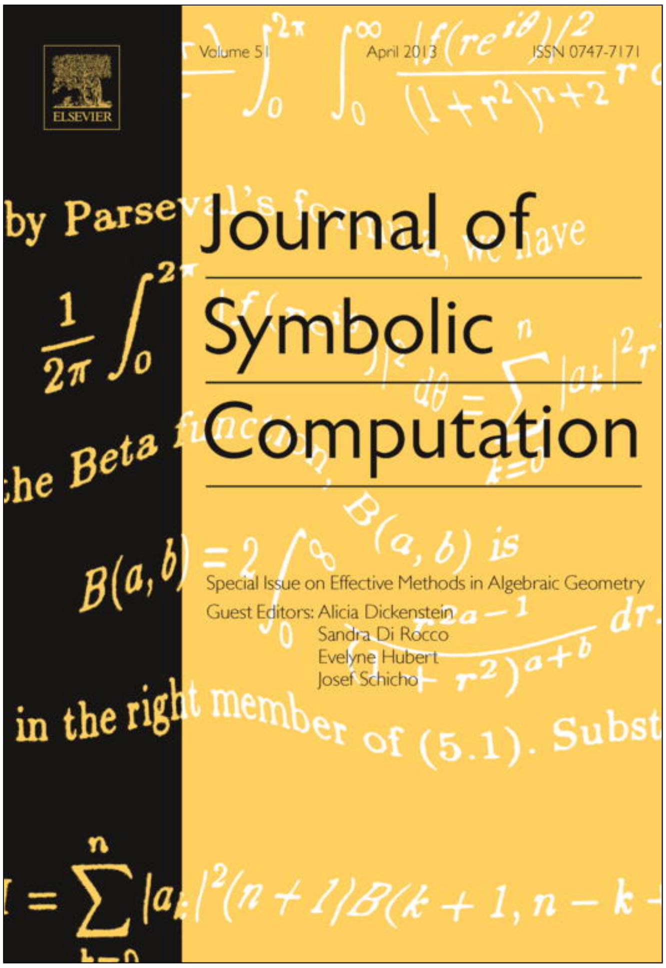
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Affine solution sets of sparse polynomial systems ☆

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A B S T R A C T

This paper focuses on the equidimensional decomposition of affine varieties defined by sparse polynomial systems. For generic systems with fixed supports, we give combinatorial conditions for the existence of positive dimensional components which characterize the equidimensional decomposition of the associated affine variety. This result is applied to design an equidimensional decomposition algorithm for generic sparse systems. For arbitrary sparse systems of n polynomials in n variables with fixed supports, we obtain an upper bound for the degree of the affine variety defined and we present an algorithm which computes finite sets of points representing its equidimensional components.

1. Introduction

The aim of this paper is to describe the affine solution set of a polynomial system taking into account the sets of exponents of the monomials with nonzero coefficients in the polynomials involved, that is, their support sets.

Bernstein (1975), Kushnirenko (1976) and Khovanskii (1978) proved that the number of isolated solutions in $(\mathbb{C}^*)^n$ of a polynomial system with n equations in n variables is bounded by a combinatorial invariant (the mixed volume) associated with their supports. This result, which may be considered the basis for the current study of sparse polynomial systems, hints at the fact that the algorithms solving these systems should have shorter computing time than the general ones.

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There are several algorithms to compute either numerically or symbolically the isolated roots of sparse polynomial systems in $(\mathbb{C}^*)^n$ (see, for example, Verschelde et al., 1994; Huber and Sturmfels, 1995; Rojas, 2003; Jeronimo et al., 2009). The efficiency of some of these algorithms relies on the use of polyhedral deformations preserving the monomial structure of the polynomial system under consideration.

The first step towards the study of the solutions of sparse systems in the *affine* case was to obtain upper bounds for the number of isolated solutions in $\mathbb{C}^n$ in terms of the structure of their supports and to design numerical algorithms to compute them (see Rojas, 1994; Li and Wang, 1996; Rojas and Wang, 1996; Huber and Sturmfels, 1997; Emiris and Verschelde, 1999; Gao et al., 1999). Symbolic algorithms performing this task were given in Jeronimo et al. (2009) and Herrero et al. (2010).

The next natural step is to characterize the components of higher dimension of the affine variety defined by a system of sparse polynomial equations taking into account their supports. In this context, in Verschelde (2009), certificates for the existence of curves are given in the numerical framework.

There are different symbolic algorithms describing the equidimensional decomposition of a variety which only take into consideration the degrees of the polynomials defining it and not their particular monomial structure. The earliest deterministic ones can be found in Chistov and Grigoriev (1983) and Giusti and Heintz (1991) (see also Gianni et al., 1988, where the more general problem of the primary decomposition of ideals is considered). Probabilistic algorithms with shorter running time are given in Elkadi and Mourrain (1999) and Jeronimo and Sabia (2002). The complexities of these probabilistic algorithms are polynomial in the Bézout number of the system, which, in the generic case, coincides with the degree of the variety the system defines. Other probabilistic algorithms are presented in Lecerf (2003) and Jeronimo et al. (2004) with complexities depending on a new invariant related to the system (the *geometric degree*) which refines the Bézout bound. Some of these algorithms can be derandomized easily via the Schwartz–Zippel lemma (Schwartz, 1980; Zippel, 1993) provided upper bounds for the degrees of the polynomials characterizing exceptional instances are known.

Algorithms dealing with the problem from the numerical point of view can be traced back to Sommese and Wampler (1996). A series of papers by Sommese, Verschelde and Wampler present successive improvements to this procedure, leading to the irreducible decomposition algorithm based on homotopy continuation described in Sommese and Wampler (2005) (see references therein).

In this paper we analyze, both from the theoretic and algorithmic points of view, the equidimensional decomposition of the affine variety defined by a sparse polynomial system.

First, we consider the case of generic sparse systems. In this context, there exists a major difference with the case of dense polynomials. The set of solutions of a generic system of n polynomials in n variables with fixed degrees consists only of isolated points. However, fixing the set of supports of the n polynomials in n variables involved in a sparse system, for generic choices of its coefficients, there may appear affine components of positive dimension (see, for instance, Examples 3 and 8 below). We show that the existence of these generic components of positive dimension depends only on the combinatorial structure of the supports: in Proposition 6 below, we give conditions that yield these components. Such conditions provide not only a theoretic description of the equidimensional decomposition of the affine variety $V(\mathbf{f})$ defined by a *generic* sparse system $\mathbf{f}$ in terms of the solution sets in the torus of smaller systems $\mathbf{f}_I$ associated to subsets $I \subset \{1,\dots,n\}$ but also a formula for the degree of $V(\mathbf{f})$ in this generic case (see Theorem 7 below). Previous results on this subject can be found in Cattani and Dickenstein (2007). There, using also a combinatorial approach, the authors analyze thoroughly the problem of deciding whether a system of n binomials in n variables has a finite number of affine solutions and, in this case, the computational complexity of the corresponding counting problem.

Our result is used to design a probabilistic algorithm which, for a generic sparse system $\mathbf{f}$, computes the equidimensional decomposition of $V(\mathbf{f})$ with a complexity depending on its degree and combinatorial invariants associated with the system (see Theorem 14). The idea of the algorithm is to compute first a family of subsets $I \subset \{1,\dots,n\}$ which may lead to components of $V(\mathbf{f})$ and solve the corresponding polynomial systems $\mathbf{f}_I$ by applying symbolic polyhedral deformations (Jeronimo et al., 2009) and a Newton–Hensel based procedure (Giusti et al., 2001). The output of the algorithm is, for each $k = 0,\dots,n-1$, a list of *geometric resolutions* representing the equidimensional component of dimension k of $V(\mathbf{f})$. A geometric resolution of an equidimensional variety is a parametric type description of the variety which is widely used in symbolic computations (see, for instance, Giusti et al., 1998; Rouillier, 1999; Giusti et al., 2001; Schost, 2003); in Section 2.1 below we give the precise definition we use.

The next step is to consider the equidimensional decomposition of the affine variety defined by an *arbitrary* system of sparse polynomials. A question to answer beforehand is which parameter should be involved in the algebraic complexity of an algorithm solving this task. From previous experience, a natural invariant expected to appear in the complexity bounds is the degree of the variety, which is, in particular, an upper bound for the number of its irreducible components.

Unlike the Bézout bound for dense polynomials, in the sparse setting, the degree of the affine variety defined by a generic square system is *not* an upper bound for the degree of the variety defined by any system with the same supports (see Example 15). In Krick et al. (2001), a bound for the degree of the affine variety defined by an arbitrary sparse polynomial system depending on a mixed volume related to the *union* of the supports of the polynomials is presented (see also Rojas, 2003, Theorem 1, for a related result). Here, we obtain a sharper bound for this degree also given by a mixed volume associated to the supports but not involving their union:

Theorem. *Let $\mathbf{f} = (f_1, \ldots, f_n)$ be n polynomials in $\mathbb{C}[X_1, \ldots, X_n]$ supported on $\mathcal{A} = (\mathcal{A}_1, \ldots, \mathcal{A}_n)$ and let $V(\mathbf{f}) = \{x \in \mathbb{C}^n \mid f_i(x) = 0$ for every $1 \leqslant i \leqslant n\}$. Then*

$$\deg\big(V(\mathbf{f})\big) \leqslant MV_n(\mathcal{A}_1 \cup \Delta, \ldots, \mathcal{A}_n \cup \Delta),$$

where $\Delta = \{0, e_1, \ldots, e_n\}$ with e_i the ith vector of the canonical basis of $\mathbb{R}^n$ and MV_n stands for the n-dimensional mixed volume.

Finally, we obtain an algorithm which, using a polyhedral deformation, describes points in every irreducible component of the affine variety defined by an arbitrary square sparse system with complexity depending on the degree bound previously stated.

The idea of the algorithm relies on the fact that cutting the variety with a generic affine linear variety of codimension k, sufficiently many points in each irreducible component of dimension k can be obtained. To keep the complexity within the desired bounds, instead of computing this intersection, we proceed in a particular way which enables us to compute a finite superset of the intersection included in the variety. Representing a positive dimensional variety by means of a finite set of points is a well-known approach in numerical algebraic geometry (see the notion of *witness point supersets* in Sommese and Wampler, 2005).

Theorem. *Let $\mathbf{f} = (f_1, \ldots, f_n)$ be n polynomials in $\mathbb{Q}[X_1, \ldots, X_n]$ supported on $\mathcal{A} = (\mathcal{A}_1, \ldots, \mathcal{A}_n)$. There is a probabilistic algorithm which, taking as input the sparse representation of $\mathbf{f}$ computes a family of n geometric resolutions $(R^{(0)}, R^{(1)}, \ldots, R^{(n-1)})$ such that, for every $0 \leqslant k \leqslant n-1$, $R^{(k)}$ represents a finite set containing $\deg V_k(\mathbf{f})$ points in the equidimensional component $V_k(\mathbf{f})$ of dimension k of $V(\mathbf{f})$. The number of arithmetic operations over $\mathbb{Q}$ performed by the algorithm is of order $O\tilde{}(n^4 dND^2)$, where $d = \max_{1\leqslant j\leqslant n}\{\deg(f_j)\}$, $N = \sum_{j=1}^{n} \#(\mathcal{A}_j \cup \Delta)$ and $D = MV_n(\mathcal{A}_1 \cup \Delta, \ldots, \mathcal{A}_n \cup \Delta)$.*

Here $O\tilde{}$ refers to the standard soft-oh notation which does not take into account logarithmic factors. Furthermore, we have ignored factors depending polynomially on the size of certain combinatorial objects associated to the polyhedral deformation. For a precise complexity statement, see Theorem 21, and for error probability considerations, see Remark 23.

The paper is organized as follows. In Section 2, the basic definitions and notations used throughout the paper are introduced. Section 3 is devoted to the equidimensional decomposition of affine varieties defined by generic sparse systems: first, we consider the solution sets in $(\mathbb{C}^*)^n$ of underdetermined systems (see Section 3.1); then, we prove our main theoretic result on equidimensional decomposition and present our algorithm to compute it (see Section 3.2). Finally, in Section 4, we consider the case of arbitrary sparse systems: we prove the upper bound for the degree of affine varieties defined by these systems in Section 4.1 and, in Section 4.2, we describe our algorithm to compute representative points of the equidimensional components.

2. Preliminaries

2.1. Basic definitions and notation

Throughout this paper, unless otherwise explicitly stated, we deal with polynomials in $\mathbb{Q}[X_1,\dots,X_n]$, that is to say polynomials with rational coefficients in n variables $X=(X_1,\dots,X_n)$. If $\mathbf{f}=(f_1,\dots,f_s)$ is a family of such polynomials, $V(\mathbf{f})$ will denote the algebraic variety of their common zeroes in $\mathbb{C}^n$, the n-dimensional affine space over the complex numbers.

The algebraic variety $V(\mathbf{f})\subset\mathbb{C}^n$ can be decomposed uniquely as a finite union of irreducible varieties in a non-redundant way. This leads to the *equidimensional decomposition* of the variety:

$$V(\mathbf{f})=\bigcup_{k=0}^{n} V_k(\mathbf{f}),$$

where, for every $0\leqslant k\leqslant n$, $V_k(\mathbf{f})$ is the (possibly empty) union of all the irreducible components of dimension k of $V(\mathbf{f})$.

The degree of each equidimensional component $V_k(\mathbf{f})$ is the number of points in its intersection with a generic affine linear variety of codimension k, and the degree of $V(\mathbf{f})$, which we denote by $\deg(V(\mathbf{f}))$, is the sum of the degrees of its equidimensional components (see Heintz, 1983).

A common way to describe zero-dimensional affine varieties defined by polynomials over $\mathbb{Q}$ is a *geometric resolution* (see, for instance, Giusti et al., 2001, and the references therein). The precise definition we are going to use is the following:

Let $V=\{\xi^{(1)},\dots,\xi^{(D)}\}\subset\mathbb{C}^n$ be a zero-dimensional variety defined by rational polynomials. Given a linear form $\ell=\ell_1X_1+\cdots+\ell_nX_n$ in $\mathbb{Q}[X_1,\dots,X_n]$ such that $\ell(\xi^{(i)})\neq\ell(\xi^{(j)})$ if $i\neq j$, the following polynomials completely characterize V:

- the minimal polynomial $q=\prod_{1\leqslant i\leqslant D}(u-\ell(\xi^{(i)}))\in\mathbb{Q}[u]$ of ℓ over the variety V (where u is a new variable),
- polynomials $v_1,\dots,v_n\in\mathbb{Q}[u]$ with $\deg(v_j)<D$ for every $1\leqslant j\leqslant n$ satisfying $V=\{(v_1(\eta),\dots,v_n(\eta))\in\mathbb{C}^n \mid \eta\in\mathbb{C}, q(\eta)=0\}$.

The family of univariate polynomials $(q,v_1,\dots,v_n)\in\mathbb{Q}[u]^{n+1}$ is called a geometric resolution of V (associated with the linear form ℓ).

An equivalent description of V can be given through the so-called *Kronecker representation* (see Giusti et al., 2001), which consists of the minimal polynomial q and polynomials $w_1,\dots,w_n\in\mathbb{Q}[u]$ such that $V=\{(\frac{w_1}{q'}(\eta),\dots,\frac{w_n}{q'}(\eta))\in\mathbb{C}^n \mid \eta\in\mathbb{C},\ q(\eta)=0\}$, where q' is the derivative of q. Either representation can be obtained from the other one in polynomial time.

The notion of geometric resolution can be extended to any equidimensional variety:

Let $V\subset\mathbb{C}^n$ be an equidimensional variety of dimension r defined by polynomials $f_1,\dots,f_{n-r}\in\mathbb{Q}[X_1,\dots,X_n]$. Assume that for each irreducible component C of V, the identity $I(C)\cap\mathbb{Q}[X_1,\dots,X_r]=\{0\}$ holds, where $I(C)$ is the ideal of all polynomials in $\mathbb{Q}[X_1,\dots,X_n]$ vanishing identically over C. Let ℓ be a nonzero linear form in $\mathbb{Q}[X_{r+1},\dots,X_n]$ and $\pi_\ell:V\to\mathbb{C}^{r+1}$ the morphism defined by $\pi_\ell(x)=(x_1,\dots,x_r,\ell(x))$. Then, there exists a unique (up to scaling by nonzero elements of $\mathbb{Q}$) polynomial $Q_\ell\in\mathbb{Q}[X_1,\dots,X_r,u]$ of minimal degree defining $\overline{\pi_\ell(V)}$. Let $q_\ell\in\mathbb{Q}(X_1,\dots,X_r)[u]$ denote the (unique) monic multiple of Q_ℓ with $\deg_u(q_\ell)=\deg_u(Q_\ell)$. In these terms, if ℓ is a *generic* linear form, a *geometric resolution* of V is $(q_\ell,v_{r+1},\dots,v_n)\in(\mathbb{Q}(X_1,\dots,X_r)[u])^{n-r+1}$, where, for $r+1\leqslant i\leqslant n$, v_i satisfies

$$\frac{\partial q_\ell}{\partial u}(\ell)X_i=v_i(\ell)\quad\text{in } \mathbb{Q}(X_1,\dots,X_r)\otimes\mathbb{Q}[V]$$

and $\deg_u(v_i)<\deg_u(q_\ell)$.

2.2. Algorithms and codification

Although we work with polynomials, our algorithms only deal with elements in $\mathbb{Q}$. The notion of *complexity* of an algorithm we consider is the number of operations and comparisons in $\mathbb{Q}$ it has to perform. We will encode multivariate polynomials in different ways:

- in sparse form, that is, by means of the list of pairs (a, c_a) where a runs over the set of exponents of the monomials appearing in the polynomial with nonzero coefficients and c_a is the corresponding coefficient,
- in the standard dense form, which encodes a polynomial as the vector of its coefficients including zeroes (we use this encoding only for univariate polynomials),
- in the *straight-line program* (slp for short) encoding. A straight-line program is an algorithm without branchings which allows the evaluation of the polynomial at a generic value (for a precise definition and properties of slp's, see Bürgisser et al., 1997).

In our complexity estimates, we will use the usual O notation: for $f, g : \mathbb{Z}_{\geqslant 0} \to \mathbb{R}$, $f(d) = O(g(d))$ if $|f(d)| \leqslant c|g(d)|$ for a positive constant c. We will also use the notation $M(d) = d\log^2(d)\log(\log(d))$, where log denotes logarithm to base 2. We recall that multipoint evaluation and interpolation of univariate polynomials of degree d with coefficients in a characteristic-0 commutative ring R can be performed with $O(M(d))$ operations and that multiplication and division with remainder of such polynomials can be done with $O(M(d)/\log(d))$ arithmetic operations in R.

We denote by Ω the exponent in the complexity estimate $O(d^{\Omega})$ for the multiplication of two $d \times d$ matrices with rational coefficients. It is known that $\Omega < 2.376$ (see von zur Gathen and Gerhard, 1999, Chapter 12). Finally, we write $\overline{\Omega}$ for the exponent ($\overline{\Omega} < 4$) in the complexity $O(d^{\overline{\Omega}})$ of operations on $d \times d$ matrices with entries in a commutative ring R.

Our algorithms are probabilistic in the sense that they make random choices of points which lead to a correct computation provided the points lie outside certain proper Zariski closed sets of suitable affine spaces. Then, using the Schwartz–Zippel lemma (Schwartz, 1980; Zippel, 1993), the error probability of our algorithms can be controlled by making these random choices within sufficiently large sets of integer numbers whose size depend on the degrees of the polynomials defining the previously mentioned Zariski closed sets.

2.3. Sparse systems

Given a family $\mathcal{A} = (\mathcal{A}_1, \dots, \mathcal{A}_s)$ of finite subsets of $(\mathbb{Z}_{\geqslant 0})^n$, a *sparse polynomial system supported on* $\mathcal{A}$ is given by polynomials $f_j = \sum_{a \in \mathcal{A}_j} c_{j,a} X^a$ in the variables $X = (X_1, \dots, X_n)$, with $c_{j,a} \in \mathbb{C} \setminus \{0\}$ for each $a \in \mathcal{A}_j$ and $1 \leqslant j \leqslant s$. We write $\mathbf{f} = (f_1, \dots, f_s)$ for this system.

Assume $s = n$. We denote by $MV_n(\mathcal{A})$ the *mixed volume* of the convex hulls of $\mathcal{A}_1, \dots, \mathcal{A}_n$ in $\mathbb{R}^n$ (see, for example, Cox et al., 1998, Chapter 7, for the definition), which is an upper bound for the number of isolated roots in $(\mathbb{C}^*)^n$ of a sparse system supported on $\mathcal{A}$ (see Bernstein, 1975).

The mixed volume $MV_n(\mathcal{A})$ can be computed as the sum of the n-dimensional volumes of the convex hulls of all the *mixed cells* in a fine mixed subdivision of $\mathcal{A}$. Such a subdivision can be obtained by means of a standard lifting process (see Huber and Sturmfels, 1995, Section 2): let $\omega = (\omega_1, \dots, \omega_n)$ be an n-tuple of generic functions $\omega_j : \mathcal{A}_j \to \mathbb{R}$ and consider the polytope P in $\mathbb{R}^{n+1}$ obtained by taking the pointwise sum of the convex hulls of the graphs of ω_j for $1 \leqslant j \leqslant n$. Then, the projection of the lower facets of P (that is, the n-dimensional faces with inner normal vector with a positive last coordinate) induces a fine mixed subdivision of $\mathcal{A}$. The dynamic enumeration procedure described in Mizutani et al. (2007) appears to be the fastest algorithm known up until now to achieve this computation of mixed cells.

The *stable mixed volume* of $\mathcal{A}$, which is denoted by $SM_n(\mathcal{A})$, is introduced in Huber and Sturmfels (1997) as an upper bound for the number of isolated roots in $\mathbb{C}^n$ of a sparse system supported on $\mathcal{A}$. Consider $\mathcal{A}^0 = (\mathcal{A}_1^0, \dots, \mathcal{A}_n^0)$ the family with $\mathcal{A}_j^0 := \mathcal{A}_j \cup \{0\}$ for every $1 \leqslant j \leqslant n$, and let $\omega^0 = (\omega_1^0, \dots, \omega_n^0)$ be a lifting for $\mathcal{A}^0$ defined by $\omega_j^0(q) = 0$ if $q \in \mathcal{A}_j$ and $\omega_j^0(0) = 1$ if $0 \notin \mathcal{A}_j$. The stable mixed volume of $\mathcal{A}$ is defined as the sum of the mixed volumes of all the cells in the subdivision of $\mathcal{A}^0$ induced by ω^0 corresponding to facets having inner normal vectors with non-negative entries.

3. Generic sparse systems

3.1. Toric components

Let n and m be positive integers and let $\mathcal{A} = (\mathcal{A}_1, \ldots, \mathcal{A}_m)$ be a family of finite subsets of $(\mathbb{Z}_{\geqslant 0})^n$. For $1 \leqslant j \leqslant m$, let

$$F_j(C_j, X) = \sum_{a \in \mathcal{A}_j} C_{j,a} X^a$$

where $X = (X_1, \ldots, X_n)$; for $a = (a_1, \ldots, a_n)$, $X^a = \prod_{1 \leqslant j \leqslant n} X_j^{a_j}$, and $C_j = (C_{j,a})_{a \in \mathcal{A}_j}$ are $N_j = \#\mathcal{A}_j$ indeterminate coefficients.

Following Sturmfels (1994), consider the incidence variety

$$\left\{(x, c) \in \left(\mathbb{C}^*\right)^n \times \left(\mathbb{P}^{N_1-1} \times \cdots \times \mathbb{P}^{N_m-1}\right) \;\middle|\; F_j(c_j, x) = 0 \text{ for every } 1 \leqslant j \leqslant m\right\}$$

and its projection to the second factor

$$Z = \left\{c \in \mathbb{P}^{N_1-1} \times \cdots \times \mathbb{P}^{N_m-1} \;\middle|\; \exists x \in \left(\mathbb{C}^*\right)^n \text{ with } F_j(c_j, x) = 0 \text{ for every } 1 \leqslant j \leqslant m\right\}.$$

Note that the elements in Z correspond essentially to coefficients of systems supported on $\mathcal{A}$ which have a solution in $(\mathbb{C}^*)^n$.

Lemma 1. *The Zariski closure of Z equals $\mathbb{P}^{N_1-1} \times \cdots \times \mathbb{P}^{N_m-1}$ if and only if, for every $J \subseteq \{1, \ldots, m\}$, $\dim(\sum_{j \in J} \mathcal{A}_j) \geqslant \# J$. In particular, if $m > n$, a generic system supported on $\mathcal{A}$ has no solutions in $(\mathbb{C}^*)^n$. Moreover, if $m \leqslant n$ and $\dim(\sum_{j \in J} \mathcal{A}_j) \geqslant \# J$ for every $J \subseteq \{1, \ldots, m\}$, the solution set in $(\mathbb{C}^*)^n$ of a generic system supported on $\mathcal{A}$ is an equidimensional variety of dimension $n - m$ and degree $MV_n(\mathcal{A}_1, \ldots, \mathcal{A}_m, \Delta^{(n-m)})$, where $\Delta = \{0, e_1, \ldots, e_n\}$ with e_i the ith vector of the canonical basis of $\mathbb{R}^n$ and the superscript $(n - m)$ indicates that it is repeated $n - m$ times.*

Proof. The first statement of the lemma follows as in Sturmfels (1994, Theorem 1.1).

Assume that $m \leqslant n$ and for every $J \subseteq \{1, \ldots, m\}$, $\dim(\sum_{j \in J} \mathcal{A}_j) \geqslant \# J$. Then, if $\widetilde{\mathcal{A}} = (\mathcal{A}_1, \ldots, \mathcal{A}_m, \Delta^{(n-m)})$, we have that for every $\widetilde{J} \subseteq \{1, \ldots, n\}$, the inequality $\dim(\sum_{j \in \widetilde{J}} \widetilde{\mathcal{A}}_j) \geqslant \# \widetilde{J}$ holds, since $\dim(\Delta) = n$. Therefore, $MV_n(\widetilde{\mathcal{A}}) > 0$, which implies that a generic system supported on $\widetilde{\mathcal{A}}$ has finitely many solutions in $(\mathbb{C}^*)^n$ (as many as $MV_n(\widetilde{\mathcal{A}})$).

Now, the solution set of a generic system supported on $\widetilde{\mathcal{A}}$ is the intersection of the solution set of a generic system of m equations supported on $\mathcal{A}$, which is a variety of dimension at least $n - m$ in $(\mathbb{C}^*)^n$, and $n - m$ generic hyperplanes. We conclude that the solution set in $(\mathbb{C}^*)^n$ of a generic system supported on $\mathcal{A}$ is an equidimensional variety of dimension $n - m$ and degree $MV_n(\widetilde{\mathcal{A}})$. □

Assume now that $\mathbf{f} = (f_1, \ldots, f_m)$ are generic polynomials in the variables $X = (X_1, \ldots, X_n)$ supported on $\mathcal{A} = (\mathcal{A}_1, \ldots, \mathcal{A}_m) \subset (\mathbb{Z}^n_{\geqslant 0})^m$, with $m \leqslant n$. The previous lemma states that the affine variety $V^*(\mathbf{f}) \subset \mathbb{C}^n$ consisting of the union of all the irreducible components of $V(\mathbf{f}) = \{x \in \mathbb{C}^n \mid f_j(x) = 0 \text{ for every } 1 \leqslant j \leqslant s\}$ that have a non-empty intersection with $(\mathbb{C}^*)^n$ is either the empty set or an equidimensional variety of dimension $n - m$.

In what follows, we extend the symbolic algorithm from Jeronimo et al. (2009, Section 5), which deals with the case $m = n$, to a procedure for the computation of a geometric resolution of $V^*(\mathbf{f})$ for arbitrary $m \leqslant n$. As in Jeronimo et al. (2009), our algorithm assumes that a fine mixed subdivision of $(\mathcal{A}, \Delta^{(n-m)})$ induced by a generic lifting function $\omega = (\omega_1, \ldots, \omega_n)$ is given by a pre-processing.

Algorithm `GenericToricSolve`

INPUT: A sparse representation of a generic system $\mathbf{f} = (f_1, \ldots, f_m)$ in the variables $X = (X_1, \ldots, X_n)$ supported on $\mathcal{A} = (\mathcal{A}_1, \ldots, \mathcal{A}_m)$, a lifting function $\omega = (\omega_1, \ldots, \omega_n)$ and the mixed cells in the subdivision of $(\mathcal{A}, \Delta^{(n-m)})$ induced by ω.

(1) If the fine mixed subdivision of $(\mathcal{A}, \Delta^{(n-m)})$ does not contain any mixed cell, return $R = \emptyset$. Otherwise, continue to Step 2.
(2) Choose randomly the entries of a matrix $A = (a_{hl}) \in \mathbb{Q}^{n\times n}$ and a vector $b = (b_1, \ldots, b_n) \in \mathbb{Q}^n$.
(3) For $1 \leqslant h \leqslant n - m$, consider the affine linear forms $L_h = \sum_{l=1}^{n} a_{hl} X_l - b_h$.
(4) Apply Jeronimo et al. (2009, Algorithm 5.1) to obtain a geometric resolution $(q(u), v_1(u), \ldots, v_n(u))$ of the isolated common zeroes of $\mathbf{f}, L_1, \ldots, L_{n-m}$ in $(\mathbb{C}^*)^n$.
(5) Obtain an slp for the polynomials $\mathbf{g} := \mathbf{f}(A^{-1}Y)$ in the new variables $Y = (Y_1, \ldots, Y_n)$.
(6) Compute $(w_1(u), \ldots, w_n(u))^t := A(v_1(u), \ldots, v_n(u))^t$.
(7) Apply the Global Newton Iterator from Giusti et al. (2001, Algorithm 1) to the polynomials $\mathbf{g}(Y)$, the geometric resolution $(q(u), w_{n-m+1}(u), \ldots, w_n(u))$ of $V(\mathbf{g}(b, Y_{n-m+1}, \ldots, Y_n))$, and precision $\kappa = MV_n(\mathcal{A}, \Delta^{(n-m)})$ to obtain a geometric resolution R_Y of an equidimensional variety of dimension $n - m$.
(8) Obtain the geometric resolution $R := A^{-1}R_Y$ of $V^*(\mathbf{f})$.

OUTPUT: The geometric resolution R of $V^*(\mathbf{f})$.

In the sequel we will justify the correctness of the above procedure and estimate its complexity.

Since $\mathbf{f}$ is a generic sparse system of m equations in n variables, as a consequence of Lemma 1, if $L_1, \ldots, L_{n-m}$ are generic linear forms, $V^*(\mathbf{f}) \cap V(L_1, \ldots, L_{n-m})$ is either the empty set (when $V^*(\mathbf{f})$ is the empty set) or a finite set consisting of $\deg(V^*(\mathbf{f}))$ points (when $V^*(\mathbf{f})$ is not the empty set). Step 1 decides whether $V^*(\mathbf{f}) \cap V(L_1, \ldots, L_{n-m})$ is empty or not, since the mixed volume of the supports of these polynomials is the sum of the volumes of the mixed cells in a fine mixed subdivision (Huber and Sturmfels, 1995). If it is not empty, this finite set of points can be regarded as a generic fiber of a generic linear surjective projection and therefore, it enables us to recover the variety $V^*(\mathbf{f})$ by deformation techniques.

Thus, the idea of the algorithm is to choose $n - m$ linear forms at random, then compute a geometric resolution of the set $V^*(\mathbf{f}) \cap V(L_1, \ldots, L_{n-m})$ and finally, apply a Newton–Hensel lifting to the finite set obtained in order to get a geometric resolution of $V^*(\mathbf{f})$.

Step 2 deals with the random choice of the entries of a matrix and a vector. This random choice does not affect the overall complexity of the procedure (see Remark 23 below). In Step 3, the sparse encoding of $n - m$ linear forms constructed from the previous data is obtained.

The idea of Step 4 is to obtain a geometric resolution of $V^*(\mathbf{f}) \cap V(L_1, \ldots, L_{n-m})$. In order to do this, the algorithm computes the isolated common zeroes in $(\mathbb{C}^*)^n$ of the *generic* system $\mathbf{f}, L_1, \ldots, L_{n-m}$ supported on $(\mathcal{A}, \Delta^{(n-m)})$. Note that, if $L_1, \ldots, L_{n-m}$ are generic, this set of points meets only the irreducible components of $V^*(\mathbf{f})$, that is, it contains no point in the irreducible components of $V(\mathbf{f})$ with vanishing coordinates. By applying the result in Jeronimo et al. (2009, Proposition 5.13), it follows that the complexity of this step is $O((n^3(N + (n - m)n)\log d + n^{1+\Omega})M(D)M(\mathfrak{M})(M(D) + M(E)))$, where

- $N := \sum_{1\leqslant j\leqslant m} \#\mathcal{A}_j$;
- $d := \max_{1\leqslant j\leqslant m}\{\deg(f_j)\}$;
- $D := MV_n(\mathcal{A}, \Delta^{(n-m)})$;
- $\mathfrak{M} := \max\{\|\mu\|\}$, where the maximum ranges over all primitive normal vectors to the mixed cells in the fine mixed subdivision of $(\mathcal{A}, \Delta^{(n-m)})$ given by ω;
- $E := MV_{n+1}(\Delta \times \{0\}, \mathcal{A}_1(\omega_1), \ldots, \mathcal{A}_m(\omega_m), \Delta(\omega_{m+1}), \ldots, \Delta(\omega_n))$, where $\mathcal{A}_j(\omega_j)$ $(1 \leqslant j \leqslant m)$ and $\Delta(\omega_l)$ $(m + 1 \leqslant l \leqslant n)$ are, respectively, the supports of $\mathbf{f}, L_1, \ldots, L_{n-m}$ lifted by ω.

Now the algorithm lifts the geometric resolution of the zero-dimensional subset of $V(\mathbf{f})$ obtained so far to a geometric resolution of the union of the irreducible components of this variety having a non-empty intersection with $(\mathbb{C}^*)^n$. In order to do this, we consider the change of variables given by $Y = A.X$ and make this change of variables in the polynomials $\mathbf{f}$ and the geometric resolution already obtained (Steps 5 and 6). A possible way of making this change of variables is by first computing A^{-1} with $O(n^\Omega)$ operations and using it to obtain an slp of length $L = O(n^2 + n\log(d)N)$ for the polynomials in $\mathbf{g}$ (note that the length of this slp depends only on the cost $O(n^2)$ of computing the product of A^{-1} times a vector, and not on the cost of inverting A). Taking into account that the degrees of the polynomials $v_1, \ldots, v_n$ are bounded by D, to write the geometric resolution in the new variables, we perform $O(n^2D)$ operations.

Note that $(w_1(u), \ldots, w_{n-m}(u)) = b$ and $(q(u), w_{n-m+1}(u), \ldots, w_n(u))$ is a geometric resolution of the isolated points in $V(\mathbf{g}(b, Y_{n-m+1}, \ldots, Y_n))$ corresponding to the isolated points in $(\mathbb{C}^*)^n$ of $V(\mathbf{f}, L_1, \ldots, L_{n-m})$. Now, the geometric resolution of $V^*(\mathbf{f})$ with respect to the linear projection given by Y consists of polynomials in $\mathbb{Q}[Y_1, \ldots, Y_{n-m}, u]$ having total degrees bounded by D. Therefore, it suffices to compute the representatives of these polynomials in $(\mathbb{Q}[Y_1, \ldots, Y_{n-m}]/\langle Y_1 - b_1, \ldots, Y_{n-m} - b_{n-m}\rangle^{D+1})[u]$. To this end, in Step 7 we apply successively the Global Newton Iterator from Giusti et al. (2001) to the polynomials $\mathbf{g}$, starting with the geometric resolution $(q(u), w_{n-m+1}(u), \ldots, w_n(u))$ obtained in Step 6, which can be regarded as a representative in $(\mathbb{Q}[Y_1, \ldots, Y_{n-m}]/\langle Y_1 - b_1, \ldots, Y_{n-m} - b_{n-m}\rangle)[u]$, up to the required precision $D = MV_n(\mathcal{A}, \Delta^{(n-m)})$. Using Giusti et al. (2001, Lemma 2) and encoding the elements of $\mathbb{Q}[Y_1, \ldots, Y_{n-m}]/\langle Y_1 - b_1, \ldots, Y_{n-m} - b_{n-m}\rangle^k$ as $(k+1)$-tuples of slp's (one slp for each homogeneous component), the complexity of Step 7 is of order $O((mL + m^{\overline{\Omega}})M(D)D^2)$.

Finally, the algorithm changes variables back in order to obtain the desired geometric resolution of $V^*(\mathbf{f})$, which adds $O(n^2D)$ to the complexity.

Taking into account the previous complexity estimates, we have the following result:

Proposition 2. *Let* $\mathbf{f} = (f_1, \ldots, f_m)$ *be a system of* $m \leqslant n$ *generic polynomials in* $\mathbb{Q}[X_1, \ldots, X_n]$ *supported on* $\mathcal{A} = (\mathcal{A}_1, \ldots, \mathcal{A}_m)$. `GenericToricSolve` *is a probabilistic algorithm that computes a geometric resolution of the affine variety* $V^*(\mathbf{f})$ *consisting of the union of all the irreducible components of* $V(\mathbf{f})$ *that have a non-empty intersection with* $(\mathbb{C}^*)^n$. *Using the previous notation, the complexity of this algorithm is of order*

$$O\big(n^3\big(N + (n-m)n\big)\log(d)M(D)\big(M(\mathfrak{M})\big(M(D) + M(E)\big) + D^2\big)\big).$$

3.2. Affine components

3.2.1. Theoretic results

This section is devoted to showing a combinatorial description of the equidimensional decomposition of the affine variety defined by a *generic* sparse polynomial system. More precisely, we prove combinatorial conditions on the supports of the polynomials that determine the existence of irreducible components of the different possible dimensions not intersecting $(\mathbb{C}^*)^n$ and give a combinatorial characterization of the set of linear subspaces where these components lie. This characterization enables us to give a combinatorial formula for the degree of these varieties.

The following example shows that generic square sparse systems may define affine varieties containing positive dimensional components. It also shows that neither the mixed volume nor the stable mixed volume of the system are upper bounds for the degree of the affine variety defined:

Example 3. Consider a generic sparse system supported on $\mathcal{A} = (\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3)$ where $\mathcal{A}_1 = \{(1,1,2), (1,1,1)\}$, $\mathcal{A}_2 = \{(2,0,1), (1,0,1)\}$ and $\mathcal{A}_3 = \{(0,2,1), (0,1,1)\}$:

$$\begin{cases} aX_1X_2X_3^2 + bX_1X_2X_3 = 0, \\ cX_1^2X_3 + dX_1X_3 = 0, \\ eX_2^2X_3 + fX_2X_3 = 0 \end{cases}$$

with a, b, c, d, e, f nonzero complex numbers. The affine variety defined by the system has 5 irreducible components of degree 1: $\{x_3 = 0\}$, $\{x_1 = 0, x_2 = -\frac{f}{e}\}$, $\{x_1 = -\frac{d}{c}, x_2 = 0\}$, $\{x_1 = 0, x_2 = 0\}$ and

$\{(-\frac{d}{c}, -\frac{f}{e}, -\frac{b}{a})\}$. However, $MV_3(\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3) = 1$ and $SM_3(\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3) \leqslant MV_3(\mathcal{A}_1 \cup \{0\}, \mathcal{A}_2 \cup \{0\}, \mathcal{A}_3 \cup \{0\}) = 4$.

Let $\mathbf{f} = (f_1, \dots, f_s)$ be generic polynomials in the variables $X = (X_1, \dots, X_n)$ supported on $\mathcal{A} = (\mathcal{A}_1, \dots, \mathcal{A}_s) \subset (\mathbb{Z}^n_{\geqslant 0})^s$.

For $I \subset \{1, \dots, n\}$, we define

$$J_I = \Big\{ j \in \{1, \dots, s\} \;\Big|\; f_j|_{\bigcap_{i \in I}\{x_i = 0\}} \not\equiv 0 \Big\},$$

that is, the set of indices of the polynomials in $\mathbf{f}$ that do not vanish identically under the specialization $X_i = 0$ for every $i \in I$, and

$$\mathbf{f}_I = \big((f_j)_I\big)_{j \in J_I}$$

where, for a polynomial $f \in \mathbb{C}[X_1, \dots, X_n]$, f_I denotes the polynomial in $\mathbb{C}[(X_i)_{i \notin I}]$ obtained from f by specializing $X_i = 0$ for every $i \in I$. Namely, $\mathbf{f}_I$ is the set of polynomials obtained by specializing the variables indexed by I to 0 in the polynomials in $\mathbf{f}$ and discarding the ones that vanish identically. We denote by $\mathcal{A}_j^I$ the support of $(f_j)_I$, by $\pi_I : \mathbb{C}^n \to \mathbb{C}^{n-\#I}$ the projection $\pi_I(x_1, \dots, x_n) = (x_i)_{i \notin I}$ onto the coordinates not in I and by $\varphi_I : \mathbb{C}^{n-\#I} \to \mathbb{C}^n$ the map that inserts zeroes in the coordinates indexed by I.

For an irreducible subvariety W of $V(\mathbf{f}) = \{x \in \mathbb{C}^n \mid f_j(x) = 0 \text{ for every } 1 \leqslant j \leqslant s\}$, let

$$I_W = \Big\{ i \in \{1, \dots, n\} \;\Big|\; W \subset \{x_i = 0\} \Big\}.$$

Lemma 4. *Under the previous assumptions, let W be an irreducible component of $V(\mathbf{f})$. Then* $\dim W = n - \#I_W - \#J_{I_W}$. *Moreover $\pi_{I_W}(W)$ is an irreducible component of $V(\mathbf{f}_{I_W}) \subset \mathbb{C}^{n-\#I_W}$ intersecting $(\mathbb{C}^*)^{n-\#I_W}$.*

Proof. Without loss of generality, we may assume that $I_W = \{r+1, \dots, n\}$ and $J_{I_W} = \{1, \dots, m\}$ for some $r > 0$ and $m \leqslant n$. Then, $\pi := \pi_{I_W} : \mathbb{C}^n \to \mathbb{C}^r$ is the projection to the first r coordinates and $\mathbf{f}_{I_W} = (f_1(x_1, \dots, x_r, \mathbf{0}), \dots, f_m(x_1, \dots, x_r, \mathbf{0}))$, where $\mathbf{0}$ is the origin of $\mathbb{C}^{n-r}$.

Note that $W = \pi(W) \times \{\mathbf{0}\}$ and $\pi(W) \subset V(\mathbf{f}_{I_W})$. If $\pi(W) \subset C \subset V(\mathbf{f}_{I_W})$ for an irreducible component C of $V(\mathbf{f}_{I_W})$, it follows that $W \subset C \times \{\mathbf{0}\} \subset V(\mathbf{f})$, with $C \times \{\mathbf{0}\}$ an irreducible variety. Since W is an irreducible component of $V(\mathbf{f})$, the equality $W = C \times \{\mathbf{0}\}$ holds. This implies that $\pi(W) = C$ is an irreducible component of $V(\mathbf{f}_{I_W})$.

In addition, by the definition of I_W, we have that $W \cap (\bigcap_{i=1}^r \{x_i \neq 0\}) \neq \emptyset$: otherwise, $W \subset \bigcup_{i=1}^r \{x_i = 0\}$, which implies that $W \subset \{x_i = 0\}$ for some $1 \leqslant i \leqslant r$ since W is an irreducible variety. Therefore, $\pi(W) \cap (\mathbb{C}^*)^r \neq \emptyset$.

By Lemma 1, we conclude that $\pi(W) \cap (\mathbb{C}^*)^r$ has dimension $r - m$ and so, $\dim(W) = \dim(\pi(W)) = n - \#I_W - \#J_{I_W}$. □

The previous lemma allows us to prove that a result established for binomials in Cattani and Dickenstein (2007, Theorem 2.6) also holds for arbitrary polynomials.

Proposition 5. *With our previous notation, assuming that $s = n$ and $0 \in V(\mathbf{f})$, we have that $V(\mathbf{f})$ consists only of isolated points if and only if for every $I \subset \{1, \dots, n\}$, $\#I + \#J_I \geqslant n$.*

Proof. Assume that there exists $I \subset \{1, \dots, n\}$ such that $\#I + \#J_I < n$. Since $0 \in V(\mathbf{f}_I) \subset \mathbb{C}^{n-\#I}$ and this variety is defined by $\#J_I$ polynomials in $n - \#I$ variables, it follows that $\dim(V(\mathbf{f}_I)) \geqslant n - \#I - \#J_I > 0$. Taking into account that $V(\mathbf{f}) \supseteq \varphi_I(V(\mathbf{f}_I))$, we conclude that $\dim(V(\mathbf{f})) > 0$.

Conversely, if $\dim(V(\mathbf{f})) > 0$ and W is a positive dimensional irreducible component of $V(\mathbf{f})$, by Lemma 4, $n - \#I_W - \#J_{I_W} > 0$. □

Now we will characterize the sets $I \subset \{1, \dots, n\}$ such that the irreducible components of $V(\mathbf{f}_I)$ intersecting $(\mathbb{C}^*)^{n-\#I}$ yield irreducible components of $V(\mathbf{f})$.

Proposition 6. *Under the previous assumptions, let $I \subset \{1,\dots,n\}$. Then $V(\mathbf{f}_I) \cap (\mathbb{C}^*)^{n-\#I}$ is not empty if and only if for every $J \subset J_I$, $\dim(\sum_{j\in J} \mathcal{A}_j^I) \geqslant \# J$ and, in this case, $V(\mathbf{f}_I) \cap (\mathbb{C}^*)^{n-\#I}$ is an equidimensional variety of dimension $n - \#I - \# J_I$. In addition, if W is an irreducible component of $V(\mathbf{f}_I) \cap (\mathbb{C}^*)^{n-\#I}$, we have that $\varphi_I(W)$ is an irreducible component of $V(\mathbf{f}) \cap \bigcap_{i\notin I}\{x_i \neq 0\}$ if and only if for every $\widetilde{I} \subset I$, $\#\widetilde{I} + \# J_{\widetilde{I}} \geqslant \#I + \# J_I$.*

Proof. The first statement of the proposition follows from Lemma 1.

Without loss of generality, assume that $I = \{r+1,\dots,n\}$. Let W be an irreducible component of $V(\mathbf{f}_I) \cap (\mathbb{C}^*)^r$.

Suppose that for a subset $\widetilde{I} \subset I$ the inequality $\#\widetilde{I} + \# J_{\widetilde{I}} < \#I + \# J_I$ holds. Assume $\widetilde{I} = \{\tilde{r}+1,\dots,n\}$ for $\tilde{r} > r$.

Note that if $\xi = (\xi_1,\dots,\xi_r) \in V(\mathbf{f}_I)$, then $(\xi, \mathbf{0}_{n-r}) \in V(\mathbf{f})$ and, therefore, $(\xi, \mathbf{0}_{\tilde{r}-r}) \in V(\mathbf{f}_{\widetilde{I}})$. Thus, we may consider $W \times \{\mathbf{0}_{\tilde{r}-r}\} \subset V(\mathbf{f}_{\widetilde{I}})$, which will be included in an irreducible component $\widetilde{W}$ of $V(\mathbf{f}_{\widetilde{I}})$. Taking into account that $\mathbf{f}_{\widetilde{I}}$ consists of $\# J_{\widetilde{I}}$ polynomials in $n - \#\widetilde{I}$ variables and applying Lemma 1 to W and $\mathbf{f}_I$, it follows that

$$\dim(\widetilde{W}) \geqslant n - \#\widetilde{I} - \# J_{\widetilde{I}} > n - \#I - \# J_I = \dim(W).$$

We conclude that $\varphi_I(W) = W \times \{\mathbf{0}_{n-r}\} \subsetneq \widetilde{W} \times \{\mathbf{0}_{n-\tilde{r}}\} \subset V(\mathbf{f})$ and, therefore, $\varphi_I(W)$ is not an irreducible component of $V(\mathbf{f}) \cap \bigcap_{i\notin I}\{x_i \neq 0\}$.

Conversely, if $\varphi_I(W) = W \times \{\mathbf{0}_{n-r}\}$ is not an irreducible component of $V(\mathbf{f})$, there is an irreducible component $\widetilde{W}$ of this variety such that $W \times \{\mathbf{0}_{n-r}\} \subsetneq \widetilde{W}$. The previous inclusion implies that $I_{\widetilde{W}} \subset I$. Assume $I_{\widetilde{W}} = \{\tilde{r}+1,\dots,n\}$ for $\tilde{r} > r$. Due to Lemma 4, $\pi_{I_{\widetilde{W}}}(\widetilde{W})$ is an irreducible component of $V(\mathbf{f}_{I_{\widetilde{W}}})$ having a non-empty intersection with $(\mathbb{C}^*)^{\tilde{r}}$. Therefore,

$$n - \#I_{\widetilde{W}} - \# J_{I_{\widetilde{W}}} = \dim\big(\pi_{I_{\widetilde{W}}}(\widetilde{W})\big) = \dim(\widetilde{W}) > \dim W = n - \#I - \# J_I,$$

and so, $\#I + \# J_I > \#I_{\widetilde{W}} + \# J_{I_{\widetilde{W}}}$. $\square$

As a consequence of Proposition 6, we have that the irreducible components of $V(\mathbf{f}) \subset \mathbb{C}^n$ are contained in the linear subspaces $\bigcap_{i\in I}\{x_i = 0\}$ associated to the subsets $I \subset \{1,\dots,n\}$ in

$$\Gamma = \left\{ I \subset \{1,\dots,n\} \;\middle|\; \forall J \subset J_I,\ \dim\left(\sum_{j\in J} \mathcal{A}_j^I\right) \geqslant \# J;\ \forall \widetilde{I} \subset I,\ \# J_{\widetilde{I}} + \#\widetilde{I} \geqslant \# J_I + \#I \right\}.$$

Note that there may be sets $I_1 \subsetneq I_2 \subset \{1,\dots,n\}$ both in Γ, as it can be seen in Example 3, where the three sets $\{1\}$, $\{2\}$ and $\{1,2\}$ give irreducible components of the variety.

If we write $V^*(\mathbf{f}_I)$ to denote the union of all the irreducible components of $V(\mathbf{f}_I)$ having a non-empty intersection with $(\mathbb{C}^*)^{n-\#I}$, from Lemma 4 and Proposition 6, we deduce that, for every $I \in \Gamma$,

$$\varphi_I\big(V^*(\mathbf{f}_I)\big) = \bigcup_{\substack{W \text{ irred. comp. of } V(\mathbf{f}) \\ \text{such that } I_W = I}} W.$$

We obtain the following characterization of the equidimensional decomposition of $V(\mathbf{f})$ and, using Lemma 1, a combinatorial expression for the degree of $V(\mathbf{f})$:

Theorem 7. *Let $\mathbf{f} = (f_1,\dots,f_s)$ be generic polynomials in n variables supported on $\mathcal{A} = (\mathcal{A}_1,\dots,\mathcal{A}_s) \subset (\mathbb{Z}^n_{\geqslant 0})^s$. For $k = 0,\dots,n$, let $V_k(\mathbf{f})$ be the equidimensional component of dimension k of $V(\mathbf{f})$. Then, using the previous notations:*

$$V_k(\mathbf{f}) = \bigcup_{\substack{I\in\Gamma, \\ \#I + \# J_I = n-k}} \varphi_I\big(V^*(\mathbf{f}_I)\big).$$

Moreover, $\deg(V(\mathbf{f})) = \sum_{I\in\Gamma} MV_{n-\#I}(\mathcal{A}^I, \Delta^{(n-\#I-\# J_I)})$.

Example 8. Consider the following system of generic polynomials in $\mathbb{Q}[X_1, X_2, X_3, X_4]$:

$$\begin{cases} a_1X_1X_4 + a_2X_1^2X_4^2 + a_3X_1X_2X_3 + a_4X_2X_3 = 0, \\ b_1X_1X_2 + b_2X_1X_2^2 + b_3X_1X_3X_4 + b_4X_3X_4 + b_5X_3X_4^2 = 0, \\ c_1X_1X_2X_4 + c_2X_1X_3X_4 + c_3X_2X_3 + c_4X_2X_3X_4 = 0, \\ d_1X_1 + d_2X_1^2 + d_3X_1X_2 + d_4X_3^2 + d_5X_3X_4 = 0. \end{cases}$$

Then, with the previous notation, $\Gamma = \{\emptyset, \{1,2\}, \{1,3\}, \{2,3\}, \{2,4\}, \{3,4\}\}$ and then,

- $V_0(\mathbf{f}) = V^*(\mathbf{f}) \cup \{(0, 0, \frac{b_4d_5}{b_5d_4}, -\frac{b_4}{b_5}), (-\frac{d_1}{d_2}, 0, 0, \frac{a_1d_2}{a_2d_1}), (-\frac{d_1}{d_2} + \frac{b_1d_3}{b_2d_2}, -\frac{b_1}{b_2}, 0, 0)\}$:
  - For $I = \emptyset$ we have 19 isolated solutions in $(\mathbb{C}^*)^4$ (this quantity is given by the mixed volume of the family of supports associated to the system).
  - For $I = \{1,2\}$ we have

$$\mathbf{f}_{\{1,2\}} = \begin{cases} b_4X_3X_4 + b_5X_3X_4^2, \\ d_4X_3^2 + d_5X_3X_4; \end{cases} \qquad V^*(\mathbf{f}_{\{1,2\}}) = \left\{\left(\frac{b_4d_5}{b_5d_4}, -\frac{b_4}{b_5}\right)\right\},$$

which gives the point $(0, 0, \frac{b_4d_5}{b_5d_4}, -\frac{b_4}{b_5})$.

  - For $I = \{2,3\}$ we have

$$\mathbf{f}_{\{2,3\}} = \begin{cases} a_1X_1X_4 + a_2X_1^2X_4^2, \\ d_1X_1 + d_2X_1^2; \end{cases} \qquad V^*(\mathbf{f}_{\{2,3\}}) = \left\{\left(-\frac{d_1}{d_2}, \frac{a_1d_2}{a_2d_1}\right)\right\},$$

which gives the point $(-\frac{d_1}{d_2}, 0, 0, \frac{a_1d_2}{a_2d_1})$.

  - For $I = \{3,4\}$ we have

$$\mathbf{f}_{\{3,4\}} = \begin{cases} b_1X_1X_2 + b_2X_1X_2^2, \\ d_1X_1 + d_2X_1^2 + d_3X_1X_2; \end{cases} \qquad V^*(\mathbf{f}_{\{3,4\}}) = \left\{\left(-\frac{d_1}{d_2} + \frac{b_1d_3}{b_2d_2}, -\frac{b_1}{b_2}\right)\right\},$$

which gives the point $(-\frac{d_1}{d_2} + \frac{b_1d_3}{b_2d_2}, -\frac{b_1}{b_2}, 0, 0)$.

- $V_1(\mathbf{f}) = \{x \in \mathbb{C}^4 \mid x_2 = 0,\ x_4 = 0,\ d_1x_1 + d_2x_1^2 + d_4x_3^2 = 0\}$:
  - For $I = \{2,4\}$ we have

$$\mathbf{f}_{\{2,4\}} = \{d_1X_1 + d_2X_1^2 + d_4X_3^2; \qquad V^*(\mathbf{f}_{\{2,4\}}) = \left\{(x_1, x_3) \,\middle|\, d_1x_1 + d_2x_1^2 + d_4x_3^2 = 0\right\},$$

which gives the curve $\{x_2 = 0, x_4 = 0, d_1x_1 + d_2x_1^2 + d_4x_3^2 = 0\}$.

- $V_2(\mathbf{f}) = \{x \in \mathbb{C}^4 \mid x_1 = 0,\ x_3 = 0\}$:
  - For $I = \{1,3\}$ we have

$$\mathbf{f}_{\{1,3\}} = \emptyset; \qquad V^*(\mathbf{f}_{\{1,3\}}) = \mathbb{C}^2,$$

which gives the plane $\{x_1 = 0, x_3 = 0\}$.

Remark 9. In the case of a generic system $\mathbf{f} = (f_1, \ldots, f_s)$ in n variables such that $\mathcal{A}_1 = \cdots = \mathcal{A}_s$, we have that the sets $I \in \Gamma$, $I \neq \emptyset$, are all $I \subset \{1, \ldots, n\}$ such that $\#J_I = 0$ and for every $\widetilde{I} \subsetneq I$, $\#J_{\widetilde{I}} > 0$; and $\emptyset \in \Gamma$ if and only if $\dim(\mathcal{A}_1) \geqslant s$.

Moreover, for every $I \in \Gamma$, $I \neq \emptyset$, $\varphi_I(V^*(\mathbf{f}_I)) = \{x_i = 0\ \forall i \in I\}$ and so, apart from the components intersecting $(\mathbb{C}^*)^n$ that correspond to $I = \emptyset$ (if $\emptyset \in \Gamma$), the only irreducible components of $V(\mathbf{f})$ are linear subspaces of $\mathbb{C}^n$.

3.2.2. Algorithmic results

According to Proposition 6, the subsets $I \subset \{1, \ldots, n\}$ which yield irreducible components of the variety $V(\mathbf{f})$ are the ones satisfying simultaneously

(1) $\forall J \subset J_I$, $\dim(\sum_{j \in J} \mathcal{A}_j^I) \geqslant \# J$,
(2) $\forall I' \subset I$, $\# J_{I'} + \# I' \geqslant \# J_I + \# I$.

Now we present an algorithm to obtain the sets I satisfying condition (2) and the inequality $\# I + \# J_I \leqslant n$, which is a necessary condition for the system $\mathbf{f}_I$ to have zeroes in $(\mathbb{C}^*)^{n-\# I}$, weaker but easier to check than condition (1). Our algorithm to find all the affine components of $V(\mathbf{f})$ (see Algorithm `GenericAffineSolve` below) checks only among these sets whether condition (1) is fulfilled or not by means of a mixed volume computation.

Algorithm `SpecialSets`

INPUT: A family of supports $\mathcal{A} = (\mathcal{A}_1, \ldots, \mathcal{A}_s) \subset (\mathbb{Z}^n_{\geqslant 0})^s$.

(1) $P_\emptyset := \min\{n, s\}$.
(2) If $P_\emptyset = s$, add $(\emptyset, \{1, \ldots, s\})$ to an empty list $\widetilde{\Gamma}$.
(3) For $k = 1, \ldots, n$:
 For every I such that $\# I = k$:
 (a) $P_I := \min\{n, \{P_{I'}\}_{I' \subset I, \# I' = k-1}, k + \# J_I\}$.
 (b) If $P_I = k + \# J_I$, add (I, J_I) to the list $\widetilde{\Gamma}$.

OUTPUT: The list $\widetilde{\Gamma}$ of all pairs of subsets (I, J_I), with $I \subset \{1, \ldots, n\}$ such that $\# I + \# J_I \leqslant n$ and for every $\widetilde{I} \subset I$, $\#\widetilde{I} + \# J_{\widetilde{I}} \geqslant \# I + \# J_I$.

First, let us prove the correctness of this algorithm:

Lemma 10. *For every $I \subset \{1, \ldots, n\}$, $P_I = \min_{\widetilde{I} \subset I}\{n, \#\widetilde{I} + \# J_{\widetilde{I}}\}$.*

Proof. By induction on $\# I$.

For $\# I = 0$, since $\# J_\emptyset = s$, we have that $P_\emptyset = \min\{n, \#\emptyset + \# J_\emptyset\}$.

Assuming the identity holds for every subset of cardinality $k - 1$, let $I \subset \{1, \ldots, n\}$ with $\# I = k$. Consider a proper subset $\widetilde{I}_0 \subsetneq I$. Then, there exists $I' \subset I$ with $\# I' = k - 1$ such that $\widetilde{I}_0 \subset I'$. By the inductive assumption,

$$P_{I'} = \min_{\widetilde{I} \subset I'}\{n, \#\widetilde{I} + \# J_{\widetilde{I}}\}$$

and so, $P_{I'} \leqslant \#\widetilde{I}_0 + \# J_{\widetilde{I}_0}$. On the other hand, by the definition of P_I, we have that $P_I \leqslant P_{I'}$. Thus, $P_I \leqslant \#\widetilde{I}_0 + \# J_{\widetilde{I}_0}$. $\square$

Now we estimate the complexity of the algorithm. Let $N = \sum_{j=1}^s \#\mathcal{A}_j$. For each I of cardinality $k \geqslant 1$, it takes $kN + \# J_I + 1$ operations to compute $k + \# J_I$. Taking the minimum among $k + 2$ numbers takes $k + 1$ comparisons. Thus, the complexity of Step 3a is $k(N + 1) + \# J_I + 2$. In Step 3b, we add one comparison. As we have to do this for each subset of $\{1, \ldots, n\}$ of cardinality $k \geqslant 1$ and for the empty set, the complexity is bounded by $2 + \sum_{k=1}^n \binom{n}{k}(k(N + 1) + s + 3) = 2 + n2^{n-1}(N + 1) + (2^n - 1)(s + 3)$ which is of order $O(nN2^n)$.

Unfortunately, the exponential complexity of the algorithm cannot be avoided, as the following examples show:

Example 11. Let $L_1, \ldots, L_n \in \mathbb{Q}[X_1, \ldots, X_n]$ be generic affine linear forms. Consider the set of generic polynomials $f_1 = X_1.L_1, \ldots, f_n = X_n.L_n$. In this case, if $\mathcal{A} = (\mathcal{A}_1, \ldots, \mathcal{A}_n)$ is their family of supports, $\Gamma = \{I \mid I \subseteq \{1, \ldots, n\}\}$ and therefore $\#\Gamma = 2^n$.

One may think the exponentiality of the cardinal of the set Γ in this example arises from the fact that the variables are factors of the polynomials. However, this is not always the case as the following example shows. This example also shows how subroutine `SpecialSets` is useful to discard subsets which do not lead to affine components: in this case, the only element in $\widetilde{\Gamma}$ which does not correspond to a set in Γ is $(\emptyset, \{1, \dots, 2n\})$.

Example 12. Consider generic polynomials $f_1, \dots, f_{2n} \in \mathbb{Q}[X_1, \dots, X_{2n}]$,

$$f_j = \sum_{1 \leqslant k \leqslant n} c_{jk} X_{2k-1} X_{2k}, \quad j = 1, \dots, 2n,$$

all supported on the set $\mathcal{A} = \{e_{2k-1} + e_{2k} \mid 1 \leqslant k \leqslant n\}$ where e_i denotes the ith vector of the canonical basis of $\mathbb{R}^{2n}$. Then, it is easy to see that, for any subset $S \subset \{1, \dots, n\}$, the set $I_S = \{2k-1 \mid k \in S\} \cup \{2k \mid k \in \{1, \dots, n\} \setminus S\}$ is in Γ, and that the sets I_S are the only ones (together with the empty set) obtained as first coordinate of elements of $\widetilde{\Gamma}$ by the previous algorithm. Therefore, in this case, we have that the number of elements of the list $\widetilde{\Gamma}$ is $2^n + 1$.

In the following example, the usefulness of subroutine `SpecialSets` is more evident:

Example 13. Let $\mathcal{A} = (\mathcal{A}_1, \dots, \mathcal{A}_n)$ be a family of finite sets of $(\mathbb{Z}_{\geqslant 0})^n$ such that, for every $1 \leqslant j \leqslant n$, there exists a non-negative integer d_{j_i} such that $d_{j_i}.e_i \in \mathcal{A}_j$ for every $1 \leqslant i \leqslant n$. Then the output of subroutine `SpecialSets` in this case is $\widetilde{\Gamma} = \{(\emptyset, \{1, \dots, n\})\}$.

The following algorithm computes a family of geometric resolutions describing the affine variety defined by a generic system $\mathbf{f}$ with given supports $\mathcal{A}$.

Algorithm `GenericAffineSolve`

INPUT: A sparse representation of the generic system $\mathbf{f} = (f_1, \dots, f_s)$ of polynomials in $\mathbb{Q}[X_1, \dots, X_n]$.

(1) Apply Algorithm `SpecialSets` to the family of the supports $\mathcal{A} = (\mathcal{A}_1, \dots, \mathcal{A}_s)$ of $\mathbf{f}$ to obtain the list $\widetilde{\Gamma}$ of pairs of sets (I, J_I) with $I \subset \{1, \dots, n\}$ such that $\#I + \#J_I \leqslant n$ and $\forall \widetilde{I} \subset I,\ \#\widetilde{I} + \#J_{\widetilde{I}} \geqslant \#I + \#J_I$.
(2) For every $(I, J_I) \in \widetilde{\Gamma}$:
    (a) For $j \in J_I$, compute $\mathcal{A}_j^I := \{\pi_I(a) \mid a \in \mathcal{A}_j$ such that $a_i = 0\ \forall i \in I\}$, and obtain the sparse representation of the system $\mathbf{f}_I$ supported on $\mathcal{A}^I = (\mathcal{A}_j^I)_{j \in J_I}$.
    (b) Apply Algorithm `GenericToricSolve` to the sparse system $\mathbf{f}_I$ to obtain a geometric resolution R^I (possibly empty) of the affine components of the set of solutions of $\mathbf{f}_I$ intersecting the torus $(\mathbb{C}^*)^{n-\#I}$.
    (c) If $R^I \neq \emptyset$:
        (i) Obtain the geometric resolution $\varphi_I(R^I)$ of the union of all irreducible components W of $V(\mathbf{f})$ such that $I_W = I$ by adding zeroes to R^I in the coordinates indexed by I.
        (ii) If $n - (\#I + \#J_I) = k$, add $\varphi_I(R^I)$ to the list $\mathcal{V}_k$.

OUTPUT: A family of lists $\mathcal{V}_k$, $0 \leqslant k \leqslant n-1$, of geometric resolutions, each list either empty or describing the equidimensional component of $V(\mathbf{f})$ of dimension k.

The correctness of this algorithm is straightforward from Proposition 6 and Theorem 7.

When applying Algorithm `GenericToricSolve` in Step 2b, we need a pre-processing obtaining a fine mixed subdivision. To do so, we may apply the dynamic enumeration procedure from Mizutani et al. (2007). This procedure proved to be very efficient even for large systems, but there are no explicit complexity bounds; for this reason, we do not include its cost in our complexity estimates. This pre-processing, in particular, decides whether a set I satisfies $\dim(\sum_{j \in J} \mathcal{A}_j^I) \geqslant \#J$ for every $J \subset J_I$ and, therefore, it discards the sets $I \in \widetilde{\Gamma} \setminus \Gamma$. For this reason, we will only consider the complexity of the computations for the sets $I \in \Gamma$. This complexity can be estimated from the complexities of the subroutines applied at the intermediate steps (see Proposition 2) and is of order

$$O\Bigg(\sum_{I\in\Gamma}(n-\#I)^3\big(N_I+(n-\#I-\#J_I)(n-\#I)\big) \times \log(d_I)M(D_I)\big(M(\mathfrak{M}_I)\big(M(D_I)+M(E_I)\big)+D_I^2\big)\Bigg)$$

where, for every $I \in \Gamma$, N_I, d_I, D_I, $\mathfrak{M}_I$ and E_I are the parameters defined in the complexity of Algorithm `GenericToricSolve`, associated to the system $\mathbf{f}_I$.

In order to estimate the overall complexity of the algorithm, note that $N_I \leqslant N := \sum_{j=1}^{s} \#\mathcal{A}_j$ and $d_I \leqslant d := \max_{1\leqslant j\leqslant s}\{\deg(f_j)\}$ for every $I \in \Gamma$. In addition, $\mathcal{D} := \sum_{I\in\Gamma} D_I = \deg V(\mathbf{f})$. Note that, if $\omega_{\max}$ is the maximum value of the lifting functions applied to the supports $\mathcal{A}^I$, for every $I \in \Gamma$ we have

$$\begin{aligned} E_I &\leqslant MV_{n-\#I+1}\big(\Delta\times\{0\}, \big(\mathcal{A}_j^I\times\{0,\omega_{\max}\}\big)_{j\in J_I}, \big(\Delta\times\{0,\omega_{\max}\}\big)^{(n-\#I-\#J_I)}\big) \\ &\leqslant \omega_{\max}\Bigg((n-\#I-\#J_I)MV_{n-\#I}\big(\mathcal{A}^I,\Delta^{(n-\#I-\#J_I)}\big) \\ &\quad + \sum_{\ell\in J_I} MV_{n-\#I}\big(\big(\mathcal{A}_j^I\big)_{j\neq\ell},\Delta^{(n-\#I-\#J_I+1)}\big)\Bigg). \end{aligned}$$

Then, if $\mathcal{E}_{\max} := \max_{I\in\Gamma}\{(n-\#I-\#J_I)MV_{n-\#I}(\mathcal{A}^I,\Delta^{(n-\#I-\#J_I)}) + \sum_{\ell\in J_I} MV_{n-\#I}((\mathcal{A}_j^I)_{j\neq\ell}, \Delta^{(n-\#I-\#J_I+1)})\}$ and $\mathfrak{M}_{\max} := \max_{I\in\Gamma}\{\mathfrak{M}_I\}$, taking into account the complexity of Step 1, we have

Theorem 14. *Let $\mathbf{f} = (f_1,\ldots,f_s)$ be a system of generic polynomials in $\mathbb{Q}[X_1,\ldots,X_n]$ supported on $\mathcal{A} = (\mathcal{A}_1,\ldots,\mathcal{A}_s)$. `GenericAffineSolve` is a probabilistic algorithm that computes a family of lists $\mathcal{V}_k$, $0 \leqslant k \leqslant n-1$, of geometric resolutions, each list either empty or describing the equidimensional component of $V(\mathbf{f})$ of dimension k. Using the previous notation, the complexity of this algorithm is of order*

$$O\big(n2^nN + n^3\big(N+n^2\big)\log(d)M(\mathcal{D})\big(M(\mathfrak{M}_{\max})\big(M(\mathcal{D})+M(\omega_{\max}\mathcal{E}_{\max})\big)+\mathcal{D}^2\big)\big).$$

4. Arbitrary sparse systems

4.1. An upper bound for the degree

The aim of this section is to show a bound for the degree of the affine variety defined by a square system of sparse polynomials which takes into account its sparsity.

The following example shows that the degree of an affine variety defined by a generic sparse system with given supports is not an upper bound for the degree of the variety defined by a particular system with the same supports. One may think the problem arises from the presence of irreducible components not intersecting the torus either for the generic or the particular systems. However, this is not the case:

Example 15. Consider the following system:

$$\begin{cases} X_1X_2 - X_1 - X_2 + 1 = (X_1-1)(X_2-1) = 0, \\ X_1X_3 - X_1 - X_3 + 1 = (X_1-1)(X_3-1) = 0, \\ X_2X_3 - X_2 - X_3 + 1 = (X_2-1)(X_3-1) = 0. \end{cases}$$

The variety defined by the system consists of 3 lines, each having a non-empty intersection with $(\mathbb{C}^*)^3$. However, if $\mathcal{A} = (\mathcal{A}_1,\mathcal{A}_2,\mathcal{A}_3)$ is the family of the supports of the polynomials, the degree of the variety defined by a generic system with the same supports is $MV_3(\mathcal{A}_1,\mathcal{A}_2,\mathcal{A}_3) = 2$.

Our bound for the degree, which is stated in the following theorem, is the mixed volume of a family of sets obtained by enlarging the supports of the polynomials involved.

Theorem 16. *Let $\mathbf{f} = (f_1, \ldots, f_n)$ be n polynomials in $\mathbb{C}[X_1, \ldots, X_n]$ supported on $\mathcal{A} = (\mathcal{A}_1, \ldots, \mathcal{A}_n)$ and let $V(\mathbf{f}) = \{x \in \mathbb{C}^n \mid f_j(x) = 0$ for every $1 \leqslant j \leqslant n\}$. Let $\Delta = \{0, e_1, \ldots, e_n\}$ where e_i the ith vector of the canonical basis of $\mathbb{R}^n$. Then*

$$\deg\bigl(V(\mathbf{f})\bigr) \leqslant MV_n(\mathcal{A}_1 \cup \Delta, \ldots, \mathcal{A}_n \cup \Delta).$$

Before proving the theorem, we will fix some notation and definitions.

Let $r_j = \#(\mathcal{A}_j \cup \Delta) - 1$ for $1 \leqslant j \leqslant n$. Consider the morphism of varieties $\varphi : \mathbb{C}^n \to \mathbb{P}^n \times \mathbb{P}^{r_1} \times \cdots \times \mathbb{P}^{r_n}$ defined by

$$\varphi(x) = \bigl((1 : x), (x^a)_{a \in \mathcal{A}_1 \cup \Delta}, \ldots, (x^a)_{a \in \mathcal{A}_n \cup \Delta}\bigr). \tag{1}$$

Let $\mathcal{X} = \overline{\varphi(\mathbb{C}^n)}$. For $1 \leqslant j \leqslant n$, we denote by L_j the linear form in $\mathbb{P}^{r_j}$ given by the coefficients of the polynomial f_j, that is to say, if $f_j = \sum_{a \in \mathcal{A}_j} c_{j,a} X^a$, then $L_j = \sum_{a \in \mathcal{A}_j} c_{j,a} X_{j,a}$.

For each integer k ($0 \leqslant k \leqslant n$) and each subset $S \subset \{1, \ldots, k\}$ we define the variety $\mathcal{X}_{k,S}$ recursively in the following way:

(1) $\mathcal{X}_{0,\emptyset} = \mathcal{X}$.
(2) Provided $\mathcal{X}_{k,S}$ is defined for every $S \subset \{1, \ldots, k\}$, we define $\mathcal{X}_{k+1,T}$ with $T \subset \{1, \ldots, k+1\}$ as follows:
  - If $k + 1 \notin T$, $\mathcal{X}_{k+1,T}$ is the union of the irreducible components of $\mathcal{X}_{k,T}$ included in $\{L_{k+1} = 0\}$.
  - If $k + 1 \in T$, $\mathcal{X}_{k+1,T}$ is the intersection of $\{L_{k+1} = 0\}$ with the union of the irreducible components of $\mathcal{X}_{k,T \setminus \{k+1\}}$ not included in $\{L_{k+1} = 0\}$.

Note that, from this definition, each $\mathcal{X}_{k,S}$ is an equidimensional variety of dimension $n - \#S$. Moreover, if $\pi : \mathbb{P}^n \times \mathbb{P}^{r_1} \times \cdots \times \mathbb{P}^{r_n} \to \mathbb{P}^n$ is the projection onto the first factor, it is easy to see inductively that, for every $1 \leqslant k \leqslant n$,

$$\bigcup_{S \subset \{1,\ldots,k\}} \pi(\mathcal{X}_{k,S}) = \overline{V(f_1, \ldots, f_k)} \subset \mathbb{P}^n.$$

For an equidimensional subvariety W of $\mathcal{X}$, we define its multidegrees $\deg_{(r,0_k,1_{n-k})}(W)$, for $r, k \in \mathbb{Z}_{\geqslant 0}$ such that $n - k + r = \dim(W)$, as

$$\deg_{(r,0_k,1_{n-k})}(W) = \max\left\{\#\left(W \cap \bigcap_{j=1}^{r} \{\ell_{0,j} = 0\} \cap \bigcap_{j=k+1}^{n} \{\ell_j = 0\}\right)\right\}$$

where the maximum is taken over all $(\ell_{0,1}, \ldots, \ell_{0,r}, \ell_{k+1}, \ldots, \ell_n)$ such that each $\ell_{0,j}$ is a linear form in $\mathbb{P}^n$ and each ℓ_j is a linear form in $\mathbb{P}^{r_j}$ and the intersection is finite. Note that the 1 subscript indicates how many projective spaces are cut by a linear form and the 0 subscript shows how many remain uncut. As in the case of the standard degree of affine or projective varieties (see Heintz, 1983), the maximum is attained generically.

In particular, it is clear that $\deg_{(0,0_0,1_n)}(\mathcal{X}) = MV_n(\mathcal{A}_1 \cup \Delta, \ldots, \mathcal{A}_n \cup \Delta)$.

Lemma 17. *Under the previous assumptions, definitions and notations,*

$$\begin{aligned}&\deg_{(k-\#S,0_k,1_{n-k})}(\mathcal{X}_{k,S}) \\ &\quad \geqslant \deg_{(k+1-\#S,0_{k+1},1_{n-k-1})}(\mathcal{X}_{k+1,S}) + \deg_{(k-\#S,0_{k+1},1_{n-k-1})}(\mathcal{X}_{k+1,S \cup \{k+1\}}).\end{aligned}$$

Proof. As the variety $\mathcal{X}_{k,S} = \mathcal{X}_{k+1,S} \cup \widetilde{\mathcal{X}_{k,S}}$, where $\widetilde{\mathcal{X}_{k,S}}$ is the union of the irreducible components of $\mathcal{X}_{k,S}$ not contained in $\{L_{k+1} = 0\}$, using genericity in the definition of multidegrees, we have that

$$\deg_{(k-\#S,0_k,1_{n-k})}(\mathcal{X}_{k,S}) = \deg_{(k-\#S,0_k,1_{n-k})}(\mathcal{X}_{k+1,S}) + \deg_{(k-\#S,0_k,1_{n-k})}(\widetilde{\mathcal{X}_{k,S}}).$$

Note that, by adding the simplex Δ to the supports $\mathcal{A}_1, \ldots, \mathcal{A}_n$, the points of the varieties in $\mathbb{P}^n \times \mathbb{P}^{r_1} \times \cdots \times \mathbb{P}^{r_n}$ we are considering have a copy of their coordinate in $\mathbb{P}^n$ in each coordinate in $\mathbb{P}^{r_j}$ for $j = 1, \ldots, n$ (see Eq. (1)). Because of this, if ℓ_0 is a linear form in $\mathbb{P}^n$ and ℓ_{k+1} is exactly the same linear form involving only the corresponding coordinates in $\mathbb{P}^{r_{k+1}}$, we have that $\mathcal{X}_{k+1,S} \cap \{\ell_0 = 0\} = \mathcal{X}_{k+1,S} \cap \{\ell_{k+1} = 0\}$ and therefore,

$$\deg_{(k-\#S,0_k,1_{n-k})}(\mathcal{X}_{k+1,S}) \geqslant \deg_{(k+1-\#S,0_{k+1},1_{n-k-1})}(\mathcal{X}_{k+1,S})$$

and

$$\begin{aligned}\deg_{(k-\#S,0_k,1_{n-k})}(\widetilde{\mathcal{X}_{k,S}}) &\geqslant \deg_{(k-\#S,0_{k+1},1_{n-k-1})}\big(\widetilde{\mathcal{X}_{k,S}} \cap \{L_{k+1} = 0\}\big)\\ &= \deg_{(k-\#S,0_{k+1},1_{n-k-1})}(\mathcal{X}_{k+1,S\cup\{k+1\}}). \qquad \square\end{aligned}$$

Now, we can prove Theorem 16:

Proof of Theorem 16. Consider the projection $\pi : \mathbb{P}^n \times \mathbb{P}^{r_1} \times \cdots \times \mathbb{P}^{r_n} \to \mathbb{P}^n$ onto the first factor. As

$$\bigcup_{S\subset\{1,\ldots,n\}} \pi(\mathcal{X}_{n,S}) = \overline{V(\mathbf{f})} \subset \mathbb{P}^n,$$

we have that

$$\begin{aligned}\deg\big(V(\mathbf{f})\big) = \deg\Bigg(\bigcup_{S\subset\{1,\ldots,n\}} \pi(\mathcal{X}_{n,S})\Bigg) &\leqslant \sum_{S\subset\{1,\ldots,n\}} \deg \pi(\mathcal{X}_{n,S})\\ &= \sum_{S\subset\{1,\ldots,n\}} \deg_{(n-\#S,0_n,1_0)}(\mathcal{X}_{n,S})\end{aligned}$$

(this last equality is nothing but our definition of multidegree).

Applying inductively Lemma 17, we get that, for each $0 \leqslant k \leqslant n$,

$$\sum_{S\subset\{1,\ldots,k\}} \deg_{(k-\#S,0_k,1_{n-k})}(\mathcal{X}_{k,S}) \leqslant \deg_{(0,0_0,1_n)} \mathcal{X}_{0,\emptyset}$$

and therefore we obtain that

$$\sum_{S\subset\{1,\ldots,n\}} \deg_{(n-\#S,0_n,1_0)}(\mathcal{X}_{n,S}) \leqslant \deg_{(0,0_0,1_n)}(\mathcal{X}_{0,\emptyset}) = MV_n(\mathcal{A}_1 \cup \Delta, \ldots, \mathcal{A}_n \cup \Delta). \qquad \square$$

In the following examples, we show that this bound may be attained:

Example 18. Let $S \subsetneq \{1, \ldots, n\}$ and let $f_1, \ldots, f_n \in \mathbb{Q}[X_1, \ldots, X_n]$ be polynomials of degree d in the variables $(X_i)_{i\in S}$ with no zero coefficients (that is to say, the monomials appearing in $f_1, \ldots, f_n$ are those of degree less or equal to d in the variables $(X_i)_{i\in S}$). If $\mathcal{A}$ is their common support, it is evident that $MV_n(\mathcal{A}^{(n)}) = 0$ and that $SM_n(\mathcal{A}^{(n)}) = 0$. However, $MV_n((\mathcal{A} \cup \Delta)^{(n)}) = d^{\#S}$ and this degree can be attained for special choices of the polynomials: If $f_1, \ldots, f_{\#S} \in \mathbb{Q}[(X_i)_{i\in S}]$ is a family with $d^{\#S}$ isolated solutions in $\mathbb{C}^{\#S}$, take linear combinations $f_{\#S+1}, \ldots, f_n$ of $f_1, \ldots, f_{\#S}$. Then, the family of polynomials $f_1, \ldots, f_{\#S}, f_{\#S+1}, \ldots, f_n \in \mathbb{Q}[X_1, \ldots, X_n]$ defines a variety formed by $d^{\#S}$ affine linear spaces of dimension $n - \#S$.

Our bound can be also attained for generic systems:

Example 19. Consider the family of generic polynomials $f_1, \ldots, f_n$ defined in Example 11 and let their supports $\mathcal{A} = (\mathcal{A}_1, \ldots, \mathcal{A}_n)$. Then, the affine variety they define consists of 2^n points, and therefore, its degree is $2^n = MV_n(\mathcal{A}_1 \cup \Delta, \ldots, \mathcal{A}_n \cup \Delta)$.

4.2. An algorithm in the non-generic case

In the sequel we will describe an algorithm that, given an *arbitrary* square system of sparse polynomials, provides a finite set of points in each irreducible component of the affine variety the system defines. The complexity of this algorithm depends on the bound for the degree of the variety obtained in the previous section.

Let $\mathbf{f} = (f_1, \ldots, f_n)$ be n polynomials in $\mathbb{Q}[X_1, \ldots, X_n]$ supported on $\mathcal{A} = (\mathcal{A}_1, \ldots, \mathcal{A}_n)$ and, for a fixed k, $1 \leqslant k \leqslant n-1$, let $L_1, \ldots, L_k$ be generic affine linear forms in $\mathbb{Q}[X_1, \ldots, X_n]$. Then, if $V_k(\mathbf{f})$ is the equidimensional component of dimension k of $V(\mathbf{f})$, the isolated common zeroes of $f_1, \ldots, f_n, L_1, \ldots, L_k$ are $\deg(V_k(\mathbf{f}))$ points in $V_k(\mathbf{f})$. The idea is to represent each equidimensional component by means of the corresponding set of points (cf. the notion of witness point set in Sommese and Wampler, 2005).

Example 20. Consider the following polynomial system:

$$\mathbf{f} = \begin{cases} X_1^3X_2X_3 - X_1X_2X_3^3 - X_1^2 + X_3^2 = (X_1X_2X_3 - 1)(X_1 - X_3)(X_1 + X_3), \\ X_1^2X_2^2X_3 - X_1^2X_2X_3 - X_1X_2^2X_3^2 + X_1X_2X_3^2 - X_1X_2 + X_1 + X_3X_2 - X_3 \\ \quad = (X_1X_2X_3 - 1)(X_1 - X_3)(X_2 - 1), \\ X_1X_2^3X_3 - X_1X_2X_3^2 - X_2^2 + X_3 = (X_1X_2X_3 - 1)(X_2^2 - X_3). \end{cases}$$

The equidimensional components of $V(\mathbf{f})$ are

$$V_0(\mathbf{f}) = \big\{(-1, 1, 1)\big\}, \qquad V_1(\mathbf{f}) = \big\{x_1 - x_3 = 0, x_2^2 - x_3 = 0\big\}, \qquad V_2(\mathbf{f}) = \{x_1x_2x_3 - 1 = 0\}.$$

Taking $L_1 = X_1 - X_2$ and $L_2 = 6X_2 - X_3 + 7$, we have

- The set of isolated points of $V(\mathbf{f}) \cap \{x_1 - x_2 = 0\}$ is $\{(1,1,1), (0,0,0)\}$, which is a set with $2 = \deg(V_1(\mathbf{f}))$ points in $V_1(\mathbf{f})$.
- $V(\mathbf{f}) \cap \{x_1 - x_2 = 0, 6x_2 - x_3 + 7 = 0\} = \{(-1,-1,1), (-\frac{1}{2}, -\frac{1}{2}, 4), (\frac{1}{3}, \frac{1}{3}, 9)\}$, which is a set with $3 = \deg(V_2(\mathbf{f}))$ points in $V_2(\mathbf{f})$.

For a fixed k, $0 \leqslant k \leqslant n-1$, in order to compute the isolated common zeroes of $f_1, \ldots, f_n$, $L_1, \ldots, L_k$, by taking n generic linear combinations of these polynomials, we obtain a system of n polynomials in n variables having these points among its isolated zeroes (see Heintz, 1983). Note that, in order to achieve this, it suffices to take linear combinations of the form

$$f_i(X) + \sum_{j=1}^{k} b_{ij} L_j(X), \quad i = 1, \ldots, n$$

for generic b_{ij} $(1 \leqslant i \leqslant n, 1 \leqslant j \leqslant k)$.

Procedure `PointsInEquidComps` described below computes a family of n geometric resolutions $R^{(k)}$, for $0 \leqslant k \leqslant n-1$, encoding a finite set of points and such that $R^{(k)}$ represents at least $\deg(W)$ points in each irreducible component W of dimension k of $V(\mathbf{f})$.

The intermediate subroutine `CleanGR` takes as input a geometric resolution $(q(u), v_1(u), \ldots, v_n(u)) \subset (\mathbb{Q}[u])^{n+1}$ of a finite set of points $\mathcal{P} \subset \mathbb{C}^n$ and a list $\mathbf{f} = (f_1, \ldots, f_n)$ of polynomials in $\mathbb{Q}[X_1, \ldots, X_n]$, and computes a geometric resolution $(Q(u), V_1(u), \ldots, V_n(u))$ of $\mathcal{P} \cap V(\mathbf{f})$:

$$Q(u) = \gcd\big(q(u), f_1\big(v_1(u), \ldots, v_n(u)\big), \ldots, f_n\big(v_1(u), \ldots, v_n(u)\big)\big),$$

$$V_i(u) = v_i(u) \bmod Q(u), \quad i = 1, \ldots, n.$$

Let $\mathcal{A}_j$ be the support of f_j, d an upper bound for $\deg(f_j)$, $j = 1, \ldots, n$, and $D = \deg q$. First, the subroutine computes slp's of length $O(nD \log D)$ for the polynomials v_i, $i = 1, \ldots, n$. The gcd $Q(u)$ is computed successively as follows: For $j = 1, \ldots, n$, the subroutine computes an slp of length $\mathcal{L}_j = O(n \log d \# \mathcal{A}_j)$ for the polynomial f_j and, by multipoint evaluation and interpolation, the dense representation of $F_j(u) = f_j(v_1(u), \ldots, v_n(u))$ within complexity $O(M(dD)(\mathcal{L}_j + nD \log D))$; then, it computes $Q_j(u) := \gcd(Q_{j-1}, F_j(u))$ within $O(M(dD))$ additional operations. Finally, the polynomials $V_i(u)$ for $i = 1, \ldots, n$ are obtained within complexity $O(nM(D))$. The overall complexity of the procedure is of order $O(M(dD)(n \log d \sum_{j=1}^{n} \# \mathcal{A}_j + n^2 D \log D))$.

Algorithm `PointsInEquidComps`

INPUT: A sparse representation of a system $\mathbf{f} = (f_1, \ldots, f_n)$ of polynomials in $\mathbb{Q}[X_1, \ldots, X_n]$ supported on $\mathcal{A} = (\mathcal{A}_1, \ldots, \mathcal{A}_n)$, a lifting function $\omega = (\omega_1, \ldots, \omega_n)$ for $\mathcal{A}_\Delta = (\mathcal{A}_1 \cup \Delta, \ldots, \mathcal{A}_n \cup \Delta)$ and the mixed cells in the induced subdivision of $\mathcal{A}_\Delta$.

(1) Choose randomly coefficients for a polynomial system $\mathbf{g} = (g_1, \ldots, g_n)$ supported on $\mathcal{A}_\Delta$.
(2) Apply the algorithm in Jeronimo et al. (2009, Section 5) to $\mathbf{g}$ to obtain a geometric resolution $R_{\mathbf{g}}$ of its zeroes in $\mathbb{C}^n$.
(3) Choose randomly $n-1$ affine linear forms $L_1, \ldots, L_{n-1}$ in the variables $X = (X_1, \ldots, X_n)$ and $n(n-1)$ integer numbers $(b_{ij})_{1 \leqslant i \leqslant n, 1 \leqslant j \leqslant n-1}$.
(4) For $k = 0, \ldots, n-1$:
   (a) Obtain the sparse representation of the polynomials $h_i^{(k)}(X) = f_i(X) + \sum_{j=1}^{k} b_{ij} L_j(X)$ for $1 \leqslant i \leqslant n$.
   (b) Apply the algorithm in Jeronimo et al. (2009, Section 6) to $\mathbf{h}^{(k)} = (h_1^{(k)}, \ldots, h_n^{(k)})$ to obtain from $R_{\mathbf{g}}$ a geometric resolution of a finite set $\mathcal{P}_k$ which contains the isolated common zeroes of $\mathbf{h}^{(k)}$ in $\mathbb{C}^n$.
   (c) Apply subroutine `CleanGR` to the previous geometric resolution and $\mathbf{f}$ to obtain a geometric resolution $R^{(k)}$ of $\mathcal{P}_k \cap V(\mathbf{f})$.

OUTPUT: The n geometric resolutions $R^{(k)}$ for $0 \leqslant k \leqslant n-1$.

In the sequel we will estimate the complexity of this procedure.

Steps 1 and 3 are fulfilled by taking a random choice of numbers. We will not consider the cost of this random choice in the overall complexity (see Remark 23). The complexity of Step 2 is

$$O\big(\big(n^3 N_\Delta \log(d) + n^{1+\Omega}\big) M(D_\Delta) M(\mathfrak{M}_\Delta) \big(M(D_\Delta) + M(E_\Delta)\big)\big)$$

where

- $N_\Delta := \sum_{j=1}^{n} \#(\mathcal{A}_j \cup \Delta)$;
- $d := \max_{1 \leqslant j \leqslant n} \{\deg f_j\}$;
- $D_\Delta := MV_n(\mathcal{A}_1 \cup \Delta, \ldots, \mathcal{A}_n \cup \Delta)$;
- $\mathfrak{M}_\Delta := \max\{\|\mu\|\}$ where the maximum ranges over all primitive normal vectors to the mixed cells in the fine mixed subdivision of $\mathcal{A}_\Delta$ induced by ω;
- $E_\Delta := MV_{n+1}(\Delta \times \{0\}, (\mathcal{A}_1 \cup \Delta)(\omega_1), \ldots, (\mathcal{A}_n \cup \Delta)(\omega_n))$ where $(\mathcal{A}_j \cup \Delta)(\omega_j)$ for every $1 \leqslant j \leqslant n$ is the set $\mathcal{A}_j \cup \Delta$ lifted by ω.

In Step 4b, we compute a finite set which includes the affine isolated zeroes of the system $\mathbf{h}^{(k)}$. By applying the result in Jeronimo et al. (2009, Proposition 6.1), we have that the complexity of this step is bounded by $O((n^2 N_\Delta \log d + n^{1+\Omega}) M(D_\Delta) M(E'_\Delta))$ where $E'_\Delta := MV_{n+1}(\{0\} \times \Delta, \{0,1\} \times (\mathcal{A}_1 \cup \Delta), \ldots, \{0,1\} \times (\mathcal{A}_n \cup \Delta))$. Finally, the complexity of Step 4c is of order $O(M(dD_\Delta)(n \log d \sum_{j=1}^{n} \# \mathcal{A}_j + n^2 D_\Delta \log D_\Delta))$.

Note that the parameters E_Δ and E'_Δ in the previous complexities can be bounded as follows:

$$E'_\Delta = \sum_{j=1}^{n} MV_n(\Delta, \mathcal{A}_1 \cup \Delta, \ldots, \widehat{\mathcal{A}_j \cup \Delta}, \ldots, \mathcal{A}_n \cup \Delta) \leqslant nD_\Delta$$

and, if $\omega_{\max} := \max_{j,a}\{\omega_j(a) \mid 1 \leqslant j \leqslant n,\ a \in \mathcal{A}_j \cup \Delta\}$,

$$E_\Delta \leqslant MV_{n+1}\big(\Delta \times \{0\}, (\mathcal{A}_1 \cup \Delta) \times \{0, \omega_{\max}\}, \ldots, (\mathcal{A}_n \cup \Delta) \times \{0, \omega_{\max}\}\big) \leqslant \omega_{\max} n D_\Delta.$$

Taking into account these bounds, we have

Theorem 21. *Let $\mathbf{f} = (f_1, \ldots, f_n)$ be n polynomials in $\mathbb{Q}[X_1, \ldots, X_n]$ supported on $\mathcal{A} = (\mathcal{A}_1, \ldots, \mathcal{A}_n)$. `PointsInEquidComps` is a probabilistic algorithm which computes a family of n geometric resolutions $(R^{(0)}, R^{(1)}, \ldots, R^{(n-1)})$ such that, for every $0 \leqslant k \leqslant n-1$, $R^{(k)}$ represents a finite set containing $\deg V_k(\mathbf{f})$ points in the equidimensional component $V_k(\mathbf{f})$ of dimension k of $V(\mathbf{f})$. Using the previous notation, the complexity of the algorithm is of order*

$$O\big(\omega_{\max} n^4 N_\Delta \log(d) M(dD_\Delta) M(D_\Delta) M(\mathfrak{M}_\Delta)\big).$$

Example 22. Consider the polynomial system introduced in Example 15, given by the polynomials

$$\mathbf{f} = \begin{cases} f_1 = X_1X_2 - X_1 - X_2 + 1, \\ f_2 = X_1X_3 - X_1 - X_3 + 1, \\ f_3 = X_2X_3 - X_2 - X_3 + 1 \end{cases}$$

supported on $\mathcal{A}_1 = \{(1,1,0), (1,0,0), (0,1,0), (0,0,0)\}$, $\mathcal{A}_2 = \{(1,0,1), (1,0,0), (0,0,1), (0,0,0)\}$ and $\mathcal{A}_3 = \{(0,1,1), (0,1,0), (0,0,1), (0,0,0)\}$ respectively.

Algorithm `PointsInEquidComps` first chooses (at random) a system supported on $(\mathcal{A}_1 \cup \Delta, \mathcal{A}_2 \cup \Delta, \mathcal{A}_3 \cup \Delta)$, for example:

$$\mathbf{g} = \begin{cases} 2X_1X_2 - 2X_1 + X_2 - X_3 + 1, \\ X_1X_3 - X_1 + 2X_2 + 2X_3 + 2, \\ X_2X_3 + X_1 - 2X_2 + X_3 - 1 \end{cases}$$

and computes a geometric resolution $R_\mathbf{g}$ of its isolated common roots in $\mathbb{C}^3$:

$$R_\mathbf{g} = \begin{cases} u^5 - \frac{9}{2}u^4 - 17u^3 + 80u^2 - 2u - \frac{155}{2} = 0, \\ X_1 = -\frac{9}{100}u^4 + \frac{7}{40}u^3 + \frac{351}{200}u^2 - \frac{543}{200}u - \frac{137}{40}, \\ X_2 = -\frac{1}{200}u^4 - \frac{1}{80}u^3 - \frac{1}{400}u^2 + \frac{233}{400}u + \frac{7}{80}, \\ X_3 = -\frac{1}{10}u^4 + \frac{3}{20}u^3 + \frac{7}{4}u^2 - \frac{51}{20}u - \frac{13}{4}. \end{cases}$$

Then, in Step 3, the algorithm takes two linear forms:

$$L_1 = X_1 + X_2 + 2X_3,$$
$$L_2 = X_1 + 2X_2.$$

In Step 4, for $k = 0, 1, 2$, the isolated roots of the system $\mathbf{h}^{(k)}$ obtained by adding to $\mathbf{f}$ generic linear combinations of L_i, $i = 0, \ldots, k$, are computed:

$$\mathbf{h}^{(0)} = \mathbf{f}, \qquad \mathbf{h}^{(1)} = \begin{cases} f_1(X) + L_1(X) = X_1X_2 + 2X_3 + 1, \\ f_2(X) - L_1(X) = X_1X_3 - 2X_1 - 3X_3 - X_2 + 1, \\ f_3(X) + 2L_1(X) = X_2X_3 + X_2 + 3X_3 + 2X_1 + 1, \end{cases}$$

$$\mathbf{h}^{(2)} = \begin{cases} f_1(X) + L_1(X) + L_2(X) = X_1X_2 + X_1 + 2X_2 + 2X_3 + 1, \\ f_2(X) - L_1(X) + 2L_2(X) = X_1X_3 + 3X_2 - 3X_3 + 1, \\ f_3(X) + 2L_1(X) + L_2(X) = X_2X_3 + 3X_2 + 3X_3 + 3X_1 + 1. \end{cases}$$

In order to do this, the algorithm deforms the geometric resolution $R_{\mathbf{g}}$ to geometric resolutions $R_{\mathbf{h}^{(k)}}$ of the sets of isolated roots of $\mathbf{h}^{(k)}$:

$$R_{\mathbf{h}^{(0)}}=\begin{cases}u^3-7u^2+2u+40=0,\\ X_1=1,\\ X_2=-\frac{1}{14}u^2+\frac{9}{14}u-\frac{3}{7},\\ X_3=\frac{1}{7}u^2-\frac{2}{7}u-\frac{1}{7},\end{cases}$$

$$R_{\mathbf{h}^{(1)}}=\begin{cases}u^5-\frac{9}{2}u^4-13u^3+68u^2-64u=0,\\ X_1=\frac{57}{200}u^4-\frac{369}{400}u^3-\frac{953}{200}u^2+\frac{647}{50}u-3,\\ X_2=-\frac{269}{600}u^4+\frac{1573}{1200}u^3+\frac{1567}{200}u^2-\frac{2599}{150}u+1,\\ X_3=\frac{367}{600}u^4-\frac{2039}{1200}u^3-\frac{2181}{200}u^2+\frac{3407}{150}u+1,\end{cases}$$

$$R_{\mathbf{h}^{(2)}}=\begin{cases}u^5+\frac{49}{2}u^4-\frac{1549}{9}u^3+\frac{538}{9}u^2+\frac{679}{6}u-\frac{769}{18}=0,\\ X_1=-\frac{101\,214}{1\,803\,049}u^4-\frac{2\,537\,721}{1\,803\,049}u^3+\frac{15\,987\,545}{1\,803\,049}u^2+\frac{3\,986\,650}{1\,803\,049}u-\frac{7\,719\,426}{1\,803\,049},\\ X_2=\frac{58\,338}{9\,015\,245}u^4+\frac{277\,533}{1\,803\,049}u^3-\frac{11\,347\,628}{9\,015\,245}u^2+\frac{5\,343\,077}{9\,015\,245}u+\frac{8\,036\,523}{9\,015\,245},\\ X_3=\frac{389\,394}{9\,015\,245}u^4+\frac{1\,982\,655}{1\,803\,049}u^3-\frac{57\,242\,469}{9\,015\,245}u^2-\frac{21\,604\,159}{9\,015\,245}u+\frac{22\,524\,084}{9\,015\,245}.\end{cases}$$

Finally, subroutine `CleanGR` removes spurious factors from $R_{\mathbf{h}^{(k)}}$ to obtain a geometric resolution $R^{(k)}$ of a finite set containing a set of representative points of the equidimensional component of dimension k

$$R^{(0)}=\begin{cases}u^3-7u^2+2u+40=0,\\ X_1=1,\\ X_2=-\frac{1}{14}u^2+\frac{9}{14}u-\frac{3}{7},\\ X_3=\frac{1}{7}u^2-\frac{2}{7}u-\frac{1}{7},\end{cases}\qquad R^{(1)}=\begin{cases}u^3+2u^2-8u=0,\\ X_1=\frac{1}{2}u^2+u-3,\\ X_2=-\frac{1}{6}u^2+\frac{1}{3}u+1,\\ X_3=-\frac{1}{6}u^2-\frac{2}{3}u+1,\end{cases}\qquad R^{(2)}=\emptyset.$$

Since $u^3-7u^2+2u+40=(u+2)(u-4)(u-5)$ and $u^3+2u^2-8u=(u+4)u(u-2)$, substituting their roots in $R^{(0)}$ and $R^{(1)}$ respectively, we get the following sets of points in $V(\mathbf{f})$:

$$\mathcal{W}_0=\{(1,-2,1),(1,1,1),(1,1,2)\},\qquad \mathcal{W}_1=\{(-3,1,1),(1,1,-1),(1,-3,1)\}.$$

Note that $\mathcal{W}_1$ contains exactly one representative point for each of the 3 lines in $V(\mathbf{f})$. Moreover, the fact that $R^{(2)}=\emptyset$ implies that $V(\mathbf{f})$ does not have irreducible components of dimension 2. However, although there are no isolated points in $V(\mathbf{f})$, $\mathcal{W}_0\subset V(\mathbf{f})$ is not empty.

Remark 23. All the random choices of points made by our algorithms lead to correct computations provided these points do not annihilate certain polynomials whose degrees can be explicitly bounded. These bounds depend polynomially on the degrees of affine varieties associated to the input systems, which in turn, can be estimated in terms of mixed volumes due to Theorem 16. The Schwartz–Zippel lemma allows us to control the bit size of the constants to be chosen at random in order that the error probability of the algorithms is less than a fixed number within the stated complexity bounds.

Although we do not include the precise probability estimates here, for a similar analysis we refer the reader to Jeronimo et al. (2009), where the genericity of zero-dimensional sparse systems is studied and the probability of success of the algorithms to compute isolated solutions of sparse systems is stated in detail. We also refer the reader to Krick et al. (2001, Proposition 4.5) for bounds on the genericity of hyperplanes intersecting an equidimensional variety in as many points as its degree, and to Jeronimo and Sabia (2002, Lemma 3 and Remark 4) for the analysis of the genericity of linear combinations of input equations required in Section 4.2.

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