}$. If $\lambda_1\lambda_2 = \zeta^7$, then $w = F_2E_2E_{12}^3v_\lambda \in M(\lambda)$ satisfies (20).

Proof. Analogous to Remark 3.5. □

Lemma 3.21. *If $\lambda \in \mathfrak{J}_9$, then $\dim L(\lambda) = 108$. A basis of $L(\lambda)$ is given by*

$$\mathrm{B}_9 = \{n_{a,b,c,d,e} | a, c \in \mathbb{I}_{0,1}, b, d, e \in \mathbb{I}_{0,2}\}.$$

Proof. Let $w = F_2E_2E_{12}^3v_\lambda$. By Remark 3.20, $\mathcal{U}w \subsetneq M(\lambda)$. Let $L'(\lambda) = M(\lambda)/\mathcal{U}w$, so B_9 is a basis of $L'(\lambda)$. Since there exists $F \in \mathcal{U}$ such that $Fn_{1,2,1,2,2} = v_\lambda$, $L'(\lambda)$ is simple. □

Exactly as for Corollary 3.2, we conclude:

Corollary 3.22. *If $\lambda \in \mathfrak{I}_9$, then $N(\lambda) \simeq L(\chi_1^3\chi_2^3\lambda)$ and $\chi_1^3\chi_2^3\lambda \in \mathfrak{I}_7$.*

3.9. The family $\mathfrak{I}_{10}$

Recall that

$$\mathfrak{I}_{10} = \{\lambda \in \widehat{\Gamma} \mid \lambda_1 \notin \mathbb{G}_{12}, \lambda_2 = 1\}.$$

Lemma 3.23. *If $\lambda \in \mathfrak{I}_{10}$, then $\dim L(\lambda) = 72$. A basis of $L(\lambda)$ is given by*

$$B_{10} = \{n_{0,b,c,d,e} | c \in \mathbb{I}_{0,1}, b \in \mathbb{I}_{0,3}, d, e \in \mathbb{I}_{0,2}\}.$$

Proof. Let $w = \widetilde{n}_{1,0,0,0,0}$ and $L'(\lambda) = M(\lambda)/\mathcal{U}^+w$. We identify B_{10} with a basis of $L'(\lambda)$. Now $F_1^2F_{112}^2F_{11212}F_{12}^3n_{0,3,1,2,2} \in \Bbbk^\times v_\lambda$, hence $L'(\lambda)$ is simple. □

Exactly as for Corollary 3.2, we conclude:

Corollary 3.24. *If $\lambda \in \mathfrak{I}_{10}$, then $N(\lambda) \simeq L(\chi_2\lambda)$ and $\chi_2\lambda \in \mathfrak{I}_{10}$.*

4. Simple $\mathcal{U}$-modules in class $\mathcal{C}_2$

We start by the method to compute $L(\lambda)$, $\lambda \in \mathfrak{I}_j$, $j \in \mathbb{I}_{11,47}$. We illustrate by considering $\mathfrak{I}_{11}$, which is small enough to allow complete details; and $\mathfrak{I}_{13}$, with less explicit yet complete enough arguments. Then we give the main features of the proofs for the other families in $\mathcal{C}_2$. Here are the steps of the method:

(1) We identify easily a proper submodule $W = \mathcal{U}w_1$ of $M(\lambda)$ as follows:
- ⋄ if $j \in \mathbb{I}_{11,17}$, then $w_1 = \widetilde{m}_{0,0,0,0,1}$, so $W = W_1(\lambda)$, see Lemma 2.2;
- ⋄ if $j \in \mathbb{I}_{18,24}$, then $w_1 = \widetilde{m}_{0,0,0,0,2}$, so $W = W_2(\lambda)$, again by Lemma 2.2;
- ⋄ if $j \in \mathbb{I}_{25,35}$, then w_1 is as in one of the Remarks 3.5, 3.8, 3.14, 3.17, 3.20;
- ⋄ if $j \in \mathbb{I}_{36,47}$, then $w_1 = \widetilde{n}_{1,0,0,0,0}$, so $W = W(\lambda)$ by Lemma 2.2.

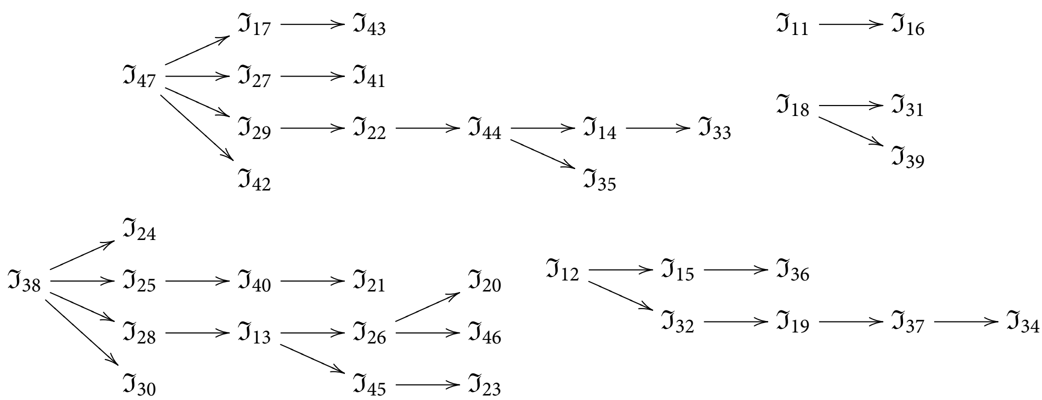
A basis of $M(\lambda)/W$ is obtained by restriction of the height of a specific PBW generator. Below we denote by w_2 an element of $M(\lambda)$ or its class modulo W, indistinctly.

(2) Next we show that there exists $\beta \neq 0$ and $w_2 \in (M(\lambda)/W)_\beta - 0$, such that $F_iw_2 = 0$, $i \in \mathbb{I}_2$; for this, we either apply one of Remarks 3.5, 3.8, 3.11, 3.14, 3.17, 3.20, or else proceed by direct computation. Hence $\mathcal{U}w_2$ is a proper submodule of $M(\lambda)/W$.

(3) Let $L'(\lambda) = M(\lambda)/(W + \mathcal{U}w_2)$. We consider a suitable set B_j inside the image of the PBW-basis in $L'(\lambda)$ that spans $L'(\lambda)$. To prove that B_j is linearly independent, we apply one of the following procedures:
   (a) For $j \in \mathbb{J} = \{11, 12, 18, 38\}$, the elements of B_j are homogeneous of different degrees.
   (b) Assume that $j \notin \mathbb{J}$. Then $\mathcal{U}w_2 \leq M(\lambda)/W$ projects onto the simple module $L(\nu)$, where ν is the weight of w_2. Also, let $u \in M(\lambda)/W$ be the element of maximal degree; then $(\mathcal{U}u)^\varphi$ projects onto a simple $L(\mu)$. Let $\mathfrak{I}_k$ and $\mathfrak{I}_\ell$ be the families containing ν and μ, respectively. At this point, we observe that we are proceeding recursively, so that we already know the simple modules in $\mathfrak{I}_k$ and $\mathfrak{I}_\ell$. With this information on hand, we check that $\mathcal{U}u = \mathcal{U}w_2 \simeq L(\nu)$. This isomorphism provides a basis of $\mathcal{U}w_2$; we conclude that there is a linear complement of $\mathcal{U}w_2$ with a basis $\widetilde{B}_j$ projecting onto B_j; thus B_j is a basis of $L'(\lambda)$.

(4) Finally we prove that $L'(\lambda)$ is simple. Let v be the element of maximal degree of $L'(\lambda)$. A short computation shows that v belongs to every submodule of $L'(\lambda)$. Applying Lemma 2.5 (or by direct computation when we have a table for the action), there exists $F \in \mathcal{U}$ such that $Fv = v_\lambda$. Hence $L'(\lambda)$ is simple.

As said, we proceed recursively, but with respect to an ad hoc partial ordering of the families in $\mathcal{C}_2$. In the quiver below, we describe this ordering; $\mathfrak{J}_{11} \longrightarrow \mathfrak{J}_{16}$ means that knowledge on $\mathfrak{J}_{11}$ is used for $\mathfrak{J}_{16}$. As we see, there is no vicious circle.



4.1. The family $\mathfrak{J}_{11}$

Recall that $\mathfrak{J}_{11} = \{\lambda \in \widehat{\Gamma} \mid \lambda_1 = 1, \lambda_2 = \zeta\}$.

Lemma 4.1. *If $\lambda \in \mathfrak{J}_{11}$, then* $\dim L(\lambda) = 11$. *A basis of $L(\lambda)$ is given by*

$$\mathrm{B}_{11} = \{m_{a,b,0,d,0} | a \in \mathbb{I}_{0,1}, b \in \mathbb{I}_{0,1}, d \in \mathbb{I}_{0,2}\} - \{m_{1,1,0,0,0}\}.$$

The action of E_i, F_i, $i \in \mathbb{I}_2$ is described in Table 2.

Proof. Let $w_1 = \widetilde{m}_{0,0,0,0,1}$, $w_2 = \widetilde{m}_{1,1,0,0,0}$; hence $F_i w_1 = 0$, $i \in \mathbb{I}_2$,

$$F_1 \widetilde{m}_{1,1,0,0,0} = 0, \qquad F_2 \widetilde{m}_{1,1,0,0,0} = (\zeta^{11} - 1)\lambda(g_2)\widetilde{m}_{1,0,0,0,1} \in W_1(\lambda) = \mathcal{U} w_1.$$

Table 2. Simple modules for $\lambda \in \mathfrak{J}_{11}$.

w	$E_1 \cdot w$	$E_2 \cdot w$	$\lambda(g_1^{-1})F_1 \cdot w$	$\lambda(g_2^{-1})F_2 \cdot w$
$v_{0,0}$	0	$v_{0,1}$	0	0
$v_{0,1}$	$v_{1,1}$	0	0	$(\zeta^{11} - 1)v_{0,0}$
$v_{1,1}$	$v_{2,1}$	0	$q_{12}(\zeta - 1)v_{0,1}$	0
$v_{2,1}$	0	$v_{2,2}$	$q_{12}\zeta^8(1 + \zeta^3)v_{1,1}$	0
$v_{2,2}$	$v_{3,2}$	0	0	$q_{21}^2(1 - \zeta)v_{2,1}$
$v_{3,2}$	$v_{4,2}$	$v_{3,3}$	$q_{12}^2(\zeta^2 - 1)v_{2,2}$	0
$v_{4,2}$	0	$v_{4,3}$	$2q_{12}^2(\zeta^2 - 1)v_{3,2}$	0
$v_{3,3}$	$q_{12}\frac{\zeta^8(\zeta^3-1)}{2}v_{4,3}$	0	0	$q_{21}^3(\zeta^2 - 1)v_{3,2}$
$v_{4,3}$	$v_{5,3}$	0	$2q_{12}^2(\zeta^2 - 1)v_{3,3}$	$q_{21}^4(\zeta^3 - 1)v_{4,2}$
$v_{5,3}$	0	$v_{5,4}$	$q_{12}^3\zeta^8(1 - \zeta^{11})v_{4,3}$	0
$v_{5,4}$	0	0	0	$q_{21}^5(\zeta^{11} + 1)v_{5,3}$

Thus $\mathcal{U}w_1 + \mathcal{U}w_2$ is a proper submodule. We claim that $L'(\lambda) = M(\lambda)/\mathcal{U}w_1 + \mathcal{U}w_2$ is simple. Consider the following elements of $L'(\lambda)$:

$$v_{0,0} = \widetilde{m}_{0,0,0,0,0}, \quad v_{0,1} = \widetilde{m}_{1,0,0,0,0}, \quad v_{1,1} = \widetilde{m}_{0,1,0,0,0}, \quad v_{2,1} = \widetilde{m}_{0,0,0,1,0},$$
$$v_{2,2} = \widetilde{m}_{1,0,0,1,0}, \quad v_{3,2} = \widetilde{m}_{0,1,0,1,0}, \quad v_{4,2} = \widetilde{m}_{0,0,0,2,0}, \quad v_{3,3} = \widetilde{m}_{1,1,0,1,0},$$
$$v_{4,3} = \widetilde{m}_{1,0,0,2,0}, \quad v_{5,3} = \widetilde{m}_{0,1,0,2,0}, \quad v_{5,4} = \widetilde{m}_{1,1,0,2,0}.$$

Notice that $v_{i,j} \in L'(\lambda)_{i\alpha_1 + j\alpha_2}$. The action of E_i, F_i on these vectors is given in Table 2, and we check that $L'(\lambda)$ is spanned by the $v_{i,j}$'s by direct computation.

For each $v_{i,j}$ there exists $E_{i,j} \in \mathcal{U}^+_{(5-i)\alpha_1+(4-j)\alpha_2}$ such that $E_{i,j}v_{i,j} = v_{5,4}$; also, there exists $F_{5,4} \in \mathcal{U}^-_{-5\alpha_1-4\alpha_2}$ such that $F_{5,4}v_{5,4} = v_\lambda$. This implies that the $v_{i,j}$'s are $\neq 0$; hence they are linearly independent, since they have different degrees, and B_{11} is identified with a basis of $L'(\lambda)$.

Let now $0 \neq U \leq L'(\lambda)$ and pick $v \in U - 0$. Expressing v in the basis B_{11}, we see that there exists $E \in \mathcal{U}^+$ such that $Ev = v_{5,4}$. But $\mathcal{U}v_{5,4} = L'(\lambda)$. Hence $L'(\lambda)$ is simple. □

Remark 4.2. If $\lambda \in \mathfrak{I}_{11}$, then $N(\lambda)/W_1(\lambda) \simeq L(\chi_1\chi_2^2\lambda)$, with $\chi_1\chi_2^2\lambda \in \mathfrak{I}_{41}$ has dimension 37. Now $W_1(\lambda)$ is a lowest weight module of lowest weight $\chi_1\lambda \in \mathfrak{I}_{43}$; since $\dim L(\chi_1\lambda) = 25$ by Lemma 4.34, the kernel of $W_1(\lambda) \twoheadrightarrow L(\chi_1\lambda)$ is a submodule of dimension 71.

4.2. *The family* $\mathfrak{I}_{12}$

Recall that $\mathfrak{I}_{12} = \{\lambda \in \widehat{\Gamma} \mid \lambda_1 = 1, \lambda_2 = \zeta^4\}$.

Lemma 4.3. *If* $\lambda \in \mathfrak{I}_{12}$*, then* $\dim L(\lambda) = 11$. *A basis of* $L(\lambda)$ *is given by*

$$\mathrm{B}_{12} = \{m_{a,b,0,d,0} : a, b, d \in \mathbb{I}_{0,1}\} \cup \{m_{0,1,1,0,0}, m_{1,0,1,1,0}, m_{0,0,1,1,0}\}.$$

The action of E_i*,* F_i*,* $i \in \mathbb{I}_2$ *is described in Table 3.*

Proof. Let $w_1 = \widetilde{m}_{0,0,0,0,1}$, $w_2 = F_2E_2E_{12}^2v_\lambda$; then $F_iw_j = 0$ for $i,j \in \mathbb{I}_2$, so $\mathcal{U}w + W_1(\lambda)$ is a proper submodule of $M(\lambda)$. Let $L'(\lambda) := M(\lambda)/\mathcal{U}w + W_1(\lambda)$. We label the elements of B_{12} as follows:

$$v_{0,0} = m_{0,0,0,0,0}, \quad v_{0,1} = m_{1,0,0,0,0}, \quad v_{1,1} = m_{0,1,0,0,0}, \quad v_{2,1} = m_{0,0,0,1,0},$$
$$v_{2,2} = m_{1,0,0,1,0}, \quad v_{1,2} = m_{1,1,0,0,0}, \quad v_{3,2} = m_{0,1,0,1,0}, \quad v_{3,3} = m_{1,1,0,1,0},$$
$$v_{4,3} = m_{0,1,1,0,0}, \quad v_{5,3} = m_{0,0,1,1,0}, \quad v_{5,4} = m_{1,0,1,1,0}.$$

The action of E_i, F_i on these vectors is given in Table and B_{12} is a basis of $L'(\lambda)$. Looking at the table, there exists $F \in \mathcal{U}^-$ such that $Fm_{1,0,1,1,0} = v_\lambda$. Then $L'(\lambda)$ is simple. □

Table 3. Simple modules for $\lambda \in \mathfrak{I}_{12}$.

w	$E_1 \cdot w$	$E_2 \cdot w$	$\lambda(g_1^{-1})F_1 \cdot w$	$\lambda(g_2^{-1})F_2 \cdot w$
$v_{0,0}$	0	$v_{0,1}$	0	0
$v_{0,1}$	$v_{1,1}$	0	0	$(\zeta^{10}+1)v_{0,0}$
$v_{1,1}$	$v_{2,1}$	$v_{1,2}$	$q_{12}(\zeta-1)v_{0,1}$	0
$v_{2,1}$	0	$v_{2,2}$	$q_{12}\zeta^8(1+\zeta^3)v_{1,1}$	0
$v_{1,2}$	$\zeta^{11}(1+\zeta^3)q_{12}v_{2,2}$	0	0	$q_{21}(1+\zeta^3)\zeta^4v_{1,1}$
$v_{2,2}$	$v_{3,2}$	0	$q_{12}(\zeta^3+1)\zeta^8v_{1,2}$	$-q_{2,1}^2v_{2,1}$
$v_{3,2}$	0	$v_{3,3}$	$q_{12}^2\zeta^{10}v_{2,2}$	0
$v_{3,3}$	0	0	0	$q_{21}^3\zeta^3(1-\zeta)v_{3,2}$
$v_{4,3}$	$\zeta^9q_{12}v_{5,3}$	0	$q_{12}^4\zeta(3)_{\zeta^{11}}v_{3,3}$	0
$v_{5,3}$	0	$v_{5,4}$	$-q_{12}^2(1+\zeta^3)v_{4,3}$	0
$v_{5,4}$	0	0	0	$q_{21}^5(1-\zeta)\zeta^4v_{5,3}$

4.3. The family $\mathfrak{J}_{13}$

Recall that $\mathfrak{J}_{13} = \{\lambda \in \widehat{\Gamma} \mid \lambda_1 = 1,\ \lambda_2 = \zeta^7\}$.

Lemma 4.4. *If $\lambda \in \mathfrak{J}_{13}$, then $\dim L(\lambda) = 23$. A basis of $L(\lambda)$ is given by*

$$\mathrm{B}_{13} = \{m_{a,b,0,d,0} | b \in \mathbb{I}_{0,2}\} \cup \{m_{a,0,1,0,0}, m_{0,3,0,d,0}, m_{1,3,0,1,0} | a \in \mathbb{I}_{0,1}, d \in \mathbb{I}_{1,2}\}.$$

Proof. Let $w_1 = \widetilde{m}_{0,0,0,0,1}$, $w_2 = F_2E_2E_{12}^3v_\lambda$. Then $W_1(\lambda) = \mathcal{U}w_1$ by Lemma 2.2, and $F_1w_2 = F_2w_2 = 0$ by Remark 4.22, so $\mathcal{U}w_1 + \mathcal{U}w_2 \lneq M(\lambda)$. We claim that $L'(\lambda) := M(\lambda)/(\mathcal{U}w_1 + \mathcal{U}w_2)$ is simple and B_{13} is a basis of $L'(\lambda)$.

Let $M = M(\lambda)/W_1(\lambda)$ and $u = m_{1,3,1,2,0} \in M$. Notice that $E_{112}^2E_{11212}E_2w_2 = -q_{12}^{18}\, u$, so $u \in \mathcal{U}w_2$. On the other hand, $E_iu = 0$, $i \in \mathbb{I}_2$, $g_1\sigma_1u = u$ and $g_2\sigma_2u = \zeta^9u$, so $(\mathcal{U}u)^\varphi$ projects over a simple module $L(\mu)$ with $\mu \in \mathfrak{J}_{14}$, see Lemma 2.3; in particular there exists $F' \in \mathcal{U}_{-7\alpha_1-5\alpha_2}$ such that $F'u \neq 0$. As $\mathcal{U}u \subseteq \mathcal{U}w_2$ and $\mathcal{U}w_2$ is a lowest weight module,

$$F'u \in (\mathcal{U}u)_{3\alpha_1+3\alpha_2} \subseteq (\mathcal{U}w_2)_{3\alpha_1+3\alpha_2} = \Bbbk w.$$

Hence we may assume that $F'u = w_2$, and $\mathcal{U}u = \mathcal{U}w_2$.

Also $g_1\sigma_1w_2 = \zeta^9w_2$, $g_2\sigma_2w_2 = \zeta^4w_2$, so $\mathcal{U}w_2$ projects over a simple module $L(\nu)$ with $\nu \in \mathfrak{J}_{28}$. For any $v \in M$, $v \neq 0$, there exists $E \in \mathcal{U}$ such that $Ev = u$. Thus we conclude that $\mathcal{U}w_2 \simeq L(\nu)$, and then $\dim L'(\lambda) = 48 - 25 = 23$ by Lemma 4.19.

Applying Lemma 2.5, there exists $F \in \mathcal{U}^-$ such that $Fm_{0,3,0,2,0} = v_\lambda$. Note that

$$E_2m_{0,3,0,2,0} = m_{1,3,0,2,0} = 0$$

since $0 = E_{12}m_{0,3,1,0,0}$ and $\Bbbk m_{1,2,1,1,0} = \Bbbk m_{1,3,0,2,0}$. Also $E_1m_{0,3,0,2,0} = 0$ because it is a scalar multiple of $m_{0,1,1,2,0}$, which is 0. Using this fact and previous relations, we are able to prove that B_{13} spans $L'(\lambda)$, but as B_{13} has 23 elements, it is a basis.

Let $0 \neq W \leq L'(\lambda)$, $w \in W - 0$. Arguing as before, there exists $E \in \mathcal{U}^+$ such that $Ew = m_{0,3,0,2,0}$, so $m_{0,3,0,2,0} \in W$, but then $v_\lambda \in W$, so $L'(\lambda)$ is simple. □

4.4. The family $\mathfrak{J}_{14}$

Recall that $\mathfrak{J}_{14} = \{\lambda \in \widehat{\Gamma} \mid \lambda_1 = 1,\ \lambda_2 = \zeta^3\}$.

Lemma 4.5. *If $\lambda \in \mathfrak{J}_{14}$, then $\dim L(\lambda) = 25$. A basis of $L(\lambda)$ is given by*

$$\mathrm{B}_{14} = \{m_{a,b,0,d,0} \mid a \in \mathbb{I}_{0,1}, b \in \mathbb{I}_{0,3}, d \in \mathbb{I}_{0,2}\} \cup \{m_{0,0,1,0,0}, m_{0,0,1,2,0}\} - \{m_{1,3,0,2,0}\}.$$

Proof. Let $w_1 = \widetilde{m}_{0,0,0,0,1}$, $w_2 = (1+\zeta^3)\widetilde{m}_{1,0,1,0,0} + q_{12}\zeta^3(1+\zeta)\widetilde{m}_{1,1,0,1,0}$. Then $W_1(\lambda) = \mathcal{U}w_1$ and $F_1w_2 = F_2w_2 = 0$ by direct computation.

Let $M = M(\lambda)/W_1(\lambda)$, $L'(\lambda) = M(\lambda)/\mathcal{U}w_2 + W_1(\lambda)$ and $u = m_{1,3,1,2,0} \in M$. Then $(\mathcal{U}u)^\varphi$ projects over $L(\mu)$ for some $\mu \in \mathfrak{J}_{13}$. Also, $\mathcal{U}w_2$ projects over $L(\nu)$ for some $\nu \in \mathfrak{J}_{44}$. Hence $\mathcal{U}u = \mathcal{U}w_2$, and moreover $\mathcal{U}w_2$ is simple, so $\dim L'(\lambda) = 48 - 25 = 23$ by Lemma 4.35. By direct computation $L'(\lambda)$ is spanned by B_{14}, so B_{14} is a basis of $L'(\lambda)$.

Moreover there exists $F \in \mathcal{U}^-$ such that $Fm_{1,0,1,2,0} = v_\lambda$, so $L'(\lambda)$ is simple. □

4.5. The family $\mathfrak{J}_{15}$

Recall that $\mathfrak{J}_{15} = \{\lambda \in \widehat{\Gamma} \mid \lambda_1 = 1,\ \lambda_2 = \zeta^9\}$.

Lemma 4.6. *If* $\lambda \in \mathfrak{I}_{15}$, *then* $\dim L(\lambda) = 37$. *A basis of* $L(\lambda)$ *is given by*

$$\begin{aligned} B_{15} = &\{m_{a,b,c,d,0} | a, c \in \mathbb{I}_{0,1}, b \in \mathbb{I}_{0,3}, d \in \mathbb{I}_{0,2}\} \\ &- \{m_{a,b,1,d,0} | a \in \mathbb{I}_{0,1}, b \in \mathbb{I}_{2,3}, d \in \mathbb{I}_{0,2}, (a,b,d) \neq (0,2,2)\}. \end{aligned}$$

Proof. Let $w_1 = \widetilde{m}_{0,0,0,0,1}$, $u = \widetilde{m}_{1,3,1,2,0}$, $w_2 = F_2 F_{12} F_{112}^2 u$. Then $W_1(\lambda) = \mathcal{U} w_1$.

Let $M = M(\lambda)/W_1(\lambda)$, so $E_1 u = E_2 u = 0$ in M, and $(\mathcal{U}u)^{\varphi} \twoheadrightarrow L(\nu)$ for some $\nu \in \mathfrak{I}_{11}$; thus $w_2 \neq 0$. By direct computation, $F_i w_2 = 0$, $i \in \mathbb{I}_2$, so $\mathcal{U} w_2$ projects over a simple module $L(\mu)$, for $\mu \in \mathfrak{I}_{12}$. From here, $\mathcal{U} w_2 \simeq L(\mu)$.

Let $L'(\lambda) = M(\lambda)/W_1(\lambda) + \mathcal{U} w_2$. Then $\dim L'(\lambda) = 37$ by Lemma 4.3, and B_{15} is a basis of $L'(\lambda)$. There exists F such that $F m_{0,2,1,2,0} = v_\lambda$, and $L'(\lambda)$ is simple. □

4.6. The family $\mathfrak{I}_{16}$

Recall that $\mathfrak{I}_{16} = \{\lambda \in \widehat{\Gamma} \mid \lambda_1 = 1,\ \lambda_2 = -1\}$.

Lemma 4.7. *If* $\lambda \in \mathfrak{I}_{16}$, *then* $\dim L(\lambda) = 37$. *A basis of* $L(\lambda)$ *is given by*

$$\begin{aligned} B_{16} = &\{m_{a,b,c,d,0} | a, c \in \mathbb{I}_{0,1}, b \in \mathbb{I}_{0,3}, d \in \mathbb{I}_{0,2}\} \\ &- \left(\{m_{a,3,c,d,0} | a, c \in \mathbb{I}_{0,1}, d \in \mathbb{I}_{1,2}\} \cup \{m_{1,2,1,2,0}, m_{0,2,1,2,0}, m_{1,2,0,2,0}\}\right). \end{aligned}$$

Proof. Let $w_1 = \widetilde{m}_{0,0,0,0,1}$, $u = \widetilde{m}_{1,3,1,2,0}$, $w_2 = F_2 F_{11212} F_{112} u$. Then $W_1(\lambda) = \mathcal{U} w_1$.

Let $M = M(\lambda)/W_1(\lambda)$, so $E_1 u = E_2 u = 0$ in M', and $(\mathcal{U}u)^{\varphi} \twoheadrightarrow L(\nu)$ for some $\nu \in \mathfrak{I}_{12}$; thus $w_2 \neq 0$. By direct computation, $F_i w_2 = 0$, $i \in \mathbb{I}_2$, so $\mathcal{U} w_2$ projects over a simple module $L(\mu)$, for $\mu \in \mathfrak{I}_{11}$. From here, $\mathcal{U} w_2 \simeq L(\mu)$.

Let $L'(\lambda) = M(\lambda)/W_1(\lambda) + \mathcal{U} w_2$. Then $\dim L'(\lambda) = 37$ by Lemma 4.1, and B_{16} is a basis of $L'(\lambda)$. There exists F such that $F m_{1,1,1,2,0} = v_\lambda$, and $L'(\lambda)$ is simple. □

4.7. The family $\mathfrak{I}_{17}$

Recall that $\mathfrak{I}_{17} = \{\lambda \in \widehat{\Gamma} \mid \lambda_1 = 1,\ \lambda_2 = \zeta^{10}\}$.

Lemma 4.8. *If* $\lambda \in \mathfrak{I}_{17}$, *then* $\dim L(\lambda) = 47$. *A basis of* $L(\lambda)$ *is given by*

$$B_{17} = \{m_{a,b,c,d,0} | a, c \in \mathbb{I}_{0,1}, b \in \mathbb{I}_{0,3}, d, e \in \mathbb{I}_{0,2}, (a,b,c,d) \neq (1,3,1,2)\}.$$

Proof. Let $w_1 = \widetilde{m}_{0,0,0,0,1}$, $w_2 = \widetilde{m}_{1,3,1,2,0}$. Then $W_1(\lambda) = \mathcal{U} w_1$, and $F_i w = 0$, $i \in \mathbb{I}_2$, so $\mathcal{U} w$ projects over a simple module $L(\mu)$, for $\mu \in \mathfrak{I}_{47}$. Let $M = M(\lambda)/W_1(\lambda)$, hence $\mathcal{U} w_2 \simeq L(\mu)$. Let $L'(\lambda) = M(\lambda)/W_1(\lambda) + \mathcal{U} w_2$. Then $\dim L'(\lambda) = 47$ by Lemma 4.38, and B_{17} is a basis of $L'(\lambda)$. There exists F such that $F m_{0,3,1,2,0} = v_\lambda$, and $L'(\lambda)$ is simple. □

4.8. The family $\mathfrak{I}_{18}$

Recall that $\mathfrak{I}_{18} = \{\lambda \in \widehat{\Gamma} \mid \lambda_1 = \zeta^8,\ \lambda_2 = \zeta^5\}$.

Lemma 4.9. *If* $\lambda \in \mathfrak{I}_{18}$, *then* $\dim L(\lambda) = 11$. *A basis of* $L(\lambda)$ *is given by*

$$\begin{aligned} B_{18} = &\{m_{a,b,1,0,1} | a, b \in \mathbb{I}_{0,1}\} \cup \{m_{0,b,0,0,e} | e \in \mathbb{I}_{0,1}, b \in \mathbb{I}_{0,3}\} \cup \{m_{1,0,0,0,0}\} \\ &- \{m_{1,1,1,0,1}, m_{3,0,0,0,1}\}. \end{aligned}$$

The action of E_i, F_i, $i \in \mathbb{I}_2$ *is described in* Table 4.

Table 4. Simple modules for $\lambda \in \mathfrak{I}_{18}$.

w	$E_1 \cdot w$	$E_2 \cdot w$	$\lambda(\sigma_1)F_1 \cdot w$	$\lambda(g_2)^{-1}F_2 \cdot w$
$v_{0,0}$	$v_{1,0}$	$v_{0,1}$	0	0
$v_{1,0}$	0	$q_{21}\zeta^9(4)_\zeta v_{1,1}$	$(1+\zeta^2)v_{0,0}$	0
$v_{0,1}$	$\zeta^8(4)_\zeta v_{1,1}$	0	0	$(\zeta^7-1)v_{0,0}$
$v_{1,1}$	$\frac{q_{12}\zeta^4(4)_{\zeta^7}}{3}v_{2,1}$	0	$q_{12}(\zeta-1)v_{0,1}$	$(\zeta^{11}-1)v_{1,0}$
$v_{2,1}$	0	$q_{21}^2\zeta^{10}(4)_\zeta v_{2,2}$	$(1-\zeta^4)v_{1,1}$	0
$v_{2,2}$	$(1-\zeta^4)v_{3,2}$	0	0	$\frac{-(1+\zeta^2)(3)_{\zeta^7}}{3}v_{2,1}$
$v_{3,2}$	$v_{4,2}$	$q_{12}\zeta^{10}(4)_\zeta v_{3,3}$	$\zeta^{10}(4)_\zeta v_{2,2}$	0
$v_{4,2}$	0	$v_{4,3}$	$q_{12}^2\zeta(\zeta+1)v_{3,2}$	0
$v_{3,3}$	$\frac{q_{12}^4\zeta^7(4)_\zeta}{3}v_{4,3}$	0	0	$\frac{\zeta^8-1}{3}v_{3,2}$
$v_{4,3}$	$v_{5,3}$	0	$q_{12}^3(\zeta^{11}+1)(4)_\zeta^2 v_{3,3}$	$q_{21}^4(\zeta^{11}-1)v_{4,2}$
$v_{5,3}$	0	0	$q_{12}^3\zeta^4 v_{4,3}$	0

Proof. $W_2(\lambda) \leq M(\lambda)$ by Lemma 2.2 and $w := F_2E_2E_{12}$ satisfies $F_1w = F_2w = 0$ by Remark 3.14. Let $L'(\lambda) = M(\lambda)/\mathcal{U}w_2 + W_2(\lambda)$. We fix the following notation for B_{18}:

$$v_{0,0} = m_{0,0,0,0,0}, \quad v_{1,0} = m_{0,0,0,0,1}, \quad v_{0,1} = m_{1,0,0,0,0}, \quad v_{1,1} = m_{0,1,0,0,0},$$
$$v_{2,1} = m_{0,1,0,0,1}, \quad v_{2,2} = m_{0,2,0,0,0}, \quad v_{3,2} = m_{0,2,0,0,1}, \quad v_{4,2} = m_{0,0,1,0,1},$$
$$v_{3,3} = m_{0,3,0,0,0}, \quad v_{4,3} = m_{1,0,1,0,1}, \quad v_{5,3} = m_{0,1,1,0,1}.$$

We check that $L'(\lambda)$ is spanned by B_{18}. From Table 4 there exist $E_{i,j} \in \mathcal{U}^+_{(5-i)\alpha_1+(3-j)\alpha_2}$, $F_{5,3} \in \mathcal{U}^-_{-5\alpha_1-3\alpha_2}$ such that $E_{i,j}v_{i,j} = v_{5,3}$, $F_{5,3}v_{5,3} = v_\lambda$. Thus $L'(\lambda)$ is simple. $\square$

4.9. The family $\mathfrak{I}_{19}$

Recall that $\mathfrak{I}_{19} = \{\lambda \in \widehat{\Gamma} \mid \lambda_1 = \zeta^8, \lambda_2 = \zeta^8\}$.

Lemma 4.10. *If* $\lambda \in \mathfrak{I}_{19}$, *then* $\dim L(\lambda) = 35$. *A basis of* $L(\lambda)$ *is given by*

$$\mathrm{B}_{19} = \{m_{0,b,0,d,e} | b \in \mathbb{I}_{0,3}, d \in \mathbb{I}_{0,2}, e \in \mathbb{I}_{0,1}\} \cup \{m_{1,b,0,0,e} | \ b, e \in \mathbb{I}_{0,1}\} \cup \{m_{0,b,1,0,0} | \ b \in \mathbb{I}_{1,3}\}$$
$$\cup \ \{m_{1,b,0,0,1} \mid b \in \mathbb{I}_{2,3}\} \cup \{m_{1,0,0,1,1}, m_{0,0,1,1,0}\}.$$

Proof. Let $w_1 = \widetilde{m}_{0,0,0,0,2}$, $w_2 = F_2E_2E_{12}^2v_\lambda$. Then $W_2(\lambda) = \mathcal{U}w_1$ and $F_1w_2 = F_2w_2 = 0$. Set $M' = M(\lambda)/W_2(\lambda)$, $u = \widetilde{m}_{1,3,1,2,1}$. Hence $\mathcal{U}w_2 \twoheadrightarrow L(\mu)$ for $\mu \in \mathfrak{I}_{32}$, and there exists $E \in \mathcal{U}$ such that $Ew_2 = u$. Moreover, there exists $F \in \mathcal{U}$ such that $Fu = w_2$, so $\mathcal{U}w_2 = \mathcal{U}u \simeq L(\mu)$. Let $L'(\lambda) = M(\lambda)/\mathcal{U}w_2 + W_2(\lambda)$, so $\dim L'(\lambda) = 96 - 61 = 35$ by Lemma 4.23, and B_{19} is a basis of $L'(\lambda)$. As in previous cases, $L'(\lambda)$ is simple. $\square$

4.10. The family $\mathfrak{I}_{20}$

Recall that $\mathfrak{I}_{20} = \{\lambda \in \widehat{\Gamma} \mid \lambda_1 = \zeta^8, \lambda_2 = \zeta^{11}\}$.

Lemma 4.11. *If* $\lambda \in \mathfrak{I}_{20}$, *then* $\dim L(\lambda) = 71$. *A basis of* $L(\lambda)$ *is given by*

$$\mathrm{B}_{20} = \{m_{a,b,c,d,e} | a, c, e \in \mathbb{I}_{0,1}, b \in \mathbb{I}_{0,3}, d \in \mathbb{I}_{0,2}\}$$
$$- \Big(\{m_{1,b,1,d,e} | b \in \mathbb{I}_{0,3}, d \in \mathbb{I}_{0,2}, e \in \mathbb{I}_{0,1}, (b,d,e) \neq (2,2,1)\} \cup \{m_{1,0,0,2,1}, m_{1,3,0,0,0}\}\Big).$$

Proof. Let $w_1 = \widetilde{m}_{0,0,0,0,2}$, $w_2 = F_2E_2E_{12}^3v_\lambda$. Then $W_2(\lambda) = \mathcal{U}w_1$ and $F_1w_2 = F_2w_2 = 0$. Set $M' = M(\lambda)/W_2(\lambda)$, $u = \widetilde{m}_{1,3,1,2,1}$. Hence $\mathcal{U}w_2 \twoheadrightarrow L(\mu)$ for $\mu \in \mathfrak{I}_{26}$, and there exists $E \in \mathcal{U}$ such that $Ew_2 = u$. Moreover, there exists $F \in \mathcal{U}$ such that $Fu = w_2$, so $\mathcal{U}w_2 = \mathcal{U}u \simeq L(\mu)$. Let $L'(\lambda) = M(\lambda)/\mathcal{U}w_2 + W_2(\lambda)$, so $\dim L'(\lambda) = 96 - 25 = 71$ by Lemma 4.17 and B_{20} is a basis of $L'(\lambda)$. As in previous cases, $L'(\lambda)$ is simple. □

4.11. The family $\mathfrak{I}_{21}$

Recall that $\mathfrak{I}_{21} = \{\lambda \in \widehat{\Gamma} \mid \lambda_1 = \zeta^8, \lambda_2 = \zeta^3\}$.

Lemma 4.12. *If $\lambda \in \mathfrak{I}_{21}$, then $\dim L(\lambda) = 61$. A basis of $L(\lambda)$ is given by*

$$\begin{aligned}\mathrm{B}_{21} &= \{m_{a,b,c,d,e} \mid a,b,c,e \in \mathbb{I}_{0,1}, d \in \mathbb{I}_{0,2}\} \cup \{m_{a,2,c,0,e} \mid a,c,e \in \mathbb{I}_{0,1}\} \\ &\quad \cup \{m_{1,3,0,0,e} \mid e \in \mathbb{I}_{0,1}\} \cup \{m_{0,3,1,0,1}, m_{1,3,1,0,1}, m_{0,2,0,1,0}\}.\end{aligned}$$

Proof. Let $w_1 = \widetilde{m}_{0,0,0,0,2}$, $u = \widetilde{m}_{1,3,1,2,1}$, $w_2 = F_1F_{11212}F_{12}u$. Then $W_2(\lambda) = \mathcal{U}w_1$.

Let $M' = M(\lambda)/W_2(\lambda)$, so $E_1u = E_2u = 0$ in M', and $(\mathcal{U}u)^\varphi \twoheadrightarrow L(\nu)$ for some $\nu \in \mathfrak{I}_{19}$; thus $w_2 \neq 0$. By direct computation, $F_iw_2 = 0$, $i \in \mathbb{I}_2$, so $\mathcal{U}w_2$ projects over a simple module $L(\mu)$, for $\mu \in \mathfrak{I}_{40}$. From here, $\mathcal{U}w_2 \simeq L(\mu)$.

Let $L'(\lambda) = M(\lambda)/W_2(\lambda) + \mathcal{U}w_2$. Then $\dim L'(\lambda) = 61$ by Lemma 4.31, and B_{21} is a basis of $L'(\lambda)$. There exists F such that $Fm_{1,1,1,2,1} = v_\lambda$, and $L'(\lambda)$ is simple. □

4.12. The family $\mathfrak{I}_{22}$

Recall that $\mathfrak{I}_{22} = \{\lambda \in \widehat{\Gamma} \mid \lambda_1 = \zeta^8, \lambda_2 = \zeta^9\}$.

Lemma 4.13. *If $\lambda \in \mathfrak{I}_{22}$, then $\dim L(\lambda) = 49$. A basis of $L(\lambda)$ is given by*

$$\begin{aligned}\mathrm{B}_{22} &= \{m_{a,b,c,d,e} \mid a,c \in \mathbb{I}_{0,1}, b \in \mathbb{I}_{0,3}, d,e \in \mathbb{I}_{0,1}\} \\ &\quad - \{m_{a,b',1,0,0}, m_{1,3,1,1,1}, m_{a,b,1,1,0} \mid a \in \mathbb{I}_{0,1}, b' \in \mathbb{I}_{0,3}, b \in \mathbb{I}_{1,3}\}.\end{aligned}$$

Proof. Let $w_1 = \widetilde{m}_{0,0,0,0,2}$, $w_2 = F_1^2E_{112}^2E_1v_\lambda$. Then $W_2(\lambda) = \mathcal{U}w_1$ and $F_1w_2 = F_2w_2 = 0$. Set $M' = M(\lambda)/W_2(\lambda)$, $u = \widetilde{m}_{1,3,1,2,1}$. Hence $\mathcal{U}w_2 \twoheadrightarrow L(\mu)$ for $\mu \in \mathfrak{I}_{29}$, and there exists $E \in \mathcal{U}$ such that $Ew_2 = u$. Moreover, there exists $F \in \mathcal{U}$ such that $Fu = w_2$, so $\mathcal{U}w_2 = \mathcal{U}u \simeq L(\mu)$. Let $L'(\lambda) = M(\lambda)/\mathcal{U}w_2 + W_2(\lambda)$, so $\dim L'(\lambda) = 96 - 47 = 49$ by Lemma 4.20, and B_{22} is a basis of $L'(\lambda)$. As in previous cases, $L'(\lambda)$ is simple. □

4.13. The family $\mathfrak{I}_{23}$

Recall that $\mathfrak{I}_{23} = \{\lambda \in \widehat{\Gamma} \mid \lambda_1 = \zeta^8, \lambda_2 = \zeta^2\}$.

Lemma 4.14. *If $\lambda \in \mathfrak{I}_{23}$, then $\dim L(\lambda) = 47$. A basis of $L(\lambda)$ is given by*

$$\begin{aligned}\mathrm{B}_{23} = \Big(&\{m_{a,b,0,d,e} \mid a,e \in \mathbb{I}_{0,1}, b \in \mathbb{I}_{0,3}, d \in \mathbb{I}_{0,2}\} \cup \{m_{a,b,1,0,0} \mid a,b \in \mathbb{I}_{0,1}\} \\ &\cup \{m_{0,2,1,0,0}, m_{1,3,1,0,0}\}\Big) - \Big(\{m_{1,b,0,1,e} \mid b \in \mathbb{I}_{0,2}, e \in \mathbb{I}_{0,1}\} \cup \{m_{0,2,0,2,0}\}\Big).\end{aligned}$$

Proof. Let $w_1 = \widetilde{m}_{0,0,0,0,2}$, $u = \widetilde{m}_{1,3,1,2,1}$, $w_2 = F_{12}^3F_{11212}F_{112}F_1u$. Then $W_2(\lambda) = \mathcal{U}w_1$.

Let $M' = M(\lambda)/W_2(\lambda)$, so $E_1u = E_2u = 0$ in M', and $(\mathcal{U}u)^\varphi \twoheadrightarrow L(\nu)$ for some $\nu \in \mathfrak{I}_{22}$; thus $w_2 \neq 0$. By direct computation, $F_iw_2 = 0$, $i \in \mathbb{I}_2$, so $\mathcal{U}w_2$ projects over a simple module $L(\mu)$, for $\mu \in \mathfrak{I}_{45}$. From here, $\mathcal{U}w_2 \simeq L(\mu)$.

Let $L'(\lambda) = M(\lambda)/W_1(\lambda) + \mathcal{U}w_2$. Then $\dim L'(\lambda) = 47$ by Lemma 4.36, and B_{23} is a basis of $L'(\lambda)$. There exists F such that $Fm_{1,3,0,2,1} = v_\lambda$, and $L'(\lambda)$ is simple. □

4.14. The family $\mathfrak{I}_{24}$

Recall that $\mathfrak{I}_{24} = \{\lambda \in \widehat{\Gamma} \mid \lambda_1 = \zeta^8, \lambda_2 = -1\}$.

Lemma 4.15. *If $\lambda \in \mathfrak{I}_{24}$, then $\dim L(\lambda) = 85$. A basis of $L(\lambda)$ is given by*

$$B_{24} = \{m_{a,b,c,d,e} | a, c, e \in \mathbb{I}_{0,1}, b \in \mathbb{I}_{0,3}, d \in \mathbb{I}_{0,2}\} - \left(\{m_{a,3,c,2,e}, m_{1,3,c,1,1} | a, c, e \in \mathbb{I}_{0,1}\} \cup \{m_{0,3,1,1,1}\}\right).$$

Proof. Let $w_1 = \widetilde{m}_{0,0,0,0,2}$, $u = \widetilde{m}_{1,3,1,2,1}$, $w_2 = F_{12}F_{11212}F_1 u$. Then $W_2(\lambda) = \mathcal{U}w_1$.

Let $M' = M(\lambda)/W_2(\lambda)$, so $E_1 u = E_2 u = 0$ in M', and $(\mathcal{U}u)^\varphi \twoheadrightarrow L(\nu)$ for some $\nu \in \mathfrak{I}_{18}$; thus $w_2 \neq 0$. By direct computation, $F_i w_2 = 0$, $i \in \mathbb{I}_2$, so $\mathcal{U}w_2$ projects over a simple module $L(\mu)$, for $\mu \in \mathfrak{I}_{38}$. From here, $\mathcal{U}w_2 \simeq L(\mu)$.

Let $L'(\lambda) = M(\lambda)/W_1(\lambda) + \mathcal{U}w_2$. Then $\dim L'(\lambda) = 85$ by Lemma 4.29, and B_{24} is a basis of $L'(\lambda)$. There exists F such that $Fm_{1,2,1,2,1} = v_\lambda$, and $L'(\lambda)$ is simple. □

4.15. The family $\mathfrak{I}_{25}$

Recall that $\mathfrak{I}_{25} = \{\lambda \in \widehat{\Gamma} \mid \lambda_1 = \zeta^{11}, \lambda_2 = \zeta^8\}$.

Lemma 4.16. *If $\lambda \in \mathfrak{I}_{25}$, then $\dim L(\lambda) = 37$. A basis of $L(\lambda)$ is given by*

$$B_{25} = B'_{25} - \left(\{m_{0,3,0,0,e} \mid e \in \mathbb{I}_{0,1}\} \cup \{m_{1,3,c,0,e}, m_{1,2,1,0,e} \mid c \in \mathbb{I}_{0,1}, e \in \mathbb{I}_{0,2}\}\right), \text{ where}$$

$$B'_{25} = \{m_{a,b,c,0,e} \mid a, c \in \mathbb{I}_{0,1}, b \in \mathbb{I}_{0,3}, e \in \mathbb{I}_{0,2}\}.$$

Proof. Let $w_1 = F_1^2 E_{112} E_1^2 v_\lambda$. By Remark 3.5, $F_i w_1 = 0$, $i \in \mathbb{I}_2$. Let $M' = M(\lambda)/\mathcal{U}w_1$, so B'_{25} is a basis of M'. Notice that $w_2 = E_2 E_{12}^3 v_\lambda$ satisfies $F_1 w_2 = F_2 w_2 = 0$. Hence $\mathcal{U}w_2 \twoheadrightarrow L(\mu)$ for $\mu \in \mathfrak{I}_{38}$, and there exists $E \in \mathcal{U}$ such that $Ew_2 = m_{1,3,1,0,2}$. Moreover, there exists $F \in \mathcal{U}$ such that $Fm_{1,3,1,0,2} = w_2$, and then $\mathcal{U}w_2 = \mathcal{U}m_{1,3,1,0,2} \simeq L(\mu)$. Let $L'(\lambda) = M(\lambda)/\mathcal{U}w_1 + \mathcal{U}w_2$, so $\dim L'(\lambda) = 48 - 11 = 37$ by Lemma 4.29 and B_{25} is a basis of $L'(\lambda)$. As in previous cases, $L'(\lambda)$ is simple. □

4.16. The family $\mathfrak{I}_{26}$

Recall that $\mathfrak{I}_{26} = \{\lambda \in \widehat{\Gamma} \mid \lambda_1 = \zeta^5, \lambda_2 = \zeta^8\}$.

Lemma 4.17. *If $\lambda \in \mathfrak{I}_{26}$, then $\dim L(\lambda) = 25$. A basis of $L(\lambda)$ is given by*

$$B_{26} = \{m_{0,b,c,0,e} | c \in \mathbb{I}_{0,1}, b \in \mathbb{I}_{0,3}, e \in \mathbb{I}_{0,2}\} \cup \{m_{1,0,0,0,0}, m_{1,0,0,0,2}\} - \{m_{0,3,1,0,0}\}.$$

Proof. Let $w_1 = F_1^2 E_{112} E_1^2 v_\lambda$, so $F_i w_1 = 0$, $i \in \mathbb{I}_2$. Let $M' = M(\lambda)/\mathcal{U}w_1$. Then B'_{25} as in Lemma 4.17 is a basis of M'. Notice that $w_2 = F_2 E_2 E_{12} v_\lambda$ satisfies $F_1 w_2 = F_2 w_2 = 0$. Hence $\mathcal{U}w_2 \twoheadrightarrow L(\mu)$ for $\mu \in \mathfrak{I}_{13}$, and there exists $E \in \mathcal{U}$ such that $Ew_2 = m_{1,3,1,0,2}$. Moreover, there exists $F \in \mathcal{U}$ such that $Fm_{1,3,1,0,2} = w$, and then $\mathcal{U}w_2 = \mathcal{U}m_{1,3,1,0,2} \simeq L(\mu)$. Let $L'(\lambda) = M(\lambda)/\mathcal{U}w_1 + \mathcal{U}w_2$, so $\dim L'(\lambda) = 48 - 23 = 25$ by Lemma 4.4, and B_{26} is a basis of $L'(\lambda)$. As in previous cases, $L'(\lambda)$ is simple. □

4.17. The family $\mathfrak{I}_{27}$

Recall that $\mathfrak{I}_{27} = \{\lambda \in \widehat{\Gamma} \mid \lambda_1 = \zeta^4, \lambda_2 = \zeta^9\}$.

Lemma 4.18. *If $\lambda \in \mathfrak{I}_{27}$, then $\dim L(\lambda) = 35$. A basis of $L(\lambda)$ is given by*

$$\mathrm{B}_{27} = \mathrm{B}'_{27} - \{n_{0,0,1,2,2}\}, \qquad \textit{where} \qquad \mathrm{B}'_{27} = \{n_{a,0,c,d,e} \mid a, c \in \mathbb{I}_{0,1}, d, e \in \mathbb{I}_{0,2}\}.$$

Proof. Let $w_1 = F_2E_{12}E_2v_\lambda$, so $F_iw_1 = 0$, $i \in \mathbb{I}_2$. Let $M' = M(\lambda)/\mathcal{U}w_1$. Then B'_{27} is a basis of M'. Notice that $w_2 = E_{11212}E_{112}^2E_1^2v_\lambda$ satisfies $F_1w_2 = F_2w_2 = 0$. Hence $\mathcal{U}w_2 \twoheadrightarrow L(\mu)$ for $\mu \in \mathfrak{I}_{47}$; as also $E_1w_2 = E_2w_2 = 0$, we have that $\mathcal{U}w_2 \simeq L(\mu)$. Let $L'(\lambda) = M(\lambda)/\mathcal{U}w_1 + \mathcal{U}w_2$, so $\dim L'(\lambda) = 36 - 1 = 35$ by Lemma 4.38, and B_{27} is a basis of $L'(\lambda)$. As in previous cases, $L'(\lambda)$ is simple. □

4.18. The family $\mathfrak{I}_{28}$

Recall that $\mathfrak{I}_{28} = \{\lambda \in \widehat{\Gamma} \mid \lambda_1 = \zeta^9, \lambda_2 = \zeta^4\}$.

Lemma 4.19. *If $\lambda \in \mathfrak{I}_{28}$, then $\dim L(\lambda) = 25$. A basis of $L(\lambda)$ is given by*

$$\mathrm{B}_{28} = \mathrm{B}'_{27} - \Big(\{n_{0,0,1,1,e}, n_{0,0,c,2,e} | c \in \mathbb{I}_{0,1}, e \in \mathbb{I}_{0,2}\} \cup \{n_{1,0,1,2,e} \mid e \in \mathbb{I}_{1,2}\}\Big).$$

Proof. Let $w_1 = F_2E_{12}E_2v_\lambda$, so $F_iw_1 = 0$, $i \in \mathbb{I}_2$. Let $M' = M(\lambda)/\mathcal{U}w_1$. Then B'_{27} is a basis of M'. Notice that $w_2 = F_1^2E_1^2E_{112}^2v_\lambda$ satisfies $F_1w_2 = F_2w_2 = 0$. Hence $\mathcal{U}w_2 \twoheadrightarrow L(\mu)$ for $\mu \in \mathfrak{I}_{38}$, and there exists $E \in \mathcal{U}$ such that $Ew_2 = m_{1,0,1,2,2}$. Moreover, there exists $F \in \mathcal{U}$ such that $Fm_{1,0,1,2,2} = w_2$, and then $\mathcal{U}w_2 = \mathcal{U}m_{1,0,1,2,2} \simeq L(\mu)$. Let $L'(\lambda) = M(\lambda)/\mathcal{U}w_1 + \mathcal{U}w_2$, so $\dim L'(\lambda) = 36 - 11 = 25$ by Lemma 4.29, and B_{28} is a basis of $L'(\lambda)$. As in previous cases, $L'(\lambda)$ is simple. □

4.19. The family $\mathfrak{I}_{29}$

Recall that $\mathfrak{I}_{29} = \{\lambda \in \widehat{\Gamma} \mid \lambda_1 = -1, \lambda_2 = -1\}$.

Lemma 4.20. *If $\lambda \in \mathfrak{I}_{29}$, then $\dim L(\lambda) = 47$. A basis of $L(\lambda)$ is given by*

$$\mathrm{B}_{29} = \mathrm{B}'_{29} - \{m_{1,3,1,0,0}\}, \qquad \textit{where} \qquad \mathrm{B}'_{29} = \{m_{a,b,c,0,e} | a, c \in \mathbb{I}_{0,1}, b \in \mathbb{I}_{0,3}, e \in \mathbb{I}_{0,2}\}.$$

Proof. Let $w_1 = F_1^2E_{112}E_1^2v_\lambda$, so $F_iw_1 = 0$, $i \in \mathbb{I}_2$. Let $M' = M(\lambda)/\mathcal{U}w_1$. Then B'_{29} is a basis of M'. Notice that $w_2 = E_2E_{12}^3E_{11212}v_\lambda$ satisfies $F_1w_2 = F_2w_2 = 0$. Hence $\mathcal{U}w_2 \twoheadrightarrow L(\mu)$ for $\mu \in \mathfrak{I}_{47}$; as also $E_1w_2 = E_2w_2 = 0$, we have that $\mathcal{U}w_2 \simeq L(\mu)$. Let $L'(\lambda) = M(\lambda)/\mathcal{U}w_1 + \mathcal{U}w_2$, so $\dim L'(\lambda) = 48 - 1 = 47$ by Lemma 4.38, and B_{29} is a basis of $L'(\lambda)$. As in previous cases, $L'(\lambda)$ is simple. □

4.20. The family $\mathfrak{I}_{30}$

Recall that $\mathfrak{I}_{30} = \{\lambda \in \widehat{\Gamma} \mid \lambda_1 = \zeta^2, \lambda_2 = \zeta^2\}$.

Lemma 4.21. *If $\lambda \in \mathfrak{I}_{30}$, then $\dim L(\lambda) = 37$. A basis of $L(\lambda)$ is given by*

$$\mathrm{B}_{30} = \mathrm{B}'_{29} - \{m_{1,b,c,0,e} \mid c \in \mathbb{I}_{0,1}, b \in \mathbb{I}_{2,3}, e \in \mathbb{I}_{0,2}, (b, c, e) \neq (3, 1, 2)\}.$$

Proof. Let $w_1 = F_1^2E_{112}E_1^2v_\lambda$, so $F_iw_1 = 0$, $i \in \mathbb{I}_2$. Let $M' = M(\lambda)/\mathcal{U}w_1$. Then B'_{29} is a basis of M'. Notice that $w_2 = E_2E_{12}^2v_\lambda$ satisfies $F_1w_2 = F_2w_2 = 0$. Hence $\mathcal{U}w_2 \twoheadrightarrow L(\mu)$ for $\mu \in \mathfrak{I}_{38}$, and there exists $E \in \mathcal{U}$ such that $Ew_2 = m_{1,3,1,0,2}$. Moreover, there exists $F \in \mathcal{U}$ such that $Fm_{1,3,1,0,2} = w_2$, and then

$\mathcal{U}w_2 = \mathcal{U}m_{1,3,1,0,2} \simeq L(\mu)$. Let $L'(\lambda) = M(\lambda)/\mathcal{U}w_1 + \mathcal{U}w_2$, so $\dim L'(\lambda) = 48 - 11 = 37$ by Lemma 4.29, and B_{30} is a basis of $L'(\lambda)$. As in previous cases, $L'(\lambda)$ is simple. □

4.21. The family $\mathfrak{I}_{31}$

Recall that $\mathfrak{I}_{31} = \{\lambda \in \widehat{\Gamma} \mid \lambda_1 = -1, \lambda_2 = \zeta^{10}\}$.

Lemma 4.22. *If $\lambda \in \mathfrak{I}_{31}$, then $\dim L(\lambda) = 61$. A basis of $L(\lambda)$ is given by*

$$\mathrm{B}_{31} = \mathrm{B}'_{31} - \big(\{n_{0,0,0,2,e} \mid e \in \mathbb{I}_{0,1}\} \cup \{n_{0,0,1,1,e}, n_{0,0,1,2,e}, n_{0,1,1,2,e} | e \in \mathbb{I}_{0,2}\}\big), \quad \textit{where}$$
$$\mathrm{B}'_{31} = \{n_{a,b,c,d,e} | a, b, c \in \mathbb{I}_{0,1}, d, e \in \mathbb{I}_{0,2}\}.$$

Proof. Let $w_1 = F_2E_2E_{12}^2 v_\lambda$. By Remark 3.17, $F_i w_1 = 0$, $i \in \mathbb{I}_2$. Let $M' = M(\lambda)/\mathcal{U}w_1$, so B'_{31} is a basis of M'. Notice that

$$w_2 = n_{0,0,0,2,1} + \tfrac{q_{21}}{3}\zeta(1+\zeta^3)(1+\zeta^2)\big(n_{0,0,1,0,2} + \zeta^4 n_{0,1,0,1,2}\big)$$

satisfies $F_1 w_2 = F_2 w_2 = 0$. Hence $\mathcal{U}w_2 \twoheadrightarrow L(\mu)$ for $\mu \in \mathfrak{I}_{18}$, and there exists $E \in \mathcal{U}$ such that $Ew_2 = n_{1,1,1,2,2}$. Moreover, there exists $F \in \mathcal{U}$ such that $Fn_{1,1,1,2,2} = w_2$, and then $\mathcal{U}w_2 = \mathcal{U}n_{1,1,1,2,2} \simeq L(\mu)$. Let $L'(\lambda) = M(\lambda)/\mathcal{U}w_1 + \mathcal{U}w_2$, so $\dim L'(\lambda) = 72 - 11 = 61$ by Lemma 4.9, and B_{31} is a basis of $L'(\lambda)$. As in previous cases, $L'(\lambda)$ is simple. □

4.22. The family $\mathfrak{I}_{32}$

Recall that $\mathfrak{I}_{32} = \{\lambda \in \widehat{\Gamma} \mid \lambda_1 = \zeta^{10}, \lambda_2 = -1\}$.

Lemma 4.23. *If $\lambda \in \mathfrak{I}_{32}$, then $\dim L(\lambda) = 61$. A basis of $L(\lambda)$ is given by*

$$\mathrm{B}_{32} = \mathrm{B}'_{31} - \Big(\{n_{a,b,1,d,2} \mid a, b \in \mathbb{I}_{0,1}, d \in \mathbb{I}_{1,2}\} \cup \{n_{0,0,1,0,2}, n_{1,0,1,0,2}, n_{1,0,0,2,2}\}\Big).$$

Proof. Let $w_1 = F_2E_2E_{12}^2 v_\lambda$. By Remark 3.17, $F_i w_1 = 0$, $i \in \mathbb{I}_2$. Let $M' = M(\lambda)/\mathcal{U}w_1$, so B'_{31} is a basis of M'. Moreover $u = n_{1,1,1,2,2} \in V_{10\alpha_1+6\alpha_2}$ satisfies that $E_1 u = E_2 u = 0$, $g_1\sigma_1 u = u$, $g_2\sigma_2 u = \zeta^8 u$, so $(\mathcal{U}w)^\varphi \twoheadrightarrow L(\nu)$, $\nu \in \mathfrak{I}_{12}$. Also $\mathcal{U}u$ is a proper submodule. Set $L'(\lambda) = M(\lambda)/\mathcal{U}w_1 + \mathcal{U}u$. By Lemma 4.3,

$$61 = \dim L(\lambda) \le \dim L'(\lambda) = \dim W - \dim \mathcal{U}w \le \dim W - \dim L(\nu) = 61,$$

so $L(\lambda) = L'(\lambda)$ and $\mathcal{U}w \simeq L(\nu)^\varphi$. In particular $w_2 := F_2F_{11212}F_{112}u \neq 0$, $F_i w_2 = 0$ and $\mathcal{U}w_2 = \mathcal{U}u$. Moreover B_{32} is a basis of $L(\lambda)$. □

4.23. The family $\mathfrak{I}_{33}$

Recall that $\mathfrak{I}_{33} = \{\lambda \in \widehat{\Gamma} \mid \lambda_1 = \zeta^2, \lambda_2 = -1\}$.

Lemma 4.24. *If $\lambda \in \mathfrak{I}_{33}$, then $\dim L(\lambda) = 71$. A basis of $L(\lambda)$ is given by*

$$\mathrm{B}_{33} = \{m_{a,b,c,d,e} \mid a, c, d \in \mathbb{I}_{0,1}, b, e \in \mathbb{I}_{0,2}\} \cup \{m_{1,3,0,0,0}\} - \{m_{0,0,1,0,0}, m_{1,2,0,1,2}\}.$$

Proof. Let $w_1 = F_1^2E_{112}^2E_1^2 v_\lambda$. By Remark 3.8, $F_1 w_1 = F_2 w_1 = 0$. By a direct computation, $\mathcal{U}w_1 \simeq L(\mu)$, with $\mu \in \mathfrak{I}_{23}$, and $\mathrm{B}' = \{m_{a,b,c,d,e} \mid d \neq 2\} \cup \{m_{0,0,0,2,2}\}$ is a basis of $W' = M(\lambda)/\mathcal{U}w_1$. Now $\mathcal{U}m_{0,0,0,2,2} = \Bbbk m_{0,0,0,2,2}$ in W', so $\mathrm{B} = \{m_{a,b,c,d,e} \mid d \neq 2\}$ is a basis of $M' = W'/\Bbbk m_{0,0,0,2,2}$.

Let $w_2 = F_1^2F_{112}^2E_{11212}E_{112}^2E_1^2 v_\lambda$. By Remark 3.11, $F_i w_2 = 0$, $i \in \mathbb{I}_2$, and $\mathcal{U}w_2 \twoheadrightarrow L(\mu)$, with $\mu \in \mathfrak{I}_{14}$, and there exists $E \in \mathcal{U}$ such that $Ew_2 = m_{1,3,1,1,2}$. Moreover, there exists $F \in \mathcal{U}$ such that $Fm_{1,3,1,1,2} = w_2$, and then $\mathcal{U}w_2 = \mathcal{U}m_{1,3,1,1,2} \simeq L(\mu)$. Let $L'(\lambda) = M(\lambda)/\mathcal{U}w_1 + \mathcal{U}m_{0,0,0,2,2} + \mathcal{U}w_2$, so $\dim L'(\lambda) = 96 - 25 = 71$ by Lemma 4.5, and B_{33} is a basis of $L'(\lambda)$. As in previous cases, $L'(\lambda)$ is simple. □

4.24. The family $\mathfrak{I}_{34}$

Recall that $\mathfrak{I}_{34} = \{\lambda \in \widehat{\Gamma} \mid \lambda_1 = \zeta^4, \lambda_2 = \zeta^3\}$.

Lemma 4.25. *If $\lambda \in \mathfrak{I}_{34}$, then $\dim L(\lambda) = 71$. A basis of $L(\lambda)$ is given by*

$$\mathrm{B}_{34} = \{n_{a,b,c,d,e} | a, c, d \in \mathbb{I}_{0,1}, b, e \in \mathbb{I}_{0,2}\} \cup \{n_{0,0,0,2,e} \mid e \in \mathbb{I}_{0,2}\} - \big(\{n_{0,0,1,0,e} | e \in \mathbb{I}_{0,2}\} \cup \{n_{0,1,1,1,0}\}\big).$$

Proof. Let $w_1 = F_2 E_{12}^3 E_2 v_\lambda$. By Remark 3.20, $F_1 w_1 = F_2 w_1 = 0$. By a direct computation, $\mathcal{U}w_1 \simeq L(\mu)$, with $\mu \in \mathfrak{I}_{36}$, and $\mathrm{B}' = \mathrm{B}'_{35} \cup \{n_{1,3,0,0,0}\}$ is a basis of $W' = M(\lambda)/\mathcal{U}w_1$. Now $\mathcal{U}n_{1,3,0,0,0} = \Bbbk n_{1,3,0,0,0}$ in W', so B'_{35} is a basis of $M' = W'/\Bbbk n_{1,3,0,0,0}$.

Let $w_2 = F_1^2 F_{112}^2 E_{11212} E_{112}^2 E_1^2 v_\lambda$. By Remark 3.11, $F_i w_2 = 0$, $i \in \mathbb{I}_2$, and $\mathcal{U}w_2 \twoheadrightarrow L(\mu)$, with $\mu \in \mathfrak{I}_{37}$, and there exists $E \in \mathcal{U}$ such that $Ew_2 = n_{1,2,1,2,2}$. Moreover, there exists $F \in \mathcal{U}$ such that $Fn_{1,2,1,2,2} = w_2$, and then $\mathcal{U}w_2 = \mathcal{U}n_{1,2,1,2,2} \simeq L(\mu)$. Let $L'(\lambda) = M(\lambda)/\mathcal{U}w_1 + \mathcal{U}n_{1,2,1,2,2} + \mathcal{U}w_2$, so $\dim L'(\lambda) = 108 - 37 = 71$ by Lemma 4.28, and B_{34} is a basis of $L'(\lambda)$. As in previous cases, $L'(\lambda)$ is simple. □

4.25. The family $\mathfrak{I}_{35}$

Recall that $\mathfrak{I}_{35} = \{\lambda \in \widehat{\Gamma} \mid \lambda_1 = \zeta^3, \lambda_2 = \zeta^4\}$.

Lemma 4.26. *If $\lambda \in \mathfrak{I}_{35}$, then $\dim L(\lambda) = 85$. A basis of $L(\lambda)$ is given by*

$$\mathrm{B}_{35} = \mathrm{B}'_{35} - \Big(\{n_{0,b,c,2,e} | c \in \mathbb{I}_{0,1}, b, e \in \mathbb{I}_{0,2}\} \cup \{n_{1,2,1,2,2}, n_{1,0,0,2,2}, n_{1,0,1,2,e} | e \in \mathbb{I}_{0,2}\}\Big) \textit{ where}$$

$$\mathrm{B}'_{35} = \{n_{a,b,c,d,e} \mid a, c \in \mathbb{I}_{0,1}, b, d, e \in \mathbb{I}_{0,2}\}$$

Proof. Let $w_1 = F_2 E_2 E_{12}^3 v_\lambda$, so $F_i w_1 = 0$, $i \in \mathbb{I}_2$. Let $M' = M(\lambda)/\mathcal{U}w_1$. Then B'_{35} is a basis of M'. Notice that $w_2 = F_1^2 E_{112}^2 E_1^2 v_\lambda$ satisfies $F_1 w_2 = F_2 w_2 = 0$. Hence $\mathcal{U}w_2 \twoheadrightarrow L(\mu)$ for $\mu \in \mathfrak{I}_{44}$, and there exists $E \in \mathcal{U}$ such that $Ew_2 = n_{1,2,1,2,2}$. Moreover, there exists $F \in \mathcal{U}$ such that $Fn_{1,2,1,2,2} = w_2$, and then $\mathcal{U}w_2 = \mathcal{U}n_{1,2,1,2,2} \simeq L(\mu)$. Let $L'(\lambda) = M(\lambda)/\mathcal{U}w_1 + \mathcal{U}w_2$, so $\dim L'(\lambda) = 108 - 23 = 85$ by Lemma 4.35, and B_{35} is a basis of $L'(\lambda)$. As in previous cases, $L'(\lambda)$ is simple. □

4.26. The family $\mathfrak{I}_{36}$

Recall that $\mathfrak{I}_{36} = \{\lambda \in \widehat{\Gamma} \mid \lambda_1 = \zeta, \lambda_2 = 1\}$.

Lemma 4.27. *If $\lambda \in \mathfrak{I}_{36}$, then $\dim L(\lambda) = 35$. A basis of $L(\lambda)$ is given by* $\mathrm{B}_{36} =$

$$\{n_{0,b,0,d,e}, n_{0,0,1,2,e}, n_{0,0,1,0,e} | b \in \mathbb{I}_{0,3}, d, e \in \mathbb{I}_{0,2}\} - \{n_{0,1,0,1,e}, n_{0,2,0,2,e}, n_{0,1,0,0,2} | e \in \mathbb{I}_{0,2}\}.$$

Proof. Let $w_1 = \widetilde{n}_{1,0,0,0,0}$, $w_2 = E_1^2 E_{12} v_\lambda$. Then $W(\lambda) = \mathcal{U}w_1$ and $F_1 w_2 = F_2 w_2 = 0$. Set $M' = M(\lambda)/W_2(\lambda)$, $u = \widetilde{n}_{0,3,1,2,2}$. Hence $\mathcal{U}w_2 \twoheadrightarrow L(\mu)$ for $\mu \in \mathfrak{I}_{15}$, and there exists $E \in \mathcal{U}$ such that $Ew_2 = u$. Moreover, there exists $F \in \mathcal{U}$ such that $Fu = w_2$, and then $\mathcal{U}w_2 = \mathcal{U}u \simeq L(\mu)$. Let $L'(\lambda) = M(\lambda)/\mathcal{U}w_2 + W(\lambda)$, so $\dim L'(\lambda) = 72 - 37 = 35$ by Lemma 4.6, and B_{36} is a basis of $L'(\lambda)$. As in previous cases, $L'(\lambda)$ is simple. □

4.27. The family $\mathfrak{I}_{37}$

Recall that $\mathfrak{I}_{37} = \{\lambda \in \widehat{\Gamma} \mid \lambda_1 = \zeta^2, \lambda_2 = 1\}$.

Lemma 4.28. *If $\lambda \in \mathfrak{I}_{37}$, then $\dim L(\lambda) = 37$. A basis of $L(\lambda)$ is given by*

$$\mathrm{B}_{37} = \{n_{0,b,0,d,e}, n_{0,0,1,0,0}, n_{0,3,1,0,e} | b \in \mathbb{I}_{0,3}, d, e \in \mathbb{I}_{0,2}\} - \{n_{0,3,0,2,e} | e \in \mathbb{I}_{0,2}\}.$$

Proof. Let $w_1 = \widetilde{n}_{1,0,0,0,0}$, $w_2 = \widetilde{n}_{0,1,0,1,1} - \zeta\widetilde{n}_{0,2,0,0,2} - \zeta^{10}(1-\zeta)^2\widetilde{n}_{0,0,1,0,1}$. Then $W(\lambda) = \mathcal{U}w_1$ and $F_1w_2 = F_2w_2 = 0$. Set $M' = M(\lambda)/W_2(\lambda)$, $u = \widetilde{n}_{0,3,1,2,2}$. Hence $\mathcal{U}w_2 \twoheadrightarrow L(\mu)$ for $\mu \in \mathfrak{I}_{19}$, and there exists $E \in \mathcal{U}$ such that $Ew_2 = u$. Moreover, there exists $F \in \mathcal{U}$ such that $Fu = w_2$, and then $\mathcal{U}w_2 = \mathcal{U}u \simeq L(\mu)$. Let $L'(\lambda) = M(\lambda)/\mathcal{U}w_2 + W(\lambda)$, so $\dim L'(\lambda) = 72 - 35 = 37$ by Lemma 4.10, and B_{37} is a basis of $L'(\lambda)$. As in previous cases, $L'(\lambda)$ is simple. □

4.28. The family $\mathfrak{I}_{38}$

Recall that $\mathfrak{I}_{38} = \{\lambda \in \widehat{\Gamma} \mid \lambda_1 = \zeta^3, \lambda_2 = 1\}$.

Lemma 4.29. *If $\lambda \in \mathfrak{I}_{38}$, then $\dim L(\lambda) = 11$. A basis of $L(\lambda)$ is given by*

$$\mathrm{B}_{38} = \{n_{0,b,c,0,e} | b, c \in \mathbb{I}_{0,1}, e \in \mathbb{I}_{0,2}\} - \{n_{0,1,1,0,2}\}.$$

The action of E_i, F_i, $i \in \mathbb{I}_2$ is described in Table 5.

Proof. Let $w_1 = \widetilde{n}_{1,0,0,0,0}$, $w_2 = F_1^2E_{112}E_1^2v_\lambda$. Then $W(\lambda) = \mathcal{U}w_1$ and $F_1w_2 = F_2w_2 = 0$. Let $L'(\lambda) = M(\lambda)/\mathcal{U}w_2 + W(\lambda)$. We label the elements of B_{38} as follows:

$$\begin{aligned}
&v_{0,0} = n_{0,0,0,0,0}, && v_{1,1} = n_{0,1,0,0,0}, && v_{3,2} = n_{0,0,1,0,0}, && v_{4,3} = n_{0,1,1,0,0},\\
&v_{1,0} = n_{0,0,0,0,1}, && v_{2,1} = n_{0,1,0,0,1}, && v_{4,2} = n_{0,0,1,0,1}, && v_{5,3} = n_{0,1,1,0,1},\\
&v_{2,0} = n_{0,0,0,0,2}, && v_{3,1} = n_{0,1,0,0,2}, && v_{5,2} = n_{0,0,1,0,2}.
\end{aligned}$$

We check that the action of E_k, F_k on $v_{i,j}$ is given by Table 5 and $L'(\lambda)$ is spanned by B_{38}. Moreover there exists $F \in \mathcal{U}^-$ such that $Fv_{5,3} = v_\lambda$, and for each pair (i,j) there is $E_{i,j} \in \mathcal{U}_{(5-i)\alpha_1+(3-j)\alpha_2}$ such that $E_{i,j}v_{i,j} = v_{5,3}$. Thus $L'(\lambda)$ is simple. □

Table 5. Simple modules for $\lambda \in \mathfrak{I}_{38}$.

w	$E_1 \cdot w$	$E_2 \cdot w$	$\lambda(g_1^{-1})F_1 \cdot w$	$\lambda(g_2^{-1})F_2 \cdot w$
$v_{0,0}$	$v_{1,0}$	0	0	0
$v_{1,0}$	$v_{2,0}$	$\zeta^7q_{21}v_{1,1}$	$(1-\zeta^3)v_{0,0}$	0
$v_{2,0}$	0	$\zeta^8q_{21}^2(1+\zeta^3)v_{2,1}$	$\zeta^7(1+\zeta)v_{1,0}$	0
$v_{1,1}$	$v_{2,1}$	0	0	$(\zeta^{11}-1)v_{1,0}$
$v_{2,1}$	$v_{3,1}$	0	$q_{12}\zeta^8v_{1,1}$	$(\zeta^{11}-1)v_{2,0}$
$v_{3,1}$	0	$q_{21}^2\zeta v_{3,2}$	$q_{12}\zeta^2v_{2,1}$	0
$v_{3,2}$	$v_{4,3}$	0	0	$q_{21}\zeta^{11}(1-\zeta^3)v_{3,1}$
$v_{4,2}$	$v_{5,2}$	$q_{21}^2\zeta^{10}v_{4,3}$	$q_{12}^2(\zeta^{11}-1)v_{3,2}$	0
$v_{5,2}$	0	$q_{21}^3(3)_\zeta v_{5,3}$	$q_{12}^2\zeta^8(1+\zeta)v_{4,2}$	0
$v_{4,3}$	$v_{5,3}$	0	0	$q_{21}^2\zeta^{10}(3)_{\zeta^{11}}v_{4,2}$
$v_{5,3}$	0	0	$q_{12}^3\zeta^8(1+\zeta^2)v_{4,3}$	$q_{21}^2\zeta^{10}(3)_{\zeta^{11}}v_{5,2}$

4.29. The family $\mathfrak{I}_{39}$

Recall that $\mathfrak{I}_{39} = \{\lambda \in \widehat{\Gamma} \mid \lambda_1 = \zeta^4,\ \lambda_2 = 1\}$.

Lemma 4.30. *If $\lambda \in \mathfrak{I}_{39}$, then $\dim L(\lambda) = 61$. A basis of $L(\lambda)$ is given by*

$$\begin{aligned} \mathrm{B}_{39} = &\{n_{0,b,c,d,e} | c \in \mathbb{I}_{0,1}, b \in \mathbb{I}_{0,3}, d, e \in \mathbb{I}_{0,2}\} \\ &- \Big(\{n_{0,3,c,2,e}, n_{0,2,1,2,e} | c \in \mathbb{I}_{0,1}, e \in \mathbb{I}_{0,2}\} \cup \{n_{0,2,0,2,e} \mid e \in \mathbb{I}_{1,2}\}\Big). \end{aligned}$$

Proof. Let $w_1 = \widetilde{n}_{1,0,0,0,0}$, $u = \widetilde{n}_{0,3,1,2,2}$, $w_2 = F_1 F_{11212} F_{12}^2 u$. Then $W_2(\lambda) = \mathcal{U} w_1$.

Let $M' = M(\lambda)/W(\lambda)$, so $E_1 u = E_2 u = 0$ in M', and $(\mathcal{U}u)^{\varphi} \twoheadrightarrow L(\nu)$ for some $\nu \in \mathfrak{I}_{38}$; thus $w_2 \neq 0$. By direct computation, $F_i w_2 = 0$, $i \in \mathbb{I}_2$, so $\mathcal{U} w_2$ projects over a simple module $L(\mu)$, for $\mu \in \mathfrak{I}_{18}$. From here, $\mathcal{U} w_2 \simeq L(\mu)$.

Let $L'(\lambda) = M(\lambda)/W_1(\lambda) + \mathcal{U} w_2$. Then $\dim L'(\lambda) = 61$ by Lemma 4.9, and B_{39} is a basis of $L'(\lambda)$. There exists F such that $Fu = v_\lambda$, and $L'(\lambda)$ is simple. □

4.30. The family $\mathfrak{I}_{40}$

Recall that $\mathfrak{I}_{40} = \{\lambda \in \widehat{\Gamma} \mid \lambda_1 = \zeta^5,\ \lambda_2 = 1\}$.

Lemma 4.31. *If $\lambda \in \mathfrak{I}_{40}$, then $\dim L(\lambda) = 35$. A basis of $L(\lambda)$ is given by*

$$\begin{aligned} \mathrm{B}_{40} = &\{n_{0,b,c,0,e} | c \in \mathbb{I}_{0,1}, b \in \mathbb{I}_{0,3}, e \in \mathbb{I}_{0,2}\} \cup \{n_{0,b,c,1,e} \mid c \in \mathbb{I}_{0,1}, b \in \mathbb{I}_{0,1}, e \in \mathbb{I}_{0,2}\} \\ &\cup \{n_{0,3,0,2,e} \mid e \in \mathbb{I}_{0,1}\} - \{n_{0,3,1,0,e} | e \in \mathbb{I}_{0,2}\}. \end{aligned}$$

Proof. Let $w_1 = \widetilde{n}_{1,0,0,0,0}$, $w_2 = F_1^2 E_{112}^2 E_1^2 v_\lambda$. Then $W(\lambda) = \mathcal{U} w_1$ and $F_1 w_2 = F_2 w_2 = 0$. Set $M' = M(\lambda)/W_2(\lambda)$, $u = \widetilde{n}_{0,3,1,2,2}$. Hence $\mathcal{U} w_2 \twoheadrightarrow L(\mu)$ for $\mu \in \mathfrak{I}_{25}$, and there exists $E \in \mathcal{U}$ such that $E w_2 = u$. Moreover, there exists $F \in \mathcal{U}$ such that $Fu = w_2$, and then $\mathcal{U} w_2 = \mathcal{U} u \simeq L(\mu)$. Let $L'(\lambda) = M(\lambda)/\mathcal{U} w_2 + W(\lambda)$, so $\dim L'(\lambda) = 72 - 37 = 35$ by Lemma 4.16, and B_{40} is a basis of $L'(\lambda)$. As in previous cases, $L'(\lambda)$ is simple. □

4.31. The family $\mathfrak{I}_{41}$

Recall that $\mathfrak{I}_{41} = \{\lambda \in \widehat{\Gamma} \mid \lambda_1 = -1,\ \lambda_2 = 1\}$.

Lemma 4.32. *If $\lambda \in \mathfrak{I}_{41}$, then $\dim L(\lambda) = 37$. A basis of $L(\lambda)$ is given by*

$$\begin{aligned} \mathrm{B}_{41} = &\{n_{0,b,c,d,0} | c \in \mathbb{I}_{0,1}, b, d \in \mathbb{I}_{0,2}\} \cup \{n_{0,b,c,d,e} | c, b \in \mathbb{I}_{0,1}, d \in \mathbb{I}_{0,2}, e \in \mathbb{I}_{1,2}\} \\ &- \{n_{0,1,c,d,2}, n_{0,0,1,2,2} | c \in \mathbb{I}_{0,1} d \in \mathbb{I}_{1,2}\}. \end{aligned}$$

Proof. Let $w_1 = \widetilde{n}_{1,0,0,0,0}$, $w_2 = F_1^2 F_{112}^2 E_{11212} E_{112}^2 E_1^2 v_\lambda$. Then $W(\lambda) = \mathcal{U} w_1$ and $F_1 w_2 = F_2 w_2 = 0$. Set $M' = M(\lambda)/W_2(\lambda)$, $u = \widetilde{n}_{0,3,1,2,2}$. Hence $\mathcal{U} w_2 \twoheadrightarrow L(\mu)$ for $\mu \in \mathfrak{I}_{27}$, and there exists $E \in \mathcal{U}$ such that $E w_2 = u$. Moreover, there exists $F \in \mathcal{U}$ such that $Fu = w_2$, and then $\mathcal{U} w_2 = \mathcal{U} u \simeq L(\mu)$. Let $L'(\lambda) = M(\lambda)/\mathcal{U} w_2 + W(\lambda)$, so $\dim L'(\lambda) = 72 - 35 = 37$ by Lemma 4.18, and B_{41} is a basis of $L'(\lambda)$. As in previous cases, $L'(\lambda)$ is simple. □

4.32. The family $\mathfrak{I}_{42}$

Recall that $\mathfrak{I}_{42} = \{\lambda \in \widehat{\Gamma} \mid \lambda_1 = \zeta^7,\ \lambda_2 = 1\}$.

Lemma 4.33. *If* $\lambda \in \mathfrak{I}_{42}$*, then* $\dim L(\lambda) = 71$*. A basis of* $L(\lambda)$ *is given by*

$$\mathrm{B}_{42} = \{n_{0,b,c,d,e} | c \in \mathbb{I}_{0,1}, b \in \mathbb{I}_{0,3}, d, e \in \mathbb{I}_{0,2}, (b,c,d,e) \neq (3,1,2,2)\}.$$

Proof. Let $w_1 = \widetilde{n}_{1,0,0,0,0}$, $w_2 = \widetilde{n}_{0,3,1,2,2}$. Then $W(\lambda) = \mathcal{U}w_1$ and $F_1w_2 = F_2w_2 = E_1w_2 = E_2w_2 = 0$, so $\mathcal{U}w_2 \simeq L(\mu)$ for $\mu \in \mathfrak{I}_{47}$. Let $L'(\lambda) = M(\lambda)/W(\lambda) + \mathcal{U}w_2$, so B_{42} is a basis of $L'(\lambda)$. There exists $F \in \mathcal{U}^-$ such that $Fn_{0,3,1,2,1} = v_\lambda$. If $n_{0,b,c,d,e} \in \mathrm{B}_{42}$, then $E_1^{1-e}E_{112}^{2-d}E_{11212}^{1-c}E_{12}^{3-b}n_{0,b,c,d,e} \in \mathbb{k}^\times n_{0,3,1,2,1}$, so $L'(\lambda)$ is simple. □

4.33. The family $\mathfrak{I}_{43}$

Recall that $\mathfrak{I}_{43} = \{\lambda \in \widehat{\Gamma} \mid \lambda_1 = \zeta^8, \lambda_2 = 1\}$.

Lemma 4.34. *If* $\lambda \in \mathfrak{I}_{43}$*, then* $\dim L(\lambda) = 25$*. A basis of* $L(\lambda)$ *is given by*

$$\mathrm{B}_{43} = \{n_{0,b,c,d,e} | c, e \in \mathbb{I}_{0,1}, b \in \mathbb{I}_{0,3}, d \in \mathbb{I}_{0,2}\} - \Big(\{n_{0,2,1,2,0}\} \cup \{n_{0,b,c,d,1} | c \in \mathbb{I}_{0,1}, b \in \mathbb{I}_{1,3}, d \in \mathbb{I}_{0,2}\} \cup \{n_{0,3,c,d,0} | c \in \mathbb{I}_{0,1}, d \in \mathbb{I}_{1,2}\}\Big).$$

Proof. Let $w_1 = \widetilde{n}_{1,0,0,0,0}$, $w_2 = E_1^2 v_\lambda$. Then $W(\lambda) = \mathcal{U}w_1$ and $F_1w_2 = F_2w_2 = 0$. Set $M' = M(\lambda)/W_2(\lambda)$, $u = \widetilde{n}_{0,3,1,2,2}$. Hence $\mathcal{U}w_2 \twoheadrightarrow L(\mu)$ for $\mu \in \mathfrak{I}_{17}$, and there exists $E \in \mathcal{U}$ such that $Ew_2 = u$. Moreover, there exists $F \in \mathcal{U}$ such that $Fu = w_2$, and then $\mathcal{U}w_2 = \mathcal{U}u \simeq L(\mu)$. Let $L'(\lambda) = M(\lambda)/\mathcal{U}w_2 + W(\lambda)$, so $\dim L'(\lambda) = 72 - 47 = 25$ by Lemma 4.8, and B_{43} is a basis of $L'(\lambda)$. As in previous cases, $L'(\lambda)$ is simple. □

4.34. The family $\mathfrak{I}_{44}$

Recall that $\mathfrak{I}_{44} = \{\lambda \in \widehat{\Gamma} \mid \lambda_1 = \zeta^9, \lambda_2 = 1\}$.

Lemma 4.35. *If* $\lambda \in \mathfrak{I}_{44}$*, then* $\dim L(\lambda) = 23$*. A basis of* $L(\lambda)$ *is given by*

$$\mathrm{B}_{44} = \{n_{0,b,0,d,e} | b \in \mathbb{I}_{0,3}, d \in \mathbb{I}_{0,2}, e \in \mathbb{I}_{0,1}\} \cup \{n_{0,0,0,0,2}\} - \{n_{0,3,0,1,1}, n_{0,3,0,2,1}\}.$$

Proof. Let $w_1 = \widetilde{n}_{1,0,0,0,0}$, $w_2 = \zeta^4\widetilde{n}_{0,0,0,1,1} + \widetilde{n}_{0,1,0,0,2}$. Then $W(\lambda) = \mathcal{U}w_1$ and $F_1w_2 = F_2w_2 = 0$. Set $M' = M(\lambda)/W_2(\lambda)$, $u = \widetilde{n}_{0,3,1,2,2}$. Hence $\mathcal{U}w_2 \twoheadrightarrow L(\mu)$ for $\mu \in \mathfrak{I}_{22}$, and there exists $E \in \mathcal{U}$ such that $Ew_2 = u$. Moreover, there exists $F \in \mathcal{U}$ such that $Fu = w_2$, and then $\mathcal{U}w_2 = \mathcal{U}u \simeq L(\mu)$. Let $L'(\lambda) = M(\lambda)/\mathcal{U}w_2 + W(\lambda)$, so $\dim L'(\lambda) = 72 - 49 = 23$ by Lemma 4.13, and B_{44} is a basis of $L'(\lambda)$. As in previous cases, $L'(\lambda)$ is simple. □

4.35. The family $\mathfrak{I}_{45}$

Recall that $\mathfrak{I}_{45} = \{\lambda \in \widehat{\Gamma} \mid \lambda_1 = \zeta^{10}, \lambda_2 = 1\}$.

Lemma 4.36. *If* $\lambda \in \mathfrak{I}_{45}$*, then* $\dim L(\lambda) = 49$*. A basis of* $L(\lambda)$ *is given by*

$$\mathrm{B}_{45} = \{n_{0,b,c,d,e} | c \in \mathbb{I}_{0,1}, b \in \mathbb{I}_{0,3}, d, e \in \mathbb{I}_{0,2}\} - \big(\{n_{0,b,c,2,e} | c \in \mathbb{I}_{0,1}, b \in \mathbb{I}_{1,3}, e \in \mathbb{I}_{0,2}\} \cup \{n_{0,0,1,2,e} | e \in \mathbb{I}_{0,2}\} \cup \{n_{0,0,1,0,2}, n_{0,3,1,1,2}\}\big).$$

Proof. Let $w_1 = \widetilde{n}_{1,0,0,0,0}$, $w_2 = n_{0,1,0,1,2} - \zeta^{11}(3)_{\zeta^7}n_{0,0,1,0,2}$. Then $W(\lambda) = \mathcal{U}w_1$ and $F_1w_2 = F_2w_2 = 0$. Set $M' = M(\lambda)/W_2(\lambda)$, $u = \widetilde{n}_{0,3,1,2,2}$. Hence $\mathcal{U}w_2 \twoheadrightarrow L(\mu)$ for $\mu \in \mathfrak{I}_{13}$, and there exists $E \in \mathcal{U}$ such that $Ew_2 = u$. Moreover, there exists $F \in \mathcal{U}$ such that $Fu = w_2$, and then $\mathcal{U}w_2 = \mathcal{U}u \simeq L(\mu)$. Let $L'(\lambda) = M(\lambda)/\mathcal{U}w_2 + W(\lambda)$, so $\dim L'(\lambda) = 72 - 23 = 49$ by Lemma 4.4, and B_{45} is a basis of $L'(\lambda)$. As in previous cases, $L'(\lambda)$ is simple. □

4.36. *The family $\mathfrak{I}_{46}$*

Recall that $\mathfrak{I}_{46} = \{\lambda \in \widehat{\Gamma} \mid \lambda_1 = \zeta^{11},\ \lambda_2 = 1\}$.

Lemma 4.37. *If $\lambda \in \mathfrak{I}_{46}$, then $\dim L(\lambda) = 47$. A basis of $L(\lambda)$ is*

$$\mathrm{B}_{46} = \{n_{0,b,c,d,e}, | c, d \in \mathbb{I}_{0,1}, b \in \mathbb{I}_{0,3}, e \in \mathbb{I}_{0,2}\} \cup \{n_{0,1,0,2,0}, n_{0,3,1,2,0}\}$$
$$-\{n_{0,1,1,0,2}, n_{0,3,0,0,1}, n_{0,1,1,0,1}\}.$$

Proof. Let $w_1 = \widetilde{n}_{1,0,0,0,0}$, $w_2 = F_1^2 E_{112}^2 E_1^2 v_\lambda$. Then $W(\lambda) = \mathcal{U} w_1$ and $F_1 w_2 = F_2 w_2 = 0$. Set $M' = M(\lambda)/W_2(\lambda)$, $u = \widetilde{n}_{0,3,1,2,2}$. Hence $\mathcal{U} w_2 \twoheadrightarrow L(\mu)$ for $\mu \in \mathfrak{I}_{26}$, and there exists $E \in \mathcal{U}$ such that $E w_2 = u$. Moreover, there exists $F \in \mathcal{U}$ such that $Fu = w_2$, and then $\mathcal{U} w_2 = \mathcal{U} u \simeq L(\mu)$. Let $L'(\lambda) = M(\lambda)/\mathcal{U} w_2 + W(\lambda)$, so $\dim L'(\lambda) = 72 - 25 = 47$ by Lemma 4.17, and B_{46} is a basis of $L'(\lambda)$. As in previous cases, $L'(\lambda)$ is simple. □

4.37. *The family $\mathfrak{I}_{47}$*

Recall that $\mathfrak{I}_{47} = \{\lambda \in \widehat{\Gamma} \mid \lambda_1 = 1,\ \lambda_2 = 1\}$.

Lemma 4.38. *If $\lambda \in \mathfrak{I}_{47}$, then $\dim L(\lambda) = 1$ and $E_i v_\lambda = 0$, $F_i v_\lambda = 0$, $g\sigma v_\lambda = \lambda(g\sigma) v_\lambda$.*

Proof. Let $N'(\lambda) = W(\lambda) + W_1(\lambda)$. By a direct computation, $N'(\lambda) = \sum_{\beta \neq 0} M(\lambda)_\beta = N(\lambda)$. Therefore $L'(\lambda) = M(\lambda)/N'(\lambda)$ is one-dimensional and simple. □

Example 4.39. Take $\Lambda = \mathbb{Z}_{12} = \langle g_2 \rangle$, $g_1 = g_2^8$ and $\sigma_1, \sigma_2 \in \widehat{\Lambda}$ such that

$$\sigma_1(g_2) = \zeta^{11}, \qquad \sigma_2(g_2) = -1; \qquad \text{hence} \qquad \sigma_1(g_1) = \zeta^4, \qquad \sigma_2(g_1) = 1. \tag{21}$$

Applying the Main Theorem, we see that there is one simple module of dimension one and exactly # different isoclasses of a given dimension as in Table 6:

Table 6. Quantity of simple modules of dimension > 1.

#	dimension	#	dimension	#	dimension	#	dimension
67	144	7	108	10	96	2	85
6	72	4	71	4	61	2	49
10	48	4	47	6	37	7	36
4	35	4	25	2	23	4	11

Note that $\mathfrak{I}_6$ and $\mathfrak{I}_{10}$ are empty.

Acknowledgements

The authors would like to the referee for pointing out several changes which improved the paper. The results are part of PhD thesis of C. Renz, defended at the Universidade Federal do Rio Grande do Sul.

Funding

The work of N. A., I. A. and A. M. was partially supported by CONICET, Secyt (UNC), the MathAmSud project GR2HOPF. The work of I. A. was partially supported by ANPCyT (Foncyt).

References

[1] Andersen, H. H., Jantzen, J. C., Soergel, W. (1994). Representations of quantum groups at a p-th root of unity and of semisimple groups in characteristic p: independence of p. *Astérisque* 220:321 (Paris: Soc. Math. France).

[2] Andruskiewitsch, N. An introduction to Nichols algebras. In: Cardona, A., Morales, P., Ocampo, H., Paycha, S., Reyes, A., eds. *Quantization, Geometry and Noncommutative Structures in Mathematics and Physics*. Mathematical Physics Studies. Springer (to appear.)

[3] Andruskiewitsch, N., Angiono, I. On finite-dimensional Nichols algebras of diagonal type. Available at: http://www.famaf.unc.edu.ar/~andrus/articulos.html. arXiv:1707.08387.

[4] Andruskiewitsch, N., Angiono, I., Rossi Bertone, F. The quantum divided power algebra of a finite-dimensional Nichols algebra of diagonal type. *Math. Res. Lett.* arXiv:1501.04518 (in press).

[5] Andruskiewitsch, N., Angiono, I., Yamane, H. (2011). On pointed Hopf superalgebras. *Contemp. Math.* 544:123–140.

[6] Andruskiewitsch, N., Radford, D., Schneider, H.-J. (2010). Complete reducibility theorems for modules over pointed Hopf algebras. *J. Algebra* 324:2932–2970.

[7] Andruskiewitsch, N., Schneider, H.-J. (2000). Finite quantum groups and Cartan matrices. *Adv. Math.* 154:1–45.

[8] Angiono, I. (2009). On Nichols algebras with standard braiding. *Algebra Number Theory* 3:35–106.

[9] Angiono, I. (2013). Nichols algebras of unidentified diagonal type. *Comm. Algebra* 41:4667–4693.

[10] Angiono, I. (2013). On Nichols algebras of diagonal type. *J. Reine Angew. Math.* 683:189–251.

[11] Angiono, I. (2015). A presentation by generators and relations of Nichols algebras of diagonal type and convex orders on root systems. *J. Eur. Math. Soc.* 17:2643–2671.

[12] Angiono, I. (2016). Distinguished Pre-Nichols algebras. *Transf. Groups* 21:1–33.

[13] Heckenberger, I. (2009). Classification of arithmetic root systems. *Adv. Math.* 220:59–124.

[14] Heckenberger, I. (2010). Lusztig isomorphism for Drinfel'd doubles of bosonizations of Nichols algebras of diagonal type. *J. Algebra* 323:2130–2182.

[15] Heckenberger, I., Yamane, H. (2010). Drinfel'd doubles and Shapovalov determinants. *Rev. UMA* 51:107–146.

[16] Kauffman, L., Radford, D. (1993). A necessary and sufficient condition for a finite dimensional Drinfel'd Double to be a ribbon Hopf algebra. *J. Algebra* 159:98–114.

[17] Lusztig, G. (2010). *Introduction to quantum groups*. Reprint of the 1994 edition. Modern Birkhäuser Classics. Birkhäuser/Springer, New York, xiv+346 pp. ISBN: 978-0-8176-4716-2.

[18] Pogorelsky, B., Vay, C. (2016). Verma and simple modules for quantum groups at non-abelian groups. *Adv. Math.* 301:423–457.

[19] Radford, D., Schneider, H.-J. (2008). On the simple representations of generalized quantum groups and quantum doubles. *J. Algebra* 319:3689–3731.