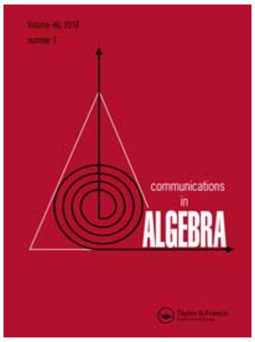

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Free 2-step nilpotent Lie algebras and indecomposable representations

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ABSTRACT

Given an algebraically closed field F of characteristic 0 and an F-vector space V, let $L(V) = V \oplus \Lambda^2(V)$ denote the free 2-step nilpotent Lie algebra associated to V. In this paper, we classify all uniserial representations of the solvable Lie algebra $\mathfrak{g} = \langle x \rangle \ltimes L(V)$, where x acts on V via an arbitrary invertible Jordan block.

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1. Introduction

We fix throughout an algebraically closed field F of characteristic zero. All Lie algebras and representations considered in this paper are assumed to be finite dimensional over F, unless explicitly stated otherwise.

It is well known that the task of classifying all indecomposable modules of an arbitrary Lie algebra is daunting (see for instance [13, 15]). Although this is also true for associative algebras, a remarkable result of Nakayama [16] states that every finitely generated module over a serial ring is a direct sum of uniserial modules (a conceptual proof can be found in [12]). Recall that uniserial modules are those having a unique composition series, and a serial ring is one whose right and left regular modules are sum of uniserials. More than 70 years have elapsed since Nakayama's paper, and even though serial rings have been intensively studied since then, very little is known about uniserial modules of Lie algebras. In order to study uniserial modules of Lie algebras in general, it is necessary to understand (and hopefully classify) the uniserial modules of various families of Lie algebras. We started this project some years ago and we succeeded for a number families [1–4, 8]. Also, there has been progress in classifying other types of indecomposable modules, that turned out to be closely related to the uniserial ones, see for instance [5–7, 9–11, 14, 17, 18].

In this paper, we make a further contribution in this direction by classifying all uniserial representations of the solvable Lie algebra $\mathfrak{g} = \langle x \rangle \ltimes L(V)$, where V is a vector space, $L(V) = V \oplus \Lambda^2(V)$ is the free 2-step nilpotent Lie algebra associated to V, and x acts on V via a single Jordan block $J_n(\lambda)$, with $\lambda \neq 0$. The case $n = 1$, when $\Lambda^2(V) = 0$, is covered in [4], so we will focus attention on the case $n > 1$.

We say that a uniserial representation $R : \mathfrak{g} \to \mathfrak{gl}(U)$ is *relatively faithful* if $\ker(R) \cap \Lambda^2(V)$ is properly contained in $\Lambda^2(V)$ and $\ker(R) \cap V = (0)$. It suffices to consider the case when R is relatively faithful, for if $\Lambda^2(V) \subseteq \ker(R)$ then [8] applies, if $V \subseteq \ker(R)$ we may appeal to [3], and if $(0) \neq \ker(R) \cap V \neq V$, we are led to consider a uniserial representation $\overline{R}$ of $\langle \overline{x} \rangle \ltimes L(\overline{V})$, where $\overline{V}$ is a factor of V by an x-invariant subspace, $\overline{x}$ acts on $\overline{V}$ via an invertible Jordan block $J_m(\lambda)$, $1 \le m < n$, and $\ker(\overline{R}) \cap \overline{V} = (0)$.

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Our main results are as follows. In Section 3 we define a family of relatively faithful uniserial representations of $\mathfrak{g}$ (the case $\lambda = 0$ being allowed). Explicitly, let $v_0, \dots, v_{n-1}$ be a basis of V such that

$$[x, v_0] = \lambda v_0 + v_1, [x, v_1] = \lambda v_1 + v_2, \dots, [x, v_{n-1}] = \lambda v_{n-1}.$$

Given a triple (a, b, c) of positive integers satisfying

$$a + b = n + 1,\ c \leq a \quad \text{or} \quad c + b = n + 1,\ a \leq c,$$

two matrices $M \in M_{a\times b}$ and $N \in M_{b\times c}$ such that

$$M_{a,1} \neq 0 \text{ and } N_{b,1} \neq 0,$$

and a scalar $\alpha \in F$, we define a representation $R = R_{a,b,c,M,N,\alpha} : \mathfrak{g} \to \mathfrak{gl}(d)$, $d = a + b + c$, in block form, in the following manner:

$$R(x) = A = \begin{pmatrix} J^a(\alpha) & 0 & 0 \\ 0 & J^b(\alpha - \lambda) & 0 \\ 0 & 0 & J^c(\alpha - 2\lambda) \end{pmatrix},$$

where $J^p(\beta)$ denotes the upper triangular Jordan block of size p and eigenvalue β,

$$R(v_k) = (\mathrm{ad}_{\mathfrak{gl}(d)} A - \lambda 1_{\mathfrak{gl}(d)})^k \begin{pmatrix} 0 & M & 0 \\ 0 & 0 & N \\ 0 & 0 & 0 \end{pmatrix}, \quad 0 \leq k \leq n - 1,$$

$$R(v \wedge w) = [R(v), R(w)], \quad v, w \in V.$$

The representation R is always uniserial. It is also relatively faithful, except for an extreme case, as described in Definition 3.2. The length of R, as defined in Definition 3.1, is equal to 3 (it coincides with the number of Jordan blocks of $R(x)$ in this case).

Conjugating all $R(y)$, $y \in \mathfrak{g}$, by a suitable block diagonal matrix commuting with A, one may normalize R, in the sense of Definition 3.2. In Section 7 we prove, for $\lambda \neq 0$, that every relatively faithful uniserial representation of $\mathfrak{g}$ is isomorphic to one and only one normalized representation $R_{a,b,c,M,N,\alpha}$ of non-extreme type. This requires, in particular, to prove that $\mathfrak{g}$ has no relatively faithful uniserial representations of length > 3. This is our most challenging obstacle, and it is proven in Theorem 7.2. The ideas behind the proof of Theorem 7.2 are somewhat subtle and are presented independently in Section 6.

We are be very interested in knowing the classification of all uniserial modules of $\mathfrak{g}$ when $\lambda = 0$ (the case when $\mathfrak{g}$ is nilpotent), but this seems to be a very difficult task.

In Section 4 we determine when $R_{a,b,c,M,N,\alpha}$ is faithful (for arbitrary λ). It turns out that $R_{a,b,c,M,N,\alpha}$ is faithful if and only if

$$(a, b, c) \in \{(n, 1, n), (n - 1, 2, n - 1), (n, 1, n - 1), (n - 1, 1, n)\}.$$

Sufficiency of this result is fairly delicate. Most of the work toward it is done in Proposition 4.5. The case $n = 3$ and $(a, b, c) = (2, 2, 2)$ is special, in the sense that it is the only faithful uniserial representation of $\mathfrak{g}$ where all blocks are square (in this case of size 2). This case is intimately related to a representation of the truncated current Lie algebra $\mathfrak{sl}(2) \otimes F[t]/(t^3)$.

In Section 5 we provide a generalization of our faithfulness result, stated without reference to Lie algebras or their representations.

Our general notation, basic concepts and preliminary material can all be found in Sections 2–4.

2. The Lie algebra $\mathfrak{g}$

We fix throughout a vector space V. There is a unique Lie algebra structure on

$$L(V) = V \oplus \Lambda^2(V)$$

such that

$$[v, w] = v \wedge w, \quad v, w \in V$$

and

$$[u, v \wedge w] = 0, \quad u, v, w \in V.$$

The Lie algebra $L(V)$ is the *free 2-step nilpotent Lie algebra associated to* V. In particular we have the following straightforward lemma.

Lemma 2.1. *Let* $\mathfrak{h}$ *be a Lie algebra and let* $\Omega : V \to \mathfrak{h}$ *be a linear map satisfying*

$$[\Omega(V), [\Omega(V), \Omega(V)]] = 0.$$

Then Ω *has a unique extension to a homomorphism of Lie algebras* $\Omega' : L(V) \to \mathfrak{h}$.

Given a Lie algebra $\mathfrak{h}$ and a representation $\mathfrak{h} \to \mathfrak{gl}(V)$, we can make $\Lambda^2(V)$ into an $\mathfrak{h}$-module via:

$$x(v \wedge w) = xv \wedge w + v \wedge xw, \quad x \in \mathfrak{h}, v, w \in V.$$

This gives a representation $\mathfrak{h} \to \mathfrak{gl}(L(V))$ whose image we readily see to be in $\mathrm{Der}(L(V))$. This produces the Lie algebra

$$\mathfrak{h} \ltimes L(V).$$

For the remainder of the paper we set

$$\mathfrak{g} = \langle x \rangle \ltimes L(V),$$

where $x \in \mathfrak{gl}(V)$.

3. Relatively faithful uniserial representations of $\mathfrak{g}$

Given $p \geq 1$ and $\alpha \in F$, we write $J_p(\alpha)$ (resp. $J^p(\alpha)$) for the lower (resp. upper) triangular Jordan block of size p and eigenvalue α.

We suppose throughout this section that $\mathfrak{g} = \langle x \rangle \ltimes L(V)$, where $x \in \mathfrak{gl}(V)$ acts on V via a single, lower triangular, Jordan block, say $J_n(\lambda)$ with $n > 1$, relative to a basis $v_0, \ldots, v_{n-1}$ of V. The case $\lambda = 0$ is allowed. Then $\mathfrak{g}$ has the following defining relations:

$$[v, w] = v \wedge w, \quad v, w \in V, \tag{3.1}$$

$$[u, v \wedge w] = 0, \quad u, v, w \in V, \tag{3.2}$$

$$[x, v_0] = \lambda v_0 + v_1, [x, v_1] = \lambda v_1 + v_2, \ldots, [x, v_{n-1}] = \lambda v_{n-1}. \tag{3.3}$$

We may translate (3.3) as

$$(\mathrm{ad}_{\mathfrak{g}}\, x - \lambda 1_{\mathfrak{g}})^k v_0 = v_k, \quad 0 \leq k \leq n-1, \tag{3.4}$$

and

$$(\mathrm{ad}_{\mathfrak{g}} x - \lambda 1_{\mathfrak{g}})^n v_0 = 0. \tag{3.5}$$

Definition 3.1. Let U be a non-zero $\mathfrak{g}$-module. Let U_1 be the subspace of U annihilated by $[\mathfrak{g}, \mathfrak{g}]$. Since $[\mathfrak{g}, \mathfrak{g}]$ is an ideal of $\mathfrak{g}$, it is clear that U_1 is a $\mathfrak{g}$-submodule of U. Moreover, since $[\mathfrak{g}, \mathfrak{g}]$ acts via nilpotent operators on U, Engel's theorem ensures that $U_1 \neq 0$. We then choose U_2 so that U_2/U_1 is the subspace of U/U_1 annihilated by $[\mathfrak{g}, \mathfrak{g}]$, and so on. This gives rise to a strictly increasing sequence of $\mathfrak{g}$-submodules of U, namely

$$0 \subset U_1 \subset U_2 \subset \cdots \subset U_\ell = U.$$

We define the *length* of U to be ℓ. Note that, since $\mathfrak{g}$ is solvable and F is algebraically closed, the length of a Jordan–Hölder composition series of U is $\dim U$.

Definition 3.2. Let (a, b, c) be a triple of positive integers satisfying

$$a+b=n+1,\ c\leq a \quad \text{or} \quad c+b=n+1,\ a\leq c, \tag{3.6}$$

let $M\in M_{a\times b}$, $N\in M_{b\times c}$ be such that

$$M_{a,1}\neq 0 \text{ and } N_{b,1}\neq 0,$$

and let $\alpha\in F$. Associated to this data we define a linear transformation $R=R_{a,b,c,M,N,\alpha}:\mathfrak{g}\to\mathfrak{gl}(d)$, $d=a+b+c$, in block form, as follows:

$$R(x)=A=\begin{pmatrix} J^a(\alpha) & 0 & 0\\ 0 & J^b(\alpha-\lambda) & 0\\ 0 & 0 & J^c(\alpha-2\lambda)\end{pmatrix}, \tag{3.7}$$

$$R(v_k)=(\mathrm{ad}_{\mathfrak{gl}(d)}A-\lambda 1_{\mathfrak{gl}(d)})^k\begin{pmatrix} 0 & M & 0\\ 0 & 0 & N\\ 0 & 0 & 0\end{pmatrix},\quad 0\leq k\leq n-1, \tag{3.8}$$

$$R(v\wedge w)=[R(v),R(w)],\quad v,w\in V. \tag{3.9}$$

We refer to M and N as *normalized*, if the last rows of M and N are equal to the first canonical vectors of F^b and F^c, respectively, and the first column of M is equal to the last canonical vector of F^a. In this case, we say that R itself is *normalized*. If R is normalized, we say that R is of *extreme type* if n is odd, $a=1$, $c=1$ and $N_{i,1}=0$ for all even i.

Conjugating all $R(y)$, $y\in\mathfrak{g}$, by a suitable block diagonal matrix commuting with A, it is always possible to normalize R, as seen in [8, Lemma 2.5].

Proposition 3.3. *The linear map $R_{a,b,c,M,N,\alpha}$ is a uniserial representation.*

Proof. It follows from Lemma 2.1 that (3.8)–(3.9) define a Lie homomorphism $L(V)\to\mathfrak{gl}(d)$. By (3.6), we have $a+b\leq n+1$ and $b+c\leq n+1$, so [8, Proposition 2.2] ensures that the relations (3.4) and (3.5) are preserved, whence R is a representation. Since $M_{a,1}\neq 0$ and $N_{b,1}\neq 0$, R is clearly uniserial. □

Proposition 3.4. *Assume $\lambda\neq 0$. The normalized representations $R_{a,b,c,M,N,\alpha}$ are non-isomorphic to each other. The normalized representation $R_{a,b,c,M,N,\alpha}$ is relatively faithful, except only for the extreme type.*

Proof. Considering the eigenvalues of the image of x as well as their multiplicities, the only possible isomorphisms are easily seen to be between $R_{a,b,c,M,N,\alpha}$ and $R_{a,b,c,M',N',\alpha}$. Suppose $T\in\mathrm{GL}(d)$, $d=a+b+c$, satisfies

$$TR_{a,b,c,M,N,\alpha}(y)T^{-1}=R_{a,b,c,M',N',\alpha}(y),\quad y\in\mathfrak{g}.$$

Then T commutes with $R_{a,b,c,M,N,\alpha}(x)=J^a(\alpha)\oplus J^b(\alpha-\lambda)\oplus J^c(\alpha-2\lambda)$, and therefore $T=T_1\oplus T_2\oplus T_3$, where T_1, T_2, T_3 are polynomials in $J^a(0), J^b(0), J^c(0)$, respectively, with non-zero constant term. This means that every superdiagonal of T_i, $1\leq i\leq 3$, has equal entries. Using this feature of T_1, T_2, T_3 in

$$TR_{a,b,c,M,N,\alpha}(v_0)=R_{a,b,c,M',N',\alpha}(v_0)T$$

together with the fact that M, N and M', N' are normalized, we readily find that T is a scalar operator, whence $M = M'$ and $N = N'$.

Since $a + b = n + 1$ or $b + c = n + 1$, [8, Proposition 2.2] yields $\ker(R) \cap V = (0)$. It remains to determine when is $\Lambda^2(V) \subseteq \ker(R)$. By [8, Theorem 3.2], this can only happen when n is odd, $a = 1$, $c = 1$, in which case direct computation forces $N_{i,1} = 0$ for all even i. □

4. Determining the faithful uniserial representations of $\mathfrak{g}$

We assume throughout this section that $\mathfrak{g} = \langle x \rangle \ltimes L(V)$, where x acts on V via a single lower Jordan block $J_n(\lambda)$, $n > 1$, relative to a basis $v_0, \dots, v_{n-1}$ of V.

Definition 4.1. Given a sequence $(d_1, \dots, d_\ell)$ of positive integers, we view every $M \in M_d$, for $d = d_1 + \cdots + d_\ell$, as partitioned into ℓ^2 blocks $M(i,j) \in M_{d_i \times d_j}$, $1 \le i, j \le \ell$. For $0 \le i \le \ell - 1$, by the ith superdiagonal of M we mean the blocks $M(1, 1+i), M(2, 2+i), \dots, M(\ell - i, \ell)$, and we say that M is an i-diagonal block matrix if all other blocks of M are equal to 0. We refer to M as block upper triangular if $M(i,j) = 0$ for all $i > j$ and as block strictly upper triangular if $M(i,j) = 0$ for all $i \ge j$.

Definition 4.2. Given an integer $\ell > 2$, a sequence of positive integers $(d_1, \dots, d_\ell)$, and a scalar $\alpha \in F$, a representation $R : \mathfrak{g} \to \mathfrak{gl}(d)$ is said to be *standard* relative to $(\ell, (d_1, \dots, d_\ell), \alpha)$ if the following conditions hold:

$$d_1 + \cdots + d_\ell = d; \quad d_i + d_{i+1} \le n + 1 \text{ for all } i;$$

$R(x)$ is the 0-diagonal block matrix

$$A = J^{d_1}(\alpha) \oplus J^{d_2}(\alpha - \lambda) \oplus \cdots \oplus J^{d_\ell}(\alpha - (\ell - 1)\lambda);$$

every $R(v)$, $v \in V$, is a 1-diagonal block matrix; every block in the first superdiagonal of $R(v_0)$ has non-zero bottom left entry.

Let $M_1, \dots, M_{\ell-1}$ denote the blocks in the first superdiagonal of $R(v_0)$. We say that R is *normalized standard* relative to $(\ell, (d_1, \dots, d_\ell), \alpha)$ if, in addition to the above conditions, the last row of each M_i is equal to the first canonical vector, and the first column of M_1 is the last canonical vector.

Note that a standard representation R is always uniserial, and its length, as defined in Definition 3.1, is equal to ℓ. Observe also that if R is a standard representation then every $R(v \wedge w)$, $v, w \in V$, is a 2-diagonal block matrix.

Lemma 4.3. *Given an integer $\ell > 2$, a sequence of positive integers $(d_1, \dots, d_\ell)$, and a scalar $\alpha \in F$, let $R : \mathfrak{g} \to \mathfrak{gl}(d)$ be a standard representation relative to them. Then* $\ker(R) \cap V = (0)$ *if and only if $d_i + d_{i+1} = n + 1$ for at least one i.*

Proof. Since the x-invariant subspaces of V form a chain, we have $\ker(R) \cap V = (0)$ if and only if $v_{n-1} \notin \ker(R)$, which is equivalent to $d_i + d_{i+1} = n + 1$ for some i, by [8, Proposition 2.2]. □

Lemma 4.4. *Given an integer $\ell > 2$, a sequence of positive integers $(d_1, \dots, d_\ell)$, and a scalar $\alpha \in F$, let $R : \mathfrak{g} \to \mathfrak{gl}(d)$ be a standard (resp. normalized standard) representation relative to them. Then the dual representation is similar to a representation $T : \mathfrak{g} \to \mathfrak{gl}(d)$ that is standard (resp. normalized standard) relative to $(\ell, (d_\ell, \dots, d_1), (\ell - 1)\lambda - \alpha)$. Moreover, R is faithful (resp. relatively faithful) if and only if T is faithful (resp. relatively faithful).*

Proof. This is straightforward. □

Proposition 4.5. *Given an integer $n \geq 2$, let $(p_1, \dots, p_{n-1}), (q_1, \dots, q_{n-1}) \in F^{n-1}$ be such that $p_j + q_j \neq 0$ for all j, and let $z, w \in F$ be non-zero. Associated to these data, we consider matrices*

$$P_0, \dots, P_{n-1} \in M_{n-1\times 2}, \qquad Q_0, \dots, Q_{n-1} \in M_{2\times n-1},$$

having the following structure:

$$P_0 = \begin{pmatrix} * & * \\ \vdots & \vdots \\ * & * \\ z & * \end{pmatrix}, \qquad P_1 = \begin{pmatrix} * & * \\ \vdots & \vdots \\ * & * \\ z & * \\ 0 & -p_1 z \end{pmatrix}, \qquad P_2 = \begin{pmatrix} * & * \\ \vdots & \vdots \\ * & * \\ z & * \\ 0 & -p_2 z \\ 0 & 0 \end{pmatrix},$$

$$P_3 = \begin{pmatrix} * & * \\ \vdots & \vdots \\ * & * \\ z & * \\ 0 & -p_3 z \\ 0 & 0 \\ 0 & 0 \end{pmatrix}, \dots, \quad P_{n-2} = \begin{pmatrix} z & * \\ 0 & -p_{n-2} z \\ 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \end{pmatrix}, \quad P_{n-1} = \begin{pmatrix} 0 & -p_{n-1} z \\ 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \end{pmatrix},$$

$$Q_0 = \begin{pmatrix} * & * & \dots & * \\ w & * & \dots & * \end{pmatrix}, \qquad Q_1 = \begin{pmatrix} q_1 w & * & * & \dots & * \\ 0 & -w & * & \dots & * \end{pmatrix},$$

$$Q_2 = \begin{pmatrix} 0 & -q_2 w & * & * & \dots & * \\ 0 & 0 & w & * & \dots & * \end{pmatrix}, \qquad Q_3 = \begin{pmatrix} 0 & 0 & q_3 w & * & \dots & * \\ 0 & 0 & 0 & -w & \dots & * \end{pmatrix}, \dots,$$

$$Q_{n-2} = \begin{pmatrix} 0 & \dots & 0 & (-1)^{n-3} q_{n-2} w & * \\ 0 & \dots & 0 & 0 & (-1)^{n-2} w \end{pmatrix}, \quad Q_{n-1} = \begin{pmatrix} 0 & \dots & 0 & (-1)^{n-2} q_{n-1} w \\ 0 & \dots & 0 & 0 \end{pmatrix}.$$

Then the matrices $T_{i,j} \in M_{n-1}$, $0 \leq i < j \leq n-1$, defined by

$$T_{i,j} = P_i Q_j - P_j Q_i, \quad 0 \leq i < j \leq n-1,$$

are linearly independent.

Proof. By induction on n. In the base case $n = 2$, we have

$$P_0 = \begin{pmatrix} z & * \end{pmatrix}, \; P_1 = \begin{pmatrix} 0 & -p_1 z \end{pmatrix}, \; Q_0 = \begin{pmatrix} * \\ w \end{pmatrix}, Q_1 = \begin{pmatrix} q_1 w \\ 0 \end{pmatrix}.$$

Therefore

$$T_{0,1} = \big((p_1 + q_1) wz\big) \neq 0.$$

Assume that $n > 2$ and that the result is true for $m = n - 1$. Let

$$\mathcal{T} = \sum_{0 \leq i < j \leq n-1} \alpha_{i,j} T_{i,j}$$

and assume $\mathcal{T} = 0$. We wish to show that

$$\alpha_{i,j} = 0, \quad 0 \leq i < j \leq n-1. \tag{4.1}$$

It suffices to show that

$$\alpha_{0,j} = 0, \quad 1 \leq j \leq n-1. \tag{4.2}$$

Indeed, assume we have proven (4.2). Since $\mathcal{T} = 0$, we obtain

$$\sum_{1\le i<j\le n-1} \alpha_{i,j} T_{i,j} = 0. \tag{4.3}$$

Let $P'_0, \dots, P'_{m-1} \in M_{m-1\times 2}$ and $Q'_0, \dots, Q'_{m-1} \in M_{2\times m-1}$ be the matrices obtained by deleting the last rows of $P_1, \dots, P_{n-1}$ and the first columns of $Q_1, \dots, Q_{n-1}$, and let $T'_{i,j} = P'_i Q'_j - P'_j Q'_i$, $0 \le i < j \le m-1$. It follows automatically from (4.3) that

$$\sum_{0\le i<j\le m-1} \alpha'_{i,j} T'_{i,j} = 0,$$

where $\alpha'_{i,j} = \alpha_{i+1,j+1}$ and, from the inductive hypothesis, we conclude

$$\alpha_{i,j} = 0, \quad 1 \le i < j \le n-1. \tag{4.4}$$

We may now obtain (4.1) from (4.2) and (4.4).

We proceed to prove (4.2). In fact we will prove by induction on $k \le n-1$ that $\alpha_{i,j} = 0$ whenever $i < j$ and $i + j \le k$.

The base case $k = 1$ is straightforward. Indeed, from $\mathcal{T}_{n-1,1} = \alpha_{0,1}(p_1 + q_1)wz$, we infer $\alpha_{0,1} = 0$.

Suppose $1 < k \le n-1$ and assume that $\alpha_{i,j} = 0$ whenever $i < j$ and $i + j \le k - 1$. Using this, a direct computation reveals that, for $i - j = n - 1 - k$, we have

$$\mathcal{T}_{i,j} = \begin{cases} (-1)^j(\alpha_{j,k-j}\, q_j - \alpha_{j-1,k+1-j}\, p_{k+1-j})\, wz, & \text{if } 1 \le j < \frac{k}{2}; \\ -(-1)^{\frac{k}{2}}\, \alpha_{\frac{k}{2}-1,\frac{k}{2}+1}\, p_{\frac{k}{2}+1}\, wz, & \text{if } j = \frac{k}{2}; \\ -(-1)^{\frac{k+1}{2}}\, \alpha_{\frac{k-1}{2},\frac{k+1}{2}} \left(q_{\frac{k+1}{2}} + p_{\frac{k+1}{2}}\right) wz, & \text{if } j = \frac{k+1}{2}; \\ -(-1)^{\frac{k+2}{2}}\, \alpha_{\frac{k}{2}-1,\frac{k}{2}+1}\, q_{\frac{k}{2}+1}\, wz, & \text{if } j = \frac{k}{2}+1; \\ -(-1)^j(\alpha_{k-j,j}\, q_j - \alpha_{k+1-j,j-1}\, p_{k+1-j})\, wz, & \text{if } \frac{k}{2}+1 < j \le n-1; \end{cases}$$

that is

i	j	$\mathcal{T}_{i,j}/wz$	
$n-k$	1	$-\alpha_{1,k-1}q_1 + \alpha_{0,k}p_k$	
$n-k+1$	2	$\alpha_{2,k-2}q_2 - \alpha_{1,k-1}p_{k-1}$	
$n-k+2$	3	$-\alpha_{3,k-3}q_3 + \alpha_{2,k-2}p_{k-2}$	
$\vdots$	$\vdots$	$\vdots$	
$n-1-\frac{k}{2}$	$\frac{k}{2}$	$-(-1)^{\frac{k}{2}}\, \alpha_{\frac{k}{2}-1,\frac{k}{2}+1}\, p_{\frac{k}{2}+1}$	(if k is even)
$n-1-\frac{k-1}{2}$	$\frac{k+1}{2}$	$-(-1)^{\frac{k+1}{2}}\, \alpha_{\frac{k-1}{2},\frac{k+1}{2}} \left(q_{\frac{k+1}{2}} + p_{\frac{k+1}{2}}\right)$	(if k is odd)
$n-1-\frac{k-2}{2}$	$\frac{k+2}{2}$	$-(-1)^{\frac{k+2}{2}}\, \alpha_{\frac{k}{2}-1,\frac{k}{2}+1}\, q_{\frac{k}{2}+1}$	(if k is even)
$\vdots$	$\vdots$	$\vdots$	
$n-3$	$k-2$	$-(-1)^{k-2}(\alpha_{2,k-2}\, q_{k-2} - \alpha_{3,k-3}\, p_3)$	
$n-2$	$k-1$	$-(-1)^{k-1}(\alpha_{1,k-1}\, q_{k-1} - \alpha_{2,k-2}\, p_2)$	
$n-1$	k	$-(-1)^{k}(\alpha_{0,k}\, q_k - \alpha_{1,k-1}\, p_1)$	

Since, by hypothesis, $p_j + q_j \neq 0$ for all j (which in turn implies that either p_j or q_j is non-zero for all j) we obtain that (4.2) holds. □

Theorem 4.6. *A representation $R_{a,b,c,M,N,\alpha}$ of $\mathfrak{g}$ is faithful if and only if*

$$(a, b, c) \in \{(n, 1, n), (n-1, 2, n-1), (n, 1, n-1), (n-1, 1, n)\}. \tag{4.5}$$

Proof. We divide the proof into two parts.

NECESSITY. Suppose the representation $R = R_{a,b,c,M,N,\alpha} : \mathfrak{g} \to \mathfrak{gl}(d)$ is faithful, where $d = a + b + c$.

Let S be the subspace of $\mathfrak{gl}(d)$ of all matrices

$$\begin{pmatrix} 0 & 0 & P \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad P \in M_{a\times c}.$$

Letting A be as in (3.7), we view S as an $F[t]$-module via $\mathrm{ad}_{\mathfrak{gl}(d)}A - 2\lambda 1_{\mathfrak{gl}(d)}$. As in [8, Proposition 2.1], we see that $\mathrm{ad}_{\mathfrak{gl}(d)}A - 2\lambda 1_{\mathfrak{gl}(d)}$ acts nilpotently on S with nilpotency degree $a + c - 1$. On the other hand, we may view $\Lambda^2(V)$ as an $F[t]$-module via $\mathrm{ad}_{\mathfrak{g}}x - 2\lambda 1_{\mathfrak{g}}$. Direct computation (alternatively, we may use the theory of $\mathfrak{sl}(2)$-modules) reveals that $\mathrm{ad}_{\mathfrak{g}}x - 2\lambda 1_{\mathfrak{g}}$ acts on $\Lambda^2(V)$ with nilpotency degree $2n - 3$. Indeed, we have

$$(\mathrm{ad}_{\mathfrak{g}}x - 2\lambda 1_{\mathfrak{g}})^m(v \wedge w) = \sum_{i+j=m} \binom{m}{i} (x - \lambda 1_V)^i v \wedge (x - \lambda 1_V)^j w. \tag{4.6}$$

Set $m = 2n-3$ in (4.6) and take $v = v_p$ and $w = v_q$ with $0 \le p < q \le n-1$. Then the right-hand side of (4.6) is equal to 0 (including the extreme case $p = 0, q = 1$, which produces $\binom{2n-3}{n-1} v_{n-1} \wedge v_{n-1} = 0$). Next set $m = 2n - 4$ in (4.6) and take $v = v_0$ and $w = v_1$. Then the right-hand side of (4.6) is equal to

$$\left[\binom{2n-4}{n-1} - \binom{2n-4}{n-2}\right] v_{n-1} \wedge v_{n-2} \neq 0.$$

Since R is faithful, restricting R to $\Lambda^2(V)$ yields a linear monomorphism $T : \Lambda^2(V) \to S$. It follows from [8, Lemma 3.1] that T commutes with the indicated actions of $F[t]$, so that T is a monomorphism of $F[t]$-modules. It follows from above that

$$2n - 3 \le a + c - 1. \tag{4.7}$$

On the other hand, by (3.6), we have $a + b = n + 1$ or $c + b = n + 1$. By duality (see Lemma 4.4), we may assume that $a + b = n + 1$. Suppose, if possible, that $b + c < n$. As the x-invariant subspaces of V form a chain, it follows from [8, Proposition 2.2] that blocks (2,3) of $R(v_{n-1})$ and $R(v_{n-2})$ are equal to 0 (alternatively, appeal to a direct computation based on (3.7) and (3.8)). Then (3.9) yields $R(v_{n-2} \wedge v_{n-1}) = 0$, a contradiction. We infer $b + c \ge n$. It follows from (3.6) that $b + c = n$ or $b + c = n + 1$. In the second case $c = a$, so (4.7) gives $a \ge n - 1$, whence

$$(a, b, c) \in \{(n, 1, n), (n-1, 2, n-1)\}.$$

In the first case $c = a - 1$, so (4.7) gives $a \ge n - \frac{1}{2}$, whence $(a, b, c) = (n, 1, n-1)$.

SUFFICIENCY. We wish to show that $R = R_{a,b,c,M,N,\alpha}$ is faithful whenever (4.5) holds.

By duality (see Lemma 4.4), we may restrict to the cases

$$(a, b, c) \in \{(n, 1, n), (n-1, 2, n-1), (n, 1, n-1)\}. \tag{4.8}$$

We will write $P(y), Q(y)$ and $T(y)$ for blocks (1, 2), (2, 3) and (1,3) of $R(y)$, $y \in \mathfrak{g}$, respectively.

By Proposition 3.4, R is relatively faithful (it follows from (4.8) that, after normalizing R, we are not in the extreme case) and thus R is faithful if and only if the matrices $T(v_i \wedge v_j)$, $0 \leq i < j \leq n-1$, are linearly independent.

- $(a, b, c) = (n-1, 2, n-1)$. Set $(p_1, \dots, p_{n-1}) = (q_1, \dots, q_{n-1}) = (1, \dots, n-1)$ and, for $i = 0, \dots, n-1$, let $P_i = P(v_i) \in M_{n-1\times 2}$ and $Q_i = Q(v_i) \in M_{2\times n-1}$. It is not difficult to see that these vectors and matrices satisfy the hypothesis of Proposition 4.5 and thus, considering (3.9), we obtain that

$$T(v_i \wedge v_j) = P(v_i)Q(v_j) - P(v_j)Q(v_i) = P_iQ_j - P_jQ_i = T_{i,j}, \quad 0 \leq i < j \leq n-1,$$

  are linearly independent.
- $(a, b, c) = (n, 1, n)$. Note that $T(v_i \wedge v_j)$, $0 \leq i < j \leq n-1$, form the canonical basis of the space $\mathfrak{so}(n)$ of all $n \times n$ skew-symmetric matrices.
- $(a, b, c) = (n, 1, n-1)$. Again, $T(v_i \wedge v_j)$, $0 \leq i < j \leq n-2$, form the canonical basis of $\mathfrak{so}(n-1)$, viewed as the subspace of $\mathfrak{so}(n)$ of matrices with zero first row and last column. On the other hand, noting that $Q(v_{n-1}) = 0$, we see that $T(v_i \wedge v_{n-1})$, $0 \leq i < n-1$, form the (opposite of the) canonical basis of F^{n-1}, viewed as top left corner, say C, of M_n. Since $\mathfrak{so}(n-1) \cap C = (0)$, the result follows.

□

Example 4.7. An interesting example occurs when $n = 3$ and $a = b = c = 2$. Then we do get a faithful module above of a very special nature: it is the only faithful uniserial module of $\mathfrak{g}$ where all the blocks are squares. Take $\lambda = \alpha = 0$ (the other cases are easy modifications).

Given a Lie algebra L and an associative commutative algebra A, we know that $L \otimes A$ is a Lie algebra under $[x \otimes a, y \otimes b] = [x, y] \otimes ab$. Moreover, if $R_1 : L \to \mathfrak{gl}(V_1)$ and $R_2 : A \to \mathfrak{gl}(V_2)$ are representations, then $R_1 \otimes R_2 : L \otimes A \to \mathfrak{gl}(V \otimes A)$ is a representation.

Now take $L = \mathfrak{sl}(2)$, with standard basis E, H, F, and $A = F[t]/(t^3)$. Let R_1 be the irreducible representation of highest weight 1 and let R_2 be the regular representation. If we restrict the representation $R_1 \otimes R_2$ to the subalgebra of $\mathfrak{sl}(2) \otimes F[t]/(t^3)$ generated by $\{E \otimes 1, F \otimes t\}$ (which is isomorphic to $\mathfrak{g}$) we obtain the case $n = 3$ and $a = b = c = 2$ of the above construction.

5. Faithfulness in purely matrix terms

The following general version of Theorem 4.6 is stated in purely matrix terms. Given integers $a, b \geq 1$, let $\Phi_{a,b} : M_{a\times b} \to M_{a\times b}$ be the nilpotent linear operator defined by

$$\Phi_{a,b}(X) = J^a(0)X - XJ^b(0).$$

We will write Φ instead of $\Phi_{a,b}$ when no confusion is possible.

Theorem 5.1. *Given a triple (a, b, c) of positive integers and a pair (P, Q) of matrices such that $P \in M_{a\times b}$, $Q \in M_{b\times c}$, we define the matrices P_i, Q_i, $T_{i,j}$ by*

$$P_i = \Phi^i(P), \; Q_i = \Phi^i(Q), \quad i \geq 0,$$

$$T_{i,j} = P_iQ_j - P_jQ_i, \quad 0 \leq i < j,$$

and set

$$n = \max\{a + b - 1, b + c - 1\}.$$

Then $P_i = Q_i = 0$ for $i \geq n$ and the set $\mathcal{T} = \{T_{i,j} : 0 \leq i < j \leq n-1\}$ is linearly independent if and only if exactly one of the following three conditions hold:

$$P_{a,1} \neq 0, Q_{b,1} \neq 0 \text{ and } (a,b,c) \in \{(n,1,n),(n-1,2,n-1),(n,1,n-1),(n-1,1,n)\},$$

$$P_{a,1} = 0, P_{a-1,1} \neq 0, Q_{b,1} \neq 0 \text{ and } (a,b,c) = (n,1,n),$$

$$P_{a,1} \neq 0, Q_{b,1} = 0, Q_{b,2} \neq 0 \text{ and } (a,b,c) = (n,1,n).$$

Proof. The case $n = 1$ is obvious, so we assume $n > 1$.

It follows from [8, Proposition 2.2] that $P_i = Q_i = 0$ for $i \geq n$. If $P_{a,1} = 0$ and $Q_{b,1} = 0$ then [8, Proposition 2.1] implies $P_{n-1} = Q_{n-1} = 0$ and thus $\mathcal{T}$ is linearly dependent.

For the remainder of the proof we assume that $P_{a,1} \neq 0$ or $Q_{b,1} \neq 0$. Three cases arise.

Case 1: $P_{a,1} \neq 0$ and $Q_{b,1} \neq 0$. By Theorem 4.6, the set $\mathcal{T}$ is linearly independent if and only if $(a,b,c) \in \{(n,1,n),(n-1,2,n-1),(n,1,n-1),(n-1,1,n)\}$.

Case 2: $P_{a,1} = 0$ and $Q_{b,1} \neq 0$. Suppose first $\mathcal{T}$ linearly independent. The necessity part of the proof of Theorem 4.6 still implies that (a,b,c) belongs to $\{(n,1,n),(n-1,2,n-1),(n,1,n-1),(n-1,1,n)\}$. We will show that $P_{a-1,1} \neq 0$ and $(a,b,c) = (n,1,n)$.

The fact that $P_{a,1} = 0$ and [8, Proposition 2.1] imply that $P_{n-1} = 0$. If $b + c < n + 1$ then $Q_{n-1} = 0$, by [8, Proposition 2.2], so $T_{i,n-1} = 0$ for all $0 \leq i < n-1$, a contradiction. Thus $b + c = n + 1$. Since $P_{a,1} = 0$, every entry of P_{n-2}, except perhaps for its top right entry, is equal to 0. By construction, Q_{n-1} shares this property. Since

$$T_{n-2,n-1} = P_{n-2}Q_{n-1} - P_{n-1}Q_{n-2} = P_{n-2}Q_{n-1} \neq 0,$$

we infer $b = 1$ and thus $c = n$. Moreover, if $a < n$ then $b = 1$, $P_{a,1} = 0$ and [8, Proposition 2.1] imply $P_{n-2} = 0$, so $T_{n-2,n-1} = 0$, a contradiction. Therefore $a = n$. Finally, if $P_{n-1,1} = 0$ we obtain again $P_{n-2} = 0$. Thus $P_{n-1,1} \neq 0$.

Finally, suppose $(a,b,c) = (n,1,n)$ and $P_{a-1,1} \neq 0$. By deleting the last row of P and arguing as in Case 1 for $(a',b',c') = (n-1,1,n)$, we obtain that $\mathcal{T}$ is linearly independent.

Case 3: $P_{a,1} \neq 0$ and $Q_{b,1} = 0$. This is completely analogous to Case 2. □

6. Lemmata

Recall the meaning of Φ given in Section 5.

Lemma 6.1. *Let $Y \in M_{a,b}$. Then $\Phi(Y) = 0$ if and only if*

$$Y = \begin{pmatrix} 0 & \cdots & 0 & \nu_1 & \nu_2 & \cdots & \nu_a \\ 0 & \cdots & 0 & 0 & \nu_1 & \ddots & \vdots \\ \vdots & & \vdots & \vdots & \vdots & \ddots & \nu_2 \\ 0 & \cdots & 0 & 0 & 0 & \cdots & \nu_1 \end{pmatrix}, \text{ if } a \leq b, \tag{6.1}$$

$$Y = \begin{pmatrix} \mu_1 & \mu_2 & \cdots & \mu_b \\ 0 & \mu_1 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \mu_2 \\ 0 & 0 & \cdots & \mu_1 \\ 0 & \cdots & \cdots & 0 \\ \vdots & \cdots & \cdots & \vdots \\ 0 & \cdots & \cdots & 0 \end{pmatrix}, \text{ if } b \leq a \tag{6.2}$$

for some $\mu_i, \nu_i \in F$.

Proof. View $M_{a,b}$ as an $\mathfrak{sl}(2)$-module as in the proof of [8, Proposition 2.1]. The nullity of Φ is the number $m = \min\{a,b\}$ of irreducible $\mathfrak{sl}(2)$-submodules of $M_{a,b}$. On the other hand, if $m = a$ (resp. $m = b$) we readily verify that Y as in (6.1) (resp. (6.2)) satisfies $\Phi(Y) = 0$. □

We say that $X \in M_{a,b}$ is a lowest matrix if $X_{a,1} = 1$.

Lemma 6.2. *Let $X_1 \in M_{a,b_1}$, $X_2 \in M_{b_2,c}$, $Y_1 \in M_{b_1,c}$ and $Y_2 \in M_{a,b_2}$. Assume that X_1 and X_2 are lowest matrices, that*

$$(Y_1, Y_2) \neq (0,0), \ \ \Phi(Y_1) = 0, \Phi(Y_2) = 0,$$

and set

$$Z = X_1Y_1 - Y_2X_2.$$

If $Z = 0$ then $a \leq b_2$, $c \leq b_1$ and

$$Y_2 = \begin{pmatrix} 0 & \cdots & 0 & \nu_1 & \nu_2 & \cdots & \nu_a \\ 0 & \cdots & 0 & 0 & \nu_1 & \ddots & \vdots \\ \vdots & & \vdots & \vdots & \vdots & \ddots & \nu_2 \\ 0 & \cdots & 0 & 0 & 0 & \cdots & \nu_1 \end{pmatrix}, \quad Y_1 = \begin{pmatrix} \mu_1 & \mu_2 & \cdots & \mu_c \\ 0 & \mu_1 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \mu_2 \\ 0 & 0 & \cdots & \mu_1 \\ 0 & \cdots & \cdots & 0 \\ \vdots & \cdots & \cdots & \vdots \\ 0 & \cdots & \cdots & 0 \end{pmatrix}, \tag{6.3}$$

with $\mu_1 = \nu_1 \neq 0$.

Proof. If $Y_1 \neq 0$, let C_i, $1 \leq i \leq c$, be the first column of Y_1 that is non-zero. By Lemma 6.1, we have

$$C_i = \begin{pmatrix} \mu \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad \mu \neq 0.$$

Since X_1 is a lowest matrix, it follows that column i of X_1Y_1 is equal to

$$\begin{pmatrix} * \\ \vdots \\ * \\ \mu \end{pmatrix}, \quad \mu \neq 0.$$

If $Y_2 \neq 0$, let R_j, $1 \leq j \leq b_2$, be the last row of Y_2 that is non-zero. By Lemma 6.1, we have

$$R_j = (0, \ldots, 0, \nu), \quad \nu \neq 0.$$

Since X_2 is a lowest matrix, it follows that row j of Y_2X_2 is equal to

$$(\nu, *, \ldots, *), \quad \nu \neq 0.$$

Since $(Y_1, Y_2) \neq 0$ and $Z = 0$, we infer from above that $Y_1 \neq 0$ and $Y_2 \neq 0$. If either if $a > b_2$ or Y_2 does not have full rank, then Lemma 6.1 implies that the last row of Y_2 is 0, so by above $Z_{a,i} = \mu$, a contradiction. Similarly, if either $c > b_1$ or Y_1 does not have full rank, then Lemma 6.1 implies that the first column of Y_1 is 0, so by above $Z_{j,1} = -\nu$, a contradiction. Thus $a \leq b_2, c \leq b_1$ and, by Lemma 6.1, Y_1 and Y_2 are as described in (6.3) with $\mu_1 \neq 0$, $\nu_1 \neq 0$. Since $Z_{a,1} = 0$, we infer $\mu_1 = \nu_1$. □

Given integers $a, b \geq 1$ and $\alpha \in F$ we consider matrices $f(\alpha), g(\alpha), h(\alpha) \in M_{a,b}$ of respective forms

$$\begin{pmatrix} 0 & \dots & 0 & \alpha \\ 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & \dots & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & \dots & 0 & * \\ \vdots & \vdots & \vdots & \vdots \\ 0 & \dots & 0 & * \\ 0 & \dots & 0 & \alpha \end{pmatrix}, \begin{pmatrix} \alpha & * & \dots & * \\ 0 & \dots & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & \dots & \dots & 0 \end{pmatrix},$$

where the entries $*$ will play no role whatsoever.

Proposition 6.3. *Given $\alpha \in F$ and a sequence (d_1, d_2, d_3, d_4) of positive integers, let $\mathfrak{h}$ be the subalgebra of $\mathfrak{gl}(d)$, $d = d_1 + d_2 + d_3 + d_4$, generated by A and X, where*
– $A \in \mathfrak{gl}(d)$ is the 0-diagonal block matrix

$$A = J^{d_1}(\alpha) \oplus J^{d_2}(\alpha - \lambda) \oplus J^{d_3}(\alpha - 2\lambda) \oplus J^{d_4}(\alpha - 3\lambda),$$

– $X \in \mathfrak{gl}(d)$ is a 1-diagonal block matrix whose blocks (1,2), (2,3), (3,4) satisfy

$$X(1,2)_{d_1,1} = X(2,3)_{d_2,1} = X(3,4)_{d_3,1} = 1.$$

Then $Y(1,4) = 0$ for all $Y \in \mathfrak{h}$ if and only if $(d_1, d_2, d_3, d_4) = (1, 1, 1, 1)$.

Proof. Sufficiency. Suppose $(d_1, d_2, d_3, d_4) = (1, 1, 1, 1)$. Then $[A, X] = \lambda X$, so $Y(1,3) = Y(2,4) = Y(1,4) = 0$ for all $Y \in \mathfrak{h}$.

Necessity. Suppose $Y(1,4) = 0$ for all $Y \in \mathfrak{h}$. Given (i,j), $1 \leq i < j \leq 4$, we set

$$D_{i,j} = (-1)^{d_j - 1}\binom{d_i + d_j - 2}{d_i - 1}.$$

Let

$$m = \max\{d_1 + d_2, d_2 + d_3, d_3 + d_4\} \text{ and } Z = (\mathrm{ad}_{\mathfrak{gl}(d)} A - \lambda 1_{\mathfrak{gl}(d)})^{m-2}(X) \in \mathfrak{h}.$$

Then Z is a 1-diagonal block matrix, where

$$Z(1,2) = \delta_{m, d_1+d_2} f(D_{1,2}),\ Z(2,3) = \delta_{m, d_2+d_3} f(D_{2,3}),\ Z(3,4) = \delta_{m, d_3+d_4} f(D_{3,4}).$$

Set $U = [X, Z]$. Then U is a 2-diagonal block matrix, where

$$U(1,3) = \delta_{m, d_2+d_3} g(D_{2,3}) - \delta_{m, d_1+d_2} h(D_{1,2}),$$

$$U(2,4) = \delta_{m, d_3+d_4} g(D_{3,4}) - \delta_{m, d_2+d_3} h(D_{2,3}).$$

Note that $U = 0$ if and only if $(d_1, d_2, d_3, d_4) = (1, 1, 1, 1)$. Suppose, if possible, that $(d_1, d_2, d_3, d_4) \neq (1, 1, 1, 1)$. Choose k as large as possible such that $V = (\mathrm{ad}_{\mathfrak{gl}(d)} A - \lambda 1_{\mathfrak{gl}(d)})^k(U) \neq 0$. By hypothesis, $[X, V] = 0$, so Lemma 6.2 implies rank $V(1,3) = d_1 \leq d_3$ and rank $V(2,4) = d_4 \leq d_2$ (∗). Several cases arise:

Case 1. $d_1+d_2 = d_2+d_3 > d_3+d_4$. We have $d_1 = d_3$, $d_4 = \operatorname{rank} V(2,4) = 1$ and $d_1 = \operatorname{rank} V(1,3) \leq 2$. From $d_4 = 1$ we infer $V = U$. Whether $d_1 = 1$ or $d_1 = 2$, we readily see that the condition $\mu_1 = \nu_1$ from Lemma 6.2 is violated.

Case 1'. $d_3 + d_4 = d_2 + d_3 > d_1 + d_2$. This is dual to Case 1, and hence impossible.

Case 2. $d_1 + d_2 = d_3 + d_4 > d_2 + d_3$. Then $d_1 > d_3$ and $d_4 > d_2$, contradicting (∗).

Case 3. $d_1 + d_2 > d_2 + d_3, d_3 + d_4$. Then $d_1 > d_3$, contradicting (∗).

Case 3'. $d_3 + d_4 > d_2 + d_3, d_1 + d_2$. Then $d_4 > d_2$, contradicting (∗).

Case 4. $d_2 + d_3 > d_1 + d_2, d_3 + d_4$. In this case, $d_1 = \operatorname{rank} V(1,3) = 1$ and $d_4 = \operatorname{rank} V(2,4) = 1$, whence $V = U$. We readily see that the condition $\mu_1 = \nu_1$ from Lemma 6.2 is violated.

Case 5. $d_1 + d_2 = d_2 + d_3 = d_3 + d_4$. We have $d_3 = d_1 = \operatorname{rank} V(1,3) \leq 2$ as well as $d_2 = d_4 = \operatorname{rank} V(2,4) \leq 2$. If $(d_1, d_2) = (2,2)$ then $k = 1$ and thus $\operatorname{rank} V(1,3) = 1 = \operatorname{rank} V(2,4)$, contradicting (*). Whether $(d_1, d_2) = (2,1)$ or $(d_1, d_2) = (1,2)$, we have $V = U$, and we readily see that the condition $\mu_1 = \nu_1$ from Lemma 6.2 is violated. □

7. Classifying the relatively faithful uniserial representations of $\mathfrak{g}$

We assume throughout this section that $\mathfrak{g} = \langle x \rangle \ltimes L(V)$, where x acts on V via a single lower Jordan block $J_n(\lambda)$, $n > 1$, relative to a basis $v_0, \dots, v_{n-1}$ of V.

Proposition 7.1. *Suppose $\lambda \neq 0$ and let $T : \mathfrak{g} \to \mathfrak{gl}(U)$ be a relatively faithful uniserial representation of dimension d. Then there is a basis $\mathcal{B}$ of U, an integer $\ell > 2$, a sequence of positive integers $(d_1, \dots, d_\ell)$ satisfying $d_1 + \cdots + d_\ell = d$, and a scalar $\alpha \in F$, such that the matrix representation $R : \mathfrak{g} \to \mathfrak{gl}(d)$ associated to $\mathcal{B}$ is normalized standard relative to $(\ell, (d_1, \dots, d_\ell), \alpha)$.*

Proof. Noting that $[\mathfrak{g}, \mathfrak{g}] = V \oplus \Lambda^2(V)$ and $[[\mathfrak{g}, \mathfrak{g}], [\mathfrak{g}, \mathfrak{g}]] = \Lambda^2(V)$, the proof of [8, Theorem 3.2] applies almost verbatim to yield the desired result. □

Theorem 7.2. *Suppose $\lambda \neq 0$. Then $\mathfrak{g}$ has no relatively faithful uniserial representations of length > 3.*

Proof. Let $T : \mathfrak{g} \to \mathfrak{gl}(U)$ be a relatively faithful representation. By Proposition 7.1, there is a basis $\mathcal{B}$ of U, an integer $\ell > 2$, a sequence of positive integers $(d_1, \dots, d_\ell)$ satisfying $d_1 + \cdots + d_\ell = d$, and a scalar $\alpha \in F$ such that the matrix representation $R : \mathfrak{g} \to \mathfrak{gl}(d)$ associated to $\mathcal{B}$ is normalized standard relative to $(\ell, (d_1, \dots, d_\ell), \alpha)$.

Suppose, if possible, that $\ell > 3$. By Lemma 4.3, there is some i such that $d_i + d_{i+1} = n + 1$. Since $\ell > 3$, we may consider the representation of $\mathfrak{g}$, say S, obtained from R by choosing any set of four contiguous indices taken from $\{1, \dots, \ell\}$ including i and $i + 1$. Then $\ker(S) \cap V = (0)$ by Lemma 4.3. Moreover, $\Lambda^2(V)$ is not contained in $\ker(S)$ because S involves a non-zero 2-diagonal block matrix, as indicated in the proof of Proposition 6.3.

We may thus assume without loss of generality that $\ell = 4$ and $(d_1, d_2, d_3, d_4) \neq (1, 1, 1, 1)$. Since R is a representation and $\Lambda^2 V$ commutes with V, it follows from the shape of the matrices in $R(\mathfrak{g})$ that block $(1, 4)$ of $R(x)$ is zero for all $x \in \mathfrak{g}$, which contradicts Proposition 6.3. □

Theorem 7.3. *Suppose $\lambda \neq 0$. Then every relatively faithful uniserial representation of $\mathfrak{g}$ is isomorphic to one and only one normalized representation $R_{a,b,c,M,N,\alpha}$ of non-extreme type.*

Proof. Let $T : \mathfrak{g} \to \mathfrak{gl}(U)$ be a relatively faithful representation of dimension d. By Proposition 7.1, there is a basis $\mathcal{B}$ of U, an integer $\ell > 2$, a sequence of positive integers $(d_1, \dots, d_\ell)$ satisfying $d_1 + \cdots + d_\ell = d$, and a scalar $\alpha \in F$ such that the matrix representation $R : \mathfrak{g} \to \mathfrak{gl}(d)$ associated to $\mathcal{B}$ is normalized standard relative to $(\ell, (d_1, \ldots, d_\ell), \alpha)$.

Theorem 7.2 gives $\ell = 3$. Set $(a, b, c) = (d_1, d_2, d_3)$. We have $a + b \leq n + 1$ and $b + c \leq n + 1$, with equality holding in at least one case, by Lemma 4.3. Thus $a + b = n + 1$ and $c \leq a$, or $b + c = n + 1$ and $a \leq c$. It follows that R is isomorphic to $R_{a,b,c,M,N,\alpha}$, where M and N are the blocks in the first superdiagonal of $R(v_0)$, and $R_{a,b,c,M,N,\alpha}$ is of non-extreme type by Proposition 3.4. Uniqueness follows from Proposition 3.4. □

8. Further cases

We assume throughout this section that $\mathfrak{g} = \langle x \rangle \ltimes L(V)$, where $x \in \mathrm{GL}(V)$. When the Jordan decomposition of x acting on V has more than one block, other representations are possible. As an illustration, let $m, n \geq 1$, let $\lambda, \mu \in F$ (we allow the case $\lambda = \mu$), and suppose $v_0, \ldots, v_{n-1}, w_0, \ldots, w_{m-1}$ is a basis of V relative to which

$$[x, v_0] = \lambda v_0 + v_1, [x, v_1] = \lambda v_1 + v_2, \ldots, [x, v_{n-1}] = \lambda v_{n-1},$$

$$[x, w_0] = \mu w_0 + w_1, [x, w_1] = \mu w_1 + w_2, \ldots, [x, w_{m-1}] = \mu w_{m-1}.$$

Let (a, b, c) be a triple of positive integers satisfying

$$a + b = n + 1, \; b + c = m + 1,$$

suppose $M \in M_{a\times b}$ and $N \in M_{b\times c}$ satisfy $M_{a,1} \neq 0$ and $N_{b,1} \neq 0$, and let $\alpha \in F$. We may then define the uniserial representation $S = S_{a,b,c,M,N,\alpha} : \mathfrak{g} \to \mathfrak{gl}(d)$, $d = a + b + c$, as follows:

$$S(x) = A = \begin{pmatrix} J^a(\alpha) & 0 & 0 \\ 0 & J^b(\alpha - \lambda) & 0 \\ 0 & 0 & J^c(\alpha - \lambda - \mu) \end{pmatrix},$$

$$S(v_k) = (\mathrm{ad}_{\mathfrak{gl}(d)} A - \lambda 1_{\mathfrak{gl}(d)})^k \begin{pmatrix} 0 & M & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad 0 \leq k \leq n - 1,$$

$$S(w_k) = (\mathrm{ad}_{\mathfrak{gl}(d)} A - \lambda 1_{\mathfrak{gl}(d)})^k \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & N \\ 0 & 0 & 0 \end{pmatrix}, \quad 0 \leq k \leq m - 1.$$

The fact that $a + b = n + 1$ and $b + c = m + 1$, together with [8, Proposition 2.2], ensure that $\ker(S) \cap V = (0)$. Moreover, since $S(v_0 \wedge w_{b-1}) \neq 0$, it follows that $\Lambda^2(V)$ is not contained in $\ker(S)$. Thus, S is relatively faithful.

We may imbed $\mathfrak{g}$ as a subalgebra of $\mathfrak{g}' = \langle x' \rangle \ltimes L(V')$, where x' has Jordan decomposition

$$J_{n_1}(\lambda) \oplus \cdots \oplus J_{n_e}(\lambda) \oplus J_{m_1}(\mu) \oplus \cdots \oplus J_{m_f}(\mu),$$

where

$$n = n_1 \geq \cdots \geq n_e, \quad m = m_1 \geq \cdots \geq m_f,$$

$$n_2 \leq n - 2, \; n_3 \leq n - 4, \; n_4 \leq n - 6, \ldots, \; n_e \leq n - 2(e - 1),$$

$$m_2 \leq m - 2, \; m_3 \leq m - 4, \; m_4 \leq m - 6, \ldots, \; m_f \leq m - 2(f - 1),$$

$$e \leq \min\{a, b\}, \; f \leq \min\{b, c\}.$$

Then [8, Theorem 4.1] ensures that we may extend the above representation S of $\mathfrak{g}$ to a uniserial representation S' of $\mathfrak{g}'$ in such that a way that we still have $\ker(S') \cap V' = (0)$. Since $\Lambda^2(V)$ is not contained in $\ker(S)$, it follows automatically that $\Lambda^2(V')$ is not contained in $\ker(S')$. Thus, S' is also relatively faithful.

If $n > 1$ (resp. $m > 1$) then S (and therefore S') is not faithful, as all wedges $v_i \wedge v_j$ (resp. $w_i \wedge w_j$) are in the kernel of S.

The case $n = 1$ and $m = 1$ lead to the representation $S_\alpha : \mathfrak{g} \to \mathfrak{gl}(3)$, given by

$$x \mapsto \begin{pmatrix} \alpha & 0 & 0 \\ 0 & \alpha - \lambda & 0 \\ 0 & 0 & \alpha - \lambda - \mu \end{pmatrix},$$

$$v_0 \mapsto \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, w_0 \mapsto \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, v_0 \wedge w_0 \mapsto \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

This is a faithful uniserial representation.

Suppose next that x acts diagonalizably on V, as in the preceding example. Depending on the nature of the eigenvalues of x, there may be other examples of relatively faithful uniserial representations. Indeed, let $\mathfrak{g} = \langle x \rangle \ltimes L(V)$, where $n > 1$, $\lambda \in F$ and $v_1, \dots, v_n$ is a basis of V such that

$$xv_1 = i_1\lambda v_1, xv_2 = i_2\lambda v_2, \dots, xv_n = i_n\lambda v_n,$$

for positive integers $1 = i_1 < i_2 < \cdots < i_n$. Setting $p = i_n + 2$ and $J = J^p(0)$, we may then define the uniserial representation $T : \mathfrak{g} \to \mathfrak{gl}(p)$, as follows:

$$x \mapsto \operatorname{diag}(\alpha, \alpha - \lambda, \dots, \alpha - (p-1)\lambda),$$

$$v_1 \mapsto J^{i_1},\ v_2 \mapsto J^{i_2}, \dots, v_{n-1} \mapsto J^{i_{n-1}}, v_n \mapsto \beta E^{1,p-1} + \gamma E^{2,p}.$$

Here we require $\beta \neq \gamma$ to ensure that $\Lambda^2(V)$ is not contained in $\ker(T)$. Since $\ker(T) \cap V = (0)$, it follows that T is relatively faithful. Note that T is only faithful when $n = 2$.

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