THE $\rho-$VARIATION AS AN OPERATOR BETWEEN MAXIMAL OPERATORS AND SINGULAR INTEGRALS

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ABSTRACT. The $\rho-$variation and the oscillation of the heat and Poisson semigroups of the Laplacian and Hermite operators (i.e Δ and $-\Delta + |x|^2$) are prove to be bounded from $L^p(\mathbb{R}^n, w(x)dx)$ into itself (from $L^1(\mathbb{R}^n, w(x)dx)$ into weak-$L^1(\mathbb{R}^n, w(x)dx)$ in the case $p = 1$) for $1 \leq p < \infty$ and w being a weight in the Muckenhoupt's A_p class.

In the case $p = \infty$ it is proved that these operators doesn't map L^∞ into itself. Even more, they map L^∞ into BMO but the range of the image is strictly smaller that the range of a general singular integral operator.

1. INTRODUCTION

Let $\mathcal{T} = \{T_t\}_{t\in\mathbb{R}}$ be a family of bounded operators acting between spaces of functions. One of the most studied problems in Harmonic Analysis is the existence of limits $\lim_{t\to 0} T_t f$ and $\lim_{t\to\infty} T_t f$, when f belongs to a certain space of functions. Typical examples of this situation are found in the study of the convergence of solutions of the heat and Poisson equations to a boundary value. Then, the question can be posed of what is the speed of convergence of the above limits. A classic method of measuring that speed is to consider square functions of the type $(\sum_{i=1}^\infty |T_{t_i} f - T_{t_{i+1}} f|^2)^{1/2}$. The problem goes back to the 30's of the last century and the names of Littlewood and Paley are associated to it.

In the last years, in order to measure this speed, other expressions such as the $\rho-$variation and the oscillation operators have been considered as well, see [1], [2], [4], [6], and the references there in. The **ρ-Variation operator** is defined by

$$\mathcal{V}_\rho(\mathcal{T})f(x) = \sup_{t_i \searrow 0} \left(\sum_{i=1}^{\infty} |T_{t_i} f(x) - T_{t_{i+1}} f(x)|^\rho \right)^{1/\rho}, \quad \rho > 2,$$

where the sup is taken over all sequences t_i that are decreasing to zero. The **Oscillation operator** can be introduced as

$$\mathcal{O}(\mathcal{T})f(x)= \left(\sum_{i=1}^{\infty} \sup_{t_{i+1} \leq \varepsilon_{i+1} < \varepsilon_i \leq t_i} |T_{\varepsilon_{i+1}} f(x) - T_{\varepsilon_i} f(x)|^2 \right)^{1/2},$$

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where t_i is a fixed sequence decreasing to zero.

Our intention in this paper is to obtain new results for those operators when the family $\mathcal{T} = \{T_t\}_{t\in\mathbb{R}}$ is either the heat semigroup or the Poisson semigroup associated to the Laplacian $\Delta = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}$ or the Hermite operator $H = -\Delta + |x|^2$.

Nowadays it is well known that the vector valued Calderón-Zygmund theory is the appropriate setting in order to study the behaviour of expressions like $(\sum_{i=1}^{\infty} |T_{t_i}f - T_{t_{i+1}}f|^2)^{1/2}$. Hence, it seems natural to use that theory in order to study the $\rho-$variation and the oscillation operators. In fact, several results for them have been obtained by that method, see for instance [1] and [6].

In this paper, using the vector valued Calderón-Zygmund theory and with the help of some previous results in [2], we prove the following

UNO **Theorem 1.1.** *Let $\mathcal{T} = \{T_t\}$ be either the heat or the Poisson semigroup associated to any of the operators Δ or H. Then the oscillation operator, $\mathcal{O}(\mathcal{T})$, and the $\rho-$variation operator, $\mathcal{V}_\rho(\mathcal{T})$, $\rho > 2$, are bounded from $L^p(\mathbb{R}^n, w(x)dx)$ into itself for $1 < p < \infty$ and $w \in A_p$. Moreover $\mathcal{O}(\mathcal{T})$ and $\mathcal{V}_\rho(\mathcal{T})$ are bounded from $L^1(\mathbb{R}^n, w(x)dx)$ into weak$-L^1(\mathbb{R}^n, w(x)dx)$ for $w \in A_1$.*

For the reader's convenience we recall that a measurable function w is said to be in the A_p class, $1 \le p < \infty$, if it satisfies the following conditions: w is positive and finite almost everywhere and the Hardy-Littlewood maximal operator is bounded from $L^p(\mathbb{R}^n, w(x)dx)$ into itself, for $1 < p < \infty$, and from $L^1(\mathbb{R}^n, w(x)dx)$ into weak$-L^1(\mathbb{R}^n, w(x)dx)$ if $p = 1$.

We suggest the reader to look at Theorem 1.1 as a result saying that the operators $\mathcal{O}(\mathcal{T})$ and $\mathcal{V}_\rho(\mathcal{T})$ behave as any standard Calderón-Zygmund operator. However due to the particular form of these operators, one could try to analyze their size in comparison with some particular operators. In this line of thought we prove that in general these operators are "bigger" than their corresponding maximal operators. In fact, we shall prove the following

menor **Theorem 1.2.** *Let $\mathcal{T}$ be the heat semigroup associated to Δ; then the operator $\mathcal{O}(\mathcal{T})$ is not bounded from $L^\infty(\mathbb{R})$ to $L^\infty(\mathbb{R})$.*

On the other hand, estimates (9) and (10) establish that, as is the case with any standard Calderón-Zygmund operator, the image by $\mathcal{O}(\mathcal{T})$ of a function in $L^\infty(\mathbb{R}^n, dx)$ with compact support will be in BMO. It is well known that a function in BMO can be unbounded and that its growth can be of logarithmic type. Moreover,

$$BMO(\mathbb{R}) = \{f_1 + \mathcal{H}f_2 : f_1, f_2 \in L^\infty(\mathbb{R})\},$$

holds, being the operator $\mathcal{H}$ the Hilbert transform. Then, one can deduce that the image of an $L^\infty(\mathbb{R})$ function by a general Calderón-Zygmund operator is a function in BMO with a logarithmic type increase. The following result shows that in some sense the oscillation and variation operators are "smaller" than a general Calderón-Zygmund operator.

mayor **Theorem 1.3.** *Let $\mathcal{T} = \{e^{t\Delta}\}$. For every function $f \in L^\infty(\mathbb{R}^n, dx)$ with support contained in the unit ball B_0, there exists a constant C such that for every ball of radius r, such that $B_r \subset B_0$, we have*

$$\frac{1}{|B_r|}\int_{B_r} |\mathcal{O}(\mathcal{T})f(x)|dx \le C\Big(\log\frac{1}{r}\Big)^{1/2}\|f\|_{L^\infty(\mathbb{R}^n,dx)}.$$

Now we shall describe the technical development of this manuscript, with especial attention to the differences between the operators associated to Δ and H.

The heat semigroup associated to Δ is defined as

$$e^{t\Delta}f(x) = \frac{1}{(\pi t)^{n/2}}\int_{\mathbb{R}^n}\exp\Big(-\frac{|x-y|^2}{t}\Big)f(y)dy.$$

The Poisson semigroup, $P_t = e^{-t\sqrt{-\Delta}}$, is introduced throughout the following subordination formula

subordinacion
$$(1)\quad P_tf(x) = \frac{1}{\sqrt{4\pi}}\int_0^\infty te^{-t^2/4s}T_sf(x)s^{-3/2}ds = \frac{1}{\sqrt{\pi}}\int_0^\infty \frac{e^{-u}}{\sqrt{u}}T_{t^2/4u}f du,$$

where $T_s = e^{s\Delta}$.

We can consider in an analogous way the heat semigroup e^{-tH} (observe that H is positive) defined as

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$$(2)\qquad e^{-tH}f(x) = (2\pi\sinh 2t)^{-n/2}\int_{\mathbb{R}^n} e^{-\frac{1}{2}|x-y|^2\coth 2t - x.y\tanh t}f(y)dy,$$

and its Poisson semigroup $(e^{-t\sqrt{H}})$ defined by the formula (1), see [9] and [8].

It is also known that the semigroups $e^{t\Delta}$, $e^{-t\sqrt{-\Delta}}$, e^{-tH}, $e^{-t\sqrt{H}}$, are contractions in $L^p(\mathbb{R}^n, dx)$, $1 \le p \le \infty$, see [9]. Moreover, if we denote by T_t any of these semigroups, the limits $\lim_{t\to 0} T_tf$ and $\lim_{t\to\infty} T_tf$ exist, in L^p-norm and almost everywhere, for functions $f \in L^p(\mathbb{R}^n, dx)$, $1 \le p < \infty$.

Before displaying the proof of Theorem 1.1 we observe that the formula (1) implies that the Poisson semigroup is a type of integral mean of the heat semigroup. This fact will allow us to prove in Theorem 2.3 the boundedness of the oscillation and $\rho-$variation operators related to the Poisson semigroup, having previously obtained the corresponding ones for the heat semigroup. Consequently, we are led to prove the L^p, $1 < p < \infty$, boundedness just for the heat semigroup. These proofs are developed in Theorem 4.1 for the Δ operator, and in Theorem 4.5 for the H operator.

Nevertheless, for $p = 1$ the situation is a little bit different, due to the fact that the space $L^{1,\infty}$ is not a Banach space. In order to save this difficulty, using again the subordination formula (1), we observe that the kernels of the operators related to the Poisson semigroup satisfy the same estimates than the corresponding ones associated to the heat semigroup (see Remark 3.1). Furthermore, as can be seen by (9) and (10), these are standard kernels. This fact, along with the L^p, $1 < p < \infty$, boundedness, allows us to apply the vector-valued Calderón-Zygmund machinery to obtain Theorem 1.1. With these ideas in mind, we introduce in Section 2 the vector-valued analogs to the oscillation and $\rho-$variation operators and we identify its corresponding vector-valued kernels. The appropriate standard estimates for the heat semigroup are proven in Section 3. The L^p boundedness results contained in Theorem 4.1 were already known in the case $e^{t\Delta}$, see [2]. Nevertheless, the results in [2] cannot be applied directly to the family e^{-tH}, reason why, in order to prove the L^p boundedness of $\mathcal{O}(\mathcal{T})$ and $\mathcal{V}_\rho(\mathcal{T})$, we need to use some sharp estimates that are the content of Section 4.

Finally, in Section 5, we show that the operators are not bounded in L^∞ but they are smaller than a standard Calderón-Zygmund operator.

approach

2. Vector valued approach

In the following we let $\{t_i\}_i$ be a given fixed decreasing sequence to 0. Consider the operator

$$\mathcal{O}'(\mathcal{T})f(x) = \left(\sum_{i=1}^{\infty} \sup_{t_{i+1}<\delta_i \le t_i} |T_{t_{i+1}}f(x) - T_{\delta_i}f(x)|^2 \right)^{1/2}.$$

It is easy to see that

$$\mathcal{O}'(\mathcal{T})f(x) \sim \mathcal{O}(\mathcal{T})f(x) \quad a.e.x.$$ sim

Consequently, it will be enough to demonstrate Theorem 1.1 for the operator $\mathcal{O}'(\mathcal{T})$ instead of $\mathcal{O}(\mathcal{T})$.

Let us denote by E_p the mixed normed Banach space of two variable functions h defined on $\mathbb{R} \times \mathbb{N}$, such that

E (3)
$$\|h\|_{E_p} \equiv \left(\sum_i (\sup_s |h(s,i)|)^p \right)^{1/p} < \infty.$$

Let $\mathcal{T} = \{T_t\}_{t>0}$ be a family of operators defined on $L^p(\mathbb{R}^n, d\mu)$, for some p in the range $1 \le p < \infty$. Let $J_i = (t_{i+1}, t_i]$ and define the operator $U(\mathcal{T}) : f \longrightarrow U(\mathcal{T})f$, where $U(\mathcal{T})f$ is the E_2-valued function given by

U (4)
$$U(\mathcal{T})f(x) = \left\{ T_{t_{i+1}}f(x) - T_s f(x) \right\} \chi_{J_i}(s).$$

Then

paso (5)
$$\mathcal{O}'(\mathcal{T})f(x) = \left\| \left\{ T_{t_{i+1}}f(x) - T_s f(x) \right\} \chi_{J_i}(s) \right\|_{E_2} = \|U(\mathcal{T})f(x)\|_{E_2}.$$

Let $\Theta = \{\varepsilon : \varepsilon = \{\varepsilon_i\}, \varepsilon_i \in \mathbb{R}, \varepsilon_i \searrow 0\}$. We consider the set $\mathbb{N} \times \Theta$ and denote by F_ρ, $1 \le \rho < \infty$, the mixed normed space of two variable functions $g(i, \varepsilon)$ such that

ro (6)
$$\|g\|_{F_\rho} \equiv \sup_\varepsilon \left(\sum_i |g(i,\varepsilon)|^\rho \right)^{1/\rho} < \infty.$$

For a family $\mathcal{T}$ as above, we also consider the operator $V(\mathcal{T}) : f \longrightarrow V(\mathcal{T})f$, acting on functions f belonging to $L^p(\mathbb{R}^n, d\mu)$, and $V(\mathcal{T})f$ being the F_ρ-valued function given by

V (7)
$$V(\mathcal{T})f(x) = \{T_{\varepsilon_{i+1}}f(x) - T_{\varepsilon_i}f(x)\}_{\varepsilon \in \Theta}.$$

As in the case of the oscillation operator it is obvious that

paso2 (8)
$$\mathcal{V}_\rho(\mathcal{T})f(x) = \|V(\mathcal{T})f(x)\|_{F_\rho}.$$

As a consequence of identities (5) and (8), to show Theorem 1.1 it is enough to prove the following Theorem.

DOS **Theorem 2.1.** *Let $\mathcal{T} = \{T_t\}$ be either the heat or the Poisson semigroup associated to any of the operators Δ or H. Then, the operator $U(\mathcal{T})$ (respectively $V(\mathcal{T})$) is bounded from $L^p(\mathbb{R}^n, w(x)dx)$ into $L^p_{E_2}(\mathbb{R}^n, w(x)dx)$ (respectively $L^p_{F_\rho}(\mathbb{R}^n, w(x)dx), \rho > 2$) for $1 < p < \infty$ and $w \in A_p$. Moreover they are bounded from $L^1(\mathbb{R}^n, w(x)dx)$ into weak$-L^1_{E_2}(\mathbb{R}^n, w(x)dx)$ (respectively weak$-L^1_{F_\rho}(\mathbb{R}^n, w(x)dx), \rho > 2$) for $w \in A_1$.*

nota **Remark 2.2.** *In the case that the family $\mathcal{T} = \{T_t\}$ is such that each operator T_t is given by integration against a kernel $M_t(x, y)$, the operator $U(\mathcal{T})$ has also an associated kernel $\mathcal{U}$, where $\mathcal{U}(x, y)$ is the element of E_2 given by*

$$(s, i) \to \mathcal{U}(x, y)(s, i) = (M_{t_{i+1}}(x, y) - M_s(x, y))\chi_{J_i}(s).$$

In other words,

$$U(\mathcal{T})f(x) = \int \mathcal{U}(x, y)f(y)dy = \int \Big\{(M_{t_{i+1}}(x, y) - M_s(x, y))\chi_{J_i}(s)\Big\}f(y)dy. \quad \text{Utotal}$$

Ucal *Analogous formulas can be given for the variation.*

The direct consequence of the last Remark is that in order to demonstrate Theorem 2.1 we can apply vector-valued Calderón-Zygmund theory. That is to say it will be enough to prove two facts: firstly, the operator $U(\mathcal{T})$ (respectively $V(\mathcal{T})$) is bounded from $L^2(\mathbb{R}^n, dx)$ into $L^2_{E_2}(\mathbb{R}^n, dx)$ (respectively from $L^2(\mathbb{R}^n, dx)$ into $L^2_{F_\rho}(\mathbb{R}^n, dx)$, for $\rho > 2$); in Section 4 this will be actually done for $p,\ 1 < p < \infty$, see the comments just after Theorem 4.1 and Theorem 4.5; and secondly, the kernels described in Remark 2.2 satisfy standard conditions; this will be done in Section 3.

The vector valued analogue of the variation and oscillation operators, allows us to prove the following Theorem announced in the introduction.

Poissoncito **Theorem 2.3.** *Let $\mathcal{P} = \{P_t\}$, the subordinated Poisson semigroup of $\mathcal{T} = \{T_t\}$, and $1 < p < \infty$. If $||\mathcal{O}(\mathcal{T})f||_{L^p(w(x)dx)} \leq C||f||_{L^p(w(x)dx)}$ then*

$$||\mathcal{O}(\mathcal{P})f||_{L^p(w(x)dx)} \leq C||f||_{L^p(w(x)dx)}.$$

A similar result can be stated for the variation operator.

Proof. We observe that

$$\begin{aligned} \mathcal{O}(\mathcal{P})f(x) &= \|U(\mathcal{P})f(x)\|_{E_2} = \|\{P_{t_{i+1}}f(x) - P_s f(x)\}\|_{E_2} \\ &\leq \frac{1}{\sqrt{\pi}}\int_0^\infty \frac{e^{-u}}{\sqrt{u}}\|\{T_{t^2_{i+1}/4u}f(x) - T_{s^2/4u}f(x)\}\|_{E_2}du \\ &\leq \frac{1}{\sqrt{\pi}}\int_0^\infty \frac{e^{-u}}{\sqrt{u}}\mathcal{O}^u(\mathcal{T})f(x)du. \end{aligned}$$

Therefore, by using Minkowsky's inequality and the boundedness for $\mathcal{O}(\mathcal{T})$, we have

$$\begin{aligned}\|\mathcal{O}(\mathcal{P})f\|_{L^p(\omega(x)dx)} &\leq C\int_0^\infty \frac{e^{-u}}{\sqrt{u}}\|\mathcal{O}^u(\mathcal{T})f\|_{L^p(\omega(x)dx)}du\\ &\leq C\int_0^\infty \frac{e^{-u}}{\sqrt{u}}\|f\|_{L^p(v(x)dx)}du\\ &\leq C\|f\|_{L^p(v(x)dx)}.\end{aligned}$$

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3. The kernels satisfy standard estimates

Let $\mathcal{T}=\{T_t\}$ be the heat semigroup of Δ. As we indicated on Remark 2.2, the kernel of the operator $U(\mathcal{T})$ is

$$\begin{aligned}(s,i)\longrightarrow \qquad & \mathcal{U}(x,y)(s,i)=(M_{t_{i+1}}(x,y)-M_s(x,y))\chi_{J_i}(s)\\ =\ & \frac{1}{\pi^{n/2}}\Big(\frac{1}{t_{i+1}^{n/2}}e^{-\frac{|x-y|^2}{t_{i+1}}}-\frac{1}{s^{n/2}}e^{-\frac{|x-y|^2}{s}}\Big)\chi_{J_i}(s)\end{aligned}$$ nucleo1

We shall show

11 (9) $$\|\mathcal{U}(x,y)(s,i)\|_{E_2}\leq \frac{C}{|x-y|^n} \quad \text{and}$$

12 (10) $$\|\nabla_x\mathcal{U}(x,y)(s,i)\|_{E_2}+\|\nabla_y\mathcal{U}(x,y)(s,i)\|_{E_2}\leq\frac{C}{|x-y|^{n+1}}.$$

Poissonremark **Remark 3.1.** *Let $\mathcal{P}$ be the subordinated Poisson semigroup of the semigroup $\mathcal{T}$. Let $\mathcal{U}(x,y)(s,i)=(M_{t_{i+1}}(x,y)-M_s(x,y))\chi_{J_i}(s)$ the kernel of the operator $U(\mathcal{T})$; by using the subordination formula (1) we get the following expression for the kernel of the operator $U(\mathcal{P})$*

$$\begin{aligned}\mathcal{W}(x,y)(s,i) &= \Big(\frac{1}{\sqrt{\pi}}\int_0^\infty\frac{e^{-u}}{\sqrt{u}}\{M_{t_{i+1}^2/4u}(x,y)-M_{s^2/4u}(x,y)\}du\Big)\chi_{J_i}(s)\\ &= \frac{1}{\sqrt{\pi}}\int_0^\infty\frac{e^{-u}}{\sqrt{u}}\Big\{M_{t_{i+1}^2/4u}(x,y)-M_{s^2/4u}(x,y)\Big\}\chi_{J_i}(s)du.\end{aligned}$$

Hence, by using Minkowski's inequality we have

$$\begin{aligned}\|\mathcal{W}(x,y)(s,i)\|_{E_2} &\leq \frac{1}{\sqrt{\pi}}\int_0^\infty\frac{e^{-u}}{\sqrt{u}}\Big\|\Big(M_{t_{i+1}^2/4u}(x,y)-M_{s^2/4u}(x,y)\Big)\chi_{J_i}(s)\Big\|_{E_2}du\\ &\leq \frac{C}{|x-y|^n}.\end{aligned}$$

A parallel reasoning could have drove us to see that the kernel $\mathcal{W}$ satisfies estimate (10).

compu **Computational Remark 3.2.** *Along the paper, but mainly along this section and the following section we shall use the following estimate. For every $N>0$, there exist positive constants C and c such that $|u|^Ne^{-|u|}\leq Ce^{-|u|/c}$, where C and c depend only on N. In general, expressions of the type $e^{-\frac{|x-y|^2}{ct}}$ should suggest to the reader that the estimate had been used in some previous calculations with $u=\frac{|x-y|^2}{t}$.*

Let $f(t)=\frac{1}{t^{n/2}}e^{-\frac{|z|^2}{t}}$; we write

$$f(s)-f(t_{i+1})=\int_{t_{i+1}}^{s} f'(t)dt=\int_0^\infty \chi_{[t_{i+1},s]}(t)f'(t)dt,$$

where

$$f'(t)=\Big(\frac{|z|^2}{t^2}-\frac{n}{2t}\Big)\frac{1}{t^{n/2}}e^{-\frac{|z|^2}{t}}.$$

Then, by making $z=x-y$, we have

$$\begin{aligned}
\|\mathcal{U}(x,y)(s,i)\|_{E_2} &\leq \|\mathcal{U}(x,y)(s,i)\|_{E_1}\leq \sum_i \sup_{t_{i+1}<s\leq t_i}\Big|\frac{1}{t_{i+1}^{n/2}}e^{-\frac{|x-y|^2}{t_{i+1}}}-\frac{1}{s^{n/2}}e^{-\frac{|x-y|^2}{s}}\Big| \\
&= \sum_i \sup_{t_{i+1}<s\leq t_i}\Big|\int_{t_{i+1}}^{s} f'(t)dt\Big| \leq \sum_i \sup_{t_{i+1}<s\leq t_i}\int_{t_{i+1}}^{s}\Big|f'(t)\Big|dt \qquad (11)\\
&= \sum_i \int_0^\infty \chi_{[t_{i+1},t_i]}(t)|f'(t)|dt \leq \int_0^\infty |f'(t)|dt\\
&= \int_0^\infty \Big|(\frac{|x-y|^2}{t^2}-\frac{n}{2t})\Big|\frac{1}{t^{n/2}}e^{-\frac{|x-y|^2}{t}}dt\\
&\leq \int_0^\infty (\frac{|x-y|^2}{t}+\frac{n}{2})\frac{1}{t^{n/2}}e^{-\frac{|x-y|^2}{t}}\frac{dt}{t}\\
&\leq C\int_0^\infty \frac{1}{t^{n/2}}e^{-\frac{|x-y|^2}{ct}}\frac{dt}{t}\\
&= \frac{C}{|x-y|^n}\int_0^\infty u^{n/2}e^{-u/c}\frac{du}{u}\\
&= \frac{C}{|x-y|^n},
\end{aligned}$$

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where we have used the Computational Remark 3.2 and in the penultimate inequality we have made the change $u=\frac{|x-y|^2}{t}$. This ends the proof of (9).

In order to prove (10) we consider $g(t)=\frac{|z|}{t^{(n/2)+1}}e^{-\frac{|z|^2}{t}}$. The proof runs along the same lines as in the case of (9), just by observing that

$$\begin{aligned}
\int_0^\infty |g'(t)|dt &\leq C\int_0^\infty \Big(\frac{|x-y|^2}{t^{(n/2)+2}}+\frac{(n/2)+1}{t^{n/2+1}}\Big)|x-y|e^{-\frac{|x-y|^2}{t}}\frac{dt}{t}\\
&\leq C|x-y|\int_0^\infty \frac{1}{t^{(n/2)+1}}e^{-\frac{|x-y|^2}{ct}}\frac{dt}{t}\\
&= \frac{C}{|x-y|^{n+1}}\int_0^\infty u^{(n/2)+1}e^{-u/c}\frac{du}{u},
\end{aligned}$$

where as usual we have made the change of variables $u=\frac{|x-y|^2}{t}$.

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Remark 3.3. *We observe that in fact we have proved the following chain of inequalities*

$$\|\mathcal{U}(x,y)(s,i)\|_{E_2}\leq \sum_i \sup_{t_{i+1}<s\leq t_i}\Big|\frac{1}{t_{i+1}^{n/2}}e^{-\frac{|x-y|^2}{t_{i+1}}}-\frac{1}{s^{n/2}}e^{-\frac{|x-y|^2}{s}}\Big|\leq \frac{C}{|x-y|^n}.$$

$$\begin{aligned}\|\nabla_x \mathcal{U}(x,y)(s,i)\|_{E_2} &\leq C\sum_i \sup_{t_{i+1}<s\leq t_i} |x-y|\Big|\frac{1}{t_{i+1}^{(n/2)+1}} e^{-\frac{|x-y|^2}{t_{i+1}}} - \frac{1}{s^{(n/2)+1}} e^{-\frac{|x-y|^2}{s}}\Big| \\ &\leq \frac{C}{|x-y|^{n+1}}.\end{aligned}$$

The constant C doesn't depend on the particular sequence of $\{t_i\}$. We left to the reader to check that for the kernels of the operator $V(\mathcal{T})$ a similar reasoning can be carried out. Hence the corresponding kernel also satisfies standard estimates considering the norm F_ρ with $\rho > 2$ given by formula (6).

Now we shall prove the standard estimates for the case of the heat semigroup of $H = -\Delta + |x|^2$. By making the change of parameter $t = t(s) = \frac{1}{2}\log\frac{1+s}{1-s}$, $0 < s < 1$, $0 < t < \infty$, we have that in order to analyze the oscillation and variation of the family $\mathcal{T} = \{T_t\}_{t=0}^{\infty}$ given by formula (2), it is enough to analyze the corresponding oscillation and variation of the family $\mathcal{R} = \{R_s\}_{0<s<1}$ given by

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$$\begin{aligned} R_s f(x) &= \int_{\mathbb{R}^n} R_s(x,y) f(y)dy \\ &= \left(\frac{1-s^2}{4\pi s}\right)^{n/2} \int_{\mathbb{R}^n} e^{-\frac{1}{4}(s|x+y|^2 + \frac{1}{s}|x-y|^2)} f(y)dy. \end{aligned} \tag{12}$$

In this case the kernel of the operator $U(\mathcal{R})$ can be expressed as

$$\{\mathcal{R}(x,y)\} = \left\{\Big(R_{s_{i+1}}(x,y) - R_s(x,y)\Big)\chi_{J_i}(s)\right\},$$

where $0 < s_{i+1} < s \leq s_i \leq 1$, $s_i \searrow 0$. In order to follow the path in the previous proofs, we consider the functions $h(s) = \dfrac{1}{4}(s|x+y|^2 + \dfrac{1}{s}|x-y|^2)$. and $f(s) = \Big(\dfrac{1-s^2}{s}\Big)^{n/2} e^{-h(s)}$. Then, proceeding analogously as we did in proving (9) for $\mathcal{T}$ we obtain

estimacion

$$\begin{aligned} &\Big\|\left\{\Big(R_{s_{i+1}}(x,y) - R_s(x,y)\Big)\chi_{J_i}(s)\right\}\Big\|_{E_2} \\ &\leq \left\|\left(\left(\frac{1-s_{i+1}^2}{s_{i+1}}\right)^{n/2} e^{-h(s_{i+1})} - \left(\frac{1-s^2}{s}\right)^{n/2} e^{-h(s)}\right)\chi_{J_i}(s)\right\|_{E_2} \\ &\leq \sum_{i=1}^{\infty} \sup_{s_{i+1}<s\leq s_i} \Big|\Big(\frac{1-s_{i+1}^2}{s_{i+1}}\Big)^{n/2} e^{-h(s_{i+1})} - \Big(\frac{1-s^2}{s}\Big)^{n/2} e^{-h(s)}\Big| \\ &= \sum_{i=1}^{\infty} \sup_{s_{i+1}<s\leq s_i} |f(s_{i+1}) - f(s)| \\ &\leq \int_0^1 |f'(s)|ds, \end{aligned} \tag{13}$$

with

$$\begin{aligned} f'(s) &= \Big\{-\frac{n}{2}\Big(\frac{1-s^2}{s}\Big)^{(n/2)-1}\frac{1+s^2}{s^2} - \Big(\frac{1-s^2}{s}\Big)^{n/2}(\frac{1}{4}|x+y|^2 - \frac{1}{4s^2}|x-y|^2)\Big\}e^{-h(s)} \\ &= \Big\{-\frac{n}{2}\Big(\frac{1-s^2}{s}\Big)^{(n/2)-1}\frac{1+s^2}{s^2}\Big\}e^{-h(s)} \\ &\quad +\Big\{-\Big(\frac{1-s^2}{s}\Big)^{n/2}(\frac{1}{4}|x+y|^2 - \frac{1}{4s^2}|x-y|^2)\Big\}e^{-h(s)} \\ &= A_1(s) + A_2(s). \end{aligned}$$

Therefore, it follows

estimacion1

$$\begin{aligned} \int_0^1 |A_1(s)|ds &\leq C\int_0^1 \Big(\frac{1-s^2}{s}\Big)^{(n/2)-1}\frac{1+s^2}{s^2}e^{-\frac{1}{4s}|x-y|^2}ds \\ &\leq C\int_0^1 \Big(\frac{1-s^2}{s}\Big)^{(n/2)}\frac{1}{1-s^2}e^{-\frac{1}{4s}|x-y|^2}\frac{ds}{s} \\ &= C\int_0^{1/2} \Big(\frac{1-s^2}{s}\Big)^{(n/2)}\frac{1}{1-s^2}e^{-\frac{1}{4s}|x-y|^2}\frac{ds}{s} \\ &\quad +C\int_{1/2}^1 \Big(\frac{1-s^2}{s}\Big)^{(n/2)}\frac{1}{1-s^2}e^{-\frac{1}{4s}|x-y|^2}\frac{ds}{s} \\ &\leq C\int_0^{1/2} \frac{1}{s^{(n/2)}}e^{-\frac{1}{4s}|x-y|^2}\frac{ds}{s} \\ &\quad +C\int_{1/2}^1 (1-s)^{(n/2)}\frac{1}{1-s}e^{-\frac{1}{4}|x-y|^2}\frac{ds}{s} \\ &\leq C\frac{1}{|x-y|^n} + Ce^{-\frac{1}{4}|x-y|^2} \leq C\frac{1}{|x-y|^n}. \end{aligned} \tag{14}$$

On the other hand

estimacion2

$$\begin{aligned} \int_0^1 |A_2(s)|ds &\leq \int_0^1 \frac{1}{s^{n/2}}\frac{1}{4}|x+y|^2 e^{-\frac{1}{4}(s|x+y|^2+\frac{1}{s}|x-y|^2)}ds \\ &\quad +\int_0^1 \frac{1}{s^{n/2}}\frac{1}{4s^2}|x-y|^2 e^{-\frac{1}{4}(s|x+y|^2+\frac{1}{s}|x-y|^2)}ds \\ &\leq C\int_0^1 \frac{1}{s^{n/2}}\frac{1}{s}e^{-\frac{1}{4}(\frac{1}{s}|x-y|^2)}ds + \int_0^1 \frac{1}{s^{n/2}}\frac{1}{s}e^{-\frac{1}{4}(\frac{1}{cs}|x-y|^2)}ds \\ &\leq C\int_0^{1/2} \frac{1}{s^{n/2}}e^{-\frac{1}{4}(\frac{1}{cs}|x-y|^2)}\frac{ds}{s} + C\int_{1/2}^1 \frac{1}{s^{n/2}}e^{-\frac{1}{4}(\frac{1}{cs}|x-y|^2)}\frac{ds}{s} \\ &\leq C\frac{1}{|x-y|^n} + Ce^{-\frac{|x-y|^2}{c}} \leq C\frac{1}{|x-y|^n}. \end{aligned} \tag{15}$$

This ends the proof of estimate (9) for the case of $U(\mathcal{R})$. In order to prove the estimate (10), we observe that

$$\begin{aligned} \Big\|\nabla_x\mathcal{R}(x,y)\Big\|_{E_2} &= \Big\|\Big\{\Big(\Big(\frac{1-s_{i+1}^2}{\pi s_{i+1}}\Big)^{n/2}\Big(-\frac{1}{2}\frac{(x-y)}{s_{i+1}}-\frac{1}{2}s_{i+1}(x+y)\Big)\,e^{-\frac{1}{4}(s_{i+1}|x+y|^2+\frac{1}{s_{i+1}}|x-y|^2)} \\ &- \Big(\frac{1-s^2}{\pi s}\Big)^{n/2}\Big(-\frac{1}{2}\frac{(x-y)}{s}-\frac{1}{2}s(x+y)\Big)\,e^{-\frac{1}{4}(s|x+y|^2+\frac{1}{s}|x-y|^2)}\Big)\chi_{J_i}(s)\Big\}\Big\|_{E_2} \\ &\le \int_0^1 \|g'(s)\|_{\mathbb{R}^n}\,ds, \end{aligned}$$

where $g(s)=\Big(\frac{1-s^2}{s}\Big)^{n/2}\Big(-\frac{1}{2}\frac{(x-y)}{s}-\frac{1}{2}s(x+y)\Big)\,e^{-h(s)}$, $x,y\in\mathbb{R}^n$, being $h(s)=\frac{1}{4}(s|x+y|^2+\frac{1}{s}|x-y|^2)$.

The derivative with respect to s of the function g is

$$\begin{aligned} g'(s) &= \Big\{-\frac{n}{2}\Big(\frac{1-s^2}{s}\Big)^{n/2-1}\Big(\frac{1+s^2}{s^2}\Big)\Big(-\frac{1}{2}\frac{(x-y)}{s}-\frac{1}{2}s(x+y)\Big) \\ &\quad+\Big(\frac{1-s^2}{s}\Big)^{n/2}\Big(\frac{1}{2}\frac{(x-y)}{s^2}-\frac{1}{2}(x+y)\Big) \\ &\quad+\Big(\frac{1-s^2}{s}\Big)^{n/2}\Big(-\frac{1}{2}\frac{(x-y)}{s}-\frac{1}{2}s(x+y)\Big)\Big(-\frac{1}{4}(|x+y|^2+\frac{1}{s^2}|x-y|^2)\Big)\Big\}e^{-h(s)} \\ &= B_1(s)+B_2(s)+B_3(s). \end{aligned}$$

Now we shall study each term of the previous sum

$$\begin{aligned} \int_0^1 |B_1(s)|ds &\le C\int_0^1 (1-s^2)^{(n/2)-1}\frac{1}{s^{n/2}}\Big(\frac{|x-y|}{s}+s|x+y|\Big)\,e^{-h(s)}\frac{ds}{s} \\ &\le C\int_0^1 (1-s^2)^{(n/2)-1}\frac{1}{s^{n/2}}\Big(\frac{1}{s^{1/2}}+s^{1/2}\Big)\,e^{-\frac{|x-y|^2}{cs}}\frac{ds}{s} \\ &\le C\int_0^{1/2}\frac{1}{s^{(n+1)/2}}e^{-\frac{|x-y|^2}{cs}}\frac{ds}{s} \\ &\quad+C\int_{1/2}^1 (1-s^2)^{(n/2)-1}e^{-\frac{|x-y|^2}{cs}}\frac{ds}{s} \\ &\le \frac{C}{|x-y|^{n+1}}+Ce^{-\frac{|x-y|^2}{c}}\le\frac{C}{|x-y|^{n+1}}. \end{aligned}$$

The terms B_2 and B_3 are easier, in a parallel way that A_2 was easier than A_1 and we leave the details to the reader. This ends the proof of estimate (10) for the case $\mathcal{R}$. A similar remark to 3.3 can be stated for this case, that is to say our proof gives that the kernel of the operator $V(\mathcal{R})$ satisfies also the standard estimates of a vector-valued Calderón-Zygmund with F_ρ norm given by the formula (6) with $\rho>2$.

acotacion

4. Boundedness in $L^2(\mathbb{R}^n, dx)$

The following Theorem was proved in [4]

masjones **Theorem 4.1.** *Let $(\Sigma, d\mu)$ a positive measure space. Let $\mathcal{T} = \{T_t\}_t$ be a symmetric diffusion semigroup if it satisfies $T^tT^s = T^{t+s}$, $T^0 = I_d$, $\lim_{t\to 0} T_t f \overset{L^2}{=} f$ and*

1. *$\|T^t f\|_p \le \|f\|_p$, for $1 \le p \le \infty$;*
2. *each T_t is a self-adjoint operator on $L^2(X)$,*
3. *each $T^t f \ge 0$ if $f \ge 0$;*
4. *for each t, $T^t(1) = 1$.*

Then the operators $\mathcal{O}(\mathcal{T})$ and $\mathcal{V}_\rho(\mathcal{T})$ are bounded in $L^p(\mathbb{R}^n, dx)$, for $1 < p < \infty$.

The family $\mathcal{T} = \{e^{t\Delta}\}_t$ is a symmetric diffusion semigroup and therefore the operators $\mathcal{O}(\mathcal{T})$ and $\mathcal{V}_\rho(\mathcal{T})$ are bounded in $L^p(\mathbb{R}^n, dx)$, for $1 < p < \infty$. In particular the vector valued operator $U(\mathcal{T})$, considered in Section 2, is bounded from $L^p(\mathbb{R}^n, dx)$ into $L^p_{E_2}(\mathbb{R}^n, dx)$. On the other hand in the previous section we showed that this operator has a (vector-valued) kernel which satisfies standard estimates. Therefore the use of vector-valued Calderón-Zygmund theory gives Theorem 2.1. An analogous reasoning can be given in order to prove the boundedness of the operator $V_\rho(\mathcal{T})$ between $L^p(\mathbb{R}^n, dx)$ into $L^p_{F_\rho}(\mathbb{R}^n, dx)$.

However, $e^{-tH}(1)(x) = e^{-t|x|^2}$; in other words, the family $\mathcal{T} = \{e^{-tH}\}_t$ is NOT a symmetric diffusion semigroup and the last Theorem can't be applied directly. In order to avoid this difficulty we shall consider the (Ornstein-Uhlenbeck) operator $\mathbf{L} = -\Delta + 2x \cdot \nabla$. It is known that the heat semigroup $\mathcal{T}_{\mathbf{OU}} = \{e^{-t\mathbf{L}}\}_t$ is a symmetric diffusion semigroup in the measure space $(\mathbb{R}^n, d\gamma(x))$, where $d\gamma(x) = \pi^{-n/2}e^{-|x|^2}dx$. In particular, by applying Theorem 4.1, the operators $\mathcal{O}(\mathcal{T}_{\mathbf{OU}})$ and $\mathcal{V}_\rho(\mathcal{T}_{\mathbf{OU}})$ are bounded in $L^2(\mathbb{R}^n, d\gamma(x))$.

There is a close relation between the operators H and $\mathbf{L}$. The eigenfunctions of $\mathbf{L}$ are the system of multidimensional Hermite polynomials $H_\alpha(x) = H_{\alpha_1}(x_1)\cdot\ldots\cdot H_{\alpha_n}(x_n)$, $x = (x_1, \ldots, x_n)$, $\alpha = (\alpha_1, \ldots, \alpha_n)$ where $H_k(s) = (-1)^k e^{s^2} \dfrac{d^k e^{-s^2}}{ds^k}$, $s \in \mathbb{R}$, in fact $\mathbf{L}H_\alpha = 2|\alpha|H_\alpha$. On the other hand, the system of multidimensional Hermite functions $h_\alpha(x) = h_{\alpha_1}(x_1)\cdot\ldots\cdot h_{\alpha_n}(x_n)$, $x = (x_1, \ldots, x_n)$, $\alpha = (\alpha_1, \ldots, \alpha_n)$ where $h_k(s) = (\pi^{1/2}2^k k!)^{-1/2} H_k(s) e^{-s^2/2}$, $s \in \mathbb{R}$, are the eigenfunctions of the operator H, satisfying $Hh_\alpha = (2|\alpha| + n)h_\alpha$. The relation between the eigenfunctions can transported to the operators associated to H and $\mathbf{L}$. The following Proposition can be found in [5].

ibrahim **Proposition 4.2.** *Let B be a normed space. The operator Λ, defined by $\Lambda f(x) = f(x)\pi^{n/4} e^{-\frac{|x|^2}{2}}$, is an isometry from $L^2_B(\mathbb{R}^n, d\gamma(x))$ into $L^2_B(\mathbb{R}^n, dx)$. Moreover if f is a linear combination of Hermite polynomials the following identities hold:*

(i) *$(H - nI_d) \circ \Lambda f(x) = \Lambda \circ \mathbf{L} f(x)$ and*
(ii) *$e^{-t(H-nI_d)} \circ \Lambda f(x) = \Lambda \circ e^{-t\mathbf{L}} f(x)$,*

with I_d the identity operator.

This Proposition has the following consequence

intermedia **Proposition 4.3.** *Consider the family $\mathcal{S} = e^{-t(H-nI_d)}$. Then the operators $\mathcal{O}(\mathcal{S})$ and $\mathcal{V}_\rho(\mathcal{S})$ are bounded in $L^2(\mathbb{R}^n, dx)$.*

Proof. Let f be a linear combination of Hermite polynomials. By applying Proposition 4.2 we have

$$\mathcal{O}(\mathcal{S})\circ\Lambda f(x)=\Lambda\circ\mathcal{O}(\mathcal{T}_{\mathbf{OU}})f(x) \quad \text{and} \quad \mathcal{V}_\rho(\mathcal{S})\circ\Lambda f(x)=\Lambda\circ\mathcal{V}_\rho(\mathcal{T}_{\mathbf{OU}})f(x).$$

Then the Proposition follows by using the boundedness of this last operators in $L^2(\mathbb{R}^n, d\gamma(x))$. □

T10 **Theorem 4.4.** *Let $\mathcal{T}=e^{-tH}$ and $\mathcal{S}=e^{-t(H-nI_d)}$. The operators $\mathcal{O}(\mathcal{S}-\mathcal{T})$ and $\mathcal{V}_\rho(\mathcal{S}-\mathcal{T})$ are bounded in $L^p(\mathbb{R}^n,dx)$, for $1<p<\infty$.*

We postpone for a while the proof of this result. In this moment we want to note that this Theorem together with Proposition 4.3 gives the following

T2 **Theorem 4.5.** *Let $\mathcal{T}=e^{-tH}$. The operators $\mathcal{O}(\mathcal{T})$ and $\mathcal{V}_\rho(\mathcal{T})$ are bounded in $L^p(\mathbb{R}^n,dx)$, for $1<p<\infty$.*

Now we can reproduce the arguments we gave just after Theorem 4.1 to obtain the results in Theorem 2.1 for the Hermite operator H.

Before beginning proof of Theorem 4.4 we present a Lemma that will be used in this section.

maximal **Lemma 4.6.** *The maximal operator $\sup_t e^{-t(H-nI_d)}f(x)$ is bounded from $L^p(\mathbb{R}^n,dx), 1<p<\infty$, into itself.*

Proof. We have $e^{-t(H-nI_d)}f=e^{tn}e^{-tH}f$. Thus

$$\begin{aligned}\sup_t\left|e^{tn}e^{-tH}f(x)\right| &\le \sup_{t\le1}\left|e^{tn}e^{-tH}f(x)\right| + \sup_{t>1}\left|e^{tn}e^{-tH}f(x)\right|\\ &\le e^n\sup_t\left|e^{-tH}f(x)\right| + \sup_{t>1}\left|e^{tn}e^{-tH}f(x)\right|\\ &= A+B.\end{aligned}$$

It is well known that $\|A\|_{L^p(\mathbb{R}^n,dx)}\le C\|f\|_{L^p(\mathbb{R}^n,dx)}$. As for B, taken a function f good enough, it follows that

$$\begin{aligned}\sup_{t\ge1}\left|e^{tn}e^{-tH}f(x)\right| &= \sup_{t\ge1}\left|\int_{\mathbb{R}^n}\sum_{k=1}^\infty\sum_{|\alpha|=k}e^{-2tk}h_\alpha(x)h_\alpha(y)f(y)dy\right|\\ &\le \sum_{k=1}^\infty\sum_{|\alpha|=k}e^{-2k}|h_\alpha(x)|\,\left|\int_{\mathbb{R}^n}h_\alpha(y)f(y)dy\right|.\end{aligned}$$

Then by Hölder's inequality

$$\sup_{t\ge1}\left|e^{tn}e^{-tH}f(x)\right|\le\sum_{k=1}^\infty\sum_{|\alpha|=k}e^{-2k}|h_\alpha(x)|\|h_\alpha\|_{L^{p'}(\mathbb{R}^n,dx)}\|f\|_{L^p(\mathbb{R}^n,dx)}.$$

Hence, an application of Minkowski's inequality renders

$$\|B\|_{L^p(\mathbb{R}^n,dx)} \le \sum_{k=1}^\infty\sum_{|\alpha|=k}e^{-2k}\|h_\alpha\|_{L^p(\mathbb{R}^n,dx)}\|h_\alpha\|_{L^{p'}(\mathbb{R}^n,dx)}\|f\|_{L^p(\mathbb{R}^n,dx)}$$

By employing [9, Lemma 1.5.2] we conclude $\|h_\alpha\|_{L^p(\mathbb{R}^n,dx)} \leq C|\alpha|^{\theta_p}, 1 \leq p \leq \infty$, for some $\theta_p > 0$. Therefore

$$\|B\|_{L^p(\mathbb{R}^n,dx)} \leq C\Big(\sum_k k^n e^{-2k} k^{\theta_p+\theta_{p'}}\Big)\|f\|_{L^p(\mathbb{R}^n,dx)} \leq C\|f\|_{L^p(\mathbb{R}^n,dx)}.$$

□

Proof. (of Theorem 4.4)
Observe that, with the notation in formula (5), taking $t^* = \frac{1}{2}\log 3$, it follows

$$\begin{aligned}\|U(\mathcal{S}-\mathcal{T})f(x)\|_{E_2} &\leq \|U(\mathcal{S}-\mathcal{T})f(x)\|_{E_2,t_i<t^*} + \|U(\mathcal{S}-\mathcal{T})f(x)\|_{E_2,t_i>t^*} \\ &\quad + \sup_t |(e^{-t(H-nI_d)} - e^{-tH})f(x)| \\ &= \mathbf{A_1}f(x) + \mathbf{A_2}f(x) + \mathbf{A_3}f(x).\end{aligned}$$

Lemma 4.6 assures that the operator $\mathbf{A_3}f$ is bounded in $L^p(\mathbb{R}^n, dx)$, $1 < p < \infty$.

Now we shall study the operator $\mathbf{A_1}$. As in the proof of the standard estimates, we make the change of parameter $t = t(s) = \frac{1}{2}\log\frac{1+s}{1-s},\ 0 < s < 1,\ 0 < t < \infty$, observe that $t^* = t(\frac{1}{2})$. Then

$$\begin{aligned}\Big(e^{-t(s)(H-nI_d)}f(x) &- e^{-t(s)(H)}f(x)\Big)\chi_{t(s)<t^*} \\ &= \Big\{\Big(\frac{1+s}{1-s}\Big)^{n/2} - 1\Big\}\chi_{(0,1/2)}(s)\int_{\mathbb{R}^n} R_s(x,y)f(y)dy \\ &= \varphi(s)\,\chi_{(0,1/2)}(s)\int_{\mathbb{R}^n} R_s(x,y)f(y)dy,\end{aligned}$$

where R_s is defined in (12), and $\varphi(s) = \Big\{\Big(\frac{1+s}{1-s}\Big)^{n/2} - 1\Big\}$. The kernel of the vector valued operator $U(\mathcal{S}-\mathcal{T})$ can be expressed as

$$\Big\{\varphi(s_{i+1})R_{s_{i+1}}(x,y) - \varphi(s)R_s(x,y)\Big)\chi_{J_i}(s)\Big\}.$$

Observe that in the range $0 \leq s \leq \frac{1}{2}$, the function φ is increasing and satisfies $\varphi(s) \sim s$. We remind, for the reader's convenience, that after the change of parameter we can restrict ourselves to the interval $[0,1/2]$. Hence

$$\begin{aligned}
\|U(\mathcal{S} \;\; - \;\; \mathcal{T})f(x)\|_{E_2,s_i<\frac12} & \\
\le\;& \Big\|\Big\{\varphi(s_{i+1})\int_{\mathbb{R}^n}(R_{s_{i+1}}(x,y)-R_s(x,y))f(y)dy\chi_{J_i}(s)\Big\}\Big\|_{E_2,s_i<\frac12}\\
&+\Big\|\Big\{(\varphi(s_{i+1})-\varphi(s))\int_{\mathbb{R}^n}R_s(x,y)f(y)dy\;\chi_{J_i}(s)\Big\}\Big\|_{E_2,s_i<\frac12}
\end{aligned}$$

29 (16)
$$\begin{aligned}
\le\;& \Big\|\Big\{\varphi(s_{i+1})\int_{\mathbb{R}^n}(R_{s_{i+1}}(x,y)-R_s(x,y))f(y)dy\chi_{J_i}(s)\Big\}\Big\|_{E_2,s_i<\frac12}\\
&+C\Big\|\Big\{(s_{i+1}-s_i)\int_{\mathbb{R}^n}R_s(x,y)f(y)dy\;\chi_{J_i}(s)\Big\}\Big\|_{E_2,s_i<\frac12}\\
\le\;& C\int_{\mathbb{R}^n}\Big\|s_{i+1}(R_{s_{i+1}}(x,y)-R_s(x,y))\chi_{J_i}(s)\Big\|_{E_2,s_i<\frac12}|f(y)|dy\\
&+C\Big\|\Big\{(s_{i+1}-s_i)\;\chi_{J_i}(s)\Big\}\Big\|_{E_2,s_i<\frac12}\sup_s\Big|\int_{\mathbb{R}^n}R_s(x,y)f(y)dy\Big|\\
\le\;& C\int_{\mathbb{R}^n}\Big\|s_{i+1}(R_{s_{i+1}}(x,y)-R_s(x,y))\chi_{J_i}(s)\Big\|_{E_2,s_i<\frac12}|f(y)|dy\\
&+C\sup_s|e^{-sH}f(x)|.
\end{aligned}$$

Following carefully the lines of (13) we can get

$$\Big\|s_{i+1}(R_{s_{i+1}}(x,y)-R_s(x,y))\chi_{J_i}(s)\Big\|_{E_2,s_i<\frac12}\le C\int_0^{1/2}sf'(s)ds.$$

In other words, we have an extra "s" in the numerator in the computations (14) and (15) which provides

$$\begin{aligned}
\Big\|s_{i+1}(R_{s_{i+1}}(x,y)-R_s(x,y))\chi_{J_i}(s)\Big\|_{E_2,s_i<\frac12} &\le C\int_0^{1/2}\frac{s}{s^{n/2}}e^{-\frac{|x-y|^2}{cs}}\frac{ds}{s}\\
&= \frac{C}{|x-y|^{n-2}}\int_{2|x-y|^2}^{\infty}u^{\frac{n-2}{2}}e^{-\frac{u}{c}}\frac{du}{u}.
\end{aligned}$$

Now we shall distinguish cases according to the size of $|x-y|$. If $|x-y|>1$, reminding Remark 3.2, we attain

$$\begin{aligned}
\Big\|s_{i+1}(R_{s_{i+1}}(x,y)-R_s(x,y))\chi_{J_i}(s)\Big\|_{E_2,s_i<\frac12} &\le \frac{Ce^{-\frac{|x-y|^2}{c}}}{|x-y|^{n-2}}\int_2^{\infty}u^{\frac{n-2}{2}}e^{-\frac{u}{c}}\frac{du}{u}\\
&\le Ce^{-\frac{|x-y|^2}{c}}.
\end{aligned}$$

If $|x-y|<1$ and $n\ge3$ we have

$$\Big\|s_{i+1}(R_{s_{i+1}}(x,y) \;\; - \;\; R_s(x,y))\chi_{J_i}(s)\Big\|_{E_2,s_i<\frac12}$$

30 (17)
$$\begin{aligned}
&\le \frac{C}{|x-y|^{n-2}}\Big(\int_{2|x-y|^2}^{1}+\int_1^{\infty}\Big)u^{\frac{n-2}{2}}e^{-\frac{u}{c}}\frac{du}{u}\\
&\le \frac{C}{|x-y|^{n-2}}.
\end{aligned}$$

In the case $|x-y|<1$ and $n<3$, we can write

$$
\begin{aligned}
&\Big\| s_{i+1}(R_{s_{i+1}}(x,y)-R_s(x,y))\chi_{J_i}(s)\Big\|_{E_2,s_i<\frac{1}{2}} \\
(18)\qquad &\le \frac{C}{|x-y|^{n-2}}\Big(\int_{2|x-y|^2}^{1}+\int_1^{\infty}\Big)u^{\frac{n-2}{2}}e^{-\frac{u}{c}}\frac{du}{u} \\
&\le \frac{C}{|x-y|^{n-2}}\Big(\int_{2|x-y|^2}^{1}u^{\frac{n-2}{2}}e^{-\frac{u}{c}}\frac{du}{u}+C\Big) \\
&\le C(\log|x-y|+1).
\end{aligned}
$$

In consequence, using (16), (17) and (18), it results

$$
\|U(\mathcal{S}-\mathcal{T})f(x)\|_{E_2,s_i<\frac{1}{2}} \le C\int_{\mathbb{R}^n}\Phi(x-y)|f(y)|dy + C\sup_t|e^{-tH}f(x)|,
$$

where $\Phi(x)$ is an integrable function. Therefore

$$
\Big\| \, \|U(\mathcal{S}-\mathcal{T})f(\cdot)\|_{E_2,s_i<\frac{1}{2}}\Big\|_{L^p(\mathbb{R}^n,dx)} \le C\|f\|_{L^p(\mathbb{R}^n,dx)}.
$$

Let us analyze $\mathbf{A_2}$. Let f be a function such that $\int_{\mathbb{R}^n}f(y)h_0(y)dy=0$, hence

$$
\begin{aligned}
&\Big(e^{-t(s)(H-nI_d)}f(x)-e^{-t(s)(H)}f(x)\Big)\chi_{t(s)>t^*} \\
&=\Big\{\Big(\frac{1+s}{1-s}\Big)^{n/2}-1\Big\} \\
&\quad\times\int_{\mathbb{R}^n}\Big\{R_s(x,y)\chi_{(1/2,1)}(s)-\Big(\frac{1-s^2}{4\pi s}\Big)^{n/2}e^{-\frac{1}{2}(|x|^2+|y|^2)}\Big\}f(y)dy \\
&=\varphi(s)\int_{\mathbb{R}^n}\tilde{R}_s(x,y)f(y)dy,
\end{aligned}
$$

where R_s is defined in (12), $\tilde{R}_s(x,y)=\Big\{R_s(x,y)\chi_{(1/2,1)}(s)-\Big(\frac{1-s^2}{4\pi s}\Big)^{n/2}e^{-\frac{1}{2}(|x|^2+|y|^2)}\Big\}$ and $\varphi(s)=\Big\{\Big(\frac{1+s}{1-s}\Big)^{n/2}-1\Big\}$. The kernel of the vector valued operator $U(\mathcal{S}-\mathcal{T})$ can then be expressed as

$$
\Big\{\varphi(s_{i+1})\tilde{R}_{s_{i+1}}(x,y)-\varphi(s)\tilde{R}_s(x,y)\Big)\chi_{J_i}(s)\Big\}.
$$

Observe that in the range $1/2<s<1$, an application of the mean value Theorem produces

$$
\Big|\exp\Big(-\frac{1}{4}(s|x+y|^2+\frac{1}{s}|x-y|^2)\Big)-\exp\Big(-\frac{|x|^2+|y|^2}{2}\Big)\Big| \le Ce^{-\frac{|x-y|^2}{c}}(1-s).
$$

In consequence, for $1/2<s<1$,

$$
|\tilde{R}_s(x,y)| \le C(1-s)^{n/2}e^{-\frac{|x-y|^2}{c}}(1-s).
$$

Moreover in the range $1/2 < s < 1$ φ and φ' are increasing and $|\varphi'(s)| \leq C(1-s)^{-(n/2)-1}$; on that account for some $s_{i+1} \leq u \leq s$,

$$\begin{aligned}
&\|U(\mathcal{S}-\mathcal{T})f(x)\|_{E_2,s_i>\frac{1}{2}} \\
&\leq \Big\|\Big\{(\varphi(s_{i+1})-\varphi(s))\int_{\mathbb{R}^n}\tilde{R}_s(x,y)f(y)dy\ \chi_{J_i}(s)\Big\}\Big\|_{E_2,s_i>\frac{1}{2}} \\
&\quad+\Big\|\Big\{\varphi(s_{i+1})\int_{\mathbb{R}^n}(\tilde{R}_{s_{i+1}}(x,y)-\tilde{R}_s(x,y))f(y)dy\chi_{J_i}(s)\Big\}\Big\|_{E_2,s_i>\frac{1}{2}} \\
&\leq \Big\|\Big\{(s_{i+1}-s)\varphi'(u)\int_{\mathbb{R}^n}(1-s)^{n/2}e^{-\frac{|x-y|^2}{c}}(1-s)|f(y)|dydy\ \chi_{J_i}(s)\Big\}\Big\|_{E_2,s_i>\frac{1}{2}} \\
&\quad+\Big\|\Big\{\varphi(s_{i+1})\int_{\mathbb{R}^n}(\tilde{R}_{s_{i+1}}(x,y)-\tilde{R}_s(x,y))f(y)dy\chi_{J_i}(s)\Big\}\Big\|_{E_2,s_i>\frac{1}{2}} \\
&\leq \Big\|\Big\{(s_{i+1}-s_i)\chi_{J_i}(s)\Big\}\Big\|_{E_2,s_i>\frac{1}{2}}\int_{\mathbb{R}^n}e^{-\frac{|x-y|^2}{c}}|f(y)|dy \\
&\quad+\Big\|\Big\{\varphi(s_{i+1})\int_{\mathbb{R}^n}(\tilde{R}_{s_{i+1}}(x,y)-\tilde{R}_s(x,y))f(y)dy\chi_{J_i}(s)\Big\}\Big\|_{E_2,s_i>\frac{1}{2}} \\
&\leq C\int_{\mathbb{R}^n}e^{-\frac{|x-y|^2}{c}}|f(y)|dy \\
&\quad+\Big\|\Big\{\varphi(s_{i+1})\int_{\mathbb{R}^n}(\tilde{R}_{s_{i+1}}(x,y)-\tilde{R}_s(x,y))f(y)dy\chi_{J_i}(s)\Big\}\Big\|_{E_2,s_i>\frac{1}{2}}.
\end{aligned}$$

For the last summand, a careful look to formula (13) provides

$$\begin{aligned}
&\Big\|\Big\{\varphi(s_{i+1})\Big(\tilde{R}_{s_{i+1}}(x,y)-\tilde{R}_s(x,y)\Big)\chi_{J_i}(s)\Big\}\Big\|_{E_2,s_i>\frac{1}{2}} \\
&\leq \int_0^1 \varphi(s)|F'(s)|ds.
\end{aligned}$$

Where

$$\begin{aligned}
F'(s) &= \Big\{-\frac{n}{2}\Big(\frac{1-s^2}{s}\Big)^{(n/2)-1}\Big(\frac{1+s^2}{s^2}\Big)\Big\}\Big(e^{-h(s)}-e^{-\frac{|x|^2+|y|^2}{2}}\Big) \\
&\quad -\Big\{\Big(\frac{1-s^2}{s}\Big)^{n/2}\Big(\frac{1}{4}|x+y|^2-\frac{1}{4s^2}|x-y|^2\Big)\Big\}e^{-h(s)} \\
&= E_1(s)+E_2(s),
\end{aligned}$$

being $h(s)$ as in (13). Due to the fact that $|e^{-h(s)}-e^{-\frac{|x|^2+|y|^2}{2}}| \leq Ce^{-\frac{|x-y|^2}{c}}(1-s)$, we can reproduce the arguments in (14) for the range $1/2 < s < 1$, arriving to the fact that the kernel above is estimated by $ce^{-\frac{|x-y|^2}{c}}$. Hence as in the case of $\mathbf{A}_1$ this gives as a consequence the boundedness in $L^p(\mathbb{R}^n, dx)$, but in this case only for functions whose first Hermite coefficient is zero. Observe that any arbitrary function $f \in L^p(\mathbb{R}^n, dx)$ can be written as

$$\begin{aligned}
f(x) &= f_1(x)+f_2(x) \\
&= \Big\{f(x)-\Big(\int_{\mathbb{R}^n}f(y)h_0(y)dy\Big)h_0(x)\Big\}+\Big(\int_{\mathbb{R}^n}f(y)h_0(y)dy\Big)h_0(x).
\end{aligned}$$

It is clear that f_1 satisfies $\int_{\mathbb{R}^n} f_1(y)h_0(y)dy = 0$. Moreover $\|f_1\|_{L^p(\mathbb{R}^n,dx)} \leq C\|f\|_{L^p(\mathbb{R}^n,dx)}$ and $\|f_2\|_{L^p(\mathbb{R}^n,dx)} \leq C\|f\|_{L^p(\mathbb{R}^n,dx)}$. On the other hand

$$\begin{aligned} e^{-t(s)(H-nI_d)}f_2(x) - e^{-t(s)(H)}f_2(x) &= \Big(e^{-tn} - 1\Big)\, e^{-t(s)(H)}f_2(x) \\ &= \Big(\int_{\mathbb{R}^n} f(y)h_0(y)dy\Big)\Big(e^{-tn} - 1\Big)h_0(x). \end{aligned}$$

Hence

$$\begin{aligned} \Big\|\|U(\mathcal{S}-\mathcal{T})f_2(\cdot)\|_{E_2}\Big\|_{L^p(\mathbb{R}^n,dx)} &\leq \Big\|\Big(\int_{\mathbb{R}^n} f(y)h_0(y)dy\Big)\, h_0(\cdot)\Big\|_{L^p(\mathbb{R}^n,dx)}\|e^{-tn}\|_{E_2} \\ &\leq C\|f\|_{L^p(\mathbb{R}^n,dx)}\|h_0\|_{L^p(\mathbb{R}^n,dx)} \leq C\|f\|_{L^p(\mathbb{R}^n,dx)}. \end{aligned}$$

This ends the proof of the boundedness in $L^p(\mathbb{R}^n, dx)$ of the operator $\mathcal{O}(\mathcal{S}-\mathcal{T})$.

A parallel argument can be given for $\mathcal{V}_\rho(\mathcal{S}-\mathcal{T})$. □

extremo

5. L^∞ RESULTS

We shall begin this section by proving that the oscillation operator associated to the heat semigroup related to Δ is not bounded from $L^\infty(\mathbb{R})$ to $L^\infty(\mathbb{R})$. In fact we shall find a function $g \in L^\infty(\mathbb{R})$ such that $\mathcal{O}(Tg)(x) = \infty$, *a.e.* Let g be the function defined as

g (19)
$$g(y) = \sum_{k\in\mathbb{Z}}(-1)^{k+1}\chi_{[a^k,a^{k+1}]}(y),$$

where $a > 0$ is a real number that will be fixed later.

lema1 **Lemma 5.1.** *For every $j \in \mathbb{Z}$*

$$g(a^j y) = (-1)^j g(y).$$

Proof.

$$\begin{aligned} g(a^j y) &= \sum_{k\in\mathbb{Z}}(-1)^{k+1}\chi_{[a^k,a^{k+1}]}(a^j y) \\ &= \sum_{k-j\in\mathbb{Z}}(-1)^{k+1-j}(-1)^j\chi_{[a^{k-j},a^{k-j+1}]}(y) \\ &= (-1)^j\sum_{l\in\mathbb{Z}}(-1)^{l+1}\chi_{[a^l,a^{l+1}]}(y) \\ &= (-1)^j g(y). \end{aligned}$$

□

lema2 **Lemma 5.2.** *Let g the function defined in (19) and $t_j = a^{2j}, j \in \mathbb{Z}$. Then*

(1) $\displaystyle \frac{1}{t_j^{1/2}}\int_{\mathbb{R}} e^{-\frac{y^2}{t_j}}g(y)dy = (-1)^j\int_0^\infty e^{-u^2}g(u)du.$

(2) $\displaystyle \left|\frac{1}{t_j^{1/2}}\int_{\mathbb{R}} e^{-\frac{y^2}{t_j}}g(y)dy - \frac{1}{t_{j+1}^{1/2}}\int_{\mathbb{R}} e^{-\frac{y^2}{t_{j+1}}}g(y)dy\right| = 2\left|\int_0^\infty e^{-u^2}g(u)du\right|.$

Proof. Use the change of variable $u = \frac{y}{t_j^{1/2}}$ and Lemma 5.1. □

lema4 **Lemma 5.3.** *Given the function g defined in (19), there exists $a > 0$ such that*
$$\left|\int_0^\infty e^{-u^2} g(u)du\right| \geq C.$$

Proof. It is very well known that $\displaystyle\int_0^\infty e^{-u^2}du = \frac{\sqrt{\pi}}{2}$. On the other hand $\int_0^1 e^{-u^2}du \geq \frac{\sqrt{\pi}}{3}$. We choose $a > 1$ such that
$$\int_{\frac{1}{a}}^1 e^{-u^2}du \geq \frac{4}{5}\frac{\sqrt{\pi}}{3}$$
. Then
$$\begin{aligned}\int_0^\infty e^{-u^2} g(u)du &= \int_0^{\frac{1}{a}} e^{-u^2} g(u)du + \int_{\frac{1}{a}}^1 e^{-u^2} g(u)du + \int_1^\infty e^{-u^2} g(u)du \\ &\geq \int_{\frac{1}{a}}^1 e^{-u^2}du - \left(\int_0^{\frac{1}{a}} e^{-u^2}du + \int_1^\infty e^{-u^2}du\right) \\ &\geq \frac{4}{5}\frac{\sqrt{\pi}}{3} - \left(\frac{\sqrt{\pi}}{2} - \frac{4}{5}\frac{\sqrt{\pi}}{3}\right) = \frac{8}{5}\frac{\sqrt{\pi}}{3} - \frac{\sqrt{\pi}}{2} = \frac{\sqrt{\pi}}{30} > 0.\end{aligned}$$
□

lema5 **Lemma 5.4.** *Given $t_j = a^{2j}, j \in \mathbb{Z}$, Then*
$$\sum_j \left|\frac{1}{t_{j+1}^{1/2}}\int_{\mathbb{R}} e^{-\frac{|x-y|^2}{t_{j+1}}} g(y)dy - \frac{1}{t_j^{1/2}}\int_{\mathbb{R}} e^{-\frac{|x-y|^2}{t_j}} g(y)dy\right| = \infty, \quad x \in \mathbb{R}^n.$$

Proof. The result is obvious for $x = 0$ from Lemmas 5.2 and 5.3. Let $x > 0$. We shall prove that the number of terms in the summatory which are bigger than a certain constant is infinity. For a j fixed, the corresponding term of the summatory may be expresed, through the changes of variable $u = (y-x)/a^{j+1}, w = (y-x)/a^j$, in the form
$$\begin{aligned}&\left|\frac{1}{a^{j+1}}\int_{\mathbb{R}} e^{-\frac{|x-y|^2}{a^{j+1}}} g(y)dy - \frac{1}{a^j}\int_{\mathbb{R}} e^{-\frac{|x-y|^2}{a^j}} g(y)dy\right| \\ = &\left|\int_{\mathbb{R}} e^{-u^2} g(a^{j+1}u + x)du - \int_{\mathbb{R}} e^{-w^2} g(a^j w + x)dw\right| \\ = &\left|\int_{\mathbb{R}} e^{-u^2}(-1)^{j+1} g(u + \frac{x}{a^{j+1}})du - \int_{\mathbb{R}} e^{-w^2}(-1)^j g(w + \frac{x}{a^j})dw\right| \\ = &\left|\int_{\mathbb{R}} e^{-u^2} g(u + \frac{x}{a^{j+1}})du + \int_{\mathbb{R}} e^{-u^2} g(u + \frac{x}{a^j})du\right|.\end{aligned}$$
Now, taking account that
$$\lim_{h\to 0}\int_{\mathbb{R}} e^{-u^2} g(u+h)du = \int_{\mathbb{R}} e^{-u^2} g(u)du,$$
there exists $\eta > 0$ such that, for $h < \eta$,
$$\int_{\mathbb{R}} e^{-u^2} g(u+h)du \geq \frac{1}{2}\int_{\mathbb{R}} e^{-u^2} g(u)du.$$

Then, for each $x \in \mathbb{R}^+$ and j such that $0 < x/a^j < \eta$,

$$\left|\int_{\mathbb{R}} e^{-u^2} g(u + \frac{x}{a^{j+1}})du + \int_{\mathbb{R}} e^{-u^2} g(u + \frac{x}{a^j})du\right| \geq \left|\int_{\mathbb{R}} e^{-u^2} g(u)du\right| = C > 0.$$

□

The last Lemma provides obviously a proof of Theorem 1.2.

We finish this section presenting the proof of Theorem 1.3

Proof. Given $B_r = B(z,r) \subset B_0$ and the function f, we write $f = f_1 + f_2$, where $f_1 = f\chi_{B(z,4r)}$. Then $\mathcal{O}(\mathcal{T})f(x) \leq \mathcal{O}(\mathcal{T})f_1(x) + \mathcal{O}(\mathcal{T})f_2(x)$. By using Theorem 1.1, we have

$$\begin{aligned}\frac{1}{|B_r|}\int_{B_r} |\mathcal{O}(\mathcal{T})f_1(x)|dx &\leq \Big(\frac{1}{|B_r|}\int_{B_r} |\mathcal{O}(\mathcal{T})f_1(x)|^2 dx\Big)^{1/2} \\ &\leq C\Big(\frac{1}{|B_r|}\int_{B(z,4r)} |f(x)|^2 dx\Big)^{1/2} \leq C\|f\|_{L^\infty}.\end{aligned}$$

On the other hand,

$$\begin{aligned} & |\int_{\mathbb{R}^n} (M_{t_{i+1}}(x,y) - M_s(x,y))f(y)dy| \\ = \;& |\int_{\mathbb{R}^n} \frac{1}{\sqrt{\pi^n}}\Big(\frac{1}{t_{i+1}^{n/2}} e^{-\frac{|x-y|^2}{t_{i+1}}} - \frac{1}{s^{n/2}} e^{-\frac{|x-y|^2}{s}}\Big) f(y)dy| \\ \leq \;& \Big(\int_{\mathbb{R}^n} \frac{1}{\sqrt{\pi^n}}\Big|\frac{1}{t_{i+1}^{1/2}} e^{-\frac{|x-y|^2}{t_{i+1}}} - \frac{1}{s^{n/2}} e^{-\frac{|x-y|^2}{s}}\Big| |f(y)|^2 dy\Big)^{1/2} \\ & \times\Big(\int_{\mathbb{R}^n} \frac{1}{\sqrt{\pi^n}}\Big|\frac{1}{t_{i+1}^{n/2}} e^{-\frac{|x-y|^2}{t_{i+1}}} - \frac{1}{s^{n/2}} e^{-\frac{|x-y|^2}{s}}\Big| dy\Big)^{1/2} \\ \leq \;& 2\Big(\int_{\mathbb{R}^n} \frac{1}{\sqrt{\pi^n}}\Big|\frac{1}{t_{i+1}^{n/2}} e^{-\frac{|x-y|^2}{t_{i+1}}} - \frac{1}{s^{n/2}} e^{-\frac{|x-y|^2}{s}}\Big| |f(y)|^2 dy\Big)^{1/2}.\end{aligned}$$

Therefore, for every $x \in B_r$, by using (11) we have

$$\begin{aligned}\|U(\mathcal{T})f_2(x)\|_{E_2} &= \Big\|\int_{\mathbb{R}^n} \mathcal{U}(x,y)f_2(y)dy\Big\|_{E_2} \leq 2\Big(\int_{\mathbb{R}^n} \|\mathcal{U}(x,y)\|_{E_1}|f_2(y)|^2 dy\Big)^{1/2} \\ &\leq C\Big(\int_{\mathbb{R}^n} \frac{1}{|x-y|^n}|f_2(y)|^2 dy\Big)^{1/2} \\ &\leq C\Big(\int_{\{2r<|x-y|<1\}} \frac{1}{|x-y|^n}|f(y)|^2 dy\Big)^{1/2} \\ &\leq C\|f\|_\infty\Big(\int_{2r}^1 \frac{dt}{t}\Big)^{1/2} \leq C\|f\|_\infty\Big(\log\frac{1}{r}\Big)^{1/2}.\end{aligned}$$

□

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