Characterizations of weighted Besov spaces

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We define a class of weighted Besov spaces and we obtain a characterization of this class by means of an appropriate class of weighted Lipschitz ϕ spaces.

1 Introduction

In this work we introduce a class of weighted Besov spaces, $\dot{B}_\infty^{\phi;\infty}(w)$, where the smoothness of the functions involved is controlled by a growth function ϕ. For the case $\phi(t) = t^\alpha$ and the weight w equal to one, these spaces coincide with the well-known Besov spaces $B_\infty^{\alpha;\infty}$, $0 < \alpha < 1$, (see for example [5]).

We obtain a characterization of these weighted spaces by means of an appropriate class of weighted-Lipschitz ϕ-spaces, $\dot{\Lambda}^\phi(w)$. This characterization holds for weights belonging to a certain class, $H^\phi(\infty)$, see Definition 2.6. We use the Calderón-type reproducing formula as one of the main tools. Our result generalizes that contained in [5] for the case $\phi(t) = t^\alpha$ and $w = 1$, (see Theorem 23 in p. 19 of [5]).

The weighted Lipschitz ϕ-space, $\dot{\Lambda}^\phi(w)$ coincides with $BMO_\phi(w)$ whenever ϕ is of positive lower type and w satisfies a doubling condition, as we state in Lemma 3.1. The spaces $BMO_\phi(w)$ consisting of functions whose mean oscillation is controlled by a weight w and a growth function ϕ, were studied in different situations and contexts, by several authors. For instance, see [13], [15], [1], [12], [11], [20], [3], [4], [2], [22], [16] and [8]. The first appearance of this kind of unweighted spaces for general ϕ, BMO_ϕ goes back to [20]. There the author characterizes these spaces in terms of a non-increasing rearrangement. Later on, in [11], Janson obtained another characterization by means of the Riesz transforms and the Lipschitz spaces Λ_ϕ generalizing the well known decomposition of BMO functions by C. Fefferman (see [3]).

The weighted BMO spaces, that is $BMO_\phi(w)$ with $\phi = 1$ in our context, were first introduced by Muckenhoupt and Wheeden in [MW1] where the authors study the boundedness of the Hilbert transform.

Finally, we remark that the Lipschitz spaces Λ_ϕ, with $\phi(t) = t^\beta$, are the spaces where L^p functions for $p > n/\alpha$, are mapped by I_α, the fractional integral operator, whenever $\beta/n = \alpha/n - 1/p$. These results, for the weighted case were proved in [7]. Likewise, the integral operator defined in [10] by

$$I_\psi = \int \frac{\psi(|x-y|)}{|x-y|^n} f(y)\, dy.$$

maps $L^p(w)$ to the weighted Lipschitz space $\dot{\Lambda}^\phi(w)$.

Other generalizations of these classical results can be found in [17], [6], [14], [7], [9], [18], [19].

Our paper is organized as follows: In Section 2 we define the spaces and the classes of weights used throughout and we state the main result concerning the characterization of weighted Besov spaces by means of weighted Lipschitz spaces. In Section 3 we include some basic lemmas that we use in the proof of our main result. Finally, the main result is proved in Section 4.

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2 Statement of the main result

Before stating the main result we will include a few definitions and auxiliary lemmas.

We are going to deal with positive functions ϕ defined and increasing on $[0,\infty)$. We also assume that the following conditions are satisfied

(a) ϕ is of lower type α, $\alpha > 0$, that is there exists a constant C such that

$$\phi(st) \le Cs^\alpha \phi(t)$$

holds for every $s \in [0,1]$ and every $t \ge 0$.

(b) ϕ is of upper type β, $0 < \beta < 1$, that is there exists a constant C such that

$$\phi(st) \le Cs^\beta \phi(t)$$

holds for every $s \ge 1$ and every $t \ge 0$.

In order to define our spaces we need the following lemma contained in [5] (see p. 7). We omit its proof.

Lemma 2.1 *Fix $N \in \mathbb{Z}_+$. Then there exists a function $\varphi : \mathbb{R}^n \to \mathbb{R}$ such that*

(i) *supp* $\varphi \subset \{x \in \mathbb{R}^n : |x| \le 1\} \equiv B_1(0)$,

(ii) φ *is radial,*

(iii) $\varphi \in C^\infty(\mathbb{R}^n)$,

(iv) $\int_{\mathbb{R}^n} x^\gamma \varphi(x)\, dx = 0$ *if* $|\gamma| \le N, \quad \gamma \in \mathbb{Z}^n_+, \quad x^\gamma = x_1^{\gamma_1} x_2^{\gamma_2} ... x_n^{\gamma_n}$,
$|\gamma| = \gamma_1 + \gamma_2 + ... + \gamma_n$,

(v) $\int_0^\infty (\hat{\varphi}(t\xi))^2\, dt/t = 1$ *if* $\xi \in \mathbb{R}^n - \{0\}$.

In the sequence, we denote by $\varphi_t(x) = t^{-n}\varphi(t^{-1}x)$ with $t > 0$ and by w a weight, that is, a non negative, locally integrable function defined on $\mathbb{R}^n$.

Definition 2.2 Let φ be a function having the properties of Lemma (2.1) and ϕ be an increasing function of positive lower type. Let f be a function such that

$$\int_{\mathbb{R}^n} \frac{f(x)}{(1+|x|)^{n+1}}\, dx < \infty. \tag{1}$$

We say that the function $f \in \dot{B}^{\phi,\infty}_\infty(w)$ if

$$\|f\|_{\dot{B}^{\phi,\infty}_\infty(w)} = \sup_{x \in \mathbb{R}^n,\, t>0} \frac{|\varphi_t * f(x)|}{w^\phi_t(x)} < \infty, \tag{2}$$

where

$$w^\phi_t(x) = \int_{|u-x|<t} w(u) \frac{\phi(|x-u|)}{|x-u|^n}\, du. \tag{3}$$

Observe that $w^\phi_t(x) < \infty$ $a.e.$ x, in fact, since ϕ is of positive lower type, for any $R > 0$

$$\begin{aligned}
\int_{|x|<R} w^\phi_t(x)\, dx &= \int_{|x|<R} \int_{|u-x|<t} w(u) \frac{\phi(|x-u|)}{|x-u|^n}\, du\, dx \\
&\le \int_{|u|<R+t} w(u) \int_{|u-x|<t} \frac{\phi(|x-u|)}{|x-u|^n}\, dx\, du = C_n \int_{|u|<R+t} w(u) \int_0^t \frac{\phi(s)}{s}\, ds \\
&= C_n \int_{|u|<R+t} w(u) \int_0^1 \frac{\phi(st)}{s}\, ds \le C\phi(t) w(B(0,R+t)) < \infty.
\end{aligned}$$

Remark 2.3 We remark that condition (1) for f will be necessary in order to state the convergence of Calderón's formula in $\mathcal{S}'$ modulus constant to the function f or, equivalently, in $\mathcal{S}'_0$, with $\mathcal{S}_0 = \{\phi \in \mathcal{S} / \int \phi = 0\}$ (see [5], p. 125 or [21], p. 164).

Definition 2.4 Let ϕ be a function of positive lower type. We say that $f \in \dot{\Lambda}^{\phi}(w)$ if

$$|f(x) - f(y)| \le C(w^{\phi}_{2|x-y|}(x) + w^{\phi}_{2|x-y|}(y)) \tag{4}$$

where w^{ϕ}_t is defined as in (3) for $t = 2|x-y|$. The least constant C will be denoted by $\|f\|_{\dot{\Lambda}^{\phi}(w)}$.

Remark 2.5 If $w = 1$ and $\phi(t) = t^{\alpha}$ this space coincides with the usual Lipschitz space, and we denote it by Λ_{α}. For more general w and $\phi(t) = t^{\alpha}$ this space coincides with that defined in [7].

Now we give the classes of weights w we are working with.

Definition 2.6 We say that a weight w belongs to $H^{\phi}(\infty)$ if there is a constant C such that

$$\frac{|B|^{1/n}}{\phi(|B|^{1/n})} \int_{\mathbb{R}^n - B} \frac{w(y)\phi(|x_B - y|)}{|x_B - y|^{n+1}}\, dy \le C \frac{w(B)}{|B|}, \tag{5}$$

for every ball B in $\mathbb{R}^n$ centered in x_B.

Remark 2.7 If we let $\phi(t) = t^{n\beta}$, with $0 < \beta < 1/n$, $H^{\phi}(\infty)$ is given by

$$|B|^{-\beta+1/n} \int_{\mathbb{R}^n - B} \frac{w(y)}{|x_B - y|^{n-n\beta+1}}\, dy \le C \frac{w(B)}{|B|} \tag{6}$$

which coincides with the class $H(\beta n, \infty)$ introduced in [7].

Now, we are in condition to state the main result of this paper.

Theorem 2.8 *Let ϕ be a function of positive lower type α and of upper type $\beta < 1$. Let w be a weight in $H^{\phi}(\infty)$. Then*

$$\dot{\Lambda}^{\phi}(w) \equiv \dot{B}^{\phi,\infty}_{\infty}(w) \tag{7}$$

and the norms $\|.\|_{\dot{\Lambda}^{\phi}(w)}$ and $\|.\|_{\dot{B}^{\phi,\infty}_{\infty}(w)}$ are equivalent.

3 Lemmas and preliminary results

In this section we introduce some basic lemmas that we are going to use in the proof of the main results. The first of them gives the relation between $\dot{\Lambda}^{\phi}(w)$ and the space $BMO_{\phi}(w)$ consisting in all the locally integrable functions f such that

$$\|f\|_{BMO_{\phi}(w)} = \sup_B \frac{1}{w(B)\phi(|B|^{1/n})} \int_B |f - m_B f|\, dx < \infty \tag{8}$$

Lemma 3.1 *Let ϕ be a function of positive lower type. If w satisfies a doubling condition, then the space $\dot{\Lambda}^{\phi}(w)$ coincides with the space $BMO_{\phi}(w)$.*

Proof: The proof follows similar lines to that in Proposition 1.3 of [7]. First, we check (4) for $f \in BMO_\phi(w)$. Given x and y in $\mathbb{R}^n$, with $x \neq y$ we take $B = B(x, |x-y|)$ and $B' = B(y, |x-y|)$. Then

$$|f(x) - f(y)| \leq |f(x) - m_B f| + |f(y) - m_{B'} f| + |m_B f - m_{B'} f|.$$

The estimate of the three terms are similar, thus, we only estimate the first of them. Letting $B_i = B(x, 2^{-i}|x-y|)$, $i \geq 1$ and $B_o = B$ and using the doubling condition of w we get

$$\begin{aligned} |f(x) - m_B f| &\leq \lim_{k\to\infty} \left(|f(x) - m_{B_k} f| + \sum_{i=0}^{k-1} \left| m_{B_{i+1}} f - m_{B_i} f \right| \right) \\ &\leq C \sum_{i=0}^{\infty} |B_i|^{-1} \int_{B_i} |f(z) - m_{B_i} f| \, dz \\ &\leq C \, ||f||_{BMO_\phi(w)} \sum_{i=0}^{\infty} \frac{w(B_i)}{|B_i|} \phi\left(|B_i|^{1/n} \right) \\ &\leq C \, ||f||_{BMO_\phi(w)} \sum_{i=0}^{\infty} \int_{B_i - B_{i+1}} \frac{w(z)\phi(|x-z|)}{|z-x|^n} dz \\ &\leq C \, ||f||_{BMO_\phi(w)} \int_{|z-x|<2|x-y|} \frac{w(z)\phi(|x-z|)}{|z-x|^n} dz \\ &\leq C \, ||f||_{BMO_\phi(w)} \, w^{\phi}_{2|x-y|}(x) \end{aligned}$$

for almost $x \in \mathbb{R}^n$.

Conversely, integrating (4) on a ball B with respect to both variables, x and y, and changing the order of integration, we obtain that f belongs to $BMO_\phi(w)$. $\square$

The following proposition asserts that $f \in \dot{\Lambda}^\phi(w)$ also satisfies condition (1).

Proposition 3.2 *Let $f \in \dot{\Lambda}^\phi(w)$ and $w \in H^\phi(\infty)$. If $B = B(x_B, t)$ then there exists a constant C such that*

$$\int_{\mathbb{R}^n} |f(x) - m_B f| \frac{t}{(t + |x_B - x|)^{n+1}} \, dx \leq C \|f\|_{\dot{\Lambda}^\phi(w)} \frac{w(B)\phi(|B|^{1/n})}{|B|} \tag{9}$$

Proof: Let $B = B(x_B, t)$

$$\begin{aligned} &\int_{\mathbb{R}^n} |f(x) - m_B f| \frac{t}{(t + |x_B - x|)^{n+1}} \, dx \\ &\leq \int_B |f(x) - m_B f| \frac{t}{(t + |x_B - x|)^{n+1}} \, dx + \int_{\mathbb{R}^n - B} |f(x) - m_B f| \frac{t}{(t + |x_B - x|)^{n+1}} \, dx \\ &= I + II. \end{aligned}$$

Let us first estimate I. Since $f \in \dot{\Lambda}^\phi(w)$, by Lemma 3.1 we obtain that

$$\begin{aligned} \int_B |f(x) - m_B f| \frac{t}{(t + |x_B - x|)^{n+1}} \, dx &\leq \frac{1}{t^n} \int_B |f(x) - m_B f| \, dx \\ &\leq C \frac{w(B)\phi(|B|^{1/n})}{|B|} \|f\|_{\dot{\Lambda}^\phi(w)}. \end{aligned}$$

To estimate II, we set $B_k = B(x_B, 2^k t)$. Since, by Lemma 3.7 below w satisfies a doubling condition, we can use Lemma 3.1 obtaining

$$\begin{aligned}
\int_{\mathbb{R}^n - B} |f(x) - m_B f| \frac{t}{(t + |x_B - x|)^{n+1}}\, dx &\leq Ct \int_{\mathbb{R}^n - B} \frac{|f(x) - m_B f|}{|x_B - x|^{n+1}}\, dx \\
&\leq Ct \sum_{k=0}^{\infty} \int_{B_{k+1} - B_k} \frac{|f(x) - m_B f|}{|x_B - x|^{n+1}}\, dx \\
&\leq Ct \sum_{k=0}^{\infty} \frac{1}{(2^{k+1} t)^{n+1}} \int_{B_{k+1}} |f(x) - m_B f|\, dx \\
&\leq C \sum_{k=0}^{\infty} \frac{1}{2^{k+1}} \sum_{j=1}^{k+1} \frac{1}{|B_j|} \int_{B_j} |f(x) - m_{B_j} f|\, dx \\
&\leq C \|f\|_{\dot{\Lambda}^{\phi}(w)} \sum_{k=0}^{\infty} \frac{1}{2^{k+1}} \sum_{j=1}^{k+1} \frac{w(B_j)}{|B_j|} \phi(|B_j|^{1/n}) \\
&\leq C \|f\|_{\dot{\Lambda}^{\phi}(w)} \sum_{j=0}^{\infty} \frac{w(B_j)}{|B_j|} \phi(|B_j|^{1/n}) \sum_{k=j}^{\infty} \frac{1}{2^k} \\
&\leq C \|f\|_{\dot{\Lambda}^{\phi}(w)} t \sum_{j=0}^{\infty} \frac{w(B_j)}{(2^j t)^{n+1}} \phi(|B_j|^{1/n})
\end{aligned}$$

Since $w \in H^{\phi}(\infty)$, using again Lemma 3.7 we get

$$\begin{aligned}
II &\leq C \|f\|_{\dot{\Lambda}^{\phi}(w)} t \sum_{j=0}^{\infty} \frac{1}{(2^j t)^{n+1}} w(B_{j+1} - B_j) \phi(|B_j|^{1/n}) \\
&\leq C \|f\|_{\dot{\Lambda}^{\phi}(w)} t \sum_{j=0}^{\infty} \int_{B_{j+1} - B_j} \frac{w(t)\phi(|x_B - x|)}{|x_B - x|^{n+1}}\, dx \\
&\leq C \|f\|_{\dot{\Lambda}^{\phi}(w)} t \int_{\mathbb{R}^n - B} \frac{w(t)\phi(|x_B - x|)}{|x_B - x|^{n+1}}\, dx \\
&\leq C \|f\|_{\dot{\Lambda}^{\phi}(w)} \frac{w(B)}{|B|} \phi(|B|^{1/n}). \square
\end{aligned}$$

Remark 3.3 From the above proposition, if $f \in \dot{\Lambda}^{\phi}(w)$ and $w \in H^{\phi}(\infty)$, by taking $B = B(0, 1)$ we obtain that the function $g(x) = f(x) - m_B f$ satisfies (1).

The following two lemmas about real functions were proved in [7] and we omit its proofs.

Lemma 3.4 *Let φ be a non negative and non decreasing function defined on $(0, \infty)$. If there exist two positive constants C_o and r such that*

$$\int_t^{\infty} \frac{\varphi(s)}{s^{r+1}}\, ds \leq C_o \frac{\varphi(t)}{t^r} \tag{10}$$

for every $t > 0$, then the function $\varphi(t)/t^r$ is quasi-decreasing with constant equal to $C_o 2^{r+1}$, that is, for any $t_1 \leq t_2$, $\varphi(t_2)/t_2^r \leq C_o 2^{r+1} \varphi(t_1)/t_1^r$.

Lemma 3.5 *Let φ be a non negative and non decreasing function defined on $(0,\infty)$ and $r>0$. Then, the following conditions are equivalent:*

(i) *The function φ satisfies (10).*

(ii) *There exists $a>1$ such that $\varphi(at)\leq \frac{1}{2}a^r\varphi(t)$ for every $t>0$.*

(iii) *There exist two positive constants C and δ such that*

$$\varphi(\theta t)\leq C\theta^{r-\delta}\varphi(t)$$

for all $t>0$ and all $\theta\geq 1$.

We also need the following properties to deal with growth functions.

Lemma 3.6 *Let ϕ be a function of upper type $\beta\leq 1$. Then for every $s<t$*

$$\frac{\phi(t)}{t^n}\leq C\frac{\phi(s)}{s^n},$$

that is $\phi(t)/t^n$ is quasi decreasing.

Proof: From the fact that ϕ has upper type β and $t/s>1$ we have

$$\phi(t)\leq C(t/s)^\beta\phi(s)\leq C(t/s)^n\phi(s).\square$$

We also note that if ϕ is of lower type $\alpha>0$ then, clearly,

$$\int_0^t\frac{\phi(\rho)}{\rho}d\rho=\int_0^1\frac{\phi(ts)}{s}ds\leq C\phi(t). \tag{11}$$

The properties for weighs in $H^\phi(\infty)$ are stated in the following two lemmas.

Lemma 3.7 *Let w be a weight belonging to $H^\phi(\infty)$ where ϕ is an increasing function of finite upper type. Then w satisfies a doubling condition.*

Proof: Let B be a ball in $\mathbb{R}^n$. Since w belongs to $H^\phi(\infty)$ and ϕ is an increasing function of finite upper type, we have

$$\begin{aligned}\frac{|B|^{1/n}}{\phi(|B|^{1/n})}\int_{\mathbb{R}^n}\frac{w(y)\phi(|x_B-y|+|B|^{1/n})}{(|x_B-y|+|B|^{1/n})^{n+1}}\,dy &\leq C\left(\frac{w(B)}{|B|}+\frac{|B|^{1/n}}{\phi(|B|^{1/n})}\int_B\frac{w(y)\phi(|B|^{1/n})}{|B|^{1+1/n}}\,dy\right)\\ &\leq C\frac{w(B)}{|B|}.\end{aligned}$$

Therefore

$$\frac{w(B)}{|B|}\geq C\frac{|B|^{1/n}}{\phi(|B|^{1/n})}\int_{2B}\frac{w(y)\phi(|x_B-y|+|B|^{1/n})}{(|x_B-y|+|B|^{1/n})^{n+1}}\,dy\geq C\frac{w(2B)}{|B|},$$

since ϕ is increasing. This completes the proof.$\square$

Lemma 3.8 *Let w be a weight. Then the following conditions are equivalent*

(i) *w belongs to $H^\phi(\infty)$*

(ii) *There exist two positive constants C and δ such that*

$$w(B(x_B,\theta t))\phi(\theta t)\leq C\theta^{n+1-\delta}w(B(x_B,t))\phi(t)$$

for every ball $B=B(x_B,t)$ and for all $\theta\geq 1$.

Proof: Let us first suppose that (i) holds. Let $B=B(x_B,t)$ and $B_k=B(x_B,2^k t)$, since $w\in H^{\phi}(\infty)$, we have

$$\begin{aligned}
\frac{w(B)}{|B|} &\geq C\frac{|B|^{1/n}}{\phi(|B|^{1/n})}\int_{\mathbb{R}^n-B}\frac{w(y)\phi(|x_B-y|)}{|x_B-y|^{n+1}}\,dy\\
&= C\frac{|B|^{1/n}}{\phi(|B|^{1/n})}\sum_{k=0}^{\infty}\int_{B_{k+1}-B_k}\frac{w(y)\phi(|x_B-y|)}{|x_B-y|^{n+1}}\,dy\\
&\geq C\frac{|B|^{1/n}}{\phi(|B|^{1/n})}\sum_{k=0}^{\infty}\frac{\phi(2^k t)}{(2^k t)^{n+1}}w(B(x_B,2^{k+1}t))\\
&\geq C\frac{|B|^{1/n}}{\phi(|B|^{1/n})}\int_t^{\infty}\frac{\phi(s)}{s^{n+1}}w(B(x_B,s))\,\frac{ds}{s},
\end{aligned}$$

where we use that ϕ is increasing and w satisfies a doubling condition. Therefore, we get

$$\int_t^{\infty}\frac{\phi(s)}{s^{n+1}}w(B(x_B,s))\,\frac{ds}{s}\leq\frac{w(B(x_B,t))}{t^{n+1}}\phi(t).$$

By Lemma 3.5 with $r=n+1$, we get that there exist two positive constants C and $\delta<1$ such that

$$w(B(x_B,\theta t))\phi(\theta t)\leq C\theta^{n+1-\delta}w(B(x_B,t))\phi(t)$$

for all $t>0$ and all $\theta\geq 1$.

Conversely, if (ii) is valid then, for $B_k=B(x_B,2^k t)$, we have

$$\begin{aligned}
\int_{\mathbb{R}^n-B}\frac{w(y)\phi(|x_B-y|)}{|x_B-y|^{n+1}}\,dy &\leq C\sum_{k=0}^{\infty}\int_{B_{k+1}-B_k}\frac{w(y)\phi(|x_B-y|)}{|x_B-y|^{n+1}}\,dy\\
&\leq C\sum_{k=0}^{\infty}\frac{w(B_{k+1})\phi(|B_{k+1}|^{1/n})}{|B_{k+1}|^{1+1/n}}\\
&\leq \sum_{k=0}^{\infty}\frac{2^{k(n+1-\delta)}w(B)\phi(|B|^{1/n})}{2^{k(n+1)}|B|^{1+1/n}}\\
&\leq C\frac{w(B)\phi(|B|^{1/n})}{|B|^{1/n+1}}\sum_{k=0}^{\infty}\frac{1}{2^{k\delta}}\\
&\leq C\frac{w(B)\phi(|B|^{1/n})}{|B|^{1/n+1}}.\square
\end{aligned}$$

4 Proof of the main theorem

Proof of Theorem 2.8*:* If $f\in\dot{B}_{\infty}^{\phi,\infty}(w)$, first we prove that

$$\int_0^{\infty}\varphi_t*\varphi_t*f(x)\,dt/t \tag{12}$$

represents an element of $\dot{\Lambda}^{\phi}(w)$ in the sense that if

$$f_\epsilon(x) = \int_\epsilon^{1/\epsilon} \varphi_t * \varphi_t * f(x)\, dt/t$$

for $\epsilon > 0$, then there exists a sequence of constants $\{c_\epsilon\}_{\epsilon>0}$ such that $f_\epsilon(x) - c_\epsilon$ converges to a function $g(x)$ for each $x \in \mathbb{R}^n$ and $g \in \dot{\Lambda}^{\phi}(w)$ with $\|g\|_{\dot{\Lambda}^{\phi}(w)} \leq C\|f\|_{\dot{B}^{\phi,\infty}_{\infty}(w)}$.

Let $x_o \in \mathbb{R}^n$ and let us consider the function

$$f_{\epsilon,x_o}(x) = \int_\epsilon^{1/\epsilon} \varphi_t * \varphi_t * f(x)\, dt/t - \int_\epsilon^{1/\epsilon} \varphi_t * \varphi_t * f(x_o)\, dt/t = f_\epsilon(x) - f_\epsilon(x_o)$$

Note that $f_\epsilon(x)$ is finite for each ϵ and x. Let us first prove that, for each fixed x, the sequence $\{f_{\epsilon,x_o}\}_\epsilon$ is a Cauchy sequence in $\mathbb{C}$. In fact, let $0 < \epsilon_1 < \epsilon_2$, then

$$\begin{aligned} |f_{\epsilon_1,x_o}(x) - f_{\epsilon_2,x_o}(x)| &\leq \left(\int_{\epsilon_1}^{\epsilon_2} + \int_{1/\epsilon_2}^{1/\epsilon_1}\right) |\varphi_t * \varphi_t * f(x) - \varphi_t * \varphi_t * f(x_o)|\, dt/t \\ &\leq \int_{\epsilon_1}^{\epsilon_2} |\varphi_t * \varphi_t * f(x)|\, dt/t + \int_{\epsilon_1}^{\epsilon_2} |\varphi_t * \varphi_t * f(x_o)|\, dt/t \\ &\quad + \int_{1/\epsilon_2}^{1/\epsilon_1} |\varphi_t * \varphi_t * f(x) - \varphi_t * \varphi_t * f(x_o)|\, dt/t \\ &= I_1^1 + I_1^2 + I_2. \end{aligned} \tag{13}$$

Let us first estimate I_1^1. The estimate for I_1^2 follows similar lines. Since $|\varphi_t| \leq Ct^{-n}$, we have that

$$\begin{aligned} &\int_{\epsilon_1}^{\epsilon_2} |\varphi_t * \varphi_t * f(x)|\, dt/t \\ &= \int_{\epsilon_1}^{\epsilon_2} \left| \int \varphi_t(x-z) \frac{\varphi_t * f(z)}{w_t^{\phi}(z)} w_t^{\phi}(z) \right| dz\, dt/t \\ &\leq C\|f\|_{\dot{B}^{\phi,\infty}_{\infty}(w)} \int_{\epsilon_1}^{\epsilon_2} \int \varphi_t(x-z) w_t^{\phi}(z)\, dz\, dt/t \\ &\leq C\|f\|_{\dot{B}^{\phi,\infty}_{\infty}(w)} \int_{\epsilon_1}^{\epsilon_2} \int_{|x-z|\leq t} t^{-n} \int_{|z-u|<t} \frac{w(u)\phi(|z-u|)}{|z-u|^n}\, du\, dz\, dt/t \\ &\leq C\|f\|_{\dot{B}^{\phi,\infty}_{\infty}(w)} \int_{|x-u|<2\epsilon_2} w(u) \int_{|x-u|/2}^{\epsilon_2} t^{-n-1} \int_{|z-u|<t} \frac{\phi(|z-u|)}{|z-u|^n}\, dz\, dt\, du \\ &\leq C\|f\|_{\dot{B}^{\phi,\infty}_{\infty}(w)} \int_{|x-u|<2\epsilon_2} w(u) \int_{|x-u|/2}^{\epsilon_2} t^{-n-1} \int_0^t \frac{\phi(\rho)}{\rho}\, d\rho\, dt\, du. \end{aligned}$$

Since ϕ is of lower type $\alpha > 0$, by (11), the expression above is bounded by

$$\begin{aligned} &C\|f\|_{\dot{B}^{\phi,\infty}_{\infty}(w)} \int_{|x-u|<2\epsilon_2} w(u) \int_{|x-u|/2}^{\epsilon_2} \frac{\phi(t)}{t^n} \frac{dt}{t}, du \\ &\quad \leq C\|f\|_{\dot{B}^{\phi,\infty}_{\infty}(w)} \int_{|x-u|<2\epsilon_2} w(u) \int_1^\infty \frac{\phi(r|x-u|/2)}{r^n|x-u|^n} \frac{dr}{r}\, du. \end{aligned}$$

Using the fact that ϕ is of upper type β with $\beta < 1$, we get that

$$\begin{aligned}\int_{\epsilon_1}^{\epsilon_2} |\varphi_t * \varphi_t * f(x)|\, dt/t &\leq C\|f\|_{\dot{B}_\infty^{\phi,\infty}(w)} \int_{|x-u|<2\epsilon_2} w(u) \frac{\phi(|x-u|)}{|x-u|^n}\, du \int_1^\infty r^{\beta-n} \frac{dr}{r} \\ &\leq C\|f\|_{\dot{B}_\infty^{\phi,\infty}(w)} \int_{|x-u|<2\epsilon_2} w(u) \frac{\phi(|x-u|)}{|x-u|^n}\, du \\ &= C\|f\|_{\dot{B}_\infty^{\phi,\infty}(w)} w_{2\epsilon_2}^{\phi}(x).\end{aligned}$$

which tends to zero when ϵ_2 tends to zero, for almost every x.

In order to estimate I_2 we first note that from the mean value Theorem we have

$$\begin{aligned}I_2 &= \int_{1/\epsilon_2}^{1/\epsilon_1} |\varphi_t * \varphi_t * f(x) - \varphi_t * \varphi_t * f(x_o)|\, dt/t \\ &\leq \int_{1/\epsilon_2}^{1/\epsilon_1} \int |\varphi_t(x-z) - \varphi_t(x_o - z)| \frac{|\varphi_t * f(z)|}{w_t^{\phi}(z)} w_t^{\phi}(z)\, dz\, dt/t \\ &\leq C\|f\|_{\dot{B}_\infty^{\phi,\infty}(w)} |x - x_o| \\ &\quad \times \int_{1/\epsilon_2}^{1/\epsilon_1} \int_{\{z:|z-x|<t|\}\cup\{z:|z-x_o|<t\}} \frac{1}{t^{n+2}} \int_{|u-z|<t} w(u) \frac{\phi(|z-u|)}{|z-u|^n}\, du\, dz\, dt \qquad (14)\end{aligned}$$

Now, let ϵ_2 such that $|x - x_o| < 1/\epsilon_2 < t$. Then if $|z - x_o| < t$ we get that $|z - x| < 2t$ and $|u - x| < 3t$ for $|u - z| < t$. Then, using (11) and (14) we obtain

$$\begin{aligned}I_2 &\leq C\|f\|_{\dot{B}_\infty^{\phi,\infty}(w)} |x - x_o| \qquad (15) \\ &\quad \times \int_{1/\epsilon_2}^{1/\epsilon_1} \frac{1}{t^{n+2}} \int_{\{u:|u-x|<3t\}} w(u) \int_{|u-z|<t} \frac{\phi(|z-u|)}{|z-u|^n}\, dz\, du\, dt. \\ &\leq C\|f\|_{\dot{B}_\infty^{\phi,\infty}(w)} |x - x_o| \int_{1/\epsilon_2}^{1/\epsilon_1} \frac{\phi(t)}{t^{n+2}} \int_{|u-x|<3t} w(u)\, du\, dt \\ &\leq C\|f\|_{\dot{B}_\infty^{\phi,\infty}(w)} |x - x_o| \int_{1/\epsilon_2}^{1/\epsilon_1} \frac{\phi(t)}{t^{n+1}} w(B(x,3t))\, dt/t \\ &\leq C\|f\|_{\dot{B}_\infty^{\phi,\infty}(w)} |x - x_o| \sum_{j=1}^{J_o} \int_{2^j/\epsilon_2 < t \leq 2^{j+1}/\epsilon_2} \frac{\phi(t)}{t^{n+1}} w(B(x,3t))\, dt/t \\ &\leq C\|f\|_{\dot{B}_\infty^{\phi,\infty}(w)} |x - x_o| \sum_{j=1}^{\infty} w(B(x, 32^j/\epsilon_2)) \frac{\phi(32^j/\epsilon_2)}{(2^j/\epsilon_2)^{n+1}}\end{aligned}$$

Since $w \in H_\infty^{\phi}$, from Lemma 3.8 the last expression is bounded by

$$\begin{aligned}&C\|f\|_{\dot{B}_\infty^{\phi,\infty}(w)} |x - x_o| w(B(x,3))\phi(3) \sum_{j=1}^{\infty} \frac{(2^j/\epsilon_2)^{n+1-\delta}}{(2^j/\epsilon_2)^{n+1}} \qquad (16) \\ &\leq C\|f\|_{\dot{B}_\infty^{\phi,\infty}(w)} |x - x_o| \epsilon_2^{\delta} w(B(x,3)),\end{aligned}$$

which also tends to zero when ϵ_2 tends to zero. Thus, as we said, $\{f_{\epsilon,x_o}(x)\}$ is a Cauchy sequence in $\mathbb{C}$ and then there exists $f_{x_o}(x) = \lim_{\epsilon\to 0} f_{\epsilon,x_o}(x)$. Let us see that $f_{x_o}(x) \in \dot{\Lambda}^{\phi}(w)$ and verifies $\|f_{x_o}\|_{\dot{\Lambda}^{\phi}(w)} \leq$

$C\|f_{x_o}\|_{\dot{B}^{\phi,\infty}_\infty(w)}$, with C independent of ϵ. In fact

$$\begin{aligned} |f_{x_o}(x)-f_{x_o}(y)| &= \lim_{\epsilon\to 0}|f_{\epsilon,x_o}(x)-f_{\epsilon,x_o}(y)| \\ &= \lim_{\epsilon\to 0}|\int_\epsilon^{1/\epsilon}\varphi_t*\varphi_t*f(x)-\varphi_t*\varphi_t*f(y)\,dt/t| \\ &= \lim_{\epsilon\to 0}|f_\epsilon(x)-f_\epsilon(y)| \\ &\leq I_1+I_2, \end{aligned} \tag{17}$$

where

$$I_1=\int_0^{|x-y|}|\varphi_t*\varphi_t*f(x)-\varphi_t*\varphi_t*f(y)|\,dt/t$$

and

$$I_2=\int_{|x-y|}^{\infty}|\varphi_t*\varphi_t*f(x)-\varphi_t*\varphi_t*f(y)|\,dt/t$$

The estimate for I_1 follows in similar way as the estimate of I_1^1 and we omit it.

In order to estimate I_2, we also proceed as in the estimate of I_2 above. After applying the mean value Theorem and (11), we obtain as in (15) that

$$I_2 \leq C\|f\|_{\dot{B}^{\phi,\infty}_\infty(w)}|x-y|\sum_{j=0}^{\infty}\frac{\phi(2^j|x-y|)}{(2^j|x-y|)^{n+1}}w(B(x,32^j|x-y|)).$$

By Lemmas 3.7 and 3.8 we get

$$\begin{aligned} I_2 &\leq C\|f\|_{\dot{B}^{\phi,\infty}_\infty(w)}w(B(x,2|x-y|))\frac{\phi(|x-y|)}{|x-y|^n}\sum_{j=0}^{\infty}2^{-j\delta} \\ &\leq C\|f\|_{\dot{B}^{\phi,\infty}_\infty(w)}\frac{\phi(|x-y|)}{|x-y|^n}\int_{|u-x|<2|x-y|}w(u)\,du. \end{aligned}$$

From Lemma 3.6 the last term is bounded by

$$C\|f\|_{\dot{B}^{\phi,\infty}_\infty(w)}\int_{|u-x|<2|x-y|}w(u)\frac{\phi(|u-x|)}{|u-x|^n}\,du=C\|f\|_{\dot{B}^{\phi,\infty}_\infty(w)}w^{\phi}_{2|x-y|}(x).$$

Thus $\|f_{x_o}\|_{\dot{\Lambda}^{\phi}(w)}\leq C\|f\|_{\dot{B}^{\phi,\infty}_\infty(w)}$. Moreover, from (17) it is clear that

$$\|f_\epsilon\|_{\dot{\Lambda}^{\phi}(w)}\leq C\|f\|_{\dot{B}^{\phi,\infty}_\infty(w)} \tag{18}$$

where C is a constant independent of ϵ.

We also have that the function f_{x_o} is unique in the sense that if $g(x)=(\lim_{\epsilon\to 0}f_\epsilon(x)-d_\epsilon)$, then $g(x)=f_{x_o}+constant$. In fact

$$f_{x_o}(x)-g(x)=\lim_{\epsilon\to 0}\left(d_\epsilon-\int_\epsilon^{1/\epsilon}\varphi_t*\varphi_t*f(x_o))\,dt/t\right)$$

which is independent of x, i.e. is a constant.

On the other hand, since f in $\dot{B}^{\phi,\infty}_\infty(w)$ satisfies (1), from Remark 2.3 we have that

$$\lim_{\epsilon\to 0}f_\epsilon=f$$

in $\mathcal{S}_0'$, and consequently $\lim_{\epsilon\to 0} f_{\epsilon,x_o} = f$ in $\mathcal{S}_0'$.

Also, since f_{ϵ,x_o} converges pointwisely to f_{x_o}, then, by (18), Proposition (3.2) and the dominated convergence Theorem we get that

$$\lim_{\epsilon\to 0} f_{\epsilon,x_o} = f_{x_o}$$

in $\mathcal{S}_0'$. Thus f coincides with f_{x_o} as a function in $\dot{\Lambda}^{\phi}(w)$, that is, modulus constants.

Conversely, if $f \in \dot{\Lambda}^{\phi}(w)$ and from the fact that $\int \varphi_t\, dt = 0$ we get

$$\begin{aligned}
|\varphi_t * f(x)| &= \left|\int \varphi_t(x-y)f(y)\,dy\right| = \left|\int \varphi_t(x-y)(f(y)-f(x))\,dy\right| \\
&\le C\|f\|_{\dot{\Lambda}^{\phi}(w)} \int_{|x-y|<t} |\varphi_t(x-y)|(w^{\phi}_{2|x-y|}(x)| + w^{\phi}_{2|x-y|}(y))\,dy \\
&\le C\|f\|_{\dot{\Lambda}^{\phi}(w)} \left(w^{\phi}_{2t}(x) + \int_{|x-y|<t} \varphi_t(x-y)\left(\int_{|u-y|<2t} \frac{w(u)}{|y-u|^n}\phi(|y-u|)\,du\right)\,dy\right) \\
&\le C\|f\|_{\dot{\Lambda}^{\phi}(w)} \left(w^{\phi}_{2t}(x) + \int_{|u-x|<3t} w(u)\,t^{-n} \int_{|u-y|<2t} \frac{\phi(|y-u|)}{|y-u|^n}\,dy\,du\right) \\
&\le C\|f\|_{\dot{\Lambda}^{\phi}(w)} \left(w^{\phi}_{2t}(x) + \int_{|u-x|<3t} w(u)\,t^{-n} \int_0^{2t} \frac{\phi(\rho)}{\rho}\,d\rho\,du\right).
\end{aligned}$$

By changing variables and using (11)

$$\begin{aligned}
\int_{|u-x|<3t} w(u)\,t^{-n} \int_0^{2t} \frac{\phi(\rho)}{\rho}\,d\rho\,du &= \int_{|u-x|<3t} w(u)\,t^{-n} \int_0^{1} \frac{\phi(2st)}{s}\,ds\,du \qquad (19)\\
&\le Ct^{-n}\phi(t) \int_{|u-x|<3t} w(u)\,du \\
&\le Ct^{-n}\phi(t) \int_{|u-x|<2t} w(u)\,du,
\end{aligned}$$

where in the last inequality we have used the fact that w satisfyies a doubling condition. Now, by Lemma 3.6 we have that the last term in (19) is bounded by $Cw^{\phi}_{2t}(x)$. From this fact and Remark (3.3) we have the result.□

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