Riesz Transforms for Laguerre expansions.*

E. Harboure, J. L. Torrea and B. Viviani

Abstract

We analyze boundedness properties of some operators related to the heat-diffusion semigroup associated to Laguerre functions systems. In particular, for any $\alpha > -1$, we introduce appropriate Laguerre Riesz Transforms and we obtain power-weighted L^p inequalities, $1 < p < \infty$. We achieve this result by taking advantage of the existing classical relationship between n-variable Hermite polynomials and Laguerre polynomials on the half line of type $\alpha = n/2 - 1$. Such connection allows us to transfer known boundedness properties for Hermite operators to Laguerre operators corresponding to those specific values of α. To extend the results to any $\alpha > -1$, we make use of transplantation and some weighted inequalities we obtain in the Hermite setting (which we beleive of independent interest).

1 Introduction

The aim of this paper is to further investigate boundedness properties of operators associated to the system of the Laguerre functions that are orthonormal with respect to the Lebesgue measure on $(0, \infty)$. For Laguerre polynomials this study was started by Muckenhoupt in the pioneering papers [5], [6].

This system of functions appears as the eigenfunctions of a second order differential operator, non negative and selfadjoint L_α, which is the infinitesimal generator of the heat

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semigroup e^{-tL_α}. In this context several operators have been studied like the maximal operator of heat and Poisson semigroups, which are known in the case $\alpha \geq 0$, to be bounded on $L^p((0,\infty),dy)$ and of weak type $(1,1)$, see [8] and [9].
Also the theory of multiplier operators with respect to this system has been developped by Thangavelu see [14], [15], see also [12] where sufficient conditions on the function m are given to guarantee the boundedness of the multiplier operator T_m on $L^p((0,\infty),y^\delta dy)$ for a certain range of δ. Also some multiplier-type Riesz transforms were considered in [15].
Motivated by the second order differential operator associated to the Laguerre system we consider the Riesz transforms following the lines of [8]. In the case of the Hermite system this way of defining Riesz transforms has already been treated in [14] and [11] . Here we define them and we prove boundedness results on $L^p((0,\infty,y^\delta dy)$, $1<p<\infty$, with δ ranging on some interval. In particular we show that for $\alpha \geq 0$ the Riesz transforms are bounded for all the power weights belonging to the Muckenhoupt class $A_p((0,\infty))$, namely y^δ with $-1<\delta<p-1$. As for $-1<\alpha<0$, the $L^p((0,\infty),y^\delta dy)-$ boundedness holds for a more restricted range of δ. For instance if we set $\delta=0$ we obtain boundedness only for $\frac{1}{1+\alpha/2}<p<-2/\alpha$.

To achieve these goals we start by studying the connection between $n-$dimensional Hermite functions and Laguerre functions of $\alpha-$type for $\alpha=\frac{n}{2}-1$. This study is two-fold: it serves as a guide in defining the Riesz transforms and at the same time it provides an instrument for proving boundedness of Laguerre operators for the particular values of $\alpha=\frac{n}{2}-1$ (see Proposition 2.25.)
The idea of relating Hermite and Laguerre expansions appears for example in [1] [2]. In [2] this relation was systematically used as a tool for studying operators associated to Laguerre polynomial expansions.

To succesfully complete our program and prove boundedness results for any value of α, we shall use a powerfull tool, namely the Kanjin transplantation Theorem [3] and its extension to power-weighted L^p- spaces [12].
We record here this Theorem as it appears in [12] for future reference. By $\{\mathcal{L}_k^\alpha\}$ we denote the orthonormal system of Laguerre functions corresponding to a fixed $\alpha>-1$, see (2.2).

Theorem 1.1 *Let τ_β^α be the transplantation operator, that is*

$$\tau_\beta^\alpha(\sum c_k\mathcal{L}_k^\alpha)=\sum c_k\mathcal{L}_k^\beta.$$

Then τ_β^α is bounded on $L^p((0,\infty),y^\delta dy)$ for $1<p<\infty$ and δ such that

$$-1-\min\{\alpha,\beta,0\}p/2<\delta<(p-1)+\min\{\alpha,\beta,0\}p/2.$$

As we mentioned above for the Riesz transforms, when $\min\{\alpha,\beta\}<0$, the range of δ given in the previous theorem is strictly less than that of the powers belonging to A_p.

However this range is as large as it could be. In fact, if we consider for a fixed t the operator e^{-tL^β} for $\beta \geq 0$, we know from [9] that it is bounded on $L^p((0,\infty), y^\delta dy)$ for $1 < p < \infty$ and $-1 < \delta < p-1$. Now, if we "transplant" this result to some $\alpha < 0$, we would get boundedness for $-1-\alpha p/2 < \delta < p-1+\alpha p/2$, but it can be checked, by inspection of the kernel, that this interval is the best possible. See the details in section 6.

The paper is organized as follows. In section 2 we give the basic notions for Laguerre and Hermite function expansions and examine their relationship and how to connect the corresponding operators. We present the definitions of the Riesz transforms for Laguerre expansions and derive the first boundedness result through the established relation with the Hermite expansions. Section 3 is devoted to multiplier operators for Hermite expansions. There we obtain the weighted version of Marcinkiewicz Theorem in this context. This result, interesting in itself, is applied to get a similar result for Laguerre expansions of type $\alpha = \frac{n}{2} - 1$. In section 4 we state and prove the main result concerning the weighted boundedness of the Laguerre-Riesz transforms. Finally in Section 5 we analyze the accuracy of the transplantation Theorem for negative values of α.

2 Preliminaries

Given a real number $\alpha > -1$, we consider L_α the Laguerre second order differential operator defined by

$$L_\alpha = -y\frac{d^2}{dy^2} - \frac{d}{dy} + \frac{y}{4} + \frac{\alpha^2}{4y}, \quad y > 0. \tag{2.1}$$

It is well known that L_α is nonnegative and selfadjoint with respect to the Lebesgue measure on $(0,\infty)$, furthermore its eigenfunctions are the Laguerre functions $\mathcal{L}_k^\alpha$ defined by

$$\mathcal{L}_k^\alpha(y) = \left(\frac{\Gamma(k+1)}{\Gamma(k+\alpha+1)}\right)^{1/2} e^{-y/2} y^{\alpha/2} L_k^\alpha(y), \tag{2.2}$$

where L_k^α are the Laguerre polynomials of type α see [10, p. 100] and [14, p. 7]. The orthogonality of Laguerre polynomials with respect to the measure $e^{-y}y^\alpha$ leads to the orthonormality of the family $\{\mathcal{L}_k^\alpha\}_k$ in $L^2((0,\infty), dy)$.
Moreover the operator satisfies

$$L_\alpha(\mathcal{L}_k^\alpha) = (k + \frac{\alpha+1}{2})\mathcal{L}_k^\alpha. \tag{2.3}$$

On the other hand we consider the Hermite second order differential operator on $\mathbb{R}^n$ given by

$$H_n = -\Delta + |x|^2 = -\sum_{i=1}^n \frac{\partial^2}{\partial x_i^2} + |x|^2. \tag{2.4}$$

This operator is nonegative and selfadjoint with respect to the Lebesgue measure on $\mathbb{R}^n$. Its eigenfunctions are the $n-$dimensional Hermite functions, that is for any multiindex $\mu \in \mathbb{Z}_+^n$

$$(2.5) \qquad \Phi_\mu(x) = \Pi_{i=1}^n h_{\mu_i}(x_i), \quad \mu = (\mu_1, \dots, \mu_n),$$

with $h_k(t) = (2^k k! \sqrt{\pi})^{-1/2} H_k(t) e^{-t^2/2}$, where H_k stands for the classical Hermite polynomial of degree k. Moreover the operator H_n satisfies

$$(2.6) \qquad H_n(\Phi_\mu) = (2|\mu| + n)\Phi_\mu, \quad |\mu| = \sum \mu_i$$

As a consequence of the orthogonality properties of Hermite polynomials, the system $\{\Phi_\mu\}$ is an orthonormal system on $L^2(\mathbb{R}^n, dx)$. See [14, p. 5] and [10, p. 105] for more details about Hermite functions.

Our starting point is the relationship between the families of Laguerre and Hermite functions given by

$$(2.7) \qquad \mathcal{L}_k^\alpha(|x|^2) = c_k^\alpha \sum_{|r|=k} \frac{a_r}{b_{2r}} \Phi_{2r}(x)|x|^\alpha, \quad x \in \mathbb{R}^n, \quad \alpha = \frac{n}{2} - 1.$$

where c_k^α and b_k are the orthonormalization coefficients given by $c_k^\alpha = (\frac{\Gamma(k+1)}{\Gamma(k+\alpha+1)})^{1/2}$, $b_k = (2^k k! \sqrt{\pi})^{-1/2}$ and $b_\mu = \Pi_{i=1}^n b_{\mu_i}$. Here the coefficients a_μ are the ones given in the known formula

$$(2.8) \qquad L_k^\alpha(|x|^2) = \sum_{|r|=k} a_r H_{2r}(x), \quad r = (r_1, \dots, r_n), \quad \alpha = \frac{n}{2} - 1.$$

See [10, formula 5.6.1)] in the case $n = 1$ and [2, Lemma 1.1] in the general case. Moreover taking derivatives in (2.8) and using the formulas for the derivatives of Laguerre and Hermite polynomials, see [10, p. 102, 106], we get the following alternative equality to (2.7)

$$(2.9) \; -2x_j \mathcal{L}_{k-1}^{\alpha+1}(|x|^2) = c_{k-1}^{\alpha+1} \sum_{|r|=k} \frac{a_r}{b_{2r-e_j}} 4r_j \Phi_{2r-e_j}(x)|x|^{\alpha+1}, \quad j = 1, \dots, n, \quad \alpha = \frac{n}{2} - 1.$$

The above formula (2.7) leads to the following relationship between the Hermite and Laguerre differential operators.

$$(2.10) \quad L_\alpha(f) \circ \phi(x) = \frac{1}{4} \; H_n\left(\frac{f \circ \phi}{|\cdot|^\alpha}\right)(x) \; |x|^\alpha, \quad x \in \mathbb{R}^n, \quad \alpha = \frac{n}{2} - 1, \quad \phi(x) = |x|^2,$$

where f is a finite linear combination of Laguerre funcions $\mathcal{L}_k^\alpha$. The validity of this equation for $f = \mathcal{L}_k^\alpha$ is a simple consequence of formulas (2.3) (2.6) and (2.7).

This basic relation can be transfered straightforwardly to other operators related to L_α and H_n. For example it is easy to check that the corresponding heat operators are related by

$$(2.11) \qquad \left[\exp(-tL_\alpha)(f) \circ \phi\right](x) = \left[\exp(-\frac{t}{4}H_n)\left(\frac{f \circ \phi}{|\cdot|^\alpha}\right)\right](x) \; |x|^\alpha$$
$$x \in \mathbb{R}^n, \quad \alpha = \frac{n}{2} - 1, \quad \phi(x) = |x|^2.$$

Let us recall that given a second order non negative and selfadjoint differential operator L and $T_t = e^{-tL}$ its heat semigroup the following operators can be considered, see [8],

(1) Maximal operator: $T^*f(x) = \sup_{t>0} |T_t f(x)|$.

(2) Maximal operator of the subordinated Poisson semigroup:
$P^*f(x) = \sup_{t>0} |P_t f(x)|$, where P_t is defined by the following subordination formula

$$P_t f(x) = \frac{1}{\sqrt{2\pi}} \int_0^\infty t e^{-t^2/4s} T_s f(x) s^{-3/2} ds.$$

(3) Riesz potentials: For $0 < \sigma$, $L^{-\sigma} f(x) = \frac{1}{\Gamma(\sigma)} \int_0^\infty t^{\sigma-1} T_t f(x) dt$, which can be derived from the identity $s^{-\sigma} = \frac{1}{\Gamma(\sigma)} \int_0^\infty t^{\sigma-1} e^{-ts} dt$.

Let us point out that by using (2.10) if we denote by T^α any of the above operators for the Laguerre case and by T'_n the corresponding Hermite one, we have up to a multiplicative constant

$$T^\alpha(f) \circ \phi(x) = T'_n\left(\frac{f \circ \phi}{|\cdot|^\alpha}\right)(x)\ |x|^\alpha, \quad x \in \mathbb{R}^n, \quad \alpha = \frac{n}{2} - 1, \quad \phi(x) = |x|^2. \tag{2.12}$$

However there are other operators which have not been included in the above list: the Riesz transforms. For the Hermite case they are defined through first order differential operators $A_j = \frac{\partial}{\partial x_j} + x_j$ and their adjoints $A_j^* = -\frac{\partial}{\partial x_j} + x_j$, $1 \le j \le n$, related to H_n by

$$H_n - n = \sum_{j=1}^n A_j^* A_j, \quad \text{and} \quad H_n = \frac{1}{2}\sum_{j=1}^n (A_j A_j^* + A_j^* A_j). \tag{2.13}$$

The action of these "derivatives" on the orthonormal system Φ_μ is given by

$$A_j \Phi_\mu = (2\mu_j)^{1/2} \Phi_{\mu - e_j}, \quad A_j^* \Phi_\mu = (2\mu_j + 2)^{1/2} \Phi_{\mu + e_j}, \tag{2.14}$$

where e_j are the coordinate vectors in $\mathbb{R}^n$. With this notation the Riesz transforms are defined by $R_j^+ = A_j (H_n)^{-1/2}$ and $R_j^- = A_j^* (H_n)^{-1/2}$, where $(H_n)^{-1/2}$ is the Riesz potential of order $\sigma = \frac{1}{2}$ defined above. For results concerning these operators we refer to [14] and [11].

In the context of Laguerre functions, in order to define the corresponding Riesz transforms, we introduce appropriate first order derivatives as follows.

$$\delta_y^\alpha = \sqrt{y}\frac{d}{dy} + \frac{1}{2}\left(\sqrt{y} - \frac{\alpha}{\sqrt{y}}\right) \tag{2.15}$$

and

$$\partial_y^\beta = -\sqrt{y}\frac{d}{dy} + \frac{1}{2}\left(\sqrt{y} - \frac{\beta}{\sqrt{y}}\right). \tag{2.16}$$

The action on the corresponding Laguerre functions is given by

$$\delta_y^\alpha(\mathcal{L}_k^\alpha) = -\sqrt{k}\ \mathcal{L}_{k-1}^{\alpha+1}, \quad \alpha > -1 \tag{2.17}$$

and

$$\partial_y^\beta(\mathcal{L}_k^\beta) = -\sqrt{k+1}\ \mathcal{L}_{k+1}^{\beta-1}, \quad \beta > 0. \tag{2.18}$$

¿From these definitions it follows that

$$(\delta_y^\alpha)^* = \partial_y^{\alpha+1}, \quad \alpha > -1 \tag{2.19}$$

and

$$L_\alpha - (\frac{\alpha+1}{2}) = (\delta_y^\alpha)^*\, \delta_y^\alpha = \partial_y^{\alpha+1}\delta_y^\alpha, \quad L_\beta - (\frac{\beta-1}{2}) = (\partial_y^\beta)^*\, \partial_y^\beta = \delta_y^{\beta-1}\partial_y^\beta\,, \tag{2.20}$$
$$\alpha > -1,\ \beta > 0.$$

With these notations we introduce the Riesz transforms for the Laguerre function expansions as

$$R_+^\alpha = \delta_y^\alpha (L_\alpha)^{-1/2},\ \alpha > -1 \quad \text{and } R_-^\beta = \partial_y^\beta (L^\beta)^{-1/2},\ \beta > 0, \tag{2.21}$$

that is

$$R_+^\alpha(\mathcal{L}_k^\alpha) = -\frac{\sqrt{k}}{\sqrt{k+\frac{\alpha+1}{2}}}\, \mathcal{L}_{k-1}^{\alpha+1} \quad \text{and} \quad R_-^\beta(\mathcal{L}_k^\beta) = -\frac{\sqrt{k+1}}{\sqrt{k+\frac{\beta+1}{2}}}\, \mathcal{L}_{k+1}^{\beta-1}. \tag{2.22}$$

In order to find a relation with the Hermite Riesz transforms, for $\alpha = \frac{n}{2} - 1$, we use (2.7), (2.14) and (2.9) to get

$$\begin{aligned}
A_j(\frac{\mathcal{L}_k^\alpha \circ \phi}{|\cdot|^\alpha})(x) &= c_k^\alpha \sum_{|r|=k} \frac{a_r}{b_{2r}} (4r_j)^{1/2} \Phi_{2r-e_j}(x) \\
&= c_k^\alpha \sum_{|r|=k} \frac{a_r}{b_{2r-e_j}} 4r_j \Phi_{2r-e_j}(x) \\
&= -2x_j\sqrt{k}\mathcal{L}_{k-1}^{\alpha+1}(|x|^2)|x|^{-\alpha-1} = 2\frac{x_j}{|x|}\delta_y^\alpha(\mathcal{L}_k^\alpha)(|x|^2)|x|^{-\alpha},
\end{aligned}$$

where in the last equality we have used (2.17). Therefore, for $\alpha = \frac{n}{2} - 1$, we have the following relation of the type (2.12)

$$2\,\frac{x_j}{|x|}\,\delta_y^\alpha(f) \circ \phi(x) = A_j(\frac{f \circ \phi}{|\cdot|^\alpha})(x)\,|x|^\alpha, \quad j = 1, \ldots, n, \tag{2.23}$$

where f is a finite linear combination of $\{\mathcal{L}_k^\alpha\}_k$. From here, by using (2.12) for the operators $(L)^{-1/2}$, it is easy to check that, up to a constant, we have for $\alpha = \frac{n}{2} - 1$

$$\begin{aligned}
|R_+^\alpha(f) \circ \phi(x)| &= \left(\sum_{j=1}^n |R_j^+(\frac{f \circ \phi}{|\cdot|^\alpha})(x)|^2\right)^{1/2} |x|^\alpha \\
&= \left\| \left\{ R_j^+(\frac{f \circ \phi}{|\cdot|^\alpha})(x) \right\}_j \right\|_{\ell^2(\{1,\ldots,n\})} |x|^\alpha.
\end{aligned} \tag{2.24}$$

Now we shall see how these pointwise relationships allow us to obtain weighted norm inequalities for Laguerre operators for the special values of $\alpha = \frac{n}{2} - 1$, out of boundedness properties for Hermite operators.

Proposition 2.25 *Let $\alpha = \frac{n}{2} - 1$ with n a positive integer and E, F two Banach spaces. Let T^α, T'_n be linear bounded operators from $L^2((0,\infty), dy)$ into $L^2_E((0,\infty), dy)$ and from $L^2(\mathbb{R}^n, dx)$ into $L^2_F(\mathbb{R}^n, dx)$ respectively. Assume further that for any function $f \in S_\alpha$, the subspace of finite linear combination of Laguerre functions $\mathcal{L}_k^\alpha$, we have*

$$\|T^\alpha(f) \circ \phi(x)\|_E = \|T'_n(\frac{f \circ \phi}{|\cdot|^\alpha})(x)\|_F |x|^\alpha, \quad x \in \mathbb{R}^n \tag{2.26}$$

Then, whenever T'_n is bounded from $L^p(\mathbb{R}^n, |x|^\gamma dx)$ into $L^p_F(\mathbb{R}^n, |x|^\gamma dx)$, $1 \le p \le \infty$, T^α is bounded from $L^p((0,\infty), y^\delta dy) \cap S_\alpha$ into $L^p_E((0,\infty), y^\delta dy)$, where δ satisfies $\gamma = \alpha(p-2) + 2\delta$.

Proof It is easy to check that, by an appropriate change of variables, for any positive function g we have

$$\int_0^\infty g(y)^p y^\delta dy = C_n \int_{\mathbb{R}^n} (g \circ \phi(x))^p |x|^{2(\delta-\alpha)} dx.$$

Therefore

$$\begin{aligned} \int_0^\infty \|T^\alpha(f)(y)\|_E^p y^\delta dy &= C_n \int_{\mathbb{R}^n} \|T^\alpha(f)(\phi(x))\|_E^p |x|^{2(\delta-\alpha)} dx \\ &= C_n \int_{\mathbb{R}^n} \|T'_n(\frac{f \circ \phi}{|\cdot|^\alpha})(x)\|_F^p |x|^{2(\delta-\alpha)+\alpha p} dx \\ &\le C_n \int_{\mathbb{R}^n} |f \circ \phi(x)|^p |x|^{2(\delta-\alpha)} dx = C_n \int_0^\infty |f(y)|^p y^\delta dy. \end{aligned} \tag{2.27}$$

■

Before illustrating the usefulness of Proposition 2.25 we would like to make some remarks about Laguerre expansions on these spaces. First, given a function $f \in L^p((0,\infty), y^\delta dy)$, the Laguerre coefficients are well defined, provided $\delta < \alpha\frac{p}{2} + (p-1)$. In fact, by Holder inequality

$$| < f, \mathcal{L}_k^\alpha > | \le \|f\|_{L^p((0,\infty), y^\delta dy)} (\int_0^\infty y^{-\delta p'/p} |\mathcal{L}_k^\alpha(y)|^{p'} dy)^{1/p'}$$

and the integral is finite near the origin because of the assumption on δ. On the other hand, the functions $\{\mathcal{L}_k^\alpha\}$ belong to $L^p((0,\infty), y^\delta dy)$ as long as $\delta > -1 - \alpha p/2$. In fact under both conditions on δ the following Lemma is true.

Lemma 2.28 *Let $\alpha > -1$ and $-1 - \alpha\frac{p}{2} < \delta < \alpha\frac{p}{2} + (p-1)$. Then S_α is dense in $L^p((0,\infty), y^\delta dy)$.*

We omit the proof and we refer the reader to the paralell results [10, Th 5.7.1] and [7, Lemma 2.29].
As a straightforward application of Proposition 2.25, relation (2.24) and the density of S_α, we have the following

Corollary 2.29 *Let $\alpha = \frac{n}{2}-1$. Then the Riesz transform R_+^α is bounded on $L^p((0,\infty), y^\delta dy)$, $1<p<\infty$, as long as*

$$-1-\alpha p/2 < \delta < (p-1) + \alpha p/2.$$

Proof We recall (see [11]) that R_j^+, are bounded on $L^p(w(x)dx)$ for p in the range $1<p<\infty$ and $w \in A_p$. In particular they are bounded for $w(x) = |x|^\gamma$, $-n<\gamma<n(p-1)$. Taking into account formula (2.24) and applying proposition 2.25 together with the density of S_α the stated result follows. ■

Remark 2.30 *We observe that for $\alpha \geq 0$ we obtain all the power weights belonging to A_p. That is no longer the case for $\alpha = -\frac{1}{2}$, where we get the more restricted range, $-1+p/4 < \delta < 3p/4-1$. In particular $R_+^{-1/2}$ is bounded on $L^p((0,\infty), dy)$ for p in the range $4/3<p<4$.*

3 A weighted multiplier theorem for Hermite expansions

In this section we give a weighted extension of Marcinkiewiz multiplier theorem for Hermite expansions. We follow the same steps of Thangavelu's proof, see [14, Theorem 4.2.1], so we only indicate the differences.

Given a bounded function m defined on the set of all multiindices, we will consider multipliers defined in $L^2(\mathbb{R}^n, dx)$ by

$$T_m f = \sum_\mu m(\mu) c_\mu \Phi_\mu,$$

where $f = \sum_\mu c_\mu \Phi_\mu$. With this notation the following result holds

Theorem 3.1 *(Weighted Marcinkiewicz multiplier theorem) Let m be a function satisfying*

$$|\Delta^\beta m(\mu)| \leq C_\beta (1+|\mu|)^{-|\beta|}$$

for all β with $|\beta| \leq k$, where $k > n/2$ is an integer. Then T_m is bounded on $L^p(\mathbb{R}^n, |x|^\gamma dx)$ for γ such that

$$-n \min\{p/2, 1\} < \gamma < n \min\{p/2, p-1\}$$

and $1<p<\infty$.

In order to prove the theorem we first observe that [14, Theorem 4.1.2, p. 87] holds true for all weights $w \in A_p$, by just following the same steps. More precisely, the Littlewood-Paley g function defined by

$$g(f)^2(x) = \int_0^\infty |\frac{\partial}{\partial t} exp(-tH_n)f(x)|^2 t\, dt$$

satisfies the inequalities

$$c_p\|f\|_{L^p(\mathbb{R}^n,w(x)dx)} \leq \|g(f)\|_{L^p(\mathbb{R}^n,w(x)dx)} \leq C_p\|f\|_{L^p(\mathbb{R}^n,w(x)dx)}, \tag{3.2}$$

for $1 < p < \infty$ and $w \in A_p$.
Next, we also need a weighted version of [14, Theorem 4.1.3] concerning the boundedness of the g_k^* Littlewood-Paley function, that is

$$g_k^*(f)^2(x) = \int_{\mathbb{R}^n}\int_0^\infty t^{-\frac{n}{2}+1}(1+\frac{|x-z|^2}{t})^{-k}|\frac{\partial}{\partial t} exp(-tH_n)(f)(z)|^2\, dt\, dz.$$

Lemma 3.3 *Let $k > n/2$ and $2 \leq p < \infty$. Then, for any weight $w \in A_{p/2}$*

$$\|g_k^*(f)\|_{L^p(\mathbb{R}^n,w(x)dx)} \leq C_p\|f\|_{L^p(\mathbb{R}^n,w(x)dx)}.$$

Proof For $p > 2$ we follow the same lines of Thangavelu's proof, using that the Hardy-Littlewood maximal function M is bounded in $L^{(p/2)'}(v)$, for $v \in A_{(p/2)'}$ and the fact that $w \in A_{(p/2)}$ if and only if $w^{-(p/2)'/(p/2)} \in A_{(p/2)'}$.
As for $p = 2$ from inequality [14, 4.1.23, p. 89]

$$\int_{\mathbb{R}^n} g_k^*(f)^2(x)h(x)dx \leq C\int_{\mathbb{R}^n} g(f)^2(x)Mh(x)dx,$$

it follows that for $w \in A_1$

$$\begin{aligned}\|g_k^*(f)\|^2_{L^2(w)} &\leq C \sup_{\|hw^{-1}\|_\infty \leq 1} \|g(f)\|^2_{L^2(w)}\|(Mh)w^{-1}\|_\infty \\ &\leq C \sup_{\|hw^{-1}\|_\infty \leq 1} \|f\|^2_{L^2(w)}\|hw^{-1}\|_\infty \leq C\|f\|^2_{L^2(w)},\end{aligned}$$

where we have used (3.2), $A_1 \subset A_2$ and that $\|(Mh)w^{-1}\|_\infty \leq C\|hw^{-1}\|_\infty$ if and only if $w \in A_1$. ■

Now we turn into the proof of the multiplier Theorem 3.1.

Following the argument given by Thangavelu in the unweighted case and using the above weighted inequalities for g and g_k^* we obtain that

$$\|T_m f\|_{L^p(w)} \leq C_p\|f\|_{L^p(w)}$$

for $p \geq 2$ and $w \in A_{p/2}$. In particular, this inequality holds for $w(x) = |x|^\gamma$ where γ is such that

$$-n < \gamma < n(p/2-1), \quad \text{if } p > 2$$

or

$$-n < \gamma \leq 0, \quad \text{ if } p = 2.$$

However, being T_m a selfadjoint operator we can extend the range of γ to $-n < \gamma < n$ if $p = 2$.
Next we enlarge the range of γ for $p > 2$ by using interpolation with change of measure. In fact we apply Stein-Weiss' interpolation Theorem, see [13], between $L^2(|x|^{\gamma_1})$ and $L^q(|x|^{\gamma_2})$ with $\gamma_1 = n\delta$, $0 < \delta < 1$, and $\gamma_2 = n(q/2-1)\varepsilon$, $\quad 0 < \varepsilon < 1$. Choosing $q > 2$ large enough and appropriate ε and δ, we get the boundedness on $L^p(|x|^\gamma)$ with γ such that $0 < \gamma < np/2$. Finally, for $p < 2$, a duality argument allows us to get the required range for γ. ■

With the aid of Proposition 2.25 and the above Theorem 3.1 we can prove a weighted multiplier theorem for Laguerre expansions for the special values of α. This result will be one of the main tools in proving the boundedness of the Riesz transforms for Laguerre expansions.

Corollary 3.4 *Let $\alpha = \frac{n}{2} - 1$ with n a positive integer. Let m be a bounded function on $[0, \infty)$ and T_m^α the multiplier operator defined by*

$$T_m^\alpha(f) = \sum_{k=0}^{\infty} m(k) c_k \mathcal{L}_k^\alpha,$$

for $f = \sum_{k=0}^\infty c_k \mathcal{L}_k^\alpha$. Assume further that m satisfies

$$|(1+t^j)\frac{d^j m(t)}{dt^j}| \leq C, \quad 1 \leq j \leq k, \quad k > \frac{n}{2}.$$

Then T_m^α, initially defined on $L^2((0,\infty)), dy)$, extends to a bounded operator on $L^p((0,\infty)), y^\delta dy)$ for δ such that

$$-\min\{1, p/2\} - \alpha \min\{p/2, p-1\} < \delta < \min\{p/2, p-1\} + \alpha \min\{1, p/2\}.$$

Proof We define $\tilde{m}(t) = m(t/2)$. Then $\tilde{m}$ satisfies the same smoothness properties as m. Moreover the function $\tilde{m}(|x|)$, $x \in \mathbb{R}^n$, when restricted to all the $n-$multiindices μ, satisfies the hypothesis of Theorem 3.1. Therefore, the operator $T_{\tilde{m}}$ defined by

$$T_{\tilde{m}}(\Phi_\mu) = \tilde{m}(|\mu|)\Phi_\mu$$

is bounded on $L^p(\mathbb{R}^n, |x|^\gamma dx)$ with

$$-n \min\{p/2, 1\} < \gamma < n \min\{p/2, p-1\}.$$

Next we show that T_m^α and $T_{\tilde{m}}$ are related as in (2.12). In fact, by (2.7)

$$\begin{aligned}
T_m^\alpha(\mathcal{L}_k^\alpha)(|x|^2) &= m(k)\mathcal{L}_k^\alpha(|x|^2) = \tilde{m}(2k)(\sum_{|r|=k} c_k^\alpha \frac{a_r}{b_{2r}} \Phi_{2r}(x))|x|^\alpha \\
&= (\sum_{|r|=k} c_k^\alpha \frac{a_r}{b_{2r}} T_{\tilde{m}}(\Phi_{2r})(x))|x|^\alpha = T_{\tilde{m}}(\sum_{|r|=k} c_k^\alpha \frac{a_r}{b_{2r}} \Phi_{2r})(x)|x|^\alpha \\
&= T_{\tilde{m}}(\frac{\mathcal{L}_k^\alpha(|\cdot|^2)}{|\cdot|^\alpha})(x)|x|^\alpha.
\end{aligned}$$

Applying now proposition 2.25 and the density of S_α the result follows.

∎

Remark 3.5 *We notice that if* $\alpha \geq \max\{p/2-1, \frac{2-p}{2(p-1)}\}$, *then all the power weights in* A_p *belong to the above range for* δ.

Remark 3.6 *We also observe that for* $\alpha \geq 0$ *corollary 3.4 is contained in the multiplier theorem appeared in [12]. However the case* $\alpha = -\frac{1}{2}$ *is new.*

4 Riesz Transforms for Laguerre expansions

In section 2 we introduced the Riesz Transforms R_+^α, R_-^β and we already obtained some boundedness properties of R_+^α for the special values $\alpha = \frac{n}{2}-1$. With the aid of this result and Corollary 3.4 we will prove some boundedness for the other Riesz Transform R_-^β with $\beta = \frac{n}{2}-1$. We state this result as a lemma since it will be a tool in the proof of our main theorem, which in particular will improve the lemma.

Lemma 4.1 *Let* $1 < p < \infty$ *and* $\alpha = \frac{n}{2}-1$. *Then* $R_-^{\alpha+1}$ *is bounded on* $L^p((0,\infty), y^\delta dy)$ *for all* A_p*−power weights provided* $\alpha \geq \max\{p/2-1, \frac{2-p}{2(p-1)}\}$.

Proof. We start by observing that

$$R_-^{\alpha+1} = T_m^\alpha \circ (R_+^\alpha)^*$$

where T_m^α is the multiplier operator associated to a C^∞-function m such that $m(t) = \dfrac{\sqrt{2t+\alpha+1}}{\sqrt{2t+\alpha}}$ for $t \geq 1$ and $m(0) = 0$. In fact, by using (2.19) we have

$$(R_+^\alpha)^* = (\delta_y^\alpha (L_\alpha)^{-1/2})^* = (L_\alpha)^{-1/2}\partial_y^{\alpha+1}.$$

Therefore, for $k \geq 0$, we have

$$\begin{aligned} T_m^\alpha \circ (R_+^\alpha)^*(\mathcal{L}_k^{\alpha+1}) &= -T_m^\alpha \circ (L_a)^{-1/2}(\sqrt{k+1}\mathcal{L}_{k+1}^\alpha) = -T_m^\alpha\Big(\frac{\sqrt{k+1}}{\sqrt{k+1+\frac{\alpha+1}{2}}}\mathcal{L}_{k+1}^\alpha\Big) \\ &= -\frac{\sqrt{k+1}}{\sqrt{k+\frac{\alpha+2}{2}}}\mathcal{L}_{k+1}^\alpha = \quad R_-^{\alpha+1}(\mathcal{L}_k^{\alpha+1}). \end{aligned}$$

Now according to Corollary 3.4, see remark 3.5, T_m^α is bounded for all A_p- powers. Also from Corollary 2.29, R_+^α is bounded for those weights and so is its adjoint. ∎

Now we are in position to state and prove our main result

Theorem 4.2 *Let $\alpha > -1$ and $1 < p < \infty$. Then the Laguerre Riesz Transforms R_+^α and $R_-^{\alpha+1}$ are bounded on $L^p((0,\infty), y^\delta dy)$ whenever*

$$-1 - \min\{\alpha, 0\}p/2 < \delta < (p-1) + \min\{\alpha, 0\}p/2.$$

Proof Let us denote by τ_β^α the transplantation operator defined in Theorem 1.1. Also, by $T_m^{\beta+1}$ we denote the multiplier operator associated to the function

$$m(t) = \frac{\sqrt{2t+\beta+3}}{\sqrt{2t+\alpha+3}}.$$

Then the following formula holds

$$R_+^\alpha = \tau_{\alpha+1}^{\beta+1} \circ T_m^{\beta+1} \circ R_+^\beta \circ \tau_\beta^\alpha.$$

In fact, for $f = \mathcal{L}_k^\alpha, k \geq 0$, we have

$$\begin{aligned}
\tau_{\alpha+1}^{\beta+1} \circ T_m^{\beta+1} \circ R_+^\beta \circ \tau_\beta^\alpha(\mathcal{L}_k^\alpha) &= \tau_{\alpha+1}^{\beta+1} \circ T_m^{\beta+1} \circ R_+^\beta(\mathcal{L}_k^\beta) \\
&= -\tau_{\alpha+1}^{\beta+1} \circ T_m^{\beta+1}\Big(\frac{\sqrt{k}}{\sqrt{k+\frac{\beta+1}{2}}}\mathcal{L}_{k-1}^{\beta+1}\Big) \\
&= -\tau_{\alpha+1}^{\beta+1}\Big(\frac{\sqrt{k}}{\sqrt{k+\frac{\alpha+1}{2}}}\mathcal{L}_{k-1}^{\beta+1}\Big) \\
&= -\frac{\sqrt{k}}{\sqrt{k+\frac{\alpha+1}{2}}}\mathcal{L}_{k-1}^{\alpha+1} = R_+^\alpha(\mathcal{L}_k^\alpha).
\end{aligned}$$

In a similar way, for $R_-^{\alpha+1}$ we obtain

$$R_-^{\alpha+1} = \tau_\alpha^\beta \circ T_{\tilde{m}}^\beta \circ R_-^{\beta+1} \circ \tau_{\beta+1}^{\alpha+1},$$

where $T_{\tilde{m}}^\beta$ is the multiplier operator associated to a $C^\infty-$function $\tilde{m}$ such that
$\tilde{m}(t) = \dfrac{\sqrt{2t+\beta}}{\sqrt{2t+\alpha}}$, $t \geq 1$.
We fix p, $1 < p < \infty$, and choose $\beta = \frac{n}{2} - 1$ large enough so that $\beta \geq \max\{\frac{p}{2} - 1, \frac{2-p}{2(p-1)}, \alpha\}$. First we assume $\alpha \geq 0$ and hence we have to show that both Riesz Transforms, R_+^α and $R_-^{\alpha+1}$, are bounded for all power weights in A_p. We notice that all the involved transplantation operators are bounded for those weights. Also, according to Corollary 2.29 (see Remark 2.30) and Lemma 4.1, the same holds for R_+^β and $R_-^{\beta+1}$ because of our choice of β. Finally, it is easy to check that the functions m and $\tilde{m}$ satisfy the hypothesis of Corollary 3.4, and again our choice of β allows us to obtain the full range of A_p- powers (see Remark 3.5).
Now we assume $-1 < \alpha < 0$. All that we have said about the multiplier and Riesz operators corresponding to β remains true. Also, the operators $\tau_{\alpha+1}^{\beta+1}$ and $\tau_{\beta+1}^{\alpha+1}$ are bounded for all A_p-powers. The only difference comes from the operators τ_β^α and τ_α^β, bounded on $L^p((0,\infty), y^\delta dy)$ with $-1 - \alpha p/2 < \delta < p - 1 + \alpha p/2$, a range which turns out to be more restricted than A_p-powers for $\alpha < 0$. ■

Remark 4.3 *We observe that when* $\alpha \geq 0$, R_+^α *and* $R_-^{\alpha+1}$ *are bounded on* $L^p((0,\infty), dy)$ *for all* $1 < p < \infty$. *However for* $-1 < \alpha < 0$, *they are bounded for a limited range of* p, *namely* $\frac{2}{2+\alpha} < p < -\frac{2}{\alpha}$.

5 The accuracy of the transplantation method for negative α.

Let us consider for $\beta \geq 0$ and a fixed $t_0 > 0$ the heat operator $e^{-t_0 L^\beta}$. From the estimates given in [9] in terms of the Hardy-Littlewood maximal function we know that it is bounded on $L^p((0,\infty), w(y)dy)$ for $w \in A_p$, $1 < p < \infty$. In particular this is true for the power weights $w(y) = y^\delta$ with $-1 < \delta < p-1$. Given $-1 < \alpha < 0$, it is easy to check that

$$e^{-t_0 L_\alpha} = \tau_\alpha^\beta \circ N \circ e^{-t_0 L^\beta} \circ \tau_\beta^\alpha,$$

where N is the multiplication operator $N(f)(y) = e^{\frac{\beta-\alpha}{2} t_0} f(y)$. By the transplantation Theorem we get that $e^{-t_0 L_\alpha}$ is bounded on $L^p((0,\infty), y^\delta dy)$ with $-1 - \alpha p/2 < \delta < p - 1 + \alpha p/2$. Now we are going to see that this interval is the best possible.
Let us recall that the heat kernel for $r_0 = e^{-t_0}$ is given by

$$T_{r_0}^\alpha(y,z) = \frac{{r_0}^{1/2}}{1-r_0} \exp\{-\frac{1}{2}\frac{1+r_0}{1-r_0}(y+z)\} I_\alpha(\frac{2(r_0 yz)^{1/2}}{1-r_0})$$

with $I_\alpha(s) = i^{-\alpha} J_\alpha(is)$, being J_α the usual Bessel function of order α. From its definition we know $w^{-\alpha} J_\alpha(w)$ is analytic on the complex plane for any real α and, for $\alpha > -1$, different from 0 at the origin. This implies that for y and z small the heat kernel behaves as $y^{\alpha/2} z^{\alpha/2}$. Therefore if we integrate against the characteristic function of the interval $(0,1)$ with respect to the $z-$variable the resulting function will not be on $L^p((0,\infty), y^\delta dy)$ unless $\delta > -1 - \alpha\, p/2$. The symmetry of the kernel and a duality argument gives the other restriction on δ.

References

[1] Dinger, U. , *Weak type* $(1,1)$ *estimates of the maximal function for the Laguerre semigroup in finite dimensions,* Rev. Mat. Iberoamericana 8 (1992) 93- 120.

[2] Gutierrez, C., Incognito, A., Torrea, J.L. , *Riesz transforms, g-functions, and multipliers for the Laguerre semigroup.* Houston J. Math. 27 (2001), 579-592.

[3] Kanjin, Y., *A transplantion theorem for Laguerre series,* Tôhoku Math. J. 43 (1991) 537-555.

[4] Lindenstrauss, J. and Tzafriri, L., *Classical Banach function spaces,* Springer Ser. Modern Surveys Math. **97** Springer-Verlag, Berlin, 1979.

[5] Muckenhoupt, B. *Hermite conjugate expansions.* Trans. Amer. Math. Soc. 139 (1969) 243-260.

[6] Muckenhoupt, B. *Poisson integrals for Hermite and Laguerre expansions.* Trans. Amer. Math. Soc. 139 (1969) 231-242.

[7] Muckenhoupt, B. *Mean convergence of Hermite and Laguerre series II.* Trans. Amer. Math. Soc. 147 (1970) 433-460.

[8] Stein, E. *Topics in Harmonic Analysis Related to the Littlewood-Paley Theory,* Ann. Annals of Mathematical Studies 63. Princeton University Press, 1970.

[9] Stempak, K. *Heat-diffusion and Poisson integrals for Laguerre expansions* Tohoku Math. J. 46 (1994), 83-104.

[10] Szego, G., *Orthogonal Polynomials,* Amer. Math. Soc. Colloquium Publications XXIII. 1939.

[11] Stempak,K. Torrea, J.L. , *Poisson integrals and Riesz transforms for Hermite function expansions with weights* J. Functional Analysis (to appear).

[12] Stempak, K. , Trebels, W., *On weighted transplantation and multipliers for Laguerre expansions* , Math. Ann. 300 (1994), 203-219.

[13] Stein, E., Weiss, G., *Interpolation of operators with change of measures,* Trans. Amer. Math. Soc. 87 (1958), 159-172.

[14] Thangavelu, S.*Lectures on Hermite and Laguerre expansions,* Mathematical Notes 42. Princeton University 1993.

[15] Thangavelu, S.*A Note on a Transplantation Theorem of Kanjin and Multiple Laguerre Expansions,* Proceedings of the Amer. Math. Soc. 119 (1993), 1135-1143.

E. Harboure and B. Viviani
IMAL-FIQ
CONICET - Universidad Nacional del Litoral
Güemes 3450, 3000 Santa Fe, Argentina
harbour@ceride.gov.ar, viviani@ceride.gov.ar

J.L. Torrea
Departamento de Matemáticas
Facultad de Ciencias
Universidad Autónoma de Madrid
28049 Madrid, Spain
Joseluis.torrea@uam.es