RECENT PROGRESS ON THE MONGE-AMPÈRE EQUATION

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ABSTRACT. We consider convex solutions to the equation $\det D^2\varphi = \mu$ when μ has a doubling property. We summarize the latest results on geometric and measure theoretic properties associated to φ. We also discuss applications such as the real analysis related to φ and Hölder regularity for $\nabla\varphi$.

1. INTRODUCTION

Let $\varphi : \mathbb{R}^n \to \mathbb{R}$ be a convex function and let $\partial\varphi$ denote its normal mapping (or sub-differential) given by

$$\partial\varphi(x) = \{p \in \mathbb{R}^n : \varphi(x) + p\cdot(y-x) \leq \varphi(y),\, \forall y \in \mathbb{R}^n\}.$$

The Monge-Ampère measure μ_φ associated to φ is (well-)defined on any Borel set E by

$$\mu_\varphi(E) = |\partial\varphi(E)|,$$

where $|\cdot|$ stands for Lebesgue measure. Given a Borel measure μ on $\mathbb{R}^n$, we say that φ is a solution (in the Alexandrov sense) to $\det D^2\varphi = \mu$ in $\mathbb{R}^n$ if $\mu_\varphi = \mu$. In particular, for $f \geq 0$, we have $\det D^2\varphi = f$ if

$$\mu_\varphi(E) = \int_E f(x)\,dx,$$

for every Borel set $E \subset \mathbb{R}^n$. See [9] for this and other basic definitions on Monge-Ampère. Now, for $x \in \mathbb{R}^n, p \in \partial\varphi(x)$ and $t > 0$, a *section* of φ centered in x at height t is the open convex set

$$S_\varphi(x,p,t) = \{y \in \mathbb{R}^n : \varphi(y) < \varphi(x) + p\cdot(y-x) + t\}.$$

If we consider the archetypal convex function $\varphi_2(x) = \frac{1}{2}|x|^2$, then $\det D^2\varphi_2 = 1$, that is, the Monge-Ampère measure associated to φ_2 is Lebesgue measure. Also for $x \in \mathbb{R}^n$ and $t > 0$

$$S_{\varphi_2}(x,t) = B(x,\sqrt{2t}).$$

Hence, the family of sections of φ_2 consists of the usual balls in $\mathbb{R}^n$. In the other end of the spectrum, we have the case of $\varphi_1(x) = |x|$, verifying

$$\det D^2\varphi_1 = |B(0,1)|\delta_0,$$

(δ_0 being Dirac's measure at 0) and whose sections can be unbounded convex sets. In order to hope for some regularity results for our solutions,

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we will disregard solutions like that. One way to avoid them is to consider functions φ whose sections are bounded sets. Geometrically, this means that the graph of φ does not contain half-lines. If φ is differentiable, then we identify $\partial\varphi(x)$ and $\nabla\varphi(x)$. In this case, we just write $S_\varphi(x,t)$ for the sections.
Inspired by some basic features of the usual balls and Lebesgue measure in $\mathbb{R}^n$, now we will define some properties for the bounded sections of φ and the measure μ_φ . We say that the sections satisfy the *engulfing property* if there exists a $K > 1$ such that for every section $S_\varphi(x,p,t)$ we have

$$y \in S_\varphi(x,p,t) \Rightarrow S_\varphi(x,p,t) \subset S_\varphi(y,q,Kt)$$

for all $q \in \partial\varphi(y)$. Also, we say that the sections satisfy the *shrinking property* if there exist $0 < \tau, \lambda < 1$ such that for all $x \in \mathbb{R}^n, p \in \partial\varphi(x)$ and $t > 0$, it holds that

$$S_\varphi(x,p,\tau t) \subset \lambda S_\varphi(x,p,t).$$

Regarding the measure μ_φ, we say that μ_φ satisfies the (DP)-*doubling property* if there exists a constant $C > 0$ such that

$$\mu_\varphi(S_\varphi(x,p,t)) \leq C\mu_\varphi(S_\varphi(x,p,\frac{t}{2})),$$

for every $x \in \mathbb{R}^n$, $p \in \partial\varphi(x)$ and $t > 0$. Similarly, we say that μ_φ satisfies the (DC)-*doubling property* if there exist constants $C > 0$ and $0 < \alpha < 1$ such that for all sections $S_\varphi(x,p,t)$, we have

$$\mu_\varphi(S_\varphi(x,p,t)) \leq C\mu_\varphi(\alpha S_\varphi(x,p,t)),$$

where $\alpha S_\varphi(x,p,t)$ denotes α-dilation of $S_\varphi(x,p,t)$ with respect to its center of mass. We are now in position to go over the main aspects of Caffarelli's regularity theory for the Monge-Ampère equation. We will begin by mentioning some old and some very recent important results of the theory which have not been summarized in the literature until now.

2. The Basic Results

In [3], Caffarelli proved two striking results: if μ_φ verifies the (DC)-doubling property, then φ is strictly convex (in the sense that every tangent hyperplane touches the graph of φ at one point only), and $\varphi \in C^{1,\beta}(\mathbb{R}^n)$ for some $\beta \in (0,1]$. Caffarelli's proof for this $C^{1,\beta}$-regularity result uses a compactness argument which does not allow to estimate β in terms of the (DC)-doubling constants. The task of finding a constructive proof for this Hölder-regularity result was posed as an open problem in Villani's recent book [12].
The results above stressed the importance of the (DC)-doubling property for μ_φ and many efforts were made to better understand it, mainly by exploring the relationship between this doubling property and geometric properties of the sections (such as the engulfing and shrinking properties). In [10], Gutiérrez and Huang proved that the (DC)-doubling property for μ_φ implies both the engulfing property for the sections of φ, and the (DP)-doubling property for μ_φ (but (DP) does not imply (DC)). Also, they gave

the first geometric characterization for the (DC)-doubling property by proving that this property is equivalent to the shrinking property of the sections of φ. In [6], the authors proved that the engulfing property also characterizes the (DC)-doubling property. The interplay between geometry and measure theory can be summarized in the following theorem (see [6, 10] for these and other equivalent conditions)

Theorem 1. *Let $S_\varphi(x,p,t)$, with $x \in \mathbb{R}^n, p \in \partial\varphi(x), t > 0$; be the bounded sections of a convex function $\varphi : \mathbb{R}^n \to \mathbb{R}$. Then the following are equivalent*

(i) The sections of φ satisfy the engulfing property.
(ii) The sections of φ satisfy the shrinking property.
(iii) The measure μ_φ satisfies the (DC)*-doubling property.*
(iv) The Monge-Ampère measure μ_φ satisfies

$$ct^n \leq |S_\varphi(x,p,t)|\mu_\varphi(S_\varphi(x,p,t)) \leq Ct^n,$$

for all sections $S_\varphi(x,p,t)$ and some positive constants c, C.

Let us denote by $\mathrm{Eng}(n,K)$ the set of all convex functions $\varphi : \mathbb{R}^n \to \mathbb{R}$ whose bounded sections satisfy the engulfing property with constant K. Let us also define

$$\mathrm{Eng}(n) = \bigcup_{K>1} \mathrm{Eng}(n,K)$$

and

$$\mathrm{Eng}_0(n) = \bigcup_{K>1} \mathrm{Eng}_0(n,K),$$

where $\mathrm{Eng}_0(n,K) = \{\varphi \in \mathrm{Eng}(n,K) : \varphi(0) = 0, \nabla\varphi(0) = 0\}$.

3. Examples of functions in $\mathrm{Eng}(n)$

Let $\varphi : \mathbb{R}^n \to \mathbb{R}$ be a strictly convex differentiable function.
(i) If $\det D^2\varphi = p$, where p is a polynomial, then $\varphi \in \mathrm{Eng}(n,K)$ for some K depending only on the degree of p (in particular, K does not depend on the coefficients of p), see [9], pp. 52.
(ii) If μ_φ verifies the μ_∞ property, i.e., given $\delta_1 \in (0,1)$, there exists $\delta_2 \in (0,1)$ such that for every section $S = S_\varphi(x,t)$ and every measurable set $E \subset S$,

$$\frac{|E|}{|S|} < \delta_2 \Rightarrow \frac{\mu_\varphi(E)}{\mu_\varphi(S)} < \delta_1,$$

then $\varphi \in \mathrm{Eng}(n)$. To see how μ_∞ implies the (DC)-doubling condition, given $\delta_1 \in (0,1)$, pick $\alpha \in (0,1)$ such that

$$\frac{|S - \alpha S|}{|S|} = 1 - \alpha^n < \delta_2,$$

then

$$\frac{\mu_\varphi(S - \alpha S)}{\mu_\varphi(S)} < \delta_1,$$

and the (DC)-doubling property follows with $C = 1/(1-\delta_1)$. By Theorem 1, we get $\varphi \in \mathrm{Eng}(n)$. This μ_∞ property plays an important role in the proof of Harnack's inequality for non-negative solutions to the linearized

Monge-Ampère equation, see [5].
(iii) If $\varphi \in C^2(\mathbb{R}^n)$ and there exist constants $\lambda, \Lambda > 0$ such that

$$\lambda \leq \det D^2\varphi \leq \Lambda, \tag{3.1}$$

then $\varphi \in \operatorname{Eng}(n)$. This follows from the fact that in this case μ_φ clearly verifies the μ_∞ property. Actually, the same is true if we only ask (3.1) to hold in the Aleksandrov sense.
(iv) If $n = 1$ and $\varphi(x) = |x|^p$ with $p > 1$, then $\varphi \in \operatorname{Eng}(1)$. In general, if μ is a doubling measure on $\mathbb{R}$, then $\varphi_\mu(x) = \int_0^x \int_0^t d\mu\, dt$ belongs to $\operatorname{Eng}_0(1)$, see [7].

4. Real Analysis Related to Convex Functions. A Space of Homogeneous Type for Monge-Ampère

In $\mathbb{R}^n$ with the usual balls and Lebesgue measure (all of them related to the convex function φ_2) we are able to develop the standard real analysis (Calderón-Zygmund decomposition, types of the Hardy-Littlewood maximal function, BMO, Hardy spaces, singular integrals, Muckenhoupt's weights, etc). The question is how far we can take this by considering a general convex function φ. As we shall see, the essential property for a convex function φ so we can produce a real analysis related to its sections and measure μ_φ is the engulfing property.
In [4], Caffarelli and Gutiérrez proved the following Besicovitch-type covering lemma for the sections of φ when these sections verify certain conditions (in [6] these conditions were proved to be equivalent to the engulfing property)

Lemma 2. *Let $A \subset \mathbb{R}^n$ be a bounded set. Suppose that for each $x \in A$ a section $S_\varphi(x,t)$ is given such that t is bounded by a fixed number M. Let us denote by $\mathcal{F}$ the family of all these sections. Then there exists a countable subfamily of $\mathcal{F}$, $\{S_\varphi(x_k,t_k) : k \in \mathbb{N}\}$ such that*

(i) $A \subset \cup_{k\in\mathbb{N}} S_\varphi(x_k,t_k)$.
(ii) $x_k \notin \cup_{j<k} S_\varphi(x_j,t_j)$, *for every* $k \geq 2$.
(iii) *For $\varepsilon > 0$ small (smallness depending on the engulfing constant K), we have that the family*

$$\mathcal{F}_\varepsilon = \{S_\varphi(x_k,(1-\varepsilon)t_k) : k \in \mathbb{N}\}$$

has bounded overlaps; more precisely

$$\sum_{k=1}^{\infty} \chi_{S_\varphi(x_k,(1-\varepsilon)t_k)}(x) \leq C_K \ln 1/\varepsilon,$$

where C_K depends only on the engulfing constant K.

Lemma 2 allows us to prove a variant of the Calderón-Zygmund decomposition adjusted to the sections $S_\varphi(x,t)$ and the measure μ_φ and then carry on a real analysis. However, there is a somewhat more natural approach to the real analysis for φ. Note that, under the presence of the engulfing property, we have a family of sections and a doubling measure. If we found a quasi-distance ρ_φ related to these objects, then we would turn $(\mathbb{R}^n, \mu_\varphi, \rho_\varphi)$ into a space of homogeneous type. Therefore, all the real analysis would follow in a nowadays standard way. We will see that this is actually the case. For $\varphi : \mathbb{R}^n \to \mathbb{R}$, strictly convex and differentiable, we set

$$\rho_\varphi(x,y) = \inf\{r : y \in S_\varphi(x,r), x \in S_\varphi(y,r)\},$$

and

$$d_\varphi(x,y) = (\nabla\varphi(x) - \nabla\varphi(y)) \cdot (x-y),$$

then it is immediate to check that

$$\rho_\varphi(x,y) \leq d_\varphi(x,y) \leq 2\rho_\varphi(x,y),$$

for every $x, y \in \mathbb{R}^n$. In [1]; Aimar, Forzani and Toledano proved that: if the sections of φ satisfy the engulfing property with constant K, then ρ_φ (as much as d_φ) is a quasi-distance on $\mathbb{R}^n$ whose balls are topologically equivalent to the sections of φ, that is, there exist positive constants $0 < \delta_1 < 1 < \delta_2$, depending only on K, such that

$$S_\varphi(x, \delta_1 t) \subset B_{\rho_\varphi}(x,t) \subset S_\varphi(x, \delta_2 t),$$

for every $x \in \mathbb{R}^n$ and $t > 0$, where

$$B_{\rho_\varphi}(x,t) = \{y \in \mathbb{R}^n : \rho_\varphi(x,y) < t\}.$$

Moreover, the quasi-triangular constant of ρ_φ depends only on K. Conversely, if ρ is any quasi-distance on $\mathbb{R}^n$ whose balls are topologically equivalent to the sections of φ, then the sections of φ have the engulfing property; this is just due to the quasi-triangular inequality for ρ. Also, since the (DC)-doubling property of μ_φ implies the (DP)-doubling property of μ_φ on the sections, we have that the engulfing property turns $(\mathbb{R}^n, \mu_\varphi, d_\varphi)$ into a space of homogeneous type. The real analysis related to $\varphi \in \mathrm{Eng}(n)$ plays a key role in the L^p-theory for solutions to the linearized Monge-Ampère equation, analogous to the role that the usual real analysis plays for solutions to $\Delta u = f$. See [4, 5].

5. New characterizations for the engulfing property

Some immediate properties of $\mathrm{Eng}(n)$ are stated in the following Lemma which was proved in [8].

Lemma 3. *Let φ be in* $\mathrm{Eng}(n, K)$.

(i) If $\lambda > 0$, then $\lambda\varphi \in \mathrm{Eng}(n, K)$.
(ii) If $\psi \in \mathrm{Eng}(n, K')$, then $\varphi + \psi \in \mathrm{Eng}(n, 2(K \vee K'))$.
(iii) If for $x, y \in \mathbb{R}^n$ we set $\varphi_{x,y}(s) = \varphi(sy + (1-s)x), s \in \mathbb{R}$, then $\varphi_{x,y} \in \mathrm{Eng}(1, K)$.
(iv) $\mathrm{Aff}(\mathbb{R}^n, \mathbb{R}^n)$ acts on $\mathrm{Eng}(n, K)$ by composition.
(v) $\mathrm{Aff}(\mathbb{R}^n, \mathbb{R})$ acts on $\mathrm{Eng}(n, K)$ by addition.

To cite some other recent applications of these ideas, let us mention that in [7], the authors proved a characterization for the engulfing property in dimension 1 which, in turn, is useful to characterize all doubling measures and quasi-symmetric mappings on $\mathbb{R}$. What follows is the n-dimensional version of that Theorem. For the complete proof see [8].

Theorem 4. *Let $\varphi : \mathbb{R}^n \to \mathbb{R}$ be a strictly convex differentiable function. The following are equivalent:*

(i) There exists a constant $K > 1$ such that if $x \in S_\varphi(y,t)$ then
$$S_\varphi(y,t) \subset S_\varphi(x,Kt),$$
for every $x, y \in \mathbb{R}^n$ and $t > 0$.(Engulfing property.)

(ii) There exists a constant $K' > 1$ such that if $x, y \in \mathbb{R}^n$ and $t > 0$ verify $x \in S_\varphi(y,t)$, then $y \in S_\varphi(x, K't)$.

(iii) There exists a constant $K'' > 1$ such that for every $x, y \in \mathbb{R}^n$
$$\begin{aligned}\frac{K''+1}{K''}&\left(\varphi(y) - \varphi(x) - \nabla\varphi(x)\cdot(y-x)\right)\\ &\leq (\nabla\varphi(x) - \nabla\varphi(y))\cdot(x-y)\\ &\leq (K''+1)\left(\varphi(y) - \varphi(x) - \nabla\varphi(x)\cdot(y-x)\right).\end{aligned}$$

Proof for (ii) $\Rightarrow$ (iii) in Theorem 4. Given $x, y \in \mathbb{R}^n$ and $\varepsilon > 0$, we have
$$\varphi(x) < \varphi(x) + \varepsilon = \varphi(y) + \nabla\varphi(y)\cdot(x-y) + \varphi(x) - \varphi(y) - \nabla\varphi(y)\cdot(x-y) + \varepsilon,$$
(note that the convexity of φ implies $\varphi(x) - \varphi(y) - \nabla\varphi(y)\cdot(x-y) \geq 0$), this means that $x \in S_\varphi(y, \varphi(x) - \varphi(y) - \nabla\varphi(y)\cdot(x-y) + \varepsilon)$. By property (ii), we must have $y \in S_\varphi(x, K'(\varphi(x) - \varphi(y) - \nabla\varphi(y)\cdot(x-y) + \varepsilon))$, which means
$$\varphi(y) \leq \varphi(x) + \nabla\varphi(x)\cdot(y-x) + K'\varphi(x) - K'\varphi(y) - K'\nabla\varphi(y)\cdot(x-y) + K'\varepsilon.$$
Letting ε go to 0 and summing up we get
$$(K'+1)\varphi(y) \leq (K'+1)\varphi(x) + (\nabla\varphi(x) + K'\nabla\varphi(y))\cdot(y-x). \tag{5.2}$$
Now interchanging the roles of x and y, we obtain
$$(K'+1)\varphi(x) \leq (K'+1)\varphi(y) + (\nabla\varphi(y) + K'\nabla\varphi(x))\cdot(x-y). \tag{5.3}$$
From (5.2) and (5.3), we get
$$\begin{aligned}\frac{1}{K'+1}&\nabla\varphi(x)\cdot(x-y) + \frac{K'}{K'+1}\nabla\varphi(y)\cdot(x-y)\\ &\leq \varphi(x) - \varphi(y)\\ &\leq \left(\frac{1}{K'+1}\nabla\varphi(y) + \frac{K'}{K'+1}\nabla\varphi(x)\right)\cdot(x-y).\end{aligned} \tag{5.4}$$
By using the first inequality in (5.4) we get
$$\frac{1}{K'+1}\left(\nabla\varphi(x) - \nabla\varphi(y)\right)\cdot(x-y) \leq \varphi(x) - \varphi(y) - \nabla\varphi(y)\cdot(x-y). \tag{5.5}$$
The second inequality in (5.4) yields
$$\varphi(x) - \varphi(y) - \nabla\varphi(x)\cdot(x-y) \leq \frac{1}{K'+1}\left(\nabla\varphi(y) - \nabla\varphi(x)\right)\cdot(x-y), \tag{5.6}$$
which implies
$$\varphi(x) - \varphi(y) - \nabla\varphi(y)\cdot(x-y) \leq \frac{K'}{K'+1}\left(\nabla\varphi(x) - \nabla\varphi(y)\right)\cdot(x-y). \tag{5.7}$$

Now (iii) follows from (5.7) and (5.5) with $K'' = K'$. □

The following result relates the Euclidean balls and the d_φ-balls, providing the quantitative behaviour of $\varphi \in \mathrm{Eng}(n, K)$.

Theorem 5. *Let $\varphi \in \mathrm{Eng}(n, K)$ and $r > 0$. For $y \in \mathbb{R}^n$ define $\varphi_y(x) = \varphi(x) - \varphi(y) - \nabla\varphi(y) \cdot (x - y)$. If $|x - y| \le r$, then*

(5.8)

$$\left(\min_{z:|z-y|=r} \varphi_y(z)\right) \left(\frac{|x-y|}{r}\right)^{1+K} \le \varphi(x) - \varphi(y) - \nabla\varphi(y) \cdot (x-y) \le \left(\max_{z:|z-y|=r} \varphi_y(z)\right) \left(\frac{|x-y|}{r}\right)^{1+1/K}.$$

If $|x - y| \ge r > 0$, then

(5.9)

$$\left(\min_{z:|z-y|=r} \varphi_y(z)\right) \left(\frac{|x-y|}{r}\right)^{1+1/K} \le \varphi(x) - \varphi(y) - \nabla\varphi(y) \cdot (x-y) \le \left(\max_{z:|z-y|=r} \varphi_y(z)\right) \left(\frac{|x-y|}{r}\right)^{1+K}.$$

Proof of Theorem 5. We shall first prove that if $\varphi \in \mathrm{Eng}_0(n, K)$ and $|x| \le r$,

$$\left(\min_{z:|z|=r} \varphi(z)\right) \left(\frac{|x|}{r}\right)^{1+K} \le \varphi(x) \le \left(\max_{z:|z|=r} \varphi(z)\right) \left(\frac{|x|}{r}\right)^{1+1/K}, \tag{5.10}$$

and, if $|x| \ge r > 0$, then

$$\left(\min_{z:|z|=r} \varphi(z)\right) \left(\frac{|x|}{r}\right)^{1+1/K} \le \varphi(x) \le \left(\max_{z:|z|=r} \varphi(z)\right) \left(\frac{|x|}{r}\right)^{1+K}. \tag{5.11}$$

Consider first a function $\phi \in \mathrm{Eng}_0(1, K)$. By Theorem 4 we know that

$$\frac{1}{K}\phi(t) \le \phi'(t)t - \phi(t) \le K\phi(t), \tag{5.12}$$

for every $t \in \mathbb{R}$. Let us work out the second inequality in the first place. For $t > 0$, we get

$$\frac{\phi'(t)}{\phi(t)} \le (1+K)\frac{1}{t},$$

recognizing the derivatives of the corresponding logarithms, we get that the function $\phi(t)/t^{1+K}$ is decreasing in $(0, \infty)$. Now, given $x \in \mathbb{R}^n$, write $x = tx_0$, where $|x_0| = 1$, and define $\phi(t) = \varphi(tx_0)$. By Lemma 3, $\phi \in \mathrm{Eng}_0(1, K)$. If $|x| \le r$, then $t \le r$ and we use the mentioned monotonicity to get

$$\phi(r)/r^{1+K} \le \phi(t)/t^{1+K},$$

which is,

$$\varphi(rx_0)\frac{1}{r^{1+K}} \le \varphi(tx_0)\frac{1}{t^{1+K}} = \varphi(x)\frac{1}{|x|^{1+K}},$$

and the first inequality in (5.8) follows. The other inequalities are proven in similar fashion, by remarking that the function $\phi(t)/t^{1+1/K}$ is increasing in $(0,\infty)$.
In order to finish the proof we need to consider the general case $\varphi \in \mathrm{Eng}(n,K)$. In this case, given $y \in \mathbb{R}^n$, define $\varphi_y(x) = \varphi(x+y) - \varphi(y) - \nabla\varphi(y)\cdot x$. Thus, by Lemma 3, $\psi_y \in \mathrm{Eng}_0(n,K)$ and we complete the proof by applying (5.11) and (5.10) to the function ψ. □

6. Eng(n) is invariant under convex conjugation

Lemma 6. *If $\varphi \in \mathrm{Eng}(n,K)$, then $\nabla\varphi : \mathbb{R}^n \to \mathbb{R}^n$ is a continuous bijection.*

Proof of Lemma 6. The continuity of $\nabla\varphi$ follows from Caffarelli's results mentioned in the Introduction. Injectivity of $\nabla\varphi$ follows from the strict convexity of φ. We could also use that $\varphi \in \mathrm{Eng}(n,K)$ to turn ρ_φ into a quasi-distance, consequently

$$\nabla\varphi(x) = \nabla\varphi(y) \Rightarrow \rho_\varphi(x,y) = 0 \Rightarrow x = y.$$

To prove that $\nabla\varphi$ is onto, note that it is enough to suppose $\varphi \in \mathrm{Eng}_0(n,K)$ (subtract a hyperplane from φ). Thus, (5.11), with $r=1$, gives

$$\lim_{|x|\to+\infty} \frac{\varphi(x)}{|x|} = +\infty.$$

Now, given $a \in \mathbb{R}^n$ we can minimize $h(x) \doteq \varphi(x) - a\cdot x$ to get that $a \in \nabla\varphi(\mathbb{R}^n)$. □

Theorem 7. *Let φ be in $\mathrm{Eng}(n,K)$. If φ^* denotes the conjugate of φ, then $\varphi^* \in \mathrm{Eng}(n,K^*)$ with $K^* = 2K(K+1)$. Moreover, the sections of φ and φ^* are related as follows: for every $x \in \mathbb{R}^n, t > 0$*

$$\nabla\varphi(S_\varphi(x,t/K)) \subset S_{\varphi^*}(\nabla\varphi(x),t) \subset \nabla\varphi(S_\varphi(x,Kt)). \tag{6.13}$$

For the proof, see [8]. Recall that

$$\varphi^*(x) = \sup_{z\in\mathbb{R}^n}(x\cdot z - \varphi(z)).$$

Since φ has the engulfing property, we know that φ is a strictly convex differentiable function. By Theorem 26.5 in [11], we get that φ^* is also a strictly convex differentiable function whose domain is $\nabla\varphi(\mathbb{R}^n)$ which, by Lemma 6, equals $\mathbb{R}^n$. We also have

$$\nabla\varphi(\nabla\varphi^*(x)) = \nabla\varphi^*(\nabla\varphi(x)) = x \quad \forall x \in \mathbb{R}^n, \tag{6.14}$$

$$(\varphi^*)^* = \varphi,$$

and

$$\varphi^*(\nabla\varphi(x)) = \nabla\varphi(x)\cdot x - \varphi(x) \quad \forall x \in \mathbb{R}^n, \tag{6.15}$$

(remark that (6.15) and (5.12) imply $\varphi^*(\nabla\varphi(x)) \simeq \varphi(x)$).

7. On Caffarelli's $C^{1,\beta}$ regularity result

As mentioned in the Introduction, Caffarelli proved the $C^{1,\beta}$ regularity of any convex function $\varphi \in \text{Eng}(n, K)$. His proof is based on a compactness argument that does not provide an estimate for β or the $C^{1,\beta}$ norm of φ on compact sets. The task of finding the explicit size of these constants was posed as an open problem in Villani's recent book (see [12], pp. 141).
In [8], the authors got such estimates, in terms of K, through Theorem 5. To illustrate the main idea, let us take a look at the case $n = 1$. Consider $\varphi \in \text{Eng}_0(1, K)$, $|x| \leq 1$, and denote by $M(\varphi, 1)$ the maximum between $\varphi(1)$ and $\varphi(-1)$. Then, by (5.8), we get $\varphi(x) \leq M(\varphi, 1)|x|^{1+1/K}$. On the other hand, by (5.12), we have $0 \leq \varphi'(x)x \leq (K+1)\varphi(x)$. Consequently, for every x with $|x| \leq 1$, we get $|\varphi'(x)| \leq (K+1)M(\varphi, 1)|x|^{1/K}$. Which is the $C^{1/K}$ regularity of φ' about 0. Before stating the general result some notation is in order. Given a convex function $\phi \in \text{Eng}(n, K), y \in \mathbb{R}^n$, and $r > 0$, set

$$M(\phi, y, r) = \max_{z:|z-y|=r} \left\{ \phi(z) - \phi(y) - \nabla\phi(y) \cdot (z - y) \right\},$$

and

$$m(\phi, y, r) = \min_{z:|z-y|=r} \left\{ \phi(z) - \phi(y) - \nabla\phi(y) \cdot (z - y) \right\}.$$

Theorem 8. *Let $\varphi \in \text{Eng}(n, K), \varphi^* \in \text{Eng}(n, K^*)$, and $y \in \mathbb{R}^n$. For every $z \in \mathbb{R}^n$ with $|z - y| \leq r$, we have*

$$\frac{|\nabla\varphi(z) - \nabla\varphi(y)|}{|z - y|^{1/1+K^*}} \leq C(r, K, m(\psi_y^*, 0, 1), M(\varphi, y, r)),$$

where ψ_y^* is the convex conjugate to

$$\psi_y(x) = \varphi(x + y) - \varphi(y) - \nabla\varphi(y) \cdot x.$$

Thus, $\nabla\varphi$ is in C^β with $\beta = 1/1 + K^*$ and $K^* = 2K(K+1)$.

8. Further remarks

If the Monge-Ampère measure μ_φ satisfies the (DC)-doubling condition with constants C and α, then $\varphi \in \text{Eng}(n, K)$ with

$$K = \frac{2^{n+2} w_n w_{n-1}}{\alpha_n^{n+1}} \frac{C}{(1-\alpha)^n} + 1,$$

where w_k is the volume of the k-dimensional unit ball and $\alpha_n = n^{-3/2}$. In the case $\lambda \leq \det D^2\varphi \leq \Lambda$, if we set $\alpha = 1/2$ we get $C = 2^n\Lambda/\lambda$. These constants can be easily followed up from [9].
Although we consider solutions to $\det D^2\varphi = \mu$ in $\mathbb{R}^n$, the main results in this paper can be proved (after slight modifications) for solutions to the Monge-Ampère equation in a bounded convex domain $\Omega \subset \mathbb{R}^n$.

References

[1] Aimar H., Forzani L., and Toledano R., Balls and quasi-metrics: a space of homogeneous type modeling the real analysis related to the Monge-Ampère equation, *J. Fourier Anal. Appl.* **4**, (1998), 377-381.

[2] Caffarelli L., Boundary regularity of maps with convex potentials, *Comm. Pure Appl. Math.* **45** (1992), 1141-1151.

[3] Caffarelli L., Some regularity properties of solutions of Monge-Ampère equation, *Comm. Pure Appl. Math.* **44** (1991), 965-969.

[4] Caffarelli L. and Gutiérrez C., Real analysis related to the Monge-Ampère equation, *Trans. Amer. Math. Soc.* **348** (1996), 1075-1092.

[5] Caffarelli L. and Gutiérrez C., Properties of the solutions of the linearized Monge-Ampère equation, *Amer. J. Math.* **119(2)** (1997), 423-465.

[6] Forzani L. and Maldonado D., On geometric characterizations for Monge-Ampère doubling measures, *J. Math. Anal. Appl.* **275(2)** (2002), 721-732.

[7] Forzani L. and Maldonado D., Doubling weights, quasi-symmetric mappings, and a class of convex functions on the real line. Submitted.

[8] Forzani L. and Maldonado D., Properties of the solutions to the Monge-Ampère equation. *Nonlinear Anal.* **57**(5-6) (2004), 815-829.

[9] Gutiérrez C., *The Monge-Ampère Equation.* Progress in Nonlinear Differential Equations and Their Applications, volume 44. Birkhäuser, 2001.

[10] Gutiérrez C. and Huang Q., Geometric properties of the sections of solutions to the Monge-Ampère equation, *Trans. Amer. Math. Soc.* **352(9)** (2000), 4381-4396.

[11] Rockafellar R.T., *Convex Analysis.* Princeton University Press, 1970.

[12] Villani C., *Topics in Optimal Transportation.* Graduate Studies in Mathematics. Volume 58. AMS, 2003.

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