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A polyhedral approach for the equitable coloring problem[☆]

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ABSTRACT

In this work we study the polytope associated with a 0,1-integer programming formulation for the Equitable Coloring Problem. We find several families of valid inequalities and derive sufficient conditions in order to be facet-defining inequalities. We also present computational evidence that shows the efficacy of these inequalities used in a cutting-plane algorithm.

1. Introduction

In graph theory, there is a large family of optimization problems having relevant practical importance, besides its theoretical interest. One of the most representative problem of this family is the *Graph Coloring Problem* (GCP), which arises in many applications such as scheduling, timetabling, electronic bandwidth allocation and sequencing problems.

Given a simple graph $G = (V, E)$, where V is the set of vertices and E is the set of edges, a coloring of G is an assignment of colors to each vertex such that the endpoints of any edge have different colors. A k-coloring is a coloring that uses k colors. Equivalently, a k-coloring can be defined as a partition of V into k subsets, called *color classes*, such that adjacent vertices belong to different classes. Given a k-coloring, color classes are denoted by $C_1, \ldots, C_k$ assuming that, for each $i \in \{1, \ldots, k\}$, vertices in C_i are colored with color i. We can also define a k-coloring of G as a mapping $c : V \rightarrow \{1, \ldots, k\}$ such that $c(u) \neq c(v)$ for all $(u, v) \in E$. The GCP consists of finding the minimum number of colors such that a coloring exists. This minimum number of colors is called the *chromatic number* of the graph G and is denoted by $\chi(G)$.

Some applications impose additional restrictions arising variations of GCP. For instance, in scheduling problems, it may be required to ensure the uniformity of the distribution of workload employees. The addition of these extra *equity* constraints gives rise to the *Equitable Coloring Problem* (ECP). An *equitable k-coloring* (or just k-eqcol) of G is a k-coloring satisfying the *equity constraints*, i.e. $\big||C_i| - |C_j|\big| \leq 1$, for $i, j \in \{1, \ldots, k\}$ or, equivalently, $\lfloor n/k \rfloor \leq |C_j| \leq \lceil n/k \rceil$ for each $j \in \{1, \ldots, k\}$. The *equitable chromatic number* of G, $\chi_{eq}(G)$, is the minimum k for which G admits a k-eqcol. The ECP consists of finding $\chi_{eq}(G)$.

The ECP was introduced in [17], motivated by an application concerning *garbage collection* [19]. Other applications of the ECP concern *load balancing problems* in multiprocessor machines [6] and results in *probability theory* [18]. An introduction to ECP and some basics results are provided in [11].

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Computing $\chi_{eq}(G)$ for arbitrary graphs is proved to be *NP*-hard and just a few families of graphs are known to be easy such as complete n-partite, complete split, wheel and tree graphs [11]. In particular, if G has a universal vertex u, the cardinality of the color classes of any equitable coloring in G is at most two and the color classes of exactly two vertices correspond to a matching in the complement of G. In other words, the ECP is polynomial when G has at least one universal vertex.

There exist some remarkable differences between GCP and ECP. Unlike GCP, a graph admitting a k-eqcol may not admit a $(k+1)$-eqcol. This leads us to define the *skip set* of G, $\mathscr{S}(G)$, as the set of $k \in \{\chi_{eq}(G), \ldots, n\}$ such that G does not admit any k-eqcol. For instance, if $G = K_{3,3}$, i.e. the *complete bipartite graph* with partitions of size 3, then G admits a 2-eqcol but does not admit a 3-eqcol. Here, $\mathscr{S}(K_{3,3}) = \{3\}$. Computing the skip set of a graph is as hard as computing the equitable chromatic number. If $\mathscr{S}(G) = \varnothing$, we say that G is *monotone*. For instance, trees are monotone graphs [12].

Another drawback emerging from ECP is that the equitable chromatic number of a graph can be smaller than the equitable chromatic number of one of its induced subgraphs. In particular, in an unconnected graph, equitable chromatic numbers of each connected component are uncorrelated with the chromatic number of the whole graph.

On the other hand, some useful properties of GCP also hold for ECP. For example, it is known that G admits k-eqcols for $k \geq \Delta(G) + 1$, where $\Delta(G)$ is the maximum degree of vertices in G. In [10] a polynomial time algorithm which produces a $(\Delta(G) + 1)$-eqcol is presented.

The *Integer linear programming* (ILP) approach together with algorithms which exploit the polyhedral structure has proved to be the best tool for dealing with coloring problems. Although many ILP formulations are known for GCP, as far as we know, just two of these models were adapted for ECP. One of them, given in [2], is based on the asymmetric representatives model for the GCP [4]. Such a model employs an ordering on the vertices to remove permutation symmetries. The first branch- and cut algorithm for the equitable coloring problem in the literature is based on this model [3]. The other one, proposed by us in [14], is based on the classic *color assignments to vertices* model [1] with further improvements stated in [16]. When compared to the model of [3], our model exhibit permutation symmetries that can be handled by considering techniques presented in [9]. In addition, contrary to the model of [3], our model does not require any linearization of non-linear inequalities.

The goal of this paper is to study the last model from a polyhedral point of view and determine families of valid inequalities which can be useful in the context of an efficient cutting-plane algorithm.

The remainder of the paper is organized as follows.

In Sections 2 and 3, we study the facial structure of the polytope associated with the formulation given in [14]. We introduce several families of valid inequalities which always define high dimensional faces. Section 4 is devoted to describe a cutting-plane algorithm for solving ECP. We expose computational evidence for reflecting the improvement in the performance when the cutting-plane algorithm uses the new inequalities as cuts. That algorithm is then used to reinforce bounds on a Branch and Bound enumeration tree. At the end, a conclusion is presented.

Some definitions and notations will be useful in the following.

Given a graph $G = (V, E)$ we consider $V = \{1, \ldots, n\}$. The complement of G is denoted by $\overline{G}$. We also denote by K_n the *complete graph* of n vertices. The percentage of density of G is $\frac{100|E|}{|V|(|V|-1)/2}$. For instance, the percentage of density of any complete graph is 100. Given $u \in V$, the *degree of* u is the number of vertices adjacent to u and is denoted by $\delta(u)$. For any $S \subset V$, $G[S]$ is the graph induced by S and $G - S$ is the graph obtained by the deletion of vertices in S, i.e. $G - S = G[V \setminus S]$. In particular, if $S = \{u\}$ we just write $G - u$ instead of $G - \{u\}$. A *stable set* is a set of vertices in G, no two of which are adjacent. We denote by $\alpha(G)$ the *stability number* of G, i.e. the maximum cardinality of a stable set of G. Given $S \subset V$, we also denote by $\alpha(S)$ the stability number of $G[S]$. We say that S is *k-maximal* if $\alpha(S) = k$ and for all $v \in V \setminus S$, $\alpha(S \cup \{v\}) = k + 1$. In particular, if S is 1-maximal, we say that S is a *maximal clique*. Given $u \in V$, the *neighborhood* of u, $N(u)$, is the set of vertices adjacent to u, and the *closed neighborhood* of u, $N[u]$, is the set $N(u) \cup \{u\}$. A vertex $u \in V$ is a *universal vertex* if $N[u] = V$. A *matching* of G is a subset of edges such that no pair of them has a common extreme point. Whenever it is clear from the context, we will write χ_{eq} rather than $\chi_{eq}(G)$. The same convention also applies for other operators that depend on G such as $\mathscr{S}$ and Δ.

Throughout the paper, we consider graphs with at least five vertices and one edge, and not containing universal vertices nor K_{n-1} as an induced subgraph. Thus, for a given graph G we assume that $2 \leq \chi_{eq}(G) \leq n - 2$. The remaining cases can be solved in polynomial time.

2. The polytope $\mathcal{ECP}$

A straightforward ILP model for GCP can be obtained by modeling colorings with two sets of binary variables: variables x_{vj} for $v \in V$ and $j \in \{1, \ldots, n\}$ where $x_{vj} = 1$ if and only if the coloring assigns color j to vertex v, and variables w_j for $j \in \{1, \ldots, n\}$ where $w_j = 1$ if and only if color j is used in the coloring. The formulation is shown below:

$$\sum_{j=1}^{n} x_{vj} = 1, \quad \forall\, v \in V \tag{1}$$

$$x_{uj} + x_{vj} \leq w_j, \quad \forall\, (u, v) \in E,\ 1 \leq j \leq n. \tag{2}$$

Constraints (1) assert that each vertex has to be colored by a unique color and constraints (2) ensure that two adjacent vertices cannot share the same color. Hence, the chromatic number can be computed by minimizing $\sum_{j=1}^{n} w_j$.

This formulation presents a disadvantage: the number of integer solutions (x, w) with the same value $\sum_{j=1}^{n} w_j$ is very large. A technique widely used in combinatorial optimization to deal with this kind of problem is the concept of *symmetry breaking* [13]. This technique is applied in [16], where the following constraints are added to the previous formulation in order to remove (partially) symmetric solutions:

$$w_{j+1} \le w_j, \quad \forall\, 1 \le j \le n-1 \tag{3}$$

which means that color $j+1$ may be used only if color j is also used.

Given a partition of V into color classes, let us observe that permutations of colors between those sets yield symmetric colorings. In [16], additional constraints are proposed in order to drop most of these colorings by sorting the color classes by the minimum label of the vertices belonging to each set and only considering the coloring that assigns color j to the jth color class. These constraints are

$$x_{vj} = 0, \quad \forall\, 1 \le v < j \le n \tag{4}$$

$$x_{vj} \le \sum_{u=j-1}^{v-1} x_{uj-1}, \quad \forall\, 2 \le j \le v \le n. \tag{5}$$

It is worth mentioning that a generalization and strengthening of these inequalities are introduced in [9] and its application in a coloring problem is discussed in [8]. Since the techniques employed do not depend on the special structure of the GCP under consideration, they can also be applied to the ECP.

On the other hand, even though the formulation consisting of constraints (1)–(5) eliminates a greater amount of symmetrical solutions, many polyhedral properties depend on the labeling of vertices [16].

From now on, we represent colorings of G as binary vectors (x, w) satisfying constraints (1)–(3) and we call *Coloring Polytope*, $\mathcal{CP}(G)$, to the convex hull of binary vectors (x, w) that represent colorings of G.

In order to characterize equitable colorings, we add the following constraints to the model:

$$x_{vj} \le w_j, \quad \forall\, v \text{ isolated},\ 1 \le j \le n \tag{6}$$

$$\sum_{v \in V} x_{vj} \ge \sum_{k=j}^{n} \left\lfloor \frac{n}{k} \right\rfloor (w_k - w_{k+1}), \quad \forall\, 1 \le j \le n-1 \tag{7}$$

$$\sum_{v \in V} x_{vj} \le \sum_{k=j}^{n} \left\lceil \frac{n}{k} \right\rceil (w_k - w_{k+1}), \quad \forall\, 1 \le j \le n-1 \tag{8}$$

where w_{n+1} is a dummy variable set to 0. Constraints (6) ensure that isolated vertices use enabled colors and (7)–(8) are precisely the equity constraints. The *Equitable Coloring Polytope* $\mathcal{ECP}(G)$ is defined as the convex hull of binary vectors (x, w) that represent equitable colorings of G, i.e. they satisfy constraints (1)–(3) and (6)–(8).

From now on, we present equitable colorings by using mappings, color classes or binary vectors, according to our convenience.

We also work with two useful operators over colorings. The first one is based on the fact that *swapping* colors in a k-eqcol produces a k-eqcol indeed.

Definition 1. Let c be a k-eqcol of G with color classes $C_1, \ldots, C_k$ and $L = (j_1, j_2, \ldots, j_r)$ be an ordered list of different colors in $\{1, \ldots, k\}$. We define $swap_L(c)$ as the k-eqcol with color classes $C'_1, \ldots, C'_k$ which satisfies $C'_{j_t} = C_{j_{t+1}}\ \forall\, 1 \le t \le r-1$, $C'_{j_r} = C_{j_1}$ and $C'_i = C_i\ \forall\, i \in \{1, 2, \ldots, k\} \backslash \{j_1, j_2, \ldots, j_r\}$.

The other operator takes a k-eqcol whose color classes have at most 2 vertices and returns a $(k+1)$-eqcol.

Definition 2. Let c be a k-eqcol of G with $\lceil n/2 \rceil \le k \le n-1$ and $v \ne v'$ such that $c(v) = c(v')$. We define $intro(c, v)$ as a $(k+1)$-eqcol c' which satisfies $c'(v) = k+1$ and $c'(i) = c(i)\ \forall\, i \in V \backslash \{v\}$.

Remark 3. Let us observe that colorings with $n-1$ and n colors are always equitable. Then, we can use Proposition 1 of [16] to prove that the following $n^2 - \chi_{eq} - |\mathscr{S}|$ equitable colorings are affinely independent.

1. A $(n-1)$-eqcol c such that C_{n-1} has two vertices, namely u_1 and u_2.
2. $swap_{n-1,j}(c)$ for each $j \in \{1, \ldots, n-2\}$.
3. The n-eqcol $c' = intro(c, u_1)$.
4. $swap_{n,j,j'}(c')$ for each $j, j' \in \{1, \ldots, n-1\}$ such that $j' \ne j$.
5. $swap_{n,j}(c')$ for each $j \in \{1, \ldots, n-1\}$.
6. An arbitrary k-eqcol of G for each $k \in \{\chi_{eq}, \ldots, n-2\} \backslash \mathscr{S}$.

Theorem 4. *The dimension of* $\mathcal{ECP}$ *is* $n^2 - (\chi_{eq} + |\mathscr{S}| + 1)$ *and a minimal equation system is defined by:*

$$\sum_{j=1}^{n} x_{vj} = 1, \quad \forall\, v \in V, \tag{9}$$

$$w_j = 1, \quad \forall\, 1 \le j \le \chi_{eq}, \tag{10}$$

$$w_j = w_{j+1}, \quad \forall\, j \in \mathscr{S}, \tag{11}$$

$$\sum_{v \in V} x_{vn} = w_n. \tag{12}$$

Proof. From Remark 3, $dim(\mathcal{ECP}) \ge n^2 - (\chi_{eq} + |\mathscr{S}| + 1)$. We only need to note that $\mathcal{ECP} \subset \mathbb{R}^{n^2+n}$ and that every equitable coloring satisfies $n + \chi_{eq} + \mathscr{S} + 1$ mutually independent equalities given in (9)–(12). □

Let us analyze the faces of $\mathcal{ECP}$ defined by restrictions of the formulation. For non-negativity constraints and inequalities (3) we adapt the proofs given in [16] for $\mathcal{CP}$.

Theorem 5. *Let* $v \in V$ *and* $1 \le j \le n$. *Constraint* $x_{vj} \ge 0$ *defines a facet of* $\mathcal{ECP}$.

Proof. We exhibit $n^2 - \chi_{eq} - |\mathscr{S}| - 1$ affinely independent colorings that lie on the face of $\mathcal{ECP}$ defined by inequality $x_{vj} \ge 0$. Let us consider the following cases:

Case $j \le n-2$. Let $u_1, u_2 \in V\backslash\{v\}$ be non adjacent vertices and let c be a $(n-1)$-eqcol such that $c(v) \ne j$ and $C_{n-1} = \{u_1, u_2\}$. We consider the set of colorings given by Remark 3 starting with c and choosing the arbitrary k-eqcols in item 6 satisfying that vertex v is not painted with color j. It is clear that all these colorings, except $swap_{n,j,c(v)}(c')$ where $c' = intro(c, u_1)$, lie in the face defined by the inequality.

Case $j = n-1$. Let S be the set of n-eqcols and $(n-1)$-eqcols presented in the previous case for $j = n-2$. We consider the colorings $swap_{n-1,n-2}(\tilde{c})$ for each $\tilde{c} \in S$ and an arbitrary k-eqcol of G for each $k \in \{\chi_{eq}, \dots, n-2\}\backslash\mathscr{S}$.

Case $j = n$. Let u_2 be a vertex not adjacent to v. We consider the set of colorings given by Remark 3 starting with a $(n-1)$-eqcol c such that $C_{n-1} = \{v, u_2\}$. It is clear that all these colorings, except $intro(c, v)$, lie in the face defined by the inequality. □

Let $1 \le j \le n-1$ and $\mathcal{F}$ be the face of $\mathcal{ECP}$ defined by constraint (3), i.e. $w_{j+1} \le w_j$. Let us notice that, if G does not admit a j-eqcol, i.e. $j \in \{1, \dots, \chi_{eq} - 1\} \cup \mathscr{S}$, then (3) is a linear combination of equations of the minimal system and, therefore, $\mathcal{F} = \mathcal{ECP}$. In addition, if $j = n-1$, the class of color $n-1$ of every coloring (x, w) satisfying $w_n = w_{n-1}$ have at most one vertex and, therefore, (x, w) verifies $\sum_{v \in V} x_{vn-1} = w_{n-1}$. Then, $\mathcal{F}$ is not a facet of $\mathcal{ECP}$. For the remaining cases, we have the following result.

Theorem 6. *If* G *admits a* j*-eqcol and* $j \le n-2$, *constraint* $w_{j+1} \le w_j$ *defines a facet of* $\mathcal{ECP}$.

Proof. Let us consider the set of colorings from Remark 3 but excluding the j-eqcol from item 6. Clearly, the remaining colorings lie on the face and (3) defines a facet of $\mathcal{ECP}$. □

The following theorems are related to the faces of $\mathcal{ECP}$ defined by the equity constraints.

Theorem 7. *Let* $1 \le j \le n-1$. *Constraint*

$$\sum_{v \in V} x_{vj} \ge \sum_{k=j}^{n} \left\lfloor \frac{n}{k} \right\rfloor (w_k - w_{k+1})$$

defines a facet of $\mathcal{ECP}$.

Proof. Let u_1, u_2 be non adjacent vertices and let c be a $(n-1)$-eqcol c such that $C_{n-1} = \{u_1, u_2\}$. We consider the set of colorings given by Remark 3 starting with c and choosing k-eqcols in item 6 satisfying $|C_j| = \lfloor n/k \rfloor$ when $k \ge j$. The proposed colorings, except the $(n-1)$-eqcol that satisfies $C_j = \{u_1, u_2\}$, lie on the face and therefore (7) defines a facet of $\mathcal{ECP}$. □

Let us observe that if $1 \le j \le n-2$, the face of $\mathcal{ECP}$ defined by (8) is not a facet. Indeed, every coloring (x, w) lying on the face satisfies $\sum_{v \in V} x_{vn-1} = w_{n-1}$. For the case $j = n-1$, the constraint (8) is $\sum_{v \in V} x_{vn-1} \le 2w_{n-1} - w_n$ and we have the following.

Theorem 8. *The inequality* $\sum_{v \in V} x_{vn-1} \le 2w_{n-1} - w_n$ *defines a facet of* $\mathcal{ECP}$.

Proof. Since $n \ge 5$ and $\chi_{eq}(G) \le n-2$ there exist $u_1, u_2, u_3, u_4, u_5 \in V$ such that u_1 is not adjacent to u_2 and u_3 is not adjacent to u_4. Let c be a $(n-1)$-eqcol c such that $c(u_1) = c(u_2) = n-1$. We consider the colorings from items 1, 3, 4, 5 in Remark 3 together with the following ones.

- The $(n-2)$-eqcol $\hat{c}$ such that $\hat{c}(u_1) = \hat{c}(u_2) = c(u_3)$, $\hat{c}(u_3) = c(u_4)$ and $\hat{c}(i) = c(i)\ \forall\, i \in V \backslash \{u_1, u_2, u_3\}$.
- $swap_{j,c(u_3)}(\hat{c})$ for each $j \in \{1, \ldots, n-2\} \backslash \{c(u_3), c(u_4)\}$.
- $swap_{\hat{c}(u_5),c(u_4)}(\hat{c})$.
- An arbitrary k-eqcol of G for each $k \in \{\chi_{eq}, \ldots, n-3\} \backslash \mathscr{S}$.

The proof for the affine independence of the previous $n^2 - \chi_{eq} - |\mathscr{S}| - 1$ colorings is similar to the one for the colorings generated in Remark 3. □

2.1. Valid inequalities from $\mathcal{CP}$

Taking into account that valid inequalities for $\mathcal{CP}$ are also valid for $\mathcal{ECP}$, in this section we analyze the faces of $\mathcal{ECP}$ defined by facet-defining inequalities of $\mathcal{CP}$.

One of the families of valid inequalities presented in [16] is the following. Given a vertex v and a color j, the (v, j)-*block inequality* is $\sum_{k=j}^{n} x_{vk} \leq w_j$.

Let us observe that the $(v, 1)$-block inequality is always satisfied by equality since every coloring (x, w) verifies constraints (1) and $w_1 = 1$. Moreover, the $(v, 2)$-block inequality defines the same facet as inequality $x_{v1} \geq 0$. For the remaining cases we have the following.

Theorem 9. *Let $v \in V$ and $3 \leq j \leq n-2$. The (v, j)-block inequality defines a facet of $\mathcal{ECP}$ if and only if G admits a $(j-1)$-eqcol.*

Proof. Let $\mathcal{F}$ be the face of $\mathcal{ECP}$ defined by the (v, j)-block inequality. To prove that $\mathcal{F}$ is a facet of $\mathcal{ECP}$ when G admits a $(j-1)$-eqcol, we can use the same affinely independent colorings proposed in the proof of Proposition 10 of [16], by imposing them to be equitable colorings.

Now, let us suppose that G does not admit a $(j-1)$-eqcol. We will prove that every equitable coloring lying on the face satisfies $x_{vj-1} = 0$. Let (x, w) be a k-eqcol lying on $\mathcal{F}$. If $k \leq j-2$, clearly $x_{vj-1} = 0$. Otherwise, $\sum_{k=j}^{n} x_{vk} = 1$ since $k \neq j-1$, and then $x_{vj-1} = 0$. □

Let us consider other family of inequalities studied in [16]. Given $S \subset V$ and a color j, $\sum_{v \in S} x_{vj} \leq \alpha(S) w_j$ is valid for $\mathcal{CP}$. The authors of [16] proved that, by applying a lifting procedure on this inequality for $j \leq n - \alpha(S)$, we can get

$$\sum_{v \in S} x_{vj} + \sum_{v \in V} \sum_{k=n-\alpha(S)+1}^{n-1} x_{vk} \leq \alpha(S) w_j + w_{n-\alpha(S)+1} - w_n.$$

We will refer to it as the (S, j)-*rank inequality*.

Let us remark that, if S is not $\alpha(S)$-maximal, i.e. if there exists $v \in V \backslash S$ such that $\alpha(S \cup \{v\}) = \alpha(S)$, the (S, j)-rank inequality is dominated by the $(S \cup \{v\})$-rank inequality. Then, from now on, we only consider (S, j)-rank inequalities where S is $\alpha(S)$-maximal.

When $\alpha(S) = 1$, the (S, j)-rank inequality takes the form $\sum_{v \in S} x_{vj} \leq w_j$ and is called (S, j)-*clique inequality*. If $|S| = 1$, i.e. $S = \{v\}$ for some v, the (S, j)-clique inequality is dominated by the (v, j)-block inequality. If $|S| \geq 2$, Propositions 5 and 6 of [16] state that the (S, j)-clique inequality defines a facet of $\mathcal{CP}$. The proof of these propositions can be easily adapted to the equitable case allowing us to prove the following result.

Theorem 10. *Let Q be a maximal clique of G with $|Q| \geq 2$ and $j \leq n-1$. The (Q, j)-clique inequality defines a facet of $\mathcal{ECP}$.*

In Theorem 33 of [15] we give sufficient conditions for the (S, j)-rank inequalities to define facets of $\mathcal{ECP}$ when $\alpha(S) = 2$.

Other valid inequalities can arise when $\alpha(S) = 2$. Let Q be the set of vertices of S that are universal in $G[S]$, i.e. $Q = \{q \in S : S \subset N[q]\}$. If Q is not empty, we may apply a different lifting procedure that one used in [16], obtaining new valid inequalities for $\mathcal{CP}$ and $\mathcal{ECP}$.

Definition 11. The (S, Q, j)-2-*rank inequality* is defined for a given $S \subset V$ such that S is 2-maximal, $Q = \{q \in S : S \subset N[q]\} \neq \varnothing$ and $j \leq n-1$, as

$$\sum_{v \in S \backslash Q} x_{vj} + 2 \sum_{v \in Q} x_{vj} \leq 2 w_j. \tag{13}$$

Lemma 12. *The (S, Q, j)-2-rank inequality is valid for $\mathcal{ECP}$.*

Proof. If some vertex of Q uses color j, no one else in S can be painted with j. Therefore, the value of the l.h.s. in (13) is at most 2 when color j is used. □

If $|Q| = 1$, the (S, Q, j)-2-rank inequality is dominated by another valid inequality presented in the next section (see Remark 17).

In Theorem 34 and Corollary 35 of [15], we give sufficient conditions for the (S, Q, j)-2-rank inequalities to define facets of $\mathcal{ECP}$ when $|Q| \geq 2$.

3. New valid inequalities for $\mathcal{ECP}$

In this section, we present new families of valid inequalities for $\mathcal{ECP}$ which are not valid for $\mathcal{CP}$.

3.1. Subneighborhood inequalities

The *neighborhood inequalities* defined in [16] for each $u \in V$ and number j, i.e. $\alpha(N(u))x_{uj} + \sum_{v\in N(u)} x_{vj} \le \alpha(N(u))w_j$, are valid inequalities for $\mathcal{CP}$. Indeed, if $S \subset N(u)$, $\alpha(S)x_{uj} + \sum_{v\in S} x_{vj} \le \alpha(S)w_j$ is valid for $\mathcal{CP}$. We can reinforce the latter inequality in the context of $\mathcal{ECP}$ to obtain the following.

Definition 13. The (u, j, S)-*subneighborhood inequality* is defined for a given $u \in V$, $S \subset N(u)$ such that S is not a clique and $j \le n - 1$, as

$$\gamma_{jS}x_{uj} + \sum_{v\in S} x_{vj} + \sum_{k=j+1}^{n} (\gamma_{jS} - \gamma_{kS})x_{uk} \le \gamma_{jS}w_j, \tag{14}$$

where $\gamma_{kS} = \min\{\lceil n/\chi_{eq}\rceil, \lceil n/k\rceil, \alpha(S)\}$.

Lemma 14. *The (u, j, S)-subneighborhood inequality is valid for $\mathcal{ECP}$.*

Proof. Let (x, w) be an r-eqcol of G. If $r < j$, both sides of (14) are equal to zero. If $r \ge j$ and $x_{uj} = 1$, the value of the l.h.s. of (14) is exactly γ_{jS}. On the other hand, if $x_{uj} = 0$, the term $\sum_{v\in S} x_{vj}$ contributes up to γ_{rS} and the term $\sum_{k=j+1}^{n}(\gamma_{jS} - \gamma_{kS})x_{uk}$ contributes up to $\gamma_{jS} - \gamma_{rS}$ regardless of the color assigned to u. Hence, the l.h.s. does not exceed γ_{jS} and (14) is valid. □

Subneighborhood inequalities always define faces of high dimension.

Theorem 15. *Let $\mathcal{F}$ be the face defined by the (u, j, S)-subneighborhood inequality. Then,*

$$dim(\mathcal{F}) \ge dim(\mathcal{ECP}) - \big(\lceil n/2\rceil - 1 - |S| + \delta(u)\big) = o(dim(\mathcal{ECP})).$$

Proof. Let $s_1, s_2 \in S$ be non adjacent vertices and let $1 \le r \le \lceil n/2\rceil - 1$ such that $r \ne j$. We propose at least $n^2 - \lceil n/2\rceil - \chi_{eq} - |\mathscr{S}| + |S| - \delta(u) + 1$ affinely independent colorings lying on $\mathcal{F}$.

- A n-eqcol c such that $c(u) = j$, $c(s_1) = n$ and $c(s_2) = r$.
- $swap_{n,j_1,j_2}(c)$ for each $j_1, j_2 \in \{1, \dots, n-1\}\backslash\{j\}$ such that $j_1 \ne j_2$.
- $swap_{c(s),n,j}(c)$ for each $s \in S\backslash\{s_1\}$.
- $swap_{n,j'}(c)$ for each $j' \in \{1, \dots, n-1\}$.
- The $(n-1)$-eqcol c' such that $c'(s_1) = r$ and $c'(i) = c(i)\ \forall\, i \in V\backslash\{s_1\}$.
- $swap_{j',r}(c')$ for each $j' \in \{1, \dots, n-1\}\backslash\{j, r\}$.
- $swap_{j,r,j'}(c')$ for each $j' \in \{1, \dots, n-1\}\backslash\{j, r\}$ and, if $j \le \lceil n/2\rceil - 1$ then $j' \ge \lceil n/2\rceil$.
- The $(n-1)$-eqcol c'' such that $c''(s_1) = c(v)$, $c''(v) = j$ and $c''(i) = c(i)\ \forall\, i \in V\backslash\{s_1, v\}$, for each $v \in V\backslash N[u]$.
- If $j \ge \chi_{eq} + 1$, an arbitrary k-eqcol of G for each $k \in \{\chi_{eq}, \dots, j-1\}\backslash\mathscr{S}$.
- $swap_{j,\hat{c}(u)}(\hat{c})$ where $\hat{c}$ is a k-eqcol of G, for each $k \in \big\{\max\{j, \chi_{eq}\}, \dots, n-2\big\}\backslash\mathscr{S}$.

The proof for the affine independence of the previous colorings is similar to the one for the colorings generated in Remark 3. □

Sufficient conditions for a (u, j, S)-subneighborhood inequality to be a facet-defining inequality of $\mathcal{ECP}$ are presented in Theorem 36 of [15] for the case $\lceil n/j\rceil \le \lceil n/\chi_{eq}\rceil$ whereas the following result allows us to study the inequality for the case $\lceil n/j\rceil > \lceil n/\chi_{eq}\rceil$.

Theorem 16. *Let j such that $\lceil n/j\rceil > \lceil n/\chi_{eq}\rceil$, $\mathcal{F}_j$ be the face defined by the (u, j, S)-subneighborhood inequality and $\mathcal{F}_{\chi_{eq}}$ be the face defined by the (u, χ_{eq}, S)-subneighborhood inequality. Then, $dim(\mathcal{F}_j) = dim(\mathcal{F}_{\chi_{eq}})$.*

Proof. Clearly, if $\alpha(S) < \lceil n/\chi_{eq}\rceil$, both inequalities coincide. So, let us assume that $\alpha(S) \ge \lceil n/\chi_{eq}\rceil$. Since $\lceil n/j\rceil > \lceil n/\chi_{eq}\rceil$, $j < \chi_{eq}$ and $w_j = w_{\chi_{eq}} = 1$. Then, both inequalities only differ in the coefficients of x_{vj} and $x_{v\chi_{eq}}$ for all $v \in V$. Moreover, the coefficient of x_{vj} in the (u, j, S)-subneighborhood is the same as the one of $x_{v\chi_{eq}}$ in the (u, χ_{eq}, S)-subneighborhood, and conversely.

Let $d = dim(\mathcal{F}_{\chi_{eq}})$ and $d' = dim(\mathcal{F}_j)$. If $c^1, c^2, \dots, c^{d+1}$ are affinely independent equitable colorings in $\mathcal{F}_{\chi_{eq}}$, colorings $swap_{j,\chi_{eq}}(c^i)$ for $1 \le i \le d+1$ are well defined and they are affinely independent too. Moreover, they lie on $\mathcal{F}_j$. Therefore, $d \le d'$.

To prove that $d' \le d$, we follow the same reasoning. □

Remark 17. Let $j \leq n-1$, $S \subset V$ such that $\alpha(S) = 2$ and $Q = \{v \in S : S \subset N[v]\} = \{q\}$. The $(q, j, S\backslash\{q\})$-subneighborhood inequality is

$$\sum_{v \in S\backslash\{q\}} x_{vj} + 2x_{qj} + x_{qn} \leq 2w_j,$$

and dominates the (S, Q, j)-2-rank inequality. In Corollary 37 of [15] we give sufficient conditions for it to be a facet-defining inequality of $\mathcal{ECP}$.

3.2. Outside-neighborhood inequalities

Definition 18. The (u, j)-*outside-neighborhood inequality* is defined for a given $u \in V$ such that $N(u)$ is not a clique and $j \leq \lfloor n/2 \rfloor$, as

$$\left(\left\lfloor \frac{n}{t_j} \right\rfloor - 1\right) x_{uj} - \sum_{v \in V\backslash N[u]} x_{vj} + \sum_{k=t_j+1}^{n} b_{jk} x_{uk} \leq \sum_{k=t_j+1}^{n} b_{jk}(w_k - w_{k+1}), \tag{15}$$

where $t_j = \max\{j, \chi_{eq}\}$ and $b_{jk} = \lfloor n/t_j \rfloor - \lfloor n/k \rfloor$.

Lemma 19. *The (u, j)-outside-neighborhood inequality is valid for $\mathcal{ECP}$.*

Proof. Let (x, w) be an r-eqcol of G. If $r < j$, both sides of (15) are equal to zero. Let us assume that $r \geq j$ and C_j denotes the color class j of (x, w). We divide the proof into two cases.
Case $r = t_j$. The terms $\sum_{k=t_j+1}^{n} b_{jk} x_{uk}$ and $\sum_{k=t_j+1}^{n} b_{jk}(w_k - w_{k+1})$ vanish from the inequality so we only need to check that $(\lfloor n/t_j \rfloor - 1)x_{uj} - \sum_{v \in V\backslash N[u]} x_{vj}$ is a non positive value. If $x_{uj} = 0$, the inequality holds. If $x_{uj} = 1$,

$$\sum_{v \in V\backslash N[u]} x_{vj} = |C_j \backslash N[u]| \geq \lfloor n/t_j \rfloor - 1$$

and (15) holds.
Case $r > t_j$. We need to check that the l.h.s. of (15) is at most b_{jr}. If $x_{uj} = 0$, then $\sum_{k=t_j+1}^{n} b_{jk} x_{uk} \leq \max\{b_{jk} : t_j + 1 \leq k \leq r\} = b_{jr}$ and the inequality holds. If $x_{uj} = 1$, $\sum_{k=t_j+1}^{n} b_{jk} x_{uk} = 0$ and

$$\sum_{v \in V\backslash N[u]} x_{vj} = |C_j \backslash N[u]| \geq \lfloor n/r \rfloor - 1$$

and (15) holds. $\square$

In order to study the faces of $\mathcal{ECP}$ defined by outside-neighborhood inequalities, let us characterize the equitable colorings that belong to those faces.

Remark 20. Let $\mathcal{F}$ be the face of $\mathcal{ECP}$ defined by the (u, j)-outside-neighborhood inequality and c be an r-eqcol. Let us observe that if $r < j$, c always lies on $\mathcal{F}$. For the case $r \geq j$, let C_j be the color class j of c. Then, c lies on $\mathcal{F}$ if and only if the following conditions hold.

- If $c(u) = j$ then $|C_j| = \lfloor n/r \rfloor$.
- If $c(u) \neq j$ then
  - $C_j \subset N(u)$ and
  - if $\left\lfloor \frac{n}{r} \right\rfloor < \left\lfloor \frac{n}{\max\{j, \chi_{eq}\}} \right\rfloor$ then $c(u) \geq \left\lfloor \frac{n}{\lfloor n/r \rfloor + 1} \right\rfloor + 1$.

Like the subneighborhood inequalities, outside-neighborhood inequalities define faces of high dimension.

Theorem 21. *Let $\mathcal{F}$ be the face defined by the (u, j)-outside-neighborhood inequality. Then,*

$$dim(\mathcal{F}) \geq dim(\mathcal{ECP}) - \big(3n - \lceil n/2 \rceil - |\mathcal{S}| - \chi_{eq} - 4 - \delta(u)\big) = o(dim(\mathcal{ECP})).$$

Proof. Let $v_1 \in V\backslash N[u]$, $v_2, v_3 \in N(u)$ such that v_2 is not adjacent to v_3 and $1 \leq r \leq \lfloor n/2 \rfloor$ such that $r \neq j$. We propose $n^2 + \lceil n/2 \rceil - 3n + 4 + \delta(u)$ affinely independent solutions lying on $\mathcal{F}$.

- A n-eqcol c such that $c(u) = j$, $c(v_1) = n$, $c(v_2) = n - 1$ and $c(v_3) = r$.
- $swap_{n,j_1,j_2}(c)$ for each $j_1, j_2 \in \{1, \ldots, n-1\}\backslash\{j\}$ such that $j_1 \neq j_2$.
- $swap_{n,j,c(v)}(c)$ for each $v \in N(u)$.

- $swap_{j,r,j'}(c)$ for each $j' \in \{\lfloor n/2 \rfloor + 1, \ldots, n-1\}$.
- $swap_{n,j'}(c)$ for each $j' \in \{1, \ldots, n-1\} \setminus \{j\}$.
- The $(n-1)$-eqcol c' such that $c'(v_1) = r$, $c'(v_3) = n-1$ and $c'(i) = c(i)\ \forall\, i \in V \setminus \{v_1, v_3\}$.
- $swap_{j',n-1}(c')$ for each $j' \in \{1, \ldots, n-2\}$.
- A $(n-2)$-eqcol c'' such that $c''(v_1) = c''(u) = n-2$ and $c''(v_2) = c''(v_3) = j$.

The proof for the affine independence of the previous colorings is similar to the one for the colorings generated in Remark 3. □

The following necessary condition for an outside-neighborhood inequality to define a facet of $\mathcal{ECP}$ will be helpful in the design of the separation routine.

Theorem 22. *If the (u, j)-outside-neighborhood inequality defines a facet of $\mathcal{ECP}$ then $\alpha(N(u)) \geq \left\lfloor \frac{n}{\max\{j, \chi_{eq}\}} \right\rfloor$.*

Proof. Let $t_j = \max\{j, \chi_{eq}\}$ and $\mathcal{F}$ be the face of $\mathcal{ECP}$ defined by the (u, j)-outside-neighborhood inequality. Let us suppose that $\alpha(N(u)) < \lfloor n/t_j \rfloor$. We will prove that every equitable coloring lying on $\mathcal{F}$ also satisfies the equality

$$\sum_{l=1}^{j-1} x_{ul} + w_j = 1. \tag{16}$$

Since this equality cannot be obtained as a linear combination of the minimal equation system for $\mathcal{ECP}$ and the (u, j)-outside-neighborhood *equality*, $\mathcal{F}$ is not a facet of $\mathcal{ECP}$.

Let c be an r-eqcol that lies on $\mathcal{F}$. Clearly, if $r < j$, $w_j = 0$ and $c(u) = l$ for some $1 \leq l \leq j-1$ and, consequently, the equality (16) holds. If $r \geq j$ then $w_j = 1$ and to see that (16) holds we only have to prove that $x_{ul} = 0$ for all $l \leq j-1$ or, equivalently, $c(u) \geq j$. According to Remark 20, if $c(u) \neq j$ then $C_j \subset N(u)$ and thus $\alpha(N(u)) \geq |C_j|$. Observe that this fact implies that $\lfloor n/r \rfloor < \lfloor n/t_j \rfloor$. Indeed, if $\lfloor n/r \rfloor = \lfloor n/t_j \rfloor$, $|C_j| \geq \lfloor n/t_j \rfloor$ and it contradicts the assumption $\alpha(N(u)) < \lfloor n/t_j \rfloor$. Then, by Remark 20, $c(u) \geq \left\lfloor \frac{n}{\lfloor n/r \rfloor + 1} \right\rfloor + 1 > j$ and (16) holds. □

For the case $j \geq \chi_{eq}$, we present sufficient conditions for the (u, j)-outside-neighborhood inequality to define a facet of $\mathcal{ECP}$ in Theorem 38 of [15]. For the other case, we have the following result whose proof follows the same ideas than in Theorem 16.

Theorem 23. *Let $j < \chi_{eq}$, $\mathcal{F}_j$ be the face defined by the (u, j)-outside-neighborhood inequality and $\mathcal{F}_{\chi_{eq}}$ be the face defined by the (u, χ_{eq})-outside-neighborhood inequality. Then, $dim(\mathcal{F}_j) = dim(\mathcal{F}_{\chi_{eq}})$.*

3.3. Clique-neighborhood inequalities

Definition 24. The (u, j, k, Q)-*clique-neighborhood inequality* is defined for a given $u \in V$, a clique Q of G such that $Q \cap N[u] = \varnothing$ and numbers j, k verifying $3 \leq k \leq \alpha(N(u)) + 1$ and $1 \leq j \leq \left\lceil \frac{n}{k-1} \right\rceil - 1$, as

$$(k-1)x_{uj} + \sum_{l=\left\lceil \frac{n}{k-1} \right\rceil}^{n-2} \left(k - \left\lceil \frac{n}{l} \right\rceil\right) x_{ul} + (k-1)\big(x_{un-1} + x_{un}\big) + \sum_{v \in N(u) \cup Q} x_{vj}$$

$$+ \sum_{v \in V \setminus \{u\}} (x_{vn-1} + x_{vn}) \leq \sum_{l=j}^{n} b_{ul}(w_l - w_{l+1}), \tag{17}$$

where

$$b_{ul} = \begin{cases} \min\{\lceil n/l \rceil, \alpha(N(u)) + 1\}, & \text{if } j \leq l \leq \lceil n/k \rceil - 1 \\ k, & \text{if } \lceil n/k \rceil \leq l \leq n-2 \\ k+1, & \text{if } l \geq n-1. \end{cases}$$

Lemma 25. *The (u, j, k, Q)-clique-neighborhood inequality is valid for $\mathcal{ECP}$.*

Proof. Let (x, w) be an r-eqcol of G. If $r < j$, both sides of (17) are zero. Let us assume that $r \geq j$ and observe that the r.h.s. of (17) is b_{ur}. Let C_j, C_{n-1} and C_n be the color class j, $n-1$ and n of (x, w) respectively. We divide the proof into the following cases.

Case $r \le \lceil n/k \rceil - 1$. We have to prove that (x, w) verifies

$$(k-1)x_{uj} + \sum_{v \in N(u) \cup Q} x_{vj} \le b_{ur} = \min\left\{ \left\lceil \frac{n}{r} \right\rceil, \alpha(N(u)) + 1 \right\}.$$

If $x_{uj} = 1$, $\sum_{v \in N(u)} x_{vj} = 0$ and $\sum_{v \in Q} x_{vj} \le 1$. Since $b_{ur} \ge k$, the inequality holds. If instead $x_{uj} = 0$, $\sum_{v \in N(u) \cup Q} x_{vj} = |C_j \cap (N(u) \cup Q)| \le \min\{\lceil n/r \rceil, \alpha(N(u) \cup Q)\} \le \min\{\lceil n/r \rceil, \alpha(N(u)) + 1\}$.

Case $\lceil n/k \rceil \le r \le n-2$. We have to prove that (x, w) verifies

$$(k-1)x_{uj} + \sum_{l=\left\lceil \frac{n}{k-1} \right\rceil}^{n-2} \left(k - \left\lceil \frac{n}{l} \right\rceil\right) x_{ul} + \sum_{v \in N(u) \cup Q} x_{vj} \le k.$$

If $x_{uj} = 1$, $\sum_{l=\lceil \frac{n}{k-1} \rceil}^{n-2} (k - \lceil n/l \rceil) x_{ul} = 0$ and $\sum_{v \in N(u) \cup Q} x_{vj} \le 1$. Therefore, the inequality holds.

If instead $x_{uj} = 0$, $\sum_{l=\lceil \frac{n}{k-1} \rceil}^{n-2} (k - \lceil n/l \rceil) x_{ul} \le k - \lceil n/r \rceil$ and $\sum_{v \in N(u) \cup Q} x_{vj} \le |C_j| \le \lceil n/r \rceil$ and the inequality holds.

Case $r \ge n-1$. Let us first notice that $|C_j| + |C_{n-1}| + |C_n| \le 3$. We have to prove that (x, w) satisfies

$$L(x) + \sum_{v \in N(u) \cup Q} x_{vj} + \sum_{v \in V \setminus \{u\}} (x_{vn-1} + x_{vn}) \le k + 1$$

where

$$L(x) = (k-1)x_{uj} + \sum_{l=\left\lceil \frac{n}{k-1} \right\rceil}^{n-2} \left(k - \left\lceil \frac{n}{l} \right\rceil\right) x_{ul} + (k-1)\big(x_{un-1} + x_{un}\big).$$

Let us observe that $L(x) \le k-1$ and $L(x) = k-1$ if and only if $u \in C_j \cup C_{n-1} \cup C_n$. Then, if $L(x) = k-1$, since $u \in C_j \cup C_{n-1} \cup C_n$ we have $\sum_{v \in N(u) \cup Q} x_{vj} + \sum_{v \in V \setminus \{u\}} (x_{vn-1} + x_{vn}) \le |C_j| + |C_{n-1}| + |C_n| - 1 \le 2$, and the inequality holds.

If $L(x) \le k-2$, the inequality holds since $\sum_{v \in N(u) \cup Q} x_{vj} + \sum_{v \in V \setminus \{u\}} (x_{vn-1} + x_{vn}) \le |C_j| + |C_{n-1}| + |C_n| \le 3$. □

Let us remark that, if Q is not maximal in $G - N[u]$, the (u, j, k, Q)-clique-neighborhood inequality is dominated by a (u, j, k, Q')-clique-neighborhood, with Q' a clique such that $Q \subsetneq Q' \subset G - N[u]$.

In order to analyze the faces of $\mathcal{ECP}$ defined by clique-neighborhood inequalities, we first explore the colorings that belong to those faces.

Remark 26. Let $\mathcal{F}$ be the face of $\mathcal{ECP}$ defined by the (u, j, k, Q)-clique-neighborhood inequality and c be an r-eqcol. Let us observe that, if $r < j$, c always lies on $\mathcal{F}$. For the case $r \ge j$, let C_j, C_{n-1} and C_n be the color class j, $n-1$ and n of c respectively. Then, c lies on $\mathcal{F}$ if and only if the following conditions hold.

- If $r \le \lceil n/k \rceil - 1$ then:
  - If $c(u) = j$ then $|C_j \cap Q| = 1$ and $k = \alpha(N(u)) + 1$.
    Otherwise, $|C_j \cap (N(u) \cup Q)| = \min\{\lceil n/r \rceil, \alpha(N(u)) + 1\}$.
- If $\lceil n/k \rceil \le r \le n-2$ then:
  - If $c(u) = j$ then $|C_j \cap Q| = 1$. Otherwise,
    - $|C_j \cap (N(u) \cup Q)| = \lceil n/r \rceil$ and
    - if $r \ge \left\lceil \frac{n}{k-1} \right\rceil$ then $c(u) \ge \left\lceil \frac{n}{\lceil n/r \rceil} \right\rceil$.
- If $r \ge n-1$ then:
  - If $c(u) \in \{j, n-1, n\}$ then $|C_j \cap Q| + |C_{n-1} \setminus \{u\}| + |C_n \setminus \{u\}| = 2$.
    Otherwise, $c(u) \ge \lceil n/2 \rceil$ and $|C_j \cap (N(u) \cup Q)| + |C_{n-1}| + |C_n| = 3$.

Clique-neighborhood inequalities also define high dimensional faces in $\mathcal{ECP}$.

Theorem 27. *Let $\mathcal{F}$ be the face defined by the (u, j, k, Q)-clique-neighborhood inequality. Then,*

$$dim(\mathcal{F}) \ge dim(\mathcal{ECP}) - \big(3n - |\mathcal{S}| - \chi_{eq} - \lfloor n/2 \rfloor - \delta(u) - |Q| - 4\big) = o(dim(\mathcal{ECP})).$$

Proof. Let $v_1, v_2 \in N(u)$ be non adjacent vertices, and $q \in Q$. We propose $n^2 + \lfloor n/2 \rfloor + 4 - 3n + \delta(u) + |Q|$ affinely independent solutions lying on $\mathcal{F}$.

- A n-eqcol c such that $c(u) = j$ and $c(q) = n$.
- $swap_{n,j_1,j_2}(c)$ for each $j_1, j_2 \in \{1, \dots, n-1\} \setminus \{j\}$ such that $j_1 \ne j_2$.

- $swap_{n,j,c(v)}(c)$ for each $v \in (N(u) \cup Q)\backslash\{q\}$.
- $swap_{j',j,n}(c)$ for each $j' \in \{\lceil n/2 \rceil, \ldots, n-1\}$.
- $swap_{j',n}(c)$ for each $j' \in \{1, \ldots, n-1\}$.
- A $(n-1)$-eqcol c' such that $c'(u) = j$, $c'(v_1) = c'(v_2) = n-1$.
- $swap_{j,n-1}(c')$.
- A $(n-2)$-eqcol c'' such that $c''(u) = c''(q) = j$ and $c''(v_1) = c''(v_2) = n-2$.
- $swap_{j',n-2}(c'')$ for each $j' \in \{1, \ldots, n-3\}\backslash\{j\}$.

The proof for the affine independence of the previous colorings is similar to the one for the colorings generated in Remark 3. □

Sufficient conditions for the clique-neighborhood inequalities to define facets of $\mathcal{ECP}$ are presented in Theorem 39 and Corollary 40 of [15].

3.4. S-color inequalities

Given a set of colors S, let us analyze how many vertices can be painted with colors from S. Let (x, w) be a k-eqcol and d_{Sk} be the number of colors in S with non-empty color class in (x, w), i.e. $d_{Sk} = |S \cap \{1, \ldots, k\}|$. It is straightforward to see that (x, w) has $n - k\lfloor \frac{n}{k} \rfloor$ classes of size $\lfloor \frac{n}{k} \rfloor + 1$ and $k - (n - k\lfloor \frac{n}{k} \rfloor)$ classes of size $\lfloor \frac{n}{k} \rfloor$. Then, the number of classes of color in S having size $\lfloor \frac{n}{k} \rfloor + 1$ is at most $\min\{d_{Sk}, n - k\lfloor \frac{n}{k} \rfloor\}$. Denoting by $b_{Sk} = d_{Sk}\lfloor \frac{n}{k} \rfloor + \min\{d_{Sk}, n - k\lfloor \frac{n}{k} \rfloor\}$ we have that $\sum_{j \in S} |C_j| \leq b_{Sk}$, which motivates the following definition.

Definition 28. Let $S \subset \{1, \ldots, n\}$. The *S-color inequality* is defined as

$$\sum_{j \in S} \sum_{v \in V} x_{vj} \leq \sum_{k=1}^{n} b_{Sk}(w_k - w_{k+1}), \tag{18}$$

where $d_{Sk} = |S \cap \{1, \ldots, k\}|$ and $b_{Sk} = d_{Sk}\lfloor \frac{n}{k} \rfloor + \min\{d_{Sk}, n - k\lfloor \frac{n}{k} \rfloor\}$.

Lemma 29. *The S-color inequality is valid for $\mathcal{ECP}$.*

Proof. Let (x, w) be a k-eqcol. If $k < j$, both sides of (18) are zero. If instead $k \geq j$, the r.h.s. of (18) is b_{Sk} which is an upper bound of $\sum_{j \in S} |C_j| = \sum_{j \in S} \sum_{v \in V} x_{vj}$. □

Remark 30. Let us present some useful facts about S-color inequalities.

1. Given $S \subset \{1, \ldots, n-1\}$, the $(S \cup \{n\})$-color inequality can be obtained by adding the S-color inequality and Eq. (12) from the minimal system. Then, both inequalities define the same face of $\mathcal{ECP}$.
2. Constraints (7) and (8) are both S-color inequalities with $S = \{1, \ldots, n-1\}\backslash\{j\}$ and $S = \{j\}$ respectively.
3. It is not hard to see that the (S, j)-rank inequality with $\alpha(S) = 2$ and $j \geq \lceil n/2 \rceil$, and (17) with $k = 2$ are both dominated by the $\{j, n-1\}$-color inequality.
4. If for every k such that G admits a k-eqcol, we have that either k divides n or $n - k\lfloor \frac{n}{k} \rfloor \geq d_{Sk}$, then the S-color inequality is obtained by adding constraints (8), i.e. $\sum_{v \in V} x_{vj} \leq \sum_{k=j}^{n} \lceil n/k \rceil (w_k - w_{k+1})$, for $j \in S$. Thus, an S-color inequality can cut off a fractional solution of the linear relaxation of the formulation only if $2 \leq |S\backslash\{n\}| \leq n-3$ and there exists $k \in \{\chi_{eq}, \ldots, n-1\}\backslash\mathscr{S}$ such that $1 \leq n - k\lfloor \frac{n}{k} \rfloor \leq d_{Sk} - 1$.

The following result shows that S-color inequalities define faces of high dimension.

Theorem 31. *Let $S \subset \{1, \ldots, n\}$ such that $|S\backslash\{n\}| \geq 1$ and let $\mathcal{F}$ be the face defined by the S-color inequality. Then,*

$$dim(\mathcal{F}) \geq dim(\mathcal{ECP}) - (n - |S\backslash\{n\}| - 1) = o(dim(\mathcal{ECP})).$$

Proof. From Remark 30.1 we can assume w.l.o.g. that $S \subset \{1, \ldots, n-1\}$. Let u_1, u_2 be non adjacent vertices and c be a $(n-1)$-eqcol such that $c(u_1) = c(u_2) = n-1$. We consider colorings from Remark 3 starting from c and choosing those ones that lie in the face defined by (18). That is, by excluding the $(n-1)$-eqcols that assign colors from $\{1, \ldots, n-1\}\backslash S$ to u_1 and u_2 simultaneously, and by choosing k-eqcols where color classes from S should have as many vertices as possible, for each $k \in \{\chi_{eq}, \ldots, n-2\}\backslash\mathscr{S}$. Hence, we get $n^2 - \chi_{eq} - |\mathscr{S}| - n + 1 + |S|$ affinely independent colorings. □

Finally, sufficient conditions for the S-color inequalities to define facets of $\mathcal{ECP}$ are presented in Theorem 41 of [15].

4. Implementation and computational experience

We present computational results concerning the efficiency of valid inequalities studied in the previous sections when they are used as cuts in a cutting-plane algorithm for solving ECP.

The main elements of our implementation are described below.

4.1. Initialization

According to our computational experience reported in [14], the ILP formulation of ECP consisting of constraints (1)–(8) performs much better than the one defining $\mathcal{ECP}$, i.e. without (4)–(5). Since every valid inequality of $\mathcal{ECP}$ is also valid for equitable colorings satisfying constraints (1)–(8), we use this tighter formulation for computational experiments, with inequalities (5) handled as lazy constraints in the implementation. This means they are not part of the initial relaxation, but they are added later as cuts whenever necessary.

We tested several criteria for labeling vertices and the one which has proved to be the best in practice is the following. We first find a maximal clique Q. Denoting by q the size of Q, we assign the first q natural numbers to vertices of Q. The labels of remaining vertices are assigned in decreasing order of degree, i.e. satisfying $\delta(v) \geq \delta(v+1)$ for all $v \in \{q+1, \ldots, n\}$.

The maximal clique Q is generated through the repeated addition of a vertex into a partial clique. Initially, the vertices are in decreasing order of their degrees. A sequential greedy heuristic is applied where the vertex to be added is chosen among candidates vertices from the order list. We repeat the procedure n times. At the end of each iteration, the first vertex in the list, which was used to initialize the clique, is removed and added to the end. Finally, we choose the largest obtained clique and we break ties by taking that clique whose sum of degrees of its vertices is the greatest.

To find an initial upper bound $\overline{\chi_{eq}}$, we use the heuristic *Naive* presented in [11]. This allows us to eliminate variables x_{vj} and w_j with $j > \overline{\chi_{eq}}$ from the model.

In addition, a lower bound $\underline{\chi_{eq}}$ is obtained by considering the maximum between the size of the maximal clique Q and the value

$$\max\left\{\left\lceil \frac{n+1}{\theta(G - N[v]) + 2} \right\rceil : v \in V\right\},$$

also proposed in [11], where $\theta(G)$ is the cardinal of a clique partition of G found greedily.

We also compute bounds of the stability number of $N(u)$ for all $u \in V$ (via heuristic procedures), which will be useful for the separation routines. We denote the upper bound as $\overline{\alpha}(N(u))$ and the lower bound as $\underline{\alpha}(N(u))$.

4.2. Description of the cutting-plane algorithm

The design of the separation routines for each family of valid inequalities is described below. Given a fractional solution (x^*, w^*) of the linear relaxation, we look for violated inequalities as follows.

- *Clique and Block inequalities*. They are handled in the same way as in [16].
- *Clique-neighborhood inequalities*. For each maximal clique Q we found during the clique separation procedure and for each $u \in V \backslash \big(\cup_{q\in Q} N[q]\big)$, $j \in \{1, \ldots, \overline{\chi_{eq}}\}$ and k such that

  $$\max\{3, \lceil n/\overline{\chi_{eq}}\rceil\} \leq k \leq \min\big\{\lceil n/j\rceil, \lceil n/\underline{\chi_{eq}}\rceil, \overline{\alpha}(N(u)) + 1\big\},$$

  we verify whether (x^*, w^*) violates a weaker version of the (u, j, k, Q)-clique-neighborhood which consists of replacing $\alpha(N(u))$ by $\overline{\alpha}(N(u))$ to compute b_{ul} in Definition 24.
- 2-*rank inequalities*. For each $j \in \{1, \ldots, \overline{\chi_{eq}}\}$, we find a pair of vertices v_1 and v_2 such that $x^*_{v_1 j} + x^*_{v_2 j}$ has the highest value, but less than 1, and we initialize $S = \{v_1, v_2\}$ and $Q = \varnothing$. Then, we fill sets S and Q by adding vertices, one by one, with the following rule. Let v be a vertex with largest fractional value of x^*_{vj}, adjacent to every vertex of Q and such that $S \cup \{v\}$ is 2-maximal. If $S \subset N[v]$ we add v to the set Q. Otherwise, we add it to S. When it is not possible to add more vertices to S or Q, we check whether the (S, Q, j)-2-rank inequality cuts off (x^*, w^*).
  We also implement an additional mechanism that prevents from generating violated cuts with similar support. Each time a (S, Q, j)-2-rank inequality is found (not necessarily violated by the fractional solution), we mark every vertex of S as *forbidden*, to mean that those vertices cannot take part of upcoming (S, Q, j)-2-inequalities. The procedure is performed over and over, until not more than 5 vertices are not forbidden. Then, we unmark all the forbidden vertices and start over with the next value of j.
- *S-color inequalities*. We first find t such that $0 < w_t < 1$ and $w_{t+1} = 0$. If t does not exist (meaning that $w^* \in \mathbb{Z}^n$), we do not generate any cut. Otherwise, we order in decreasing way the color classes $j \in \{1, \ldots, t\}$ according to the number of fractional variables x^*, i.e. $|\{v : x^*_{vj} \notin \mathbb{Z}^n \ \forall\, v \in V\}|$. Then, for each $s \in \{2, \ldots, t-2\}$ such that

  $$s \geq 1 + \min\{n - k\lfloor n/k\rfloor : k \in \{1, \ldots, t\} \wedge k \text{ does not divide } n\}$$

  (see Remark 30.4), we scan S-color inequalities with $|S| = s$ and S having the most fractional classes, looking for the inequality that maximizes violation. Once the best S-color inequality is identified we check whether it cuts off (x^*, w^*). The procedure given before allows us to produce only one inequality. In order to generate more inequalities we do the following. Each time a S-color inequality is identified (regardless of the inequality is violated or not), we mark one color class belonging to S as *forbidden*, to mean that it cannot take part of upcoming S-color inequalities. Then, we repeat the procedure until only two color classes are not forbidden.

Table 1
Strategies.

Strategy name	Clique	2-rank	Block	S-color	Sub-neighbor.	Outside-neighbor.	Clique-neighbor.
S1	•						
S2	•	•					
S3	•	•	•				
S4	•	•	•	•			
S5	•	•	•	•	•		
S6	•	•	•	•	•	•	
S7	•	•	•	•	•	•	•

Table 2
Average of *Time* and *Cuts* for strategies S2–S7.

% density graph	Time						Cuts					
	S2	S3	S4	S5	S6	S7	S2	S3	S4	S5	S6	S7
30	77	**75**	**74**	82	82	98	**2034**	2053	2053	2093	2093	3203
50	**241**	248	248	267	267	252	**3694**	3796	3796	4065	4065	3944
70	648	**601**	632	700	738	735	6182	5805	**5670**	6306	6405	6377
90	720	763	**612**	658	**610**	**610**	5443	5493	**5065**	5187	5143	5143

- *Subneighborhood and Outside-neighborhood inequalities*. They are handled by enumeration: for each $j \in \{1, \dots, \overline{\chi_{eq}}\}$ and u such that $\underline{\alpha}(N(u)) \geq 3$ (because vertices u with $\alpha(N(u)) \leq 2$ lead to clique and 2-rank cuts), we check whether (x^*, w^*) violates a weaker version of these inequalities, defined as follows. For the subneighborhood inequalities, we compute $\xi_k = \min\{\lceil n/\underline{\chi_{eq}}\rceil, \lceil n/k\rceil, \overline{\alpha}(N(u))\}$, and then we consider inequalities of the form:

$$\xi_j x_{uj} + \sum_{v \in N(u)} x_{vj} + \sum_{k=j+1}^{n} (\xi_j - \xi_k) x_{uk} \leq \xi_j w_j.$$

For the outside-neighborhood inequalities, we first check the condition of Theorem 22, i.e. $\underline{\alpha}(N(u)) \geq \lfloor n/\max\{j, \underline{\chi_{eq}}\}\rfloor$ and then we use the inequality that results from replacing t_j with $\max\{j, \underline{\chi_{eq}}\}$ in (15).

4.3. Performance of cuts at root node

In order to evaluate the quality of a cutting-plane algorithm, we analyze the increase of the lower bound when cuts are added progressively to the LP-relaxation.

In this experiment, we compare the performance of seven strategies given in Table 1, where each one is a combination of separation routines that determine the behavior of the cutting-plane algorithm.

The experiment was carried out on a server equipped with an Intel i5 2.67 GHz over Linux Operating System. The server also has the well known general-purpose IP-solver CPLEX 12.2 which is used for solving linear relaxations. We consider 50 randomly generated graphs with 150 vertices and different densities of edges. For each graph and each strategy, we ran 30 iterations of the cutting-plane algorithm.

In order to compare the strategies involved, we call LB_i to the objective value of the linear relaxation after the ith iteration and we compute:

- Percentage of lower bound improvement: $Impr = 100\frac{\lceil LB_{30}\rceil - \lceil LB_0\rceil}{\lceil LB_0\rceil}$.
- Time elapsed up to reach the best lower bound, i.e. at iteration $\min\{i : \lceil LB_i\rceil = \lceil LB_{30}\rceil\}$. We denote it as *Time*.
- Number of cuts generated up to reach the best lower bound. We denote it as *Cuts*.

For graphs having 10% of density, all the strategies showed no improvement in the lower bound. For graphs having at least 30% of density, all the strategies except S1 reaches the same bound in every instance, while S1 attains worse bounds. In Fig. 1, we display the average of *Impr* over instances having the same density.

As we have mentioned, strategies S2–S7 reached the same bound in every instance. One way to tie them is by inspecting the average of *Time*, i.e. the time elapsed, and *Cuts*, i.e. the number of cuts generated. The smaller *Time* is, the sooner the algorithm reaches the best bound. On the other hand, the less *Cuts* is, the better the quality of the cuts involved are. Table 2 resumes these results. Best values are emphasized with boldface font.

As we can see from Table 2, strategy S4 reaches the best lower bound with fewer cuts for graphs having at least 70% of density and the amount of cuts generated is relatively acceptable for graphs having at most 50% of density. Strategy S4 also has the best balance between number of cuts generated and time consumed. Therefore, this strategy is a good candidate for our cutting-plane algorithm.

From the previous results we conclude that the cuts obtained from the polyhedral study are indeed effective. They appear to be strong in practice, increasing significantly the initial lower bound.

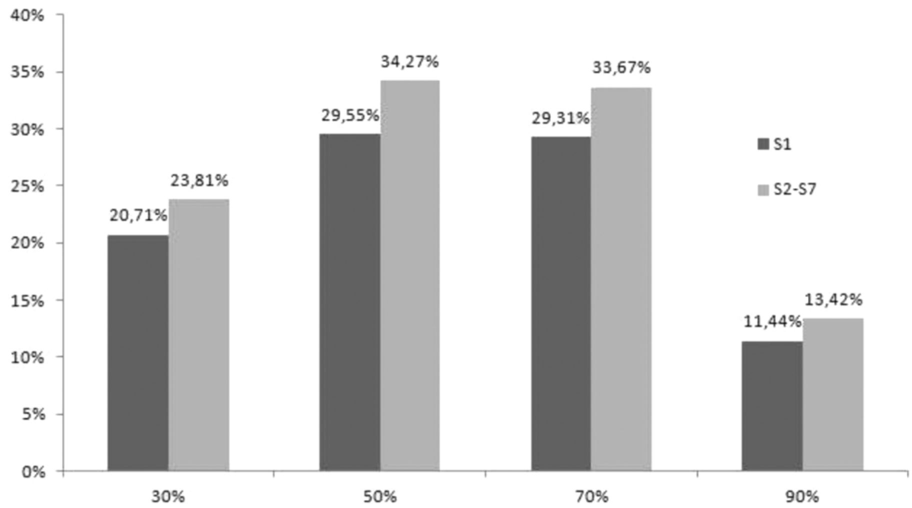

Fig. 1. Average of *Impr* for strategies S1 and S2–S7.

Table 3
Performance of different strategies.

Num. of vertices	% density graph	% solved				Nodes				Time			
		B&B	S1	S4	S7	B&B	S1	S4	S7	B&B	S1	S4	S7
90	10	100	100	100	100	2 933	3 050	**1 718**	**1 718**	33	33	**21**	**21**
60	30	100	100	100	100	7 515	2 976	**1 050**	6 567	129	52	**35**	130
60	50	100	100	100	100	29 490	20 639	21 232	**15 786**	974	1065	**812**	**812**
60	70	87.5	**100**	**100**	**100**	19 811	12 891	**5 330**	6 454	734	508	**327**	340
90	90	62.5	62.5	**100**	**100**	52 545	35 538	**12 645**	15 536	2332	2404	**689**	1088

Nevertheless, the long-term efficiency of cuts cannot be appreciated here and require further experimentation. This topic is covered in the next section.

4.4. Long-term efficiency of cuts

The purpose of the following experiments is to compare the Branch and Bound (B&B) algorithm of CPLEX with a Cut and Branch. The algorithm consists of applying 30 iterations of the cutting-plane algorithm to the initial relaxation. Then, we run a Branch and Bound enumeration until the optimal solution is found or a time limit of 2 h is reached.

Preliminary experiments showed that strategies S2–S6 have a similar behavior each other, although S4 presents the best performance among them. This led us to deepen the analysis of strategies S1, S4 and S7.

First, we apply both algorithms to 40 randomly generated graphs with different number of vertices and densities of edges. Since instances having 10% and 90% of density are easier to solve, we increased the number of vertices of them. Table 3 reports:

- Percentage of solved instances within 2 h of execution.
- Average of nodes evaluated over solved instances.
- Average of total CPU time in seconds over solved instances.

The new inequalities show again a substantial improvement and, in particular, strategy S4 is established as the best one. It is worth mentioning that strategy S7 evaluated fewer nodes than S4 when solving instances of 50% of density, but this reduction on the number of nodes was not enough to counteract the CPU time elapsed.

The second experiment consists of solving 35 benchmark instances from the DIMACS challenge [7] and 5 Kneser graphs [5].

Instances `anna`, `games120`, `homer`, `jean` and `kneser5_2` have been solved at the initial stage, i.e. the initial lower and upper bounds coincide.

Instances `1-FullIns_3`, `2-FullIns_3`, `3-FullIns_3`, `5-FullIns_3`, `david`, `kneser7_2`, `kneser7_3`, `kneser9_4`, `miles750`, `miles1000`, `miles1500`, `mug88_1`, `mug88_25`, `mug100_1`, `mug100_25`, `mulsol.i.1`, `myciel3`, `myciel4`, `queen6_6`, `queen7_7` and `zeroin.i.1` have been solved in at most 2 s of CPU time. The performance over the other instances considered is presented in Table 4. In this table, a bar "–" means that the instance has not been solved by the strategy in two hours of execution.

Let us observe that S7 outperforms B&B in 5 instances (`1-FullIns_4`, `fpsol2.i.3`, `le450_5a`, `myciel5` and `queen8_12`) while B&B outperforms S7 only for `queen8_8`. In addition, S4 is the only one able to solve `le450_5b`.

Table 4
Performance of strategies over benchmark instances.

Name	Vert.	Edges	χ_{eq}	Nodes				Time			
				B&B	S1	S4	S7	B&B	S1	S4	S7
1-FullIns_4	93	593	5	1175	1 175	2 339	**494**	17	18	31	**8**
4-FullIns_3	114	541	7	**23**	**23**	71	29	2	2	4	3
fpsol2.i.1	496	11 654	65	1	1	1	1	3	4	4	4
fpsol2.i.2	451	8 691	47	8	8	8	**5**	9	10	9	11
fpsol2.i.3	425	8 688	55	47	47	**26**	48	32	32	**26**	**25**
inithx.i.1	864	18 707	54	1	1	1	1	34	34	35	35
kneser11_5	462	1 386	3	104	104	104	104	9	9	9	9
le450_5a	450	5 714	5	–	2	**0**	**0**	–	7324	5970	**3929**
le450_5b	450	5 734	5	–	–	**0**	–	–	–	**5447**	–
myciel5	47	236	6	763 580	763 580	**325 458**	**325 458**	954	850	**321**	**321**
queen8_12	96	1 368	12	146	177	110	**56**	21	20	15	**10**
queen8_8	64	728	9	**28 956**	105 698	49 182	43 286	**445**	1292	570	907
zeroin.i.2	211	3 541	36	21	21	18	**13**	3	3	3	3
zeroin.i.3	206	3 540	36	33	33	**12**	22	2	2	2	4
Average[a]				66 175	72 572	31 444	**30 793**	128	190	**86**	112

[a] le450_5a and le450_5b are not considered in the average since they are not solved by B&B.

Comparing averages, strategy S7 evaluates fewer nodes than S4, as well as for random graphs, but S4 needs less solving time.

From the reported computational experience we conclude that the new inequalities used as cuts are good enough to be considered as part of the implementation of a further competitive Branch and Cut algorithm that solves the ECP.

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