A sharp weighted transplantation theorem for Laguerre function expansions*

G. Garrigós, E. Harboure, T. Signes, J.L. Torrea, B. Viviani

October 5, 2006

Abstract

We find the sharp range of boundedness for transplantation operators associated with Laguerre function expansions in L^p spaces with power weights. Namely, the operators interchanging $\{\mathcal{L}_k^\alpha\}$ and $\{\mathcal{L}_k^\beta\}$ are bounded in $L^p(y^{\delta p})$ if and only if $-\frac{\rho}{2} - \frac{1}{p} < \delta < 1 - \frac{1}{p} + \frac{\rho}{2}$, where $\rho = \min\{\alpha, \beta\}$. This improves a previous partial result by Stempak and Trebels, which was only sharp for $\rho \leq 0$. Our approach is based on new multiplier estimates for Hermite expansions, weighted inequalities for local singular integrals and a careful analysis of Kanjin's original proof of the unweighted case. As a consequence we obtain new results on multipliers, Riesz transforms and g-functions for Laguerre expansions in $L^p(y^{\delta p})$.

1 Introduction

In $\mathbb{R}_+ = (0, \infty)$, we consider the system of Laguerre functions defined by

$$\mathcal{L}_k^\alpha(y) = c_{k,\alpha}\, y^{\frac{\alpha}{2}}\, e^{-\frac{y}{2}}\, L_k^{(\alpha)}(y), \quad k = 0, 1, 2, \dots \tag{1.1}$$

where $L_k^{(\alpha)}(y) = (y^{\alpha+k} e^{-y})^{(k)} / (k! y^\alpha e^{-y})$ is the usual Laguerre polynomial of degree k. For each $\alpha > -1$, this system is an orthonormal basis of $L^2(\mathbb{R}_+)$ when we choose the normalizing constants

$$c_{k,\alpha} = \sqrt{\Gamma(k+1)/\Gamma(\alpha+k+1)}, \quad k = 0, 1, 2, \dots$$

*Research partially supported by *Banco Santander-UAM* grant "*IALE: 2004-2005*" and by *Ministerio de Educación y Ciencia* (Spain) grants: "MTM2005-08350-C03-01 ", "MTM2004/01888", "MTM2004-00678". First author also supported by *Programa Ramón y Cajal*, MCyT (Spain), second and fifth authors by grants from CONICET and UNL (Argentina), and third author by *Fundación Séneca* PB/21/FS/02.
2000 Mathematical Subject Classification: 42A45, 42B15, 42B20, 42B25, 42C10.
Keywords: Laguerre semigroup, transplantation, multiplier, Littlewood-Paley theory.

(see e.g. [19]). This produces a formal expansion $f = \sum_{k=0}^{\infty} \langle f, \mathcal{L}_k^\alpha \rangle \mathcal{L}_k^\alpha$, which is convergent in norm at least for $f \in L^2(\mathbb{R}_+)$.

A main object in the theory of Laguerre function expansions are the so-called *transplantation operators*, defined for $\alpha, \beta > -1$ and $f \in L^2(\mathbb{R}_+)$ by

$$T_\beta^\alpha f = \sum_{k=0}^{\infty} \langle f, \mathcal{L}_k^\alpha \rangle \mathcal{L}_k^\beta. \tag{1.2}$$

The L^p boundedness of such operators was first established in a celebrated theorem of Kanjin [10]. Namely T_β^α is bounded in $L^p(\mathbb{R}_+)$ whenever $\frac{|\gamma|}{2} < \frac{1}{p} < 1 - \frac{|\gamma|}{2}$, where $\gamma := \min\{\alpha, \beta, 0\}$. In particular, boundedness holds for all $1 < p < \infty$ when $\alpha, \beta \geq 0$. We refer to Ch. 6 of [21] for a discussion and several applications of transplantation in problems involving Laguerre function expansions (see also [18, 8]).

In this paper we shall be interested in extensions of Kanjin's result to power weighted Lebesgue spaces $L_\delta^p = L^p(\mathbb{R}_+, y^{\delta p} dy)$. The main theorem in this setting is due to Stempak and Trebels [18], which have established the boundedness of T_β^α in L_δ^p whenever

$$\frac{|\gamma|}{2} - \frac{1}{p} < \delta < 1 - \frac{1}{p} - \frac{|\gamma|}{2}, \quad \text{where} \quad \gamma := \min\{\alpha, \beta, 0\}. \tag{1.3}$$

Power weighted estimates for T_β^α appear naturally in the study of multiplier and transplantation theorems for several well-known variants of the Laguerre system, as noticed by Thangavelu in [20] (see also [18, 1], and §6 below).

Our goal in this paper is to improve the result of Stempak and Trebels with a new transplantation theorem in a range of weights strictly larger than (1.3), and which is in fact optimal for the operators T_β^α. As we shall see, this result transfers to other systems, producing as well optimal power weighted inequalities for the corresponding transplantation and multiplier operators (see Corollary 6.19). More precisely, our main result can be stated as follows.

Theorem 1.4 *Let $-1 < \alpha < \beta$ and $1 < p < \infty$. Then the operators T_β^α and T_α^β admit a bounded extension to L_δ^p if and only if*

$$-\frac{\alpha}{2} - \frac{1}{p} < \delta < 1 - \frac{1}{p} + \frac{\alpha}{2}. \tag{1.5}$$

We point out that (1.3) coincides with (1.5) precisely when $\min\{\alpha, \beta\} \leq 0$ (see Figure 1.1). Such a constraint in p and δ for negative parameters is well known in Laguerre systems. However, the fact that the range $-\frac{1}{p} < \delta < 1 - \frac{1}{p}$ can be improved for positive parameters seems to come as a surprise.

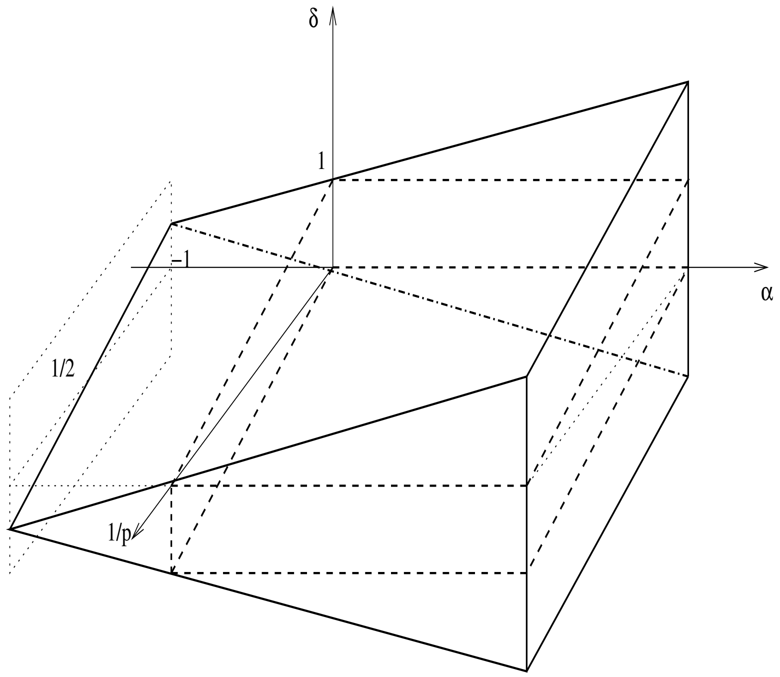

Figure 1.1: Region of L^p_δ boundedness for T^α_β when $\beta > \alpha$, according to Theorem 1.4. The region in dashed lines corresponds to earlier results for $\alpha > 0$ in [18].

That such behavior should be possible was suggested to the authors by recent results about Riesz transforms and other operators, which have a better behavior for special α's due to properties of Hermite function expansions (see [9, 12] or §5.1 below). In fact, a phenomenon of similar type was recently discovered by Nowak and Stempak for the Hankel transform transplantation operator [13].

We should nevertheless point out that the range in (1.5) is the natural one suggested by examples. Indeed, it is straightforward to verify that this is precisely the range where both $\mathcal{L}^\alpha_k$ and $\mathcal{L}^\beta_k$ belong to $L^p_\delta \cap L^{p'}_{-\delta}$, so that each of the individual summands $\langle f, \mathcal{L}^\alpha_k \rangle \mathcal{L}^\beta_k(y)$ in (1.2) is well-defined in L^p_δ. An appropriate modification of this argument as in [8, §5] is enough to obtain the necessity statement of Theorem 1.4. Moreover, it is also easy to see that T^α_β does not admit $(L^p_{\delta_1}, L^p_{\delta_2})$ inequalities when $\delta_1 \neq \delta_2$ (see Remark 6.20 below).

The main contribution of the paper is therefore the sufficient condition in the theorem, which requires some new ideas compared to [18], plus a few refinements in certain estimates of Kanjin's original proof [10]. The key argument is a new multiplier theorem for Hermite function expansions in $\mathbb{R}^n$, which can be stated as follows (see section 2 for a precise definition of the Hermite functions $\mathfrak{h}_{\mathbf{k}}$):

Theorem 1.6 *Let $1 < p < \infty$ and $m \in \ell^\infty(\mathbb{N}^n)$ such that*

$$|\Delta^{\boldsymbol{\alpha}} m(\mathbf{k})| \leq C\,(1+|\mathbf{k}|)^{-|\boldsymbol{\alpha}|}, \quad \mathbf{k} \in \mathbb{N}^n, \quad \forall\, |\boldsymbol{\alpha}| \leq n+1. \tag{1.7}$$

Consider the operator $\mathcal{T}_m f = \sum_{\mathbf{k}} m(\mathbf{k}) \langle f, \mathfrak{h}_{\mathbf{k}} \rangle \mathfrak{h}_{\mathbf{k}}$, defined at least for $f \in L^2(\mathbb{R}^n)$. Then, $\mathcal{T}_m$ admits a bounded extension to $L^p(w)$ whenever the weight w belongs to the Muckenhoupt class $A_p(\mathbb{R}^n)$.

This improves a previous result in [8, Th. 3.1], where only $A_{p/2}$ for $p \geq 2$ was obtained (which in turn was an adaptation of an earlier argument by Thangavelu; see [21, Th. 4.2.1]). We observe that this multiplier theorem can be transferred to Laguerre function expansions for the special parameters $\alpha = \frac{n-2}{2}$, using the method developed in [7] (see also [8]). All these results will be presented in section 2.

In section 3 we study the transplantation operators $T_\alpha^{\alpha+i\theta}$ introduced by Kanjin [10]. Appropriately modified with a multiplier, Kanjin found for these operators an explicit expression, which can be further estimated by a positive operator (of Hardy type) and a singular integral. Here we shall refine the estimates of the positive operator to show boundedness in $L^p_\delta(\mathbb{R}_+)$ for all $\delta > -\frac{1}{p} - \frac{\alpha}{2}$. On the other hand, the oscillating part is only a *local* singular integral, so that, as noticed by Nowak and Stempak [13], it is a bounded operator in $L^p_\delta(\mathbb{R}_+)$ for all $\delta \in \mathbb{R}$. Finally, the multiplier which appears in Kanjin's explicit expression of $T_\alpha^{\alpha+i\theta}$ can be handled with Theorem 1.6 for the special parameters $\alpha = \frac{n-2}{2}$. With these ideas and complex interpolation we shall prove a new multiplier theorem for Laguerre function expansions, which is the main result in section 3.

Theorem 1.8 *Let $\alpha > -1$, $1 < p < \infty$ and $m \in C^\infty[0,\infty)$ such that*

$$|D^\ell m(\xi)| \leq C_\ell\,(1+\xi)^{-\ell}, \quad \xi \geq 0, \quad \ell = 0, 1, 2, \ldots \tag{1.9}$$

Consider the operator $\mathcal{T}_m f = \sum_{k\geq 0} m(k) \langle f, \mathcal{L}_k^\alpha \rangle \mathcal{L}_k^\alpha$, defined at least for $f \in L^2(\mathbb{R}_+)$. Then, $\mathcal{T}_m$ admits a bounded extension to L^p_δ whenever $-\frac{\alpha}{2} - \frac{1}{p} < \delta < 1 - \frac{1}{p} + \frac{\alpha}{2}$.

The range of power weights is sharp for each p and α, and improves the one given in the multiplier theorem of Stempak and Trebels for all $\alpha > 0$ (see Theorem 1.1 and Corollary 4.3 in [18]). We observe that the Mihlin-type version we have stated above suffices for our applications, but the same conclusions hold with less smoothness required on the multiplier $m(\xi)$ (see Remark 3.24 below).

Armed with Theorem 1.8, it will be easy to conclude the proof of Theorem 1.4. Indeed, we can now handle the multiplier which appears in Kanjin's explicit expression of $T_\alpha^{\alpha+i\theta}$ for any $\alpha > -1$, and obtain as a consequence the boundedness of this operator in L^p_δ in the whole range $-\frac{\alpha}{2}-\frac{1}{p}<\delta<1-\frac{1}{p}+\frac{\alpha}{2}$. Then, a clever use of complex interpolation with three parameters (p, α and δ) will be enough to establish the desired result. It should be observed that the use we make of complex interpolation produces in addition a simplification of Kanjin's original proof of the unweighted case, since there is no need to appeal to the operators $T_\alpha^{\alpha+2}$. This program is carried out in section 4. As an illustration, we present in section 5 an application of the above theorems to the boundedness of Riesz transforms and Littlewood-Paley g-functions associated with the Laguerre system.

Finally, in section 6, we state the corresponding versions of the transplantation and multiplier theorems for modified Laguerre systems (see Corollary 6.19).

Acknowledgements: This research was carried out during a visit of the first and third authors to CONICET and the Institute of Mathematics of Universidad Nacional de Litoral, Santa Fe, Argentina, in October 2005. We wish to thank people in both institutions for their support and hospitality.

2 Multipliers for Hermite expansions

Following [21, Ch. 1], Hermite functions in $\mathbb{R}^n$ are defined by

$$\mathfrak{h}_{\mathbf{k}}(x) \;=\; d_{\mathbf{k},n}\, e^{-|x|^2/2} \prod_{i=1}^{n} H_{k_i}(x_i)\,, \quad \mathbf{k}=(k_1,\ldots,k_n), \quad k_i\geq 0, \tag{2.1}$$

where $H_k(t)=(-1)^k e^{t^2} D^{(k)}(e^{-t^2})$ is the usual Hermite polynomial in $\mathbb{R}$. Normalizing with $d_{\mathbf{k},n}=\prod_{i=1}^n (2^{k_i}k_i!\sqrt{\pi})^{-1/2}$, $\{\mathfrak{h}_{\mathbf{k}}\}_{\mathbf{k}\geq\mathbf{0}}$ is an orthonormal basis of $L^2(\mathbb{R}^n)$ and a complete system of eigenvectors for the Hermite operator $-\Delta+|x|^2$.

2.1 Proof of Theorem 1.6

In order to prove Theorem 1.6, we follow the usual approach adapted from the Euclidean case [15]. Most steps are contained in Ch. 4 of [21], so we only sketch them here.

We define respectively the *Hermite g-function* and *g^*-function* by

$$\begin{aligned} g_\ell(f)(x) &= \Big[\int_0^\infty \big|s^\ell\,\partial_s^\ell T_s f(x)\big|^2\,\frac{ds}{s}\Big]^{\frac12}, \qquad\qquad \ell=1,2,\ldots \\ g^*_\lambda(f)(x) &= \Big[\int_{\mathbb{R}^n}\int_0^\infty \frac{s^{-\frac{n}{2}}}{(1+\frac{|x-y|}{\sqrt{s}})^{n\lambda}}\,\big|s\,\partial_s T_s f(y)\big|^2\,\frac{dsdy}{s}\Big]^{\frac12}, \quad \lambda>1, \end{aligned}$$

where $T_s=e^{-s(-\Delta+|x|^2)}$ denotes the *Hermite heat semigroup*. We shall denote the kernel of

T_s by $T_s(y,z)$, so that we can write

$$s^\ell\,\partial_s^\ell T_s f(y) = \int_{\mathbb{R}^n} s^\ell [\tfrac{\partial^\ell T_s(y,z)}{\partial s^\ell}]\, f(z)\,dz.$$

For convenience, we shall change variables $s = t^2$ in the definition of g and g^*, and denote by $Q_t(y,z)$ the new (normalized) kernels $t^{2\ell}[\frac{\partial^\ell T_s(y,z)}{\partial s^\ell}]|_{s=t^2}$ for $\ell \geq 1$. It is well known that these kernels are symmetric and satisfy the estimates

$$\begin{aligned} &(a)\quad |Q_t(y,z)| \,\leq\, C_N\, \frac{t^{-n}}{\big(1+|y-z|/t\big)^N} \\ &(b)\quad |Q_t(y+h,z)-Q_t(y,z)| \,\leq\, C_N\, \frac{|h|}{t}\, \frac{t^{-n}}{\big(1+|y-z|/t\big)^N}, \qquad \forall\, |h| \leq t \end{aligned} \tag{2.2}$$

for some $C_N > 0$ and any positive integer N (see e.g. [21, p. 87]). From these estimates and the theory of vector-valued singular integrals it is not difficult to obtain the following proposition (see e.g. [21, Th. 4.1.2]).

Proposition 2.3 *Let $1 < p < \infty$, $\ell = 1, 2, \ldots$ and $w \in A_p(\mathbb{R}^n)$. Then, there exists $C > 0$ such that*

$$C^{-1}\,\|g_\ell(f)\|_{L^p(w)} \,\leq\, \|f\|_{L^p(w)} \,\leq\, C\,\|g_\ell(f)\|_{L^p(w)}.$$

The second required result is the following pointwise estimate, that can be found in [21, p. 91].

Proposition 2.4 *Let $\lambda > 1$ and m be a bounded sequence so that (1.7) holds for all $|\boldsymbol{\alpha}| \leq \lceil \lambda n/2 \rceil = \min\{k \in \mathbb{N} \;:\; k \geq \lambda n/2\}$. Then, for all $\ell \geq \lambda n/2 + 1$ we have*

$$g_\ell(\mathcal{T}_m f)(x) \,\leq\, C'\, g^*_\lambda(f)(x), \quad a.e.\, x \in \mathbb{R}^n.$$

At this point, combining the previous two results we have, for $f \in C_c(\mathbb{R}^n)$,

$$\|\mathcal{T}_m f\|_{L^p(w)} \,\leq\, C\,\|g_{n+2}(\mathcal{T}_m f)\|_{L^p(w)} \,\leq\, C'\|g^*_\lambda(f)\|_{L^p(w)},$$

provided condition (1.7) is satisfied and λ is bigger but close enough to 2. The only remaining step to establish Theorem 1.6 is the $L^p(w)$ boundedness of the g^*-function for A_p weights. This result seems to be new in the literature, so we shall state and prove it in detail in the next subsection.

2.2 Weighted inequalities for g^*-functions

THEOREM 2.5 *Let $1 < p < \infty$ and $w \in A_p(\mathbb{R}^n)$. Then, for each $\lambda > 2$ there is a constant $C > 0$ so that*

$$\|g^*_\lambda(f)\|_{L^p(w)} \leq C\,\|f\|_{L^p(w)}.$$

REMARK 2.6 The unweighted version of this theorem for $p \geq 2$ can be found in [21, Th. 4.1.3]. In the weighted case, a variant of the previous for $p \geq 2$ and $w \in A_{p/2}(\mathbb{R}^n)$ appears in [8, Lemma 3.3]. We shall make use of these facts later on.

REMARK 2.7 As we will see in the proof, this theorem is actually valid for any kernel $Q_t(x, y)$ satisfying the estimates (a) and (b) in (2.2) above. Thus, it will hold as well for semigroups with more general potentials $-\Delta + V(x)$ (see e.g. [5]).

To prove the theorem it is convenient to look at g^* as a *vector-valued singular integral.* Let X denote the Hilbert space $L^2(\mathbb{R}_+ \times \mathbb{R}^n, dtdy/t^{n+1})$, and consider the operator $\mathbf{G}$: $L^2(\mathbb{R}^n) \to L^2_X(\mathbb{R}^n)$ defined by

$$\mathbf{G}f(x) = \int_{\mathbb{R}^n} \mathbf{K}(x, z) f(z)\, dz,$$

where $\mathbf{K}(x, z)$ is the X-valued kernel

$$\mathbf{K}(x, z) = \Big\{ (1 + \tfrac{|x-y|}{t})^{-\frac{n\lambda}{2}}\, Q_t(y, z) \Big\}_{(t,y)}.$$

Observe that $|\mathbf{G}f(x)|_X = g^*_\lambda(f)(x)$. Therefore, the boundedness of g^*_λ in $L^p(w)$ is equivalent to the boundedness of $\mathbf{G}$ from $L^p(w)$ into $L^p_X(w)$. Moreover, by Remark 2.6 boundedness holds in the unweighted case at least for $2 \leq p < \infty$. The crucial estimate to establish the theorem is contained in the following lemma.

LEMMA 2.8 *Let $\lambda > 2$. Then, there exists $\delta > 0$ such that*

$$|\mathbf{K}(x, z) - \mathbf{K}(x, z_0)|_X \leq C\, \frac{|z - z_0|^\delta}{|x - z|^{n+\delta}}, \quad \textit{whenever} \quad |z - z_0| < \tfrac{1}{2}|x - z|.$$

This lemma says that $\mathbf{G}$ is a Calderón-Zygmund vector-valued operator with a variable kernel satisfying a strong Hörmander condition in the second variable. Hence the classical theory applies (see e.g. [14, p. 30]), and $\mathbf{G}$ admits a bounded extension from $L^p(\mathbb{R}^n; w(x)dx)$ into $L^p_X(\mathbb{R}^n; w(x)dx)$ for all $1 < p < \infty$ and all $w \in A_p(\mathbb{R}^n)$. We observe that the L^p boundedness of $\mathbf{G}$ for $p \geq 2$ asserted in Remark 2.6 is used strongly in order to obtain the full weighted result (see the hypotheses of Th. III.1.2 in [14]).

At this point Theorem 2.5 is completely proved except for Lemma 2.8. We devote the rest of the section to obtain this estimate.

PROOF of Lemma 2.8:

Throughout the proof we shall use the fact that $|x-z| \sim |x-z_0|$, meaning that $c_1|x-z| \leq |x-z_0| \leq c_2|x-z|$ for some constants $c_1, c_2 > 0$ which can be estimated by the triangle inequality.

The main difficulty is to split the domain of integration into a relevant number of regions. We do this as follows:

$$
\begin{aligned}
|\mathbf{K}(x,z)-\mathbf{K}(x,z_0)|_X^2 &= \int_0^\infty \int_{\mathbb{R}^n} \left(1+\tfrac{|x-y|}{t}\right)^{-n\lambda} \left| Q_t(y,z)-Q_t(y,z_0) \right|^2 \tfrac{dydt}{t^{n+1}} \\
&= \int_0^{|z-z_0|/2} \int_{|y-z|>\frac{2}{3}|x-z|} + \int_{|z-z_0|/2}^{\infty} \int_{|y-z|>\frac{2}{3}|x-z|} + \\
&\quad + \int_0^{|z-z_0|/2} \int_{|y-z|\leq\frac{2}{3}|x-z|} + \int_{|z-z_0|/2}^{\frac{2}{3}|x-z|} \int_{|y-z|\leq\frac{2}{3}|x-z|} + \int_{\frac{2}{3}|x-z|}^{\infty} \int_{|y-z|\leq\frac{2}{3}|x-z|} \\
&= I + II + III + IV + V.
\end{aligned}
$$

We start with the first two integrals. Observe that in this region $y \in \mathbb{R}^n \setminus B_{\frac{2}{3}|x-z|}(z)$, and therefore $|y-z| \sim |y-z_0|$. For the first integral we use the crude estimate in (a) of (2.2) and disregard the factor raised to λ:

$$
\begin{aligned}
I &\leq \int_0^{\frac{|z-z_0|}{2}} \int_{|y-z|>\frac{2}{3}|x-z|} \left| Q_t(y,z)-Q_t(y,z_0) \right|^2 \tfrac{dydt}{t^{n+1}} \\
&\lesssim \int_0^{\frac{|z-z_0|}{2}} \int_{|y-z|>\frac{2}{3}|x-z|} t^{-2n} \left(1+\tfrac{|y-z|}{t}\right)^{-N} \tfrac{dydt}{t^{n+1}} \\
&\leq \int_0^{\frac{|z-z_0|}{2}} t^{N-3n} \int_{|y-z|>\frac{2}{3}|x-z|} |y-z|^{-N}\, dy\, \tfrac{dt}{t} \\
&= c\,\frac{|z-z_0|^{N-3n}}{|x-z|^{N-n}} \leq c\,\frac{|z-z_0|^{2\delta}}{|x-z|^{2n+2\delta}},
\end{aligned}
$$

provided we take $N > 3n+2\delta$. To compute the second integral we still disregard the factor raised to λ, but use instead the estimate (b) in (2.2):

$$\begin{aligned} II &\leq \int_{\frac{|z-z_0|}{2}}^{\infty} \int_{|y-z|>\frac{2}{3}|x-z|} \left(\tfrac{|z-z_0|}{t}\right)^2 t^{-2n} \left(1+\tfrac{|y-z|}{t}\right)^{-N} \tfrac{dydt}{t^{n+1}} \\ \text{(choose } N=3n+1\text{)} \quad &\leq |z-z_0|^2 \int_{\frac{|z-z_0|}{2}}^{\infty} t^{-1} \int_{|y-z|>\frac{2}{3}|x-z|} |y-z|^{-(3n+1)}\, dy\, \tfrac{dt}{t} \\ &= c\, \frac{|z-z_0|}{|x-z|^{2n+1}} \leq c\, \frac{|z-z_0|^{2\delta}}{|x-z|^{2n+2\delta}}, \end{aligned}$$

for any $\delta \leq 1/2$. Passing to integrals III, IV and V, observe that in these regions $y \in B_{\frac{2}{3}|x-z|}(z)$ and therefore $|x-y| \sim |x-z|$. So, we shall estimate $(1+\frac{|x-y|}{t})^{-n\lambda} \sim (1+\frac{|x-z|}{t})^{-n\lambda}$, which can be taken outside the integral in dy. As before for III we use estimate (a) to obtain:

$$III \lesssim \int_0^{\frac{|z-z_0|}{2}} \left(\frac{t}{|x-z|}\right)^{n\lambda} \int_{|y-z|\leq\frac{2}{3}|x-z|} t^{-2n} \left[(1+\tfrac{|y-z|}{t})^{-N} + (1+\tfrac{|y-z_0|}{t})^{-N}\right] \tfrac{dydt}{t^{n+1}}.$$

For the integration in dy it is enough to enlarge the domain to $\mathbb{R}^n$, which easily gives

$$\begin{aligned} III &\lesssim |x-z|^{-n\lambda} \int_0^{\frac{|z-z_0|}{2}} t^{n\lambda-2n}\, \tfrac{dt}{t} \\ \text{(use } \lambda>2\text{)} \quad &= c\, \frac{|z-z_0|^{n(\lambda-2)}}{|x-z|^{n\lambda}} \leq c\, \frac{|z-z_0|^{2\delta}}{|x-z|^{2n+2\delta}}, \end{aligned}$$

provided we choose $\delta \leq n(\lambda/2-1)$. To treat IV we can also enlarge the integration in dy to $\mathbb{R}^n$, which using (b) instead of (a) leads to:

$$\begin{aligned} IV &\lesssim \int_{\frac{|z-z_0|}{2}}^{\frac{2}{3}|x-z|} \frac{t^{n(\lambda-2)}}{|x-z|^{n\lambda}} \left(\frac{|z-z_0|}{t}\right)^2 \int_{u\in\mathbb{R}^n} (1+|u|)^{-N}\, du \tfrac{dt}{t} \\ \text{(use } |z-z_0|<2t\text{)} \quad &\leq c\, \frac{|z-z_0|^{2\delta}}{|x-z|^{n\lambda}} \int_0^{\frac{2}{3}|x-z|} t^{n(\lambda-2)-2\delta}\, \tfrac{dt}{t} = c'\, \frac{|z-z_0|^{2\delta}}{|x-z|^{2n+2\delta}}, \end{aligned}$$

provided we choose $\delta < n(\lambda/2-1)$. Finally, V is estimated with (b) but disregarding the λ-factor, which gives

$$\begin{aligned} V &\lesssim \int_{\frac{2}{3}|x-z|}^{\infty} \int_{|y-z|\leq\frac{2}{3}|x-z|} \left(\tfrac{|z-z_0|}{t}\right)^2 t^{-2n} \left(1+\tfrac{|y-z|}{t}\right)^{-N} \tfrac{dydt}{t^{n+1}} \\ &= |z-z_0|^2 \int_{\frac{2}{3}|x-z|}^{\infty} t^{-2n-2} \int_{|u|\leq\frac{2}{3}\frac{|x-z|}{t}} (1+|u|)^{-N}\, du\, \tfrac{dt}{t} \\ &\lesssim |z-z_0|^2 \int_{\frac{2}{3}|x-z|}^{\infty} t^{-2n-2} \left(\tfrac{|x-z|}{t}\right)^n \tfrac{dt}{t} = c\, \frac{|z-z_0|^2}{|x-z|^{2n+2}}, \end{aligned}$$

which is smaller than the desired expression when $\delta \leq 1$. The lemma is now proved with any positive $\delta < \min\{n(\lambda/2-1), 1/2\}$.

□

2.3 Laguerre multipliers for special α's

Theorem 1.6 has an immediate counterpart for Laguerre expansions when $\alpha = \frac{n}{2} - 1$, by using the same transference principle as in [8, Cor. 3.4]. Since it is an important step in this paper, we describe the procedure in some detail in this subsection. The key point is the following formula which relates Laguerre and Hermite functions (see [7, Lemma 1.1]). Below, we use the notation $|\mathbf{k}| = k_1 + \ldots + k_n$ for every multi-index $\mathbf{k} = (k_1, \ldots, k_n) \in \mathbb{N}^n$.

Lemma 2.9 *Let $\alpha = \frac{n-2}{2}$ where $n \in \mathbb{Z}_+$. Then, for some constants $a_{\mathbf{k}} \in \mathbb{R}$ the following formula holds*

$$\mathcal{L}_k^\alpha(|z|^2) = \sum_{|\mathbf{k}|=k} a_{\mathbf{k}}\, \mathfrak{h}_{2\mathbf{k}}(z)\, |z|^\alpha, \quad \forall\, z \in \mathbb{R}^n,\ k = 0, 1, 2, \ldots \tag{2.10}$$

We shall also use the following elementary fact.

Lemma 2.11 *For every $f \in L^1(0, \infty)$ we have*

$$\int_{\mathbb{R}^n} f(|z|^2)\, |z|^{-(n-2)}\, dz \;=\; c_n \int_0^\infty f(t)\, dt. \tag{2.12}$$

Proof: Use first polar coordinates $|z| = r$, and then change variables $r^2 = t$. □

Corollary 2.13 *Theorem 1.8 holds when $\alpha = \frac{n-2}{2}$ and n is a positive integer.*

Proof: Let $m(\xi)$ be as in the statement of Theorem 1.8. The function $M(\boldsymbol{\xi}) = m((\xi_1 + \ldots + \xi_n)/2)$ restricted to the lattice $\mathbb{N}^n$ defines a multiplier $\{M(\mathbf{k})\}$ which satisfies the smoothness conditions in (1.7). This is in fact an easy consequence of the following lemma.

Lemma 2.14 *Let $M \in C^L([0, \infty)^n)$. Then for each $\boldsymbol{\ell} \in \mathbb{N}^n$ with $|\boldsymbol{\ell}| \leq L$ we have*

$$|\Delta^{\boldsymbol{\ell}} M(\mathbf{k})| \;\leq \sup_{\xi_i \in (k_i, k_i+\ell_i)} |D^{(\boldsymbol{\ell})} M(\boldsymbol{\xi})|, \quad \forall\, \mathbf{k} \in \mathbb{N}^n. \tag{2.15}$$

Proof: When $n = 1$ one has the formula

$$\Delta^\ell M(k) \;=\; \int_0^1 \int_{s_1}^{s_1+1} \cdots \int_{s_{\ell-1}}^{s_{\ell-1}+1} D^{(\ell)} M(s_\ell + k)\, ds_\ell \ldots ds_1, \quad k \geq 0, \tag{2.16}$$

which can be easily verified by induction on ℓ. In $\mathbb{R}^n$, by repeated composition of (2.16) one can represent $\Delta^{\boldsymbol{\ell}} M(\mathbf{k})$ in terms of a similar integral, from which (2.15) is obtained easily. □

Continuing with the proof of Corollary 2.13, we can use (2.10) to write

$$(\mathcal{T}_m f)(|z|^2) \;=\; \sum_{k=0}^{\infty} \sum_{|\mathbf{k}|=k} m(k)\,\langle f, \mathcal{L}_k^\alpha\rangle\, a_{\mathbf{k}}\, \mathfrak{h}_{2\mathbf{k}}(z)\, |z|^\alpha, \quad z \in \mathbb{R}^n.$$

Then, changing variables as in (2.12) and using (2.10) we have

$$\begin{aligned}
\|\mathcal{T}_m f\|^p_{L^p(w)} &= \int_0^\infty |(\mathcal{T}_m f)(t)|^p\, w(t)\, dt \\
&= c_n \int_{\mathbb{R}^n} \Big|\sum_{k=0}^{\infty} \sum_{|\mathbf{k}|=k} M(2\mathbf{k})\,\langle f, \mathcal{L}_k^\alpha\rangle\, a_{\mathbf{k}}\, \mathfrak{h}_{2\mathbf{k}}(z)\Big|^p\, |z|^{\alpha p-(n-2)}\, w(|z|^2)\, dz \\
&\leq c' \int_{\mathbb{R}^n} \Big|\sum_{k=0}^{\infty} \sum_{|\mathbf{k}|=k} \langle f, \mathcal{L}_k^\alpha\rangle\, a_{\mathbf{k}}\, \mathfrak{h}_{2\mathbf{k}}(z)\Big|^p\, |z|^{(n-2)(\frac{p}{2}-1)}\, w(|z|^2)\, dz \\
&= c''\, \|f\|^p_{L^p(w)}.
\end{aligned}$$

Of course, in the inequality we are using Theorem 1.6, for which we have the required smoothness on $\{M(\mathbf{k})\}$ but we also need

$$|z|^{(n-2)(\frac{p}{2}-1)}\, w(|z|^2) \in A_p(\mathbb{R}^n).$$

Now it is well-known that $|z|^\gamma \in A_p(\mathbb{R}^n)$ if and only if $-n < \gamma < n(p-1)$. Recall that we are interested in the case $w(y) = y^{p\delta}$. Therefore, writing $\gamma = (n-2)(\frac{p}{2}-1) + 2p\delta$ we easily see that the above condition is equivalent to $-\frac{\alpha}{2} - \frac{1}{p} < \delta < 1 - \frac{1}{p} + \frac{\alpha}{2}$, establishing the result.

□

REMARK 2.17 Observe from Theorem 1.6 that, letting $\alpha = \frac{n-2}{2}$ the corollary also holds for multipliers $m \in C^{2\alpha+3}[0,\infty)$ which satisfy the hypothesis (1.9) whenever $\ell \leq 2\alpha + 3$.

3 Multipliers for Laguerre expansions

In this section we prove Theorem 1.8. Recall that the cases $\alpha \leq 0$ in Theorem 1.8 were already proved by Stempak and Trebels (see Theorem 1.1 and Corollary 4.3 in [18]). We shall concentrate mainly in $\alpha > 0$, which is also what produces the new results in Theorem 1.4 (see however Remark 3.23). For later use of complex interpolation, it is important to fix throughout the paper the inner product notation $\langle f, g\rangle = \int f\,\overline{g}$.

The strategy is to obtain the result from the special cases in Corollary 2.13, by interpolation of the analytic family of operators

$$\mathcal{T}_m^z f = \sum_{k=0}^{\infty} m_k \langle f, \mathcal{L}_k^{\overline{z}}\rangle\, \mathcal{L}_k^z, \quad \text{where } z \in \mathbb{C} \text{ with } \Re e\, z > -1. \tag{3.1}$$

In order to give a precise meaning to this expression and make the whole argument work, we first need to recall the definition of Kanjin's operators $T_\alpha^{\alpha+i\theta}$ and extend its boundedness to the full range of $L^p_\delta(\mathbb{R}_+)$.

Throughout this section we shall use the following notation from [10]. We write $M(\theta)$ for any function of the form $M(\theta) = (1+|\theta|)^N\, e^{c|\theta|}$ for suitably large constants N and c. Other constants appearing in the paper such as C, c or N may depend (continuously) on α, p and δ, but are independent of $\theta \in \mathbb{R}$. Finally, it is also convenient to denote the admissible range of indices by

$$\mathcal{A} = \big\{ (\tfrac{1}{p}, \alpha, \delta) \in (0,1) \times (-1,\infty) \times \mathbb{R} \;:\; -\tfrac{\alpha}{2} - \tfrac{1}{p} < \delta < 1 - \tfrac{1}{p} + \tfrac{\alpha}{2} \big\} \tag{3.2}$$

(see Figure 1.1).

3.1 Boundedness of $T_\alpha^{\alpha+i\theta}$ in $L^p_\delta(\mathbb{R}_+)$ for special $\boldsymbol{\alpha}$'s

Recall from [10, p. 539] that Laguerre polynomials can be extended to complex parameters $z \in \mathbb{C}$ with $\Re e\, z > -1$ by the formula

$$L_k^{(z)}(y) \;=\; \frac{D_y^{(k)}[y^{z+k}e^{-y}]}{k!\, y^z\, e^{-y}} \;=\; \sum_{j=0}^{k} \frac{\Gamma(k+z+1)}{\Gamma(k-j+1)\Gamma(j+z+1)}\, \frac{(-y)^j}{j!}, \quad y>0,$$

and likewise for the corresponding Laguerre functions

$$\mathcal{L}_k^z(y) \;=\; \big(\tfrac{\Gamma(k+1)}{\Gamma(z+k+1)}\big)^{\frac12}\, y^{z/2}\, e^{-y/2}\, L_k^{(z)}(y), \quad y>0.$$

Moreover, the following lemma due to Kanjin holds (see [10, Lemma 1]).

Lemma 3.3 *Let $\alpha > -1$ and $f \in C_c^\infty(0,\infty)$. Then, for each $N \geq 1$ there exist constants $C > 0$ and $k_0 \in \mathbb{N}$ (depending on N, f and α) such that*

$$\big|\langle f, \mathcal{L}_k^{\alpha+i\theta}\rangle\big| \;\leq\; C\,(1+|\theta|)^{4N+\alpha}\, e^{\frac{\pi}{2}|\theta|}\,(1+|k|)^{-N}, \quad k \geq k_0, \tag{3.4}$$

for all $\theta \in \mathbb{R}$.

Using this lemma one can define the complex transplantation operators

$$T_\alpha^z f = \sum_{k=0}^{\infty} \langle f, \mathcal{L}_k^z\rangle\, \mathcal{L}_k^\alpha, \quad \Re e\, z > -1, \quad \alpha > -1,$$

at least for functions $f \in C_c^\infty(\mathbb{R}_+)$. Kanjin has shown the boundedness of $T_\alpha^{\alpha+i\theta}$ in $L^p(\mathbb{R}_+)$ for all $1 < p < \infty$ and $\alpha \geq 0$ (see [10, Prop. 2]). Stempak and Trebels extended the result to the weighted spaces L^p_δ for $\alpha \geq 0$ and $\max\{-\frac{1}{p}, -\frac12\} < \delta < \min\{1-\frac{1}{p}, \frac12\}$ (see [18, Prop. 4.2]). The purpose of this section is to improve the range of validity of such result to all $\alpha > -1$ and all admissible weights $-\frac{\alpha}{2} - \frac{1}{p} < \delta < 1 - \frac{1}{p} + \frac{\alpha}{2}$.

Theorem 3.5 *Let $\alpha > -1$ and $\theta \in \mathbb{R}$. Then, $T_\alpha^{\alpha+i\theta}$ can be boundedly extended to $L^p_\delta(\mathbb{R}_+)$ for all $1 < p < \infty$ and $-\frac{\alpha}{2} - \frac{1}{p} < \delta < 1 - \frac{1}{p} + \frac{\alpha}{2}$. Moreover, there exist constants $C, c > 0$ and $N \in \mathbb{N}$ (depending only on α, p, δ) so that*

$$\|T_\alpha^{\alpha+i\theta} f\|_{L^p_\delta} \le C\,(1+|\theta|)^N\, e^{c|\theta|}\, \|f\|_{L^p_\delta}, \quad \forall\, \theta \in \mathbb{R}. \tag{3.6}$$

The proof of the theorem will follow the scheme proposed by Kanjin in [10], except for a few refinements leading to the new results. For every $\alpha > -1$ and $\theta \in \mathbb{R}$ we define a multiplier by

$$\lambda(\xi) = \lambda_{\alpha,\theta}(\xi) = \Big(\frac{\Gamma(\xi+\alpha+1+i\theta)}{\Gamma(\xi+\alpha+1)}\Big)^{\frac{1}{2}}, \quad \xi \ge 0. \tag{3.7}$$

Observe that λ is an analytic function of ξ when $\Re e\,\xi > -1-\alpha$. The following result is a slight modification of Lemma 2 in [10], which is valid with exactly the same proof.

Lemma 3.8 *Let $\alpha > -1$. Then the function $\lambda(\xi)$ defined in (3.7) belongs to $C^\infty[0,\infty)$ and satisfies*

$$\sup_{\xi\in[0,\infty)} (1+|\xi|)^\ell\, |D^\ell \lambda(\xi)| \le C_\ell\,(1+|\theta|)^\ell, \quad \forall\, \theta \in \mathbb{R}, \quad \ell = 0,1,2,\dots$$

where the constant C_ℓ is independent of θ.

We shall prove Theorem 3.5 under the following assumption on $(\frac{1}{p}, \alpha, \delta)$.

Assumption (A): *The point $(\frac{1}{p}, \alpha, \delta) \in \mathcal{A}$ is so that the multiplier operator $\mathcal{T}_\lambda f = \sum_{k=0}^\infty \lambda(k)\langle f, \mathcal{L}_k^\alpha\rangle \mathcal{L}_k^\alpha$, with $\lambda = \lambda_{\alpha,\theta}$ as in (3.7), is bounded on $L^p_\delta(\mathbb{R}_+)$ and moreover*

$$\|\mathcal{T}_\lambda f\|_{L^p_\delta} \le C\,(1+|\theta|)^N\, e^{c|\theta|}\, \|f\|_{L^p_\delta}, \quad \forall\, \theta \in \mathbb{R} \tag{A}$$

for some constants $C, c, N > 0$.

Remark 3.9 Observe that, by Corollary 2.13 and Lemma 3.8, Assumption (A) is already known to hold for parameters in $\mathcal{A}$ of the form $(\frac{1}{p}, \frac{n-2}{2}, \delta)$, whenever $n \in \mathbb{Z}_+$. Moreover, the assumption also holds trivially for $(\frac{1}{2}, \alpha, 0)$ and all $\alpha > -1$, while by duality it holds for a fixed $(\frac{1}{p}, \alpha, \delta)$ if and only if it does for $(\frac{1}{p'}, \alpha, -\delta)$. Finally, as we observed before, Assumption (A) holds for all $(\frac{1}{p}, \alpha, \delta) \in \mathcal{A}$ with $\alpha \le 0$, by the results of Stempak and Trebels in [18].

Clearly, under Assumption (A) it suffices to show (3.6) with $T_\alpha^{\alpha+i\theta}$ replaced by the operator

$$\widetilde{T}_\alpha^{\alpha+i\theta} f = \sum_{k=0}^\infty \big(\tfrac{\Gamma(k+\alpha+1)}{\Gamma(k+\alpha+1+i\theta)}\big)^{\frac{1}{2}} \langle f, \mathcal{L}_k^{\alpha+i\theta}\rangle \mathcal{L}_k^\alpha.$$

This new write up of $T_\alpha^{\alpha+i\theta}$ is due to Kanjin and it leads to a remarkable explicit formula in terms of an oscillatory integral. More precisely, following [10, §3] we can define for $\varepsilon > 0$ the operators

$$G_{\theta,\varepsilon}(f) \;=\; \sum_{k=0}^{\infty} \big(\tfrac{\Gamma(k+\alpha+1)}{\Gamma(k+\alpha+1+\varepsilon+i\theta)}\big)^{\frac12} \langle f, \mathcal{L}_k^{\alpha+\varepsilon+i\theta}\rangle \mathcal{L}_k^\alpha$$

so that $\widetilde{T}_\alpha^{\alpha+i\theta} f(x) \;=\; \lim_{\varepsilon\to 0} G_{\theta,\varepsilon} f(x)$, for all $x > 0$, at least when $f \in C_c^\infty(0,\infty)$ (by Lemma 3.3). Moreover, the following remarkable formula holds [10, (3.10)]:

$$G_{\theta,\varepsilon} f(x) \;=\; \tfrac{1}{\Gamma(\varepsilon+i\theta)} \int_x^\infty f(t)\, e^{-\frac{t-x}{2}} \big(1-\tfrac{x}{t}\big)^{\varepsilon-1+i\theta} \big(\tfrac{x}{t}\big)^{\frac{\alpha}{2}}\, t^{\frac{\varepsilon+i\theta}{2}}\, \tfrac{dt}{t}. \tag{3.10}$$

The rest of this section is devoted to the proof of the following proposition.

Proposition 3.11 *Let $\alpha > -1$ and p, δ so that $\delta > -\frac1p - \frac{\alpha}{2}$. Then, there exist constants $C, c > 0$ and $N \in \mathbb{N}$ (depending only on α, p, δ) so that*

$$\|G_{\theta,\varepsilon} f\|_{L^p_\delta} \;\le\; C\,(1+|\theta|)^N\, e^{c|\theta|} \,\big(\, \|f(x) x^{\frac{\varepsilon}{2}}\|_{L^p_\delta} + \|f(x) x^{-\frac{\varepsilon}{2}}\|_{L^p_\delta}\big), \tag{3.12}$$

for all $\theta \in \mathbb{R}$ and all $0 < \varepsilon \le 1$.

Remark 3.13 We remark that under Assumption (A), Theorem 3.5 follows immediately from the last proposition and Fatou's lemma. Indeed, using these facts we have

$$\begin{aligned}
\|T_\alpha^{\alpha+i\theta} f\|_{L^p_\delta} \;&=\; \|\mathcal{T}_\lambda \widetilde{T}_\alpha^{\alpha+i\theta} f\|_{L^p_\delta} \lesssim M(\theta)\, \|\widetilde{T}_\alpha^{\alpha+i\theta} f\|_{L^p_\delta} \\
&\le\; M(\theta)\, \varliminf_{\varepsilon\to 0}\, \|G_{\theta,\varepsilon} f\|_{L^p_\delta} \\
&\lesssim\; M(\theta)\, \varliminf_{\varepsilon\to 0} \big(\|f(x) x^{\frac{\varepsilon}{2}}\|_{L^p_\delta} + \|f(x) x^{-\frac{\varepsilon}{2}}\|_{L^p_\delta}\big) \\
&=\; M(\theta)\, \|f\|_{L^p_\delta}, \quad f \in C_c^\infty(0,\infty),
\end{aligned}$$

where in the last step we have implicitly used that the constants in Proposition 3.11 are independent of ε.

Proof of Proposition 3.11:

As noticed by Kanjin, $|\Gamma(\varepsilon+i\theta)|^{-1} \lesssim (1+|\theta|) e^{\frac{\pi|\theta|}{2}}$ (see [10, p. 547]), so in the rest of the proof we only look at the integral defining $G_{\theta,\varepsilon} f(x)$ in (3.10). We shall prove (3.12) by splitting this integral into "local" and "global" parts: $\int_x^{2x}$ and $\int_{2x}^\infty$. The last part can be crudely estimated by a positive operator, since no singularity is present there:

$$\begin{aligned}
\Big| \int_{2x}^\infty f(t)\, e^{-\frac{t-x}{2}} \big(1-\tfrac{x}{t}\big)^{\varepsilon-1+i\theta} \big(\tfrac{x}{t}\big)^{\frac{\alpha}{2}}\, t^{\frac{\varepsilon+i\theta}{2}}\, \tfrac{dt}{t} \Big| \;&\le\; C\, x^{\frac{\alpha}{2}}\, e^{x/2} \int_{2x}^\infty |f(t)|\, e^{-t/2}\, t^{\frac{\varepsilon-\alpha}{2}}\, \tfrac{dt}{t} \\
&=:\; G_\varepsilon^1 f(x),
\end{aligned}$$

where we used $t > 2x$ to control $1 - \frac{x}{t} \ge \frac12$. The next lemma takes care of this part.

LEMMA 3.14 *Let $\delta > -\frac{1}{p} - \frac{\alpha}{2}$. Then there exists a constant $C > 0$, independent of $\varepsilon \in (0,1]$, so that*

$$\left\|G^1_\varepsilon f\right\|_{L^p_\delta} \le C \left\| \frac{f(x)x^{\varepsilon/2}}{1+x} \right\|_{L^p_\delta}, \quad \forall f \in C^\infty_c(0,\infty). \tag{3.15}$$

PROOF:

Let $\gamma \in \mathbb{R}$ be a fixed number to be specified later. Multiplying and dividing by t^γ inside the integral defining $G^1_\varepsilon f(x)$, and using Hölder's inequality we have

$$\left\|G^1_\varepsilon f\right\|^p_{L^p_\delta} \le \int_0^\infty x^{(\frac{\alpha}{2}+\delta)p}\, e^{\frac{px}{2}} \int_{2x}^\infty |f(t)|^p\, t^{\gamma p}\, e^{-\frac{t}{2}}\, t^{\frac{\varepsilon-\alpha}{2}}\, \frac{dt}{t} \left[\int_{2x}^\infty s^{-\gamma p'}\, e^{-\frac{s}{2}}\, s^{\frac{\varepsilon-\alpha}{2}}\, \frac{ds}{s}\right]^{\frac{p}{p'}} dx.$$

The integral inside the brackets can easily be estimated (separating the cases $x \ge 1$ and $x \le 1$) by

$$c_\varepsilon\, x^{-\gamma p'}\, x^{\frac{\varepsilon-\alpha}{2}}\, \frac{e^{-x/2}}{1+x},$$

provided we have $-\gamma p' + \frac{\varepsilon-\alpha}{2} < 0$. Observe that growth of the constant c_ε is of the order $1/(\gamma p' - \frac{\varepsilon-\alpha}{2})$. Inserting this expression to the power p/p' in the above inequality, and using Fubini we have

$$\begin{aligned} \left\|G^1_\varepsilon f\right\|^p_{L^p_\delta} &\lesssim \int_0^\infty |f(t)|^p\, t^{\gamma p}\, e^{-\frac{t}{2}}\, t^{\frac{\varepsilon-\alpha}{2}} \int_0^{t/2} x^{-\gamma p}\, x^{\frac{\varepsilon-\alpha}{2}\frac{p}{p'}}\, \frac{e^{\frac{x}{2}}}{(1+x)^{p-1}}\, x^{(\frac{\alpha}{2}+\delta)p}\, dx\, \frac{dt}{t} \\ &= \int_0^1 \dots \tfrac{dt}{t} + \int_1^\infty \dots \tfrac{dt}{t} = I_1 + I_2. \end{aligned} \tag{3.16}$$

In the first case we can easily estimate the integral in dx, provided that the exponent $\kappa := -\gamma p + \frac{\varepsilon-\alpha}{2}\frac{p}{p'} + (\frac{\alpha}{2}+\delta)p + 1 > 0$. This leads to

$$\begin{aligned} I_1 &\le c'_\varepsilon \int_0^1 |f(t)|^p\, t^{\gamma p}\, t^{\frac{\varepsilon-\alpha}{2}} \left[t^{-\gamma p + \frac{\varepsilon-\alpha}{2}\frac{p}{p'} + (\frac{\alpha}{2}+\delta)p + 1}\right] \frac{dt}{t} \\ &= c'_\varepsilon \int_0^1 |f(t)\, t^{\frac{\varepsilon}{2}+\delta}|^p\, dt. \end{aligned}$$

Here the constant c'_ε is of the order $1/\kappa$. We can estimate I_2 similarly, except that the integral in dx takes a different form, leading to

$$\begin{aligned} I_2 &\le c''_\varepsilon \int_1^\infty |f(t)|^p\, t^{\gamma p}\, e^{-\frac{t}{2}}\, t^{\frac{\varepsilon-\alpha}{2}} \left[t^{-\gamma p + \frac{\varepsilon-\alpha}{2}\frac{p}{p'} + (\frac{\alpha}{2}+\delta)p}\, \frac{e^{t/2}}{(1+t)^{p-1}}\right] \frac{dt}{t} \\ &\le 2c''_\varepsilon \int_1^\infty \left| f(t)\, \frac{t^{\frac{\varepsilon}{2}+\delta}}{1+t}\right|^p dt, \end{aligned}$$

again provided $\kappa > 0$ and with $c''_\varepsilon \lesssim 1/\kappa$. Therefore, for all these computations to be valid we only need to choose $\gamma \in \mathbb{R}$ so that

$$\tfrac{\varepsilon-\alpha}{2}\tfrac{1}{p'} < \gamma < \tfrac{\varepsilon-\alpha}{2}\tfrac{1}{p'} + (\tfrac{\alpha}{2}+\delta) + \tfrac{1}{p}.$$

This is clearly always possible when $\delta > -\frac{1}{p} - \frac{\alpha}{2}$. Moreover, choosing γ close to the right hand point all the constants c_ε, c'_ε and c''_ε are bounded by C independently of ε. Thus, inserting the previous estimates for I_1 and I_2 in (3.16) we obtain (3.15).

□

Going back to (3.12), it remains to look at the part of $G_{\theta,\varepsilon}f(x)$ defined by the local integral $\int_x^{2x}$. Proceeding as in [10, p. 547], we add and subtract 1 to the factor $(\frac{x}{t})^{\frac{\alpha}{2}}$, so that we can write

$$\begin{aligned} \Big|\int_x^{2x} \dots dx\Big| \;&\le\; \int_x^{2x} e^{-\frac{t-x}{2}}\,|f(t)|\,\big|1-\tfrac{x}{t}\big|^{\varepsilon-1}\,\big|(\tfrac{x}{t})^{\frac{\alpha}{2}}-1\big|\,t^{\frac{\varepsilon}{2}}\,\tfrac{dt}{t} \;+ \\ &\quad + \Big|\int_x^{2x} f(t)\,e^{-\frac{t-x}{2}}\,(t-x)^{\varepsilon-1+i\theta}\,t^{-\frac{\varepsilon+i\theta}{2}}\,dt\Big| \\ &=:\; G^2_\varepsilon f(x) \,+\, \big|\,G^3_\varepsilon f(x)\,\big|\,. \end{aligned} \tag{3.17}$$

Using the Taylor expansion $(1+w)^\gamma = 1+\gamma w+O(|w|^2)$, valid for all $\gamma\in\mathbb{R}$ when $|w|\le 1/2$, we must have that

$$\big(\tfrac{x}{t}\big)^{\frac{\alpha}{2}} - 1 \;=\; \tfrac{\alpha}{2}\,\big(\tfrac{x}{t}-1\big) \,+\, O\big(\tfrac{x}{t}-1\big)^2, \quad t\in(x,2x).$$

Thus, in the first integral we can kill the singularity, since

$$\big|1-\tfrac{x}{t}\big|^{\varepsilon-1}\,\big|(\tfrac{x}{t})^{\frac{\alpha}{2}}-1\big| \;\le\; \tfrac{\alpha}{2}\,\big|1-\tfrac{x}{t}\big|^{\varepsilon} \,+\, O\big(1-\tfrac{x}{t}\big)^{1+\varepsilon} \;\le\; C, \quad t\in(x,2x).$$

Bounding as well the exponential by 1 we obtain

$$G^2_\varepsilon f(x) \;\lesssim\; \int_x^{2x} |f(t)|\,t^{\frac{\varepsilon}{2}-1}\,dt \;\lesssim\; x^{\frac{\varepsilon}{2}-1}\int_x^{2x}|f(t)|\,dt.$$

It is now easy to compute the L^p_δ-norm of these expressions:

$$\begin{aligned} \int_0^\infty \big|G^2_\varepsilon f(x)\,x^\delta\big|^p\,dx \;&\lesssim\; \int_0^\infty \Big|x^{\frac{\varepsilon}{2}-1+\delta}\int_x^{2x}|f(t)|\,dt\Big|^p\,dx \\ \text{(by Hölder)}\;&\le\; \int_0^\infty x^{(\frac{\varepsilon}{2}-1+\delta)p}\int_x^{2x}|f(t)|^p\,dt\,x^{p/p'}\,dx \\ \text{(by Fubini)}\;&\le\; \int_0^\infty |f(t)|^p\int_{t/2}^t x^{(\frac{\varepsilon}{2}-1+\delta)p}\,x^{p-1}\,dx\,dt \\ &\lesssim\; \int_0^\infty |f(t)|^p\,t^{(\frac{\varepsilon}{2}+\delta)p}\,dt. \end{aligned}$$

As noticed by Kanjin, the remaining term $G^3_\varepsilon f(x)$ in (3.17) can be expressed in terms of a singular integral kernel. Namely, letting

$$K_{\varepsilon,\theta}(u) = e^{-\frac{|u|}{2}}\,|u|^{\varepsilon-1+i\theta}\chi_{(-\infty,0)}(u), \quad u\in\mathbb{R},$$

it is easily verified that

$$|K_{\varepsilon,\theta}(u)| \leq \frac{C}{|u|} \quad \text{and} \quad |K'_{\varepsilon,\theta}(u)| \leq C\,\frac{1+|\theta|}{|u|^2}, \quad \forall\, u \neq 0,$$

with a constant C independent of $\theta \in \mathbb{R}$ and $\varepsilon \in (0,1]$. Moreover, letting $g(t) = f(t)t^{-\frac{\varepsilon+i\theta}{2}}$, we can write

$$G^3_\varepsilon f(x) \,=\, \int_x^{2x} g(t)\, K_{\varepsilon,\theta}(x-t)\, dt, \quad x > 0. \tag{3.18}$$

The right hand side of (3.18) will then be a *local Calderón-Zygmund operator in* $\mathbb{R}_+$ (in the sense of Nowak and Stempak [13]) if we can show the following lemma.

Lemma 3.19 *There exists a constant C independent of $\varepsilon \in (0,1]$ so that*

$$\Big[\int_0^\infty \Big|\int_x^{2x} g(t)\, K_{\varepsilon,\theta}(x-t)\, dt\Big|^2 dx\Big]^{\frac12} \,\leq\, C\,(1+|\theta|)e^{\frac{\pi}{2}|\theta|}\,\|g\|_2, \quad \forall\, g \in C_c^\infty(0,\infty).$$

Assuming the lemma, we can use Theorem 4.3 in [13] to obtain

$$\begin{aligned}\big\|G^3_\varepsilon f\big\|_{L^p_\delta} \;&=\; \Big[\int_0^\infty \Big|\int_x^{2x} g(t)\, K_{\varepsilon,\theta}(x-t)\, dt\Big|^p x^{p\delta}\, dx\Big]^{\frac1p} \\ &\lesssim\; M(\theta)\,\|g\|_{L^p_\delta} \,=\, M(\theta)\,\big\|f(t)t^{-\frac{\varepsilon}{2}}\big\|_{L^p_\delta}.\end{aligned}$$

This argument is valid for all $\delta \in \mathbb{R}$ and $1 < p < \infty$, since power weights x^δ always belong to the local Muckenhoupt classes $A^p_{loc}(0,\infty)$ (see [13]). Thus, the proof of Proposition 3.11 will be finished once we establish Lemma 3.19.

Proof of Lemma 3.19:

For each $x > 0$ we write

$$\int_x^{2x} g(t)\, K_{\varepsilon,\theta}(x-t)\, dt \,=\, \int_x^\infty \ldots\, dt \;-\; \int_{2x}^\infty \ldots\, dt \;=:\; T_1 g(x) \,+\, T_2 g(x).$$

Observe that $T_1 g(x) = g * K_{\varepsilon,\theta}(x)$, and as was shown by Kanjin [10, p. 547] we have

$$\sup_{\xi \in \mathbb{R}} \big|\hat{K}_{\varepsilon,\theta}(\xi)\big| \,\leq\, C\, e^{\frac{\pi}{2}|\theta|}.$$

Thus, extending $g \equiv 0$ in $(-\infty, 0)$ and using Plancherel, it immediately follows that

$$\|T_1 g\|_2 \,\leq\, C\, e^{\frac{\pi}{2}|\theta|}\, \|g\|_2.$$

To estimate the term $T_2 g(x)$, we use Lemma 3.14 with $\alpha = 0$ to obtain

$$\begin{aligned}\int_0^\infty \big|T_2 g(x)\big|^2 dx \;&\leq\; \int_0^\infty \Big|\int_{2x}^\infty |g(t)|\, e^{-\frac{t-x}{2}}\,(t-x)^{\varepsilon-1}\, dt\Big|^2 dx \\ \text{(since } t-x \geq t/2) \;&\lesssim\; \int_0^\infty \Big|\int_{2x}^\infty |t^{\frac{\varepsilon}{2}} g(t)|\, e^{-\frac{t-x}{2}}\, t^{\frac{\varepsilon}{2}-1}\, dt\Big|^2 dx \\ \text{(Lemma 3.14 with } \alpha = 0) \;&\leq\; C\,\Big\|\frac{t^\varepsilon g(t)}{1+t}\Big\|_2^2 \,\leq\, C\,\|g\|_2^2,\end{aligned}$$

whenever $\varepsilon \leq 1$.

$\square$

3.2 Proof of Theorem 1.8

We may assume $\alpha > 0$, since the cases $\alpha \leq 0$ are contained in [18]. We shall obtain Theorem 1.8 by complex interpolation from Corollary 2.13 and the knowledge we presently have of Theorem 3.5. That is, the L^p_δ boundedness of $T_\alpha^{\alpha+i\theta}$ when $\alpha = \frac{n-2}{2}$, and also the L^2 boundedness for all $\alpha > -1$ (see Remarks 3.9 and 3.13).

Lemma 3.20 *Let $P_0 = (\frac{1}{p_0}, \alpha_0, \delta_0)$ and $P_1 = (\frac{1}{p_1}, \alpha_1, \delta_1)$ be two fixed points in $\mathcal{A}$ for which Theorem 1.8 is known to hold*. Then the theorem must also hold for all points $P = (\frac{1}{p}, \alpha, \delta)$ of the form*

$$P = (1-t)P_0 + tP_1, \quad t \in (0,1). \tag{3.21}$$

Proof: We shall use the convenient notation

$$\alpha(z) = (1-z)\alpha_0 + z\alpha_1 \quad \text{and} \quad \delta(z) = (1-z)\delta_0 + z\delta_1,$$

for complex $z = s + i\theta$ such that $0 \leq s \leq 1$. Recall that $M(\theta)$ denotes a function of the form $M(\theta) = (1+|\theta|)^N e^{c|\theta|}$ for suitably large constants N and c. Also, observe from Lemma 3.3 and Remarks 3.9 and 3.13 that the operator

$$\mathcal{T}_m^{\sigma+i\tau} f = \sum_{k=0}^{\infty} m(k) \langle f, \mathcal{L}_k^{\sigma-i\tau} \rangle \, \mathcal{L}_k^{\sigma+i\tau} \; = \; (T_\sigma^{\sigma+i\tau})^* \, \mathcal{T}_m^\sigma \, T_\sigma^{\sigma-i\tau} f$$

is well-defined and bounded at least when $f \in L^2(\mathbb{R}_+)$. We define an analytic family of operators by letting

$$S_z F(y) \; = \; y^{\delta(z)} \, \mathcal{T}_m^{\alpha(z)} (F(x) x^{-\delta(z)})(y),$$

at least for $F \in L^2_c(0,\infty)$. We must show that $\{S_z\}$ satisfies the conditions of Stein's interpolation theorem (see [3]). First of all, given any two subsets E_1, E_2 compactly contained in $(0,\infty)$, the function

$$z \longmapsto \Phi(z) = \langle S_z(\chi_{E_1}), \chi_{E_2} \rangle,$$

defined whenever $0 \leq \Re\mathfrak{e}\, z \leq 1$, satisfies

$$\begin{aligned} \big|\Phi(z)\big| \;&\leq\; \big\|\mathcal{T}_m^{\alpha(z)} (x^{-\delta(z)} \chi_{E_1})\big\|_2 \, \big\|y^{\delta(z)} \chi_{E_2}\big\|_2 \\ &\leq\; C_{E_2} \big\|(T_{\alpha(s)}^{\alpha(s)+i(\alpha_1-\alpha_0)\theta})^* \, \mathcal{T}_m^{\alpha(s)} \, T_{\alpha(s)}^{\alpha(s)-i(\alpha_1-\alpha_0)\theta} \, (x^{-\delta(z)} \chi_{E_1})\big\|_2 \\ &\leq\; C_{E_2} \, M(\theta) \, \big\|x^{-\delta(z)} \chi_{E_1}\big\|_2 \leq C_{E_1} C_{E_2} \, M(\theta), \end{aligned} \tag{3.22}$$

*These may be any of points we discussed in Remark 3.9

by the L^2 boundedness of $T_\sigma^{\sigma+i\tau}$, $\forall\ \sigma > -1$. We next show that Φ is holomorphic in a neighborhood of the strip $\overline{S} := \{0 \leq \Re z \leq 1\}$. Since $\|\mathcal{T}_m^{\alpha(z)}\|_{L^2\to L^2}$ is uniformly bounded in compact sets of $\overline{S}$, by a standard approximation argument (with estimates similar to (3.22)) it will suffice to show the holomorphy of $z \mapsto \langle S_z F, G\rangle$ for all $F, G \in C_c^\infty(0,\infty)$. Now, if we denote $f(x) = x^{-\delta(z)}F(x)$, $g(y) = y^{\delta(\overline{z})}G(y)$ and $\alpha(z) = \sigma + i\tau$, we can write

$$
\begin{aligned}
\langle S_z F, G\rangle &= \langle \mathcal{T}_m^{\alpha(z)}(f), g\rangle = \langle \mathcal{T}_m^{(\sigma)}\, T_\sigma^{\sigma-i\tau}(f), T_\sigma^{\sigma+i\tau}(g)\rangle \\
&= \sum_k m_k\, \langle f, \mathcal{L}_k^{\sigma-i\tau}\rangle\, \overline{\langle g, \mathcal{L}_k^{\sigma+i\tau}\rangle} \\
&= \sum_k m_k\, \langle x^{-\delta(z)}F, \mathcal{L}_k^{\alpha(\overline{z})}\rangle\, \langle y^{\delta(z)}\overline{G}, \mathcal{L}_k^{\alpha(\overline{z})}\rangle\,.
\end{aligned}
$$

Since, by Lemma 3.4, the series converges uniformly when z belongs to a compact set of $\overline{S}$, it suffices to show the holomorphy of the map

$$
z \in \overline{S} \;\mapsto\; \langle x^{\pm\delta(z)}F, \mathcal{L}_k^{\alpha(\overline{z})}\rangle \;=\; \int_0^\infty x^{\pm\delta(z)}F(x)\,\mathcal{L}_k^{\alpha(z)}(x)\,dx,
$$

for all $F \in C_c^\infty(0,\infty)$. But this is simple a consequence of the holomorphy and uniform boundedness of the integrand in the (compact) support of F.

Combining this with (3.22) we see that Φ is holomorphic in the strip $\{0 < \Re z < 1\}$, continuous in the closure and has admissible growth for complex interpolation. To verify the conditions of Stein's interpolation theorem we only need to show the boundedness of the operator S_z at the limiting bands

$$
S_{i\theta}\colon L^{p_0}(\mathbb{R}_+) \longrightarrow L^{p_0}(\mathbb{R}_+) \quad \text{and} \quad S_{1+i\theta}\colon L^{p_1}(\mathbb{R}_+) \longrightarrow L^{p_1}(\mathbb{R}_+).
$$

When $\Re z = 0$ we use the assumption that Theorem 1.8 (and hence Assumption (A) in §3.1) hold for the point P_0. Then, both $\mathcal{T}_m^{\alpha_0}$ and $T_{\alpha_0}^{\alpha_0+i\tau}$ are bounded in $L^{p_0}_{\delta_0}$ (and in $L^{p_0'}_{-\delta_0}$, by Remark 3.9), which implies

$$
\begin{aligned}
\|S_{i\theta}F\|_{p_0} &= \big\|(T_{\alpha_0}^{\alpha_0+i(\alpha_1-\alpha_0)\theta})^*\, \mathcal{T}_m^{\alpha_0}\, T_{\alpha_0}^{\alpha_0-i(\alpha_1-\alpha_0)\theta}\,(x^{-\delta(i\theta)}F)\big\|_{L^{p_0}_{\delta_0}} \\
&\leq M(\theta)\,\|x^{-\delta_0-i(\delta_1-\delta_0)\theta}F(x)\|_{L^{p_0}_{\delta_0}} = M(\theta)\,\|F\|_{p_0}.
\end{aligned}
$$

One proves similarly the boundedness for $\Re z = 1$. Thus, by Stein's theorem S_s must be bounded in $L^{p_s}(\mathbb{R}_+)$ for $\frac{1}{p_s} = \frac{1-s}{p_0} + \frac{s}{p_1}$ and all $s \in (0,1)$. Letting $s = t$ and using (3.21) we see that $p_t = p$, $\alpha(t) = \alpha$ and $\delta(t) = \delta$. Moreover such boundedness translates into

$$
\begin{aligned}
\big\|\mathcal{T}_m^\alpha f\big\|_{L^p_\delta} &= \big\|\,y^{\delta(t)}\,\mathcal{T}_m^{\alpha(t)}(x^{\delta(t)}f(x)\,x^{-\delta(t)})\,\big\|_{L^p} = \big\|S_t(x^{\delta(t)}f(x))\big\|_{L^p} \\
&\leq M\,\big\|\,x^{\delta(t)}f(x)\big\|_{L^p} = M\,\|f\|_{L^p_\delta}.
\end{aligned}
$$

Thus Theorem 1.8 holds for the point $P = (\frac{1}{p}, \alpha, \delta)$, which establishes the lemma.

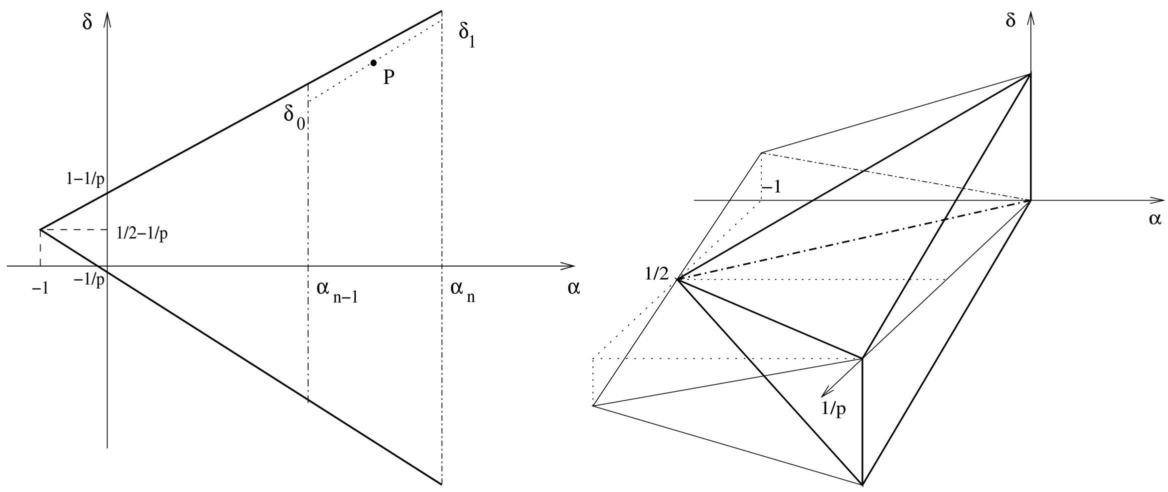

Figure 3.2: Interpolation diagram for points $P \in \mathcal{A}$, when $\alpha > 0$ or $\alpha < 0$.

□

End of the proof of Theorem 1.8:

We need to show that $\mathcal{T}_m^\alpha$ is bounded in L^p_δ for every fixed $P = (\frac{1}{p}, \alpha, \delta) \in \mathcal{A}$. We may assume that $\alpha > 0$ (otherwise see [18]), and $\alpha \neq \alpha_n := \frac{n-2}{2}$ (by Corollary 2.13). Let n be the integer so that $\alpha_{n-1} < \alpha < \alpha_n$. Then it is an elementary exercise to find two points in $\mathcal{A}$ of the form $P_0 = (\frac{1}{p}, \alpha_{n-1}, \delta_0)$, $P_1 = (\frac{1}{p}, \alpha_n, \delta_1)$ and some $t \in (0,1)$ so that $P = (1-t)P_0 + tP_1$ (see left hand side of Figure 3.2). Now, Theorem 1.8 holds for P_0 and P_1 by Corollary 2.13, and therefore it must also hold for P by Lemma 3.20.

□

Remark 3.23 It should be noted that, when $-1 < \alpha < 0$, one can choose α_0 close enough to -1 and interpolate between the points $P_0 = (\frac{1}{2}, \alpha_0, 0)$ and $P_1 = (\frac{1}{p_1}, 0, \delta_1)$ without making use of the results in [10] or [18] (see Remark 3.9). In the unweighted case $\delta = \delta_1 = 0$ this fills the admissible range of indices $\frac{|\alpha|}{2} < \frac{1}{p} < 1 - \frac{|\alpha|}{2}$, and therefore can be used to simplify Kanjin's original proof of the transplantation theorem (see Remark 4.2 below). In the weighted case, however, this only fills a star-shaped region with vertex at $(\frac{1}{2}, -1, 0)$ (see right hand side of Figure 3.2).

Remark 3.24 Observe also that the above proof works as well requiring less smoothness on $m(\xi)$. Indeed, when $\frac{n-3}{2} < \alpha \leq \frac{n-2}{2}$ we have only used Corollary 2.13, which in view of Remark 2.17 holds provided that $m \in C^{\lceil 2\alpha \rceil + 3}[0, \infty)$ and $D^\ell m(\xi)$ satisfies the hypothesis (1.9) for $\ell \leq \lceil 2\alpha \rceil + 3$, where $\lceil 2\alpha \rceil = \min\{k \in \mathbb{N} \;:\; k \geq 2\alpha\}$.

End of the proof of Theorem 3.5:

By Lemma 3.8 all multipliers $\lambda = \lambda_{\alpha,\theta}$ in (3.7) satisfy the conditions of Theorem 1.8. Hence, Assumption (A) is satisfied for all $(\frac{1}{p}, \alpha, \delta) \in \mathcal{A}$, and Theorem 3.5 follows from Remark 3.13.

$\square$

4 Proof of the transplantation theorem

As announced in the introduction, the proof will be directly obtained from the boundedness of $T_\alpha^{\alpha+i\theta}$, without appeal to the operators $T_\alpha^{\alpha+2}$ used by Kanjin in [10]. The procedure is based on complex interpolation, as we did in section 3.2 to establish the multiplier theorem. We shall also use the following elementary result, which is an easy consequence of the boundedness of $T_\alpha^{\alpha+i\theta}$.

Lemma 4.1 *Let $\alpha > -1$ and $z = \sigma + i\tau$ with $\sigma > -1$. Then the operator T_α^z is bounded in $L^2(\mathbb{R}_+)$.*

Proof:

Let $f \in C_c^\infty(0,\infty)$, so that $T_\alpha^z f = \sum_k \langle f, \mathcal{L}_k^z \rangle \mathcal{L}_k^\alpha$ is well defined by Lemma 3.4. Then, using orthogonality we have

$$\begin{aligned} \left\| T_\alpha^{\sigma+i\tau} f \right\|_2^2 &= \sum_{k=0}^\infty \left| \langle f, \mathcal{L}_k^{\sigma+i\tau} \rangle \right|^2 = \Big\| \sum_{k=0}^\infty \langle f, \mathcal{L}_k^{\sigma+i\tau} \rangle \, \mathcal{L}_k^\sigma \Big\|_2^2 \\ &= \left\| T_\sigma^{\sigma+i\tau} f \right\|_2^2 \le M(\tau) \, \|f\|_2^2, \end{aligned}$$

where in the last step we have used Theorem 3.5.

$\square$

Now we fix $\beta > \alpha_0 > -1$ so that $-\frac{\alpha_0}{2} - \frac{1}{p} < \delta < 1 - \frac{1}{p} + \frac{\alpha_0}{2}$. By condition (1.5) we need to show that $T_{\alpha_0}^\beta$ and $T_\beta^{\alpha_0}$ are bounded in $L_\delta^p(\mathbb{R}_+)$. We let $P := (\frac{1}{p}, \beta, \delta)$, which clearly belongs to $\mathcal{A}$. It is not difficult to see that there exists two other points in $\mathcal{A}$ of the form $P_0 = (\frac{1}{p_0}, \alpha_0, \delta_0)$ and $P_1 = (\frac{1}{2}, \alpha_1, 0)$ and some $t \in (0,1)$ such that $P = (1-t)P_0 + tP_1$ (see Figure 1.1). This can be done explicitly if α_1 is chosen sufficiently large, by taking $\delta_0 = \delta/(1-t)$ and $t = \frac{\beta - \alpha_0}{\alpha_1 - \alpha_0}$. As in §3.2 we use the notation $\alpha(z) = (1-z)\alpha_0 + z\alpha_1$ and $\delta(z) = (1-z)\delta_0$ for $z \in \mathbb{C}$.

By Lemma 4.1 we can define the analytic family of operators

$$S_z(y) = y^{\delta(z)} \, T_{\alpha_0}^{\alpha(\bar{z})} (F(x) x^{-\delta(z)})(y), \quad 0 \le \Re z \le 1,$$

at least for $F \in L^2_c(0,\infty)$. Then, exactly the same reasoning as in §3.2 shows that S_z satisfies the conditions of Stein's theorem, where the boundedness of

$$S_{i\theta}\colon L^{p_0}(\mathbb{R}_+) \longrightarrow L^{p_0}(\mathbb{R}_+) \quad \text{and} \quad S_{1+i\theta}\colon L^2(\mathbb{R}_+) \longrightarrow L^2(\mathbb{R}_+),$$

follows this time from, respectively, Theorem 3.5 and Lemma 4.1. Thus, S_t must be bounded in $L^{p_t} = L^p$, which translates into

$$\big\|T^{\beta}_{\alpha_0} f\big\|_{L^p_\delta} \,=\, \big\|S_t(x^{(1-t)\delta_0} f(x))\big\|_{L^p} \,\le\, M \,\big\|\, x^{(1-t)\delta_0} f(x)\big\|_{L^p} \,=\, M\, \|f\|_{L^p_\delta}.$$

This proves the required L^p_δ boundedness for the operators $T^{\beta}_{\alpha_0}$, and any $\beta > \alpha_0 > -1$. The boundedness of $T^{\alpha_0}_{\beta}$ follows by duality. Indeed, if $(\frac{1}{p}, \alpha_0, \delta) \in \mathcal{A}$, then an elementary algebraic manipulation shows that also $(\frac{1}{p'}, \alpha_0, -\delta) \in \mathcal{A}$, where $\frac{1}{p'} = 1 - \frac{1}{p}$. Then, for all $f \in C^\infty_c(0,\infty)$ we have

$$\begin{aligned}
\big\|T^{\alpha_0}_{\beta} f\big\|_{L^p_\delta} &= \sup_{\|g\|_{p'}=1} \Big| \int_0^\infty T^{\alpha_0}_{\beta} f(x)\, x^\delta\, g(x)\, dx \Big| \\
&= \sup_{\|g\|_{p'}=1} \Big| \int_0^\infty f(y)\, T^{\beta}_{\alpha_0}\big(x^\delta\, g(x)\big)\, dx \Big| \\
&\le \big\|y^\delta f(y)\big\|_{L^p} \sup_{\|g\|_{p'}=1} \big\|T^{\beta}_{\alpha_0}(x^\delta\, g)\big\|_{L^{p'}_{-\delta}} \\
\text{(previous case)} \quad &\le \big\|f\big\|_{L^p_\delta}\, M \sup_{\|g\|_{p'}=1} \big\|x^\delta\, g\big\|_{L^{p'}_{-\delta}} \,=\, M\, \|f\|_{L^p_\delta}.
\end{aligned}$$

The proof of Theorem 1.4 is now complete.

□

Remark 4.2 We point out that this approach to obtain the L^p_δ-boundedness of T^{α}_{β} only depends on the corresponding result for the operators $T^{\alpha+i\theta}_{\alpha}$ when $(\frac{1}{p}, \alpha, \delta) \in \mathcal{A}$. In particular, as was observed in Remark 3.23, it does not make use of the results in [10] or [18] when $\alpha \ge 0$, or when $\alpha < 0$ and $(\frac{1}{p}, \alpha, \delta)$ belongs to the region on the right of Figure 3.2. Therefore, in the unweighted situation studied by Kanjin, our approach gives a slightly simpler and self-contained proof which avoids dealing with the operators $T^{\alpha+2}_{\alpha}$.

5 Some applications

5.1 Riesz transforms for the Laguerre semigroup

Consider the Laguerre differential operator $L = L^{(\alpha)} = -y\frac{d^2}{dy^2} - \frac{d}{dy} + \frac{y}{4} - \frac{\alpha^2}{4y}$, which is non negative and symmetric in $L^2(0,\infty)$. For every $\alpha > -1$, the Laguerre functions $\{\mathcal{L}^\alpha_k\}_{k\ge 0}$ form a complete system of eigenvectors for $L^{(\alpha)}$, with eigenvalues given by

$$L^{(\alpha)}(\mathcal{L}^\alpha_k) \,=\, (k + \tfrac{\alpha+1}{2})\, \mathcal{L}^\alpha_k, \quad k = 0, 1, 2, \ldots \tag{5.1}$$

The Laguerre operator can be factored as $L^{(\alpha)} = \mathbf{d}^*\mathbf{d} + \frac{\alpha+1}{2} I$, where

$$\mathbf{d} = \mathbf{d}^{(\alpha)} \;=\; \sqrt{y}\,\tfrac{d}{dy} \;+\; \tfrac{1}{2}\big(\sqrt{y} - \tfrac{\alpha}{\sqrt{y}}\big).$$

Following [8], this leads to a definition of Riesz transform as:

$$R \;=\; R^{(\alpha)} \;=\; \mathbf{d} \circ L^{-1/2}\,, \qquad \text{when} \quad \alpha > -1.$$

In [8, Th. 4.2] it was shown that these operators are bounded in L^p_δ whenever $-\frac{\gamma}{2} - \frac{1}{p} < \delta < 1 - \frac{1}{p} + \frac{\gamma}{2}$, where $\gamma = \min\{\alpha, 0\}$. The proof was based on transplantation from the special cases $\alpha = \frac{n-2}{2}$. In those cases the result was obtained from the boundedness of the Riesz transforms associated with the Hermite semigroup in $\mathbb{R}^n$ (due to Stempak and Torrea; see [17]). However, as was pointed out in [8, Cor. 2.29], the Hermite setting implies a larger range of indices in these special Laguerre cases, namely $-\frac{n-2}{4} - \frac{1}{p} < \delta < 1 - \frac{1}{p} + \frac{n-2}{4}$, which suggests that in the general case one could replace γ by α. We show here that this is indeed the case.

Corollary 5.2 *Let $\alpha > -1$ and $1 < p < \infty$. Then, the Riesz transform $R^{(\alpha)}$ is bounded in $L^p_\delta(0,\infty)$ if and only if $-\frac{\alpha}{2} - \frac{1}{p} < \delta < 1 - \frac{1}{p} + \frac{\alpha}{2}$.*

Proof: The proof is exactly the same as Theorem 4.2 in [8], and follows by writing $R^{(\alpha)} = T^{\beta+1}_{\alpha+1} \circ \mathcal{M} \circ R^{(\beta)} \circ T^{\alpha}_{\beta}$, where $\beta = \frac{n-2}{2} \geq \alpha$ and $\mathcal{M}$ is a certain multiplier operator satisfying the hypothesis of Theorem 1.8. The boundedness of $R^{(\alpha)}$ then follows from the above remarks and Theorems 1.4 and 1.8.

$\square$

5.2 Littlewood-Paley g-functions for the Laguerre semigroup

Consider the heat diffusion semigroup e^{-tL} associated with the Laguerre operator $L = L^{(\alpha)}$. Following the classical approach in [16], g-functions of order $\ell = 1, 2, \ldots$ can be defined by

$$\mathfrak{g}^{(\alpha)}_\ell(f) \;=\; \Big\{ \int_0^\infty |t^\ell \frac{\partial^\ell}{\partial t^\ell}(e^{-tL^{(\alpha)}} f)|^2 \, \frac{dt}{t} \Big\}^{1/2}. \tag{5.3}$$

When $\alpha \geq 0$, the semigroup $e^{-tL^{(\alpha)}}$ is known to be contractive in $L^p(0,\infty)$ for all $1 \leq p \leq \infty$ (see e.g. [6]), and therefore the L^p boundedness of g-functions can be obtained from the classical theory in [16, 4]. However, this is not the case when $-1 < \alpha < 0$, where $e^{-tL^{(\alpha)}}$ is not even bounded in L^p unless $\frac{2}{2-|\alpha|} < p < \frac{2}{|\alpha|}$ (see [12]). Different methods must be used in such cases to study the corresponding g-functions, and moreover, no results seem to appear in the literature concerning weighted inequalities, even when $\alpha \geq 0$ (see however, [18, Prop. 2.1]). The main result of this section covers this gap, and will be obtained as an application of our transplantation Theorem 1.4 and the corresponding result of Thangavelu for Hermite functions (see Proposition 2.3 above).

THEOREM 5.4 *Let $\alpha > -1$, $1 < p < \infty$, and δ such that $-1/p - \alpha/2 < \delta < 1 - 1/p + \alpha/2$. Then for every $\ell = 1, 2, \dots,$ there is a constant $c > 0$ so that*

$$\tfrac{1}{c}\,\|f\|_{L^p_\delta} \;\leq\; \|\mathfrak{g}^{(\alpha)}_\ell(f)\|_{L^p_\delta} \;\leq\; c\,\|f\|_{L^p_\delta}, \qquad f \in C^\infty_c(0,\infty).$$

PROOF: We shall only prove the right hand inequality, since the left hand case follows from the usual polarization argument. We first consider the case $\ell = 1$. For simplicity we write $\mathfrak{g}(f) = \mathfrak{g}^{(\alpha)}_1(f)$, and drop the superscripts (α) when reference to such index is clear. Also, recall that the kernel $h_t(x, y)$ of e^{-tL} is explicitly given by the formula

$$\begin{aligned} h_t(y,z) \;&=\; \sum_{k=0}^{\infty} e^{-t(k+\frac{\alpha+1}{2})}\mathcal{L}^\alpha_k(y)\mathcal{L}^\alpha_k(z) \\ \text{(letting } r = e^{-t}) \quad &=\; \frac{r^{1/2}}{1-r}\,\exp\{-\tfrac{1}{2}\tfrac{1+r}{1-r}(y+z)\}\, I_\alpha\big(\tfrac{2(ryz)^{1/2}}{1-r}\big), \end{aligned} \tag{5.5}$$

where $I_\alpha(s) = i^{-\alpha}J_\alpha(is)$ and J_α is the usual Bessel function of order α (see e.g. [12]).

First we claim that the theorem is true when $\alpha = \frac{n-2}{2}$. Indeed, denoting $\Phi(x) = |x|^2$, from (2.10) one easily sees that $e^{-tL}(f)(|x|^2) = e^{-\frac{t}{4}(-\Delta+|x|^2)}(\frac{f\circ\Phi}{|\cdot|^\alpha})(x)$, $x \in \mathbb{R}^n$ (see [12]). Hence $\mathfrak{g}(f)(|x|^2) = 4g_1(\frac{f\circ\Phi}{|\cdot|^\alpha})(x)|x|^\alpha$, where g_1 was defined at the beginning of section 2. From here the claim can be obtained from Proposition 2.3, following exactly the same lines as in the proof of Corollary 2.13.

At this point we would like to use transplantation to reach all indices $\alpha > -1$ from the known cases $\beta = \frac{n-2}{2}$. This, however, will not be so simple since, as we shall see, an undesired factor $e^{t(\beta-\alpha)}$ appears in the process. To deal with this, we split the operator into two parts:

$$\mathfrak{g}^\sharp(f) \;=\; \Big\{\int_{t_0}^\infty |t\frac{\partial}{\partial t}(e^{-tL}f)|^2\,\frac{dt}{t}\Big\}^{1/2} \quad\text{and}\quad \mathfrak{g}_\sharp(f) \;=\; \Big\{\int_0^{t_0} |t\frac{\partial}{\partial t}(e^{-tL}f)|^2\,\frac{dt}{t}\Big\}^{1/2}, \tag{5.6}$$

where t_0 is a sufficiently large number to be chosen later. We begin with the first part, for which we need the following lemma.

LEMMA 5.7 *There exists a small number $r_0 \in (0,1)$ and $C = C(\alpha, r_0) > 0$ such that*

$$\sup_{0<r\leq r_0}\Big|\frac{\partial}{\partial r}[h_{\ln 1/r}(y,z)]\Big| \;\leq\; C\, r_0^{\frac{\alpha-1}{2}}\; y^{\frac{\alpha}{2}}\, z^{\frac{\alpha}{2}}\, e^{-\frac{y+z}{8}}, \quad \forall\, y,z > 0. \tag{5.8}$$

Proof: Taking derivatives in the explicit expression for $h_{\ln 1/r}(y,z)$ in (5.5), and using the relation $I'_\alpha(s) = \frac{\alpha}{s} I_\alpha(s) + I_{\alpha+1}(s)$ (see [2]), we see that

$$\begin{aligned}
\frac{\partial}{\partial r}[h_{\ln 1/r}(y,z)] &= \tfrac{1}{2}\tfrac{1+r}{\sqrt{r}\,(1-r)^2}\,\exp\{-\tfrac{1}{2}\tfrac{1+r}{1-r}(y+z)\}\,I_\alpha(\tfrac{2(ryz)^{1/2}}{1-r}) \\
&\quad -\tfrac{\sqrt{r}}{(1-r)^3}(y+z)\,\exp\{-\tfrac{1}{2}\tfrac{1+r}{1-r}(y+z)\}\,I_\alpha(\tfrac{2(ryz)^{1/2}}{1-r}) \\
&\quad +\tfrac{1+r}{(1-r)^3}(yz)^{1/2}\,\exp\{-\tfrac{1}{2}\tfrac{1+r}{1-r}(y+z)\}\,\Big(\tfrac{\alpha(1-r)}{2(ryz)^{1/2}}\,I_\alpha(\tfrac{2(ryz)^{1/2}}{1-r}) + I_{\alpha+1}(\tfrac{2(ryz)^{1/2}}{1-r})\Big) \\
&= \sum_{i=1}^{4} K_i(r,y,z).
\end{aligned}$$

From the size estimates of Bessel functions (see e.g. [2]) we know that $I_\alpha(s) \sim s^\alpha$ for $s \le 1$ and $I_\alpha(s) \sim \frac{e^s}{s^{1/2}}$ for $s \ge 1$, which gives the crude estimate

$$I_\alpha(s) \le C_\alpha s^\alpha e^{2s} \quad \forall\, s > 0. \tag{5.9}$$

Therefore, for all $r \in (0, r_0]$ we have

$$\begin{aligned}
\exp\{-\tfrac{1}{2}\tfrac{1+r}{1-r}(y+z)\}\,I_\alpha(\tfrac{2(ryz)^{1/2}}{1-r}) &\le C\,\tfrac{r^{\alpha/2}y^{\alpha/2}z^{\alpha/2}}{(1-r)^\alpha}\,\exp\{-\tfrac{1}{2}\tfrac{1+r}{1-r}(y+z) + \tfrac{4\sqrt{r}}{1-r}\,y^{1/2}z^{1/2}\}, \\
&\le C\,r^{\alpha/2}y^{\alpha/2}z^{\alpha/2}e^{-\frac{1}{4}(y+z)},
\end{aligned}$$

provided we choose r_0 small enough so that $\frac{\sqrt{r_0}}{1-r_0} \le 1/8$. Thus, $K_1(r,y,z)$ and $K_3(r,y,z)$ are readily estimated by the right hand side of (5.8). Similarly,

$$K_2(r,y,z) \le C\,(y+z)\,r^{\alpha/2}y^{\alpha/2}z^{\alpha/2}e^{-\frac{1}{4}(y+z)} \le C\,r^{\alpha/2}y^{\alpha/2}z^{\alpha/2}e^{-\frac{1}{8}(y+z)}$$

and

$$K_4(r,y,z) \le C\,\sqrt{yz}\,r^{\frac{\alpha+1}{2}}\,y^{\frac{\alpha+1}{2}}\,z^{\frac{\alpha+1}{2}}\,e^{-\frac{1}{4}(y+z)} \le C\,(y+z)^2\,r^{\frac{\alpha+1}{2}}\,y^{\frac{\alpha}{2}}\,z^{\frac{\alpha}{2}}\,e^{-\frac{1}{4}(y+z)},$$

from which a similar bound follows.

□

Going back to $\mathfrak{g}^\sharp(f)$, and choosing t_0 large so that $e^{-t_0} = r_0$, we have

$$\begin{aligned}
\mathfrak{g}^\sharp(f)(y) &\le \Big\{\int_{t_0}^{\infty}\Big[\int_{\mathbb{R}_+}\big|\tfrac{\partial}{\partial t}[h_t(y,z)]\big|\,|f(z)|dz\Big]^2 t\,dt\Big\}^{1/2} \\
&\le \int_{\mathbb{R}_+}\Big\{\int_{t_0}^{\infty}\big|\tfrac{\partial}{\partial t}[h_t(y,z)]\big|^2 t\,dt\Big\}^{1/2}|f(z)|\,dz = \int_{\mathbb{R}_+} Q(y,z)\,|f(z)|\,dz,
\end{aligned} \tag{5.10}$$

where, using (5.8) we have for all $y, z > 0$

$$Q(y,z) = \Big\{\int_0^{r_0}\big|\tfrac{\partial}{\partial r}[h_{\ln 1/r}(y,z)]\big|^2\,r\,\ln\tfrac{1}{r}\,dr\Big\}^{1/2} \le C\,y^{\frac{\alpha}{2}}\,z^{\frac{\alpha}{2}}\,e^{-\frac{y+z}{8}}. \tag{5.11}$$

Thus, taking L^p_δ norms in (5.10) and using Hölder's inequality

$$\big\|\mathfrak{g}^\sharp(f)\big\|_{L^p_\delta} \le C\Big[\int_{\mathbb{R}_+} e^{-\frac{py}{8}}\, y^{(\frac{\alpha}{2}+\delta)p}\, dy\Big]^{1/p}\Big[\int_{\mathbb{R}_+} e^{-\frac{p'z}{8}} z^{(\frac{\alpha}{2}-\delta)p'} dz\Big]^{1/p'}\, \|f\|_{L^p_\delta},$$

and both integrals are finite since $-\frac{1}{p}-\frac{\alpha}{2}<\delta<1-\frac{1}{p}+\frac{\alpha}{2}$. Thus, we have established the following proposition.

Proposition 5.12 *Let $\alpha>-1$, $-\frac{1}{p}-\frac{\alpha}{2}<\delta<1-\frac{1}{p}+\frac{\alpha}{2}$ and $\mathfrak{g}^\sharp$ defined as in (5.6). Then, $\big\|\mathfrak{g}^\sharp(f)\big\|_{L^p_\delta}\le C\,\|f\|_{L^p_\delta}$.*

We now turn to the operator $\mathfrak{g}_\sharp$ in (5.6), which we need to write as a linear vector-valued operator in order to use transplantation. We let H denote the Hilbert space $L^2((0,\infty),\frac{dt}{t})$ and set $\mathbf{G}: L^2(\mathbb{R}_+)\to L^2(\mathbb{R}_+;H)$ defined by

$$\mathbf{G}(f)=\mathbf{G}^{(\alpha)}(f)=\Big\{\,t\frac{\partial}{\partial t}(e^{-tL^{(\alpha)}}f)\,\Big\}_{t>0},\quad f\in L^2(\mathbb{R}_+). \tag{5.13}$$

Since $\mathfrak{g}(f)=\big|\mathbf{G}(f)\big|_H$, the L^p_δ boundedeness of $\mathfrak{g}$ is equivalent to boundedness of $\mathbf{G}$ from L^p_δ into $L^p_\delta(\mathbb{R}_+;H)$. We shall denote analogously

$$\mathbf{G}_\sharp(f)=\mathbf{G}_\sharp^{(\alpha)}(f)=\Big\{\,t\frac{\partial}{\partial t}(e^{-tL^{(\alpha)}}f)\,\chi_{(0,t_0]}(t)\Big\}_{t>0}.$$

Finally, we denote by $\overline{T^\alpha_\beta}$ the obvious vector-valued extension of the transplantation operator to $L^2(\mathbb{R}_+;H)$ as

$$\overline{T^\alpha_\beta}\big(\{f_t\}_{t>0}\big)\;=\;\Big\{T^\alpha_\beta(f_t)\Big\}_{t>0},\quad \{f_t\}_{t>0}\in L^2(\mathbb{R}_+;H).$$

By Krivine's theorem (see e.g. [11]), the vector-valued operator $\overline{T^\alpha_\beta}$ is bounded in $L^p_\delta(\mathbb{R}_+;H)$ if and only if T^α_β is bounded in $L^p_\delta(\mathbb{R}_+)$. Similarly, we denote by $\overline{\mathcal{M}}$ the vector-valued extension of the multiplier operator $\mathcal{M}f=\sum_{k\ge0}m(k)\langle f,\mathcal{L}^\beta_k\rangle\mathcal{L}^\beta_k$, where $m(s)=\frac{2s+\alpha+1}{2s+\beta+1}$. Observe that this multiplier trivially satisfies the conditions in (1.9).

Now, given $\alpha>-1$ we choose $\beta=\frac{n}{2}-1$, for some positive integer n such that $\beta\ge\alpha$. We claim that

$$\mathbf{G}_\sharp^{(\alpha)}=\overline{T^\beta_\alpha}\circ\overline{N}_{\beta-\alpha}\circ\overline{\mathcal{M}}\circ\mathbf{G}^{(\beta)}\circ T^\alpha_\beta, \tag{5.14}$$

where $\overline{N}_{\beta-\alpha}$ stands for the pointwise multiplication operator defined by

$$\overline{N}_{\beta-\alpha}\big(\{f_t\}_{t>0}\big)\;=\;\Big\{e^{\frac{\beta-\alpha}{2}t}\,\chi_{(0,t_0]}(t)\,f_t\,\Big\}_{t>0},\qquad \{f_t\}_{t>0}\in L^2(\mathbb{R}_+;H).$$

Indeed, by density and linearity it suffices to check (5.14) for $f=\mathcal{L}_k^\alpha$, $k=0,1,2,\ldots$. But this is an elementary exercise:

$$\mathcal{L}_k^\alpha \xrightarrow{T_\beta^\alpha} \mathcal{L}_k^\beta \xrightarrow{\mathbf{G}^{(\beta)}} \big\{-t(k+\tfrac{\beta+1}{2})e^{-(k+\frac{\beta+1}{2})t}\mathcal{L}_k^\beta\big\}_{t>0} \xrightarrow{\overline{\mathcal{M}}}$$
$$\big\{-t(k+\tfrac{\alpha+1}{2})e^{-(k+\frac{\beta+1}{2})t}\mathcal{L}_k^\beta\big\}_{t>0} \xrightarrow{\overline{N}_{\beta-\alpha}} \big\{-t(k+\tfrac{\alpha+1}{2})e^{-(k+\frac{\alpha+1}{2})t}\,\chi_{(0,t_0]}(t)\,\mathcal{L}_k^\beta\big\}_{t>0}$$
$$\xrightarrow{\overline{T_\alpha^\beta}} \big\{-t(k+\tfrac{\alpha+1}{2})e^{-(k+\frac{\alpha+1}{2})t}\,\chi_{(0,t_0]}(t)\,\mathcal{L}_k^\alpha\big\}_{t>0} \;=\; \mathbf{G}_\sharp^{(\alpha)}(\mathcal{L}_k^\alpha).$$

Finally the boundedness of each of such operators in L^p_δ or $L^p_\delta(\mathbb{R}_+;H)$ when $-\frac{1}{p}-\frac{\alpha}{2}<\delta<1-\frac{1}{p}+\frac{\alpha}{2}$ follows from Theorems 1.4 and 1.8, and the above mentioned remarks. Combining this result with Proposition 5.12 completes the proof of Theorem 5.4 for $\ell=1$.

To conclude the proof of the theorem we turn to the L^p_δ-boundedness of $\mathfrak{g}_\ell$ when $\ell\ge 2$. This will follow from a repeated use of Krivine's theorem. Indeed, from the previous result we know the boundedness of $\mathbf{G}:L^p_\delta\to L^p_\delta(\mathbb{R}_+;H)$, which by Krivine's theorem implies the boundedness of the vector-valued extension $\overline{\mathbf{G}}:L^p_\delta(H)\to L^p_\delta(H\times H)$ given by

$$\big\{f_s\big\}_{s>0} \longmapsto \big\{\mathbf{G}f_s\big\}_{s>0} \;=\; \big\{t\tfrac{\partial}{\partial t}[e^{-tL}f_s]\,\big\}_{(t,s)}$$

Thus, we obtain boundedness for the composition operator $\overline{\mathbf{G}}\circ\mathbf{G}: L^p_\delta \longrightarrow L^p_\delta(H\times H)$. Now, the semigroup property of e^{-tL} gives

$$t\frac{\partial}{\partial t}\Big[\,e^{-tL}\,s\,\tfrac{\partial}{\partial s}\big(e^{-sL}f\big)\,\Big] \;=\; ts\,\frac{\partial^2}{\partial u^2}\big[e^{-uL}f\big]_{|u=s+t}\,.$$

Also, by changing of variables $\sigma=s+t$ in the integrals below we see that

$$\begin{aligned}
\big|\overline{\mathbf{G}}\circ\mathbf{G}f\big|^2_{H\times H} &= \int_0^\infty\int_0^\infty st\,\Big|\frac{\partial^2}{\partial u^2}\big[e^{-uL}f\big]_{|u=s+t}\Big|^2\,ds\,dt\\
&= \int_0^\infty\int_t^\infty (\sigma-t)\,t\,\big|\tfrac{\partial^2}{\partial u^2}\big[e^{-uL}f\big]_{|u=\sigma}\big|^2\,d\sigma\,dt\\
&= \int_0^\infty \big|\tfrac{\partial^2}{\partial \sigma^2}\big[e^{-\sigma L}f\big]\big|^2\int_0^\sigma (\sigma-t)\,t\,dt\,d\sigma \;=\; \tfrac{1}{6}\,\mathfrak{g}_2(f)^2.
\end{aligned}$$

Combining all these facts we obtain the wished estimate $\|\mathfrak{g}_2(f)\|_{L^p_\delta}\le C\,\|f\|_{L^p_\delta}$. Similar arguments and induction will give the same result for $\mathfrak{g}_\ell$ and all $\ell\ge 1$, completing the proof of Theorem 5.4.

□

6 Multipliers and transplantation for related systems

Throughout this section, given a measure μ we use the notation $L^p_\delta(\mu)=L^p((0,\infty),y^{\delta p}d\mu(y))$. For fixed $\alpha>-1$ we consider the following orthonormal systems

(i) $\{\varphi_k^\alpha(y) := \sqrt{2y}\,\mathcal{L}_k^\alpha(y^2)\}_{k\geq 0}$ in $L^2(0,\infty)$;

(ii) $\{\ell_k^\alpha(y) := y^{-\alpha/2}\,\mathcal{L}_k^\alpha(y)\}_{k\geq 0}$ in $L^2(\mu_\alpha)$, where $d\mu_\alpha(y) = y^\alpha dy$;

(iii) $\{\psi_k^\alpha(y) := \sqrt{2}\,y^{-\alpha}\,\mathcal{L}_k^\alpha(y^2)\}_{k\geq 0}$ in $L^2(\nu_\alpha)$, where $d\nu_\alpha(y) = y^{2\alpha+1}dy$.

These are complete eigenvector systems of certain modifications of the Laguerre operator, for which multiplier and transplantation estimates have been studied by various authors (see [6, 21, 18] and references therein). In this section we show how to obtain such results from the power weighted estimates of the standard Laguerre system $\{\mathcal{L}_k^\alpha\}$ (see also [1]). To this end, we define the following operators:

$$V f(y) = \sqrt{2y}\,f(y^2), \quad W^\alpha f(y) = y^{-\alpha/2}\,f(y) \quad \text{and} \quad Z^\alpha f(y) = \sqrt{2}\,y^{-\alpha}\,f(y^2).$$

The proof of the next lemma is completely elementary and left to the reader.

Lemma 6.15 *Let $\alpha > -1$, and $\gamma, \sigma, \zeta \in \mathbb{R}$.*

(i) If $\delta = \frac{\gamma}{2} + \frac{1}{4} - \frac{1}{2p}$, then $\|Vf\|_{L^p_\gamma(dy)} = 2^{\frac{1}{2}-\frac{1}{p}}\|f\|_{L^p_\delta}$;

(ii) If $\delta = \sigma + \alpha(\frac{1}{p} - \frac{1}{2})$, then $\|W^\alpha f\|_{L^p_\sigma(\mu_\alpha)} = \|f\|_{L^p_\delta}$.

(iii) If $\delta = \frac{\zeta}{2} + \alpha(\frac{1}{p} - \frac{1}{2})$ then $\|Z^\alpha f\|_{L^p_\zeta(\nu_\alpha)} = 2^{\frac{1}{2}-\frac{1}{p}}\|f\|_{L^p_\delta}$.

The following is an immediate corollary of the previous lemma and Theorem 1.8.

Corollary 6.16 *Let $\alpha > -1$, $1 < p < \infty$ and consider a multiplier $m \in C^\infty[0,\infty)$ as in Theorem 1.8. Then, there exists $C > 0$ so that, for every finite sequence $\{c_k\}$*

(i) $\quad \left\|\sum_{k=0}^\infty m(k)c_k\varphi_k^\alpha\right\|_{L^p_\gamma} \leq C\left\|\sum_{k=0}^\infty c_k\varphi_k^\alpha\right\|_{L^p_\gamma}, \qquad$ *if* $\; -\alpha-\frac{1}{2}-\frac{1}{p} < \gamma < \frac{1}{p'}+\alpha+\frac{1}{2}$;

(ii) $\quad \left\|\sum_{k=0}^\infty m(k)c_k\ell_k^\alpha\right\|_{L^p_\sigma(\mu_\alpha)} \leq C\left\|\sum_{k=0}^\infty c_k\ell_k^\alpha\right\|_{L^p_\sigma(\mu_\alpha)}, \qquad$ *if* $\; -\frac{1+\alpha}{p} < \sigma < \frac{1+\alpha}{p'}$;

(iii) $\quad \left\|\sum_{k=0}^\infty m(k)c_k\psi_k^\alpha\right\|_{L^p_\zeta(\nu_\alpha)} \leq C\left\|\sum_{k=0}^\infty c_k\psi_k^\alpha\right\|_{L^p_\zeta(\nu_\alpha)}, \qquad$ *if* $\; -\frac{2(1+\alpha)}{p} < \zeta < \frac{2(1+\alpha)}{p'}$.

Remark 6.17 For the system $\{\ell_k^\alpha\}$, this result improves an earlier sufficient condition $-\min\{\frac{\alpha+1}{p}, \frac{\alpha+1}{2}\} < \sigma < \min\{\frac{\alpha+1}{p'}, \frac{\alpha+1}{2}\}$, obtained by Stempak and Trebels for $\alpha \geq 0$ under weaker smoothness assumptions on the multiplier (see [18, Th1.1]). The condition on the indices is also necessary, as is easily seen testing with $m(0) = 1$ and $m(k) = 0$, $k \geq 1$.

We state below the corresponding optimal transplantation estimates, which follow combining Lemma 6.15 with Theorem 1.4. We use $\langle .,. \rangle_\mu$ to denote the scalar product in $L^2(\mu)$.

Definition 6.18 *Given $\alpha, \beta > -1$, we define the following transplantation operators:*

$$\tau^\alpha_\beta f = \sum_{k=0}^{\infty} \langle f, \varphi^\alpha_k\rangle \varphi^\beta_k, \qquad \mathbb{T}^\alpha_\beta f = \sum_{k=0}^{\infty} \langle f, \ell^\alpha_k\rangle_{\mu_\alpha} \ell^\beta_k \quad \text{and} \quad \mathbf{T}^\alpha_\beta f = \sum_{k=0}^{\infty} \langle f, \psi^\alpha_k\rangle_{\nu_\alpha} \psi^\beta_k.$$

Corollary 6.19 *Let $1 < p < \infty$ and $-1 < \alpha < \beta$. Then:*

(i) The operators τ^α_β and τ^β_α admit a bounded extension to $L^p_\gamma \to L^p_\gamma$ if and only if

$$-\alpha - \frac{1}{2} - \frac{1}{p} < \gamma < \frac{1}{p'} + \alpha + \frac{1}{2}.$$

(ii) Let $\sigma_0 \in \mathbb{R}$ and $\sigma_1 = \sigma_0 + (\alpha - \beta)(\frac{1}{p} - \frac{1}{2})$. Then, $\mathbb{T}^\alpha_\beta \colon L^p_{\sigma_0}(\mu_\alpha) \to L^p_{\sigma_1}(\mu_\beta)$ and $\mathbb{T}^\beta_\alpha \colon L^p_{\sigma_1}(\mu_\beta) \to L^p_{\sigma_0}(\mu_\alpha)$ are bounded operators if and only if

$$-\frac{1+\alpha}{p} < \sigma_0 < \frac{1+\alpha}{p'}.$$

(iii) Let $\zeta_0 \in \mathbb{R}$ and $\zeta_1 = \zeta_0 + 2(\alpha - \beta)(\frac{1}{p} - \frac{1}{2})$. Then, $\mathbf{T}^\alpha_\beta \colon L^p_{\zeta_0}(\nu_\alpha) \to L^p_{\zeta_1}(\nu_\beta)$ and $\mathbf{T}^\beta_\alpha \colon L^p_{\zeta_1}(\nu_\beta) \to L^p_{\zeta_0}(\nu_\alpha)$ are bounded operators if and only if

$$-\frac{2(1+\alpha)}{p} < \zeta_0 < \frac{2(1+\alpha)}{p'}.$$

Proof: This is a straightforward consequence of Lemma 6.15, Theorem 1.4, and the identities $\tau^\alpha_\beta = V \cdot T^\alpha_\beta \cdot V^*$, $\mathbb{T}^\alpha_\beta = W^\beta \cdot T^\alpha_\beta \cdot (W^\alpha)^*$ and $\mathbf{T}^\alpha_\beta = Z^\beta \cdot T^\alpha_\beta \cdot (Z^\alpha)^*$. We leave details to the reader.

□

Remark 6.20 The relation between σ_0 and σ_1 in (ii) of the previous corollary is also a necessary condition. Indeed, in other case it would imply the boundedness of T^α_β from $L^p_{\delta_1}$ into $L^p_{\delta_2}$, for some numbers $\delta_1 \neq \delta_2$. Such boundedness, however, can never hold when $(\frac{1}{p}, \rho, \delta_i) \in \mathcal{A}$, since composition with $T^\beta_\alpha : L^p_{\delta_2} \to L^p_{\delta_2}$ (which is bounded by Theorem 1.4), would lead to a continuous inclusion $L^p_{\delta_1} \hookrightarrow L^p_{\delta_2}$, and hence a contradiction.

References

[1] I. Abu-Falahah and J.L. Torrea "Transferring strong boundedness among Laguerre function systems" Preprint.

[2] Bateman Manuscript Proyect. A. Erderlyi (editor), W. Magnus, F. Oberhettinger, F. G. Tricomi, Research Associates, *Higher Transcendental Functions*, Vol. II. McGraw-Hill Inc., 1953.

[3] C. Bennett and R. Sharpley. *Interpolation of operators.* Academic Press Inc., Boston, 1988.

[4] R. Coifman, R. Rochberg and G. Weiss "The L^p version of von Neumann's inequality and Littlewood-Paley-Stein theory", in *Linear spaces and Approximation (P.L. Butzer and B. Sz.-Nagy eds.), Birkhäuser-Verlag (Basel,1978), pp. 53-67.*

[5] J. Dziubański, G. Garrigós, T. Martínez, J.L. Torrea and J. Zienkiewicz. "*BMO* spaces related to Schrödinger operators with potentials satisfying a reverse Hölder inequality". Math. Z. **249** (2) (2005), 329–356.

[6] E. Görlich and C. Markett "A convolution structure for Laguerre series". Nederl. Akad. Wetensch. Indag. Math. 44 **1982** (2) 161–171.

[7] C. Gutiérrez, A. Incognito and J.L. Torrea. "Riesz transforms, g-functions, and multipliers for the Laguerre semigroup". Houston J. Math. **27** (3) (2001), 579–592.

[8] E. Harboure, J.L. Torrea and B. Viviani. "Riesz transforms for Laguerre expansions". Indiana Univ. Math. Jour. **55** (2006) 999-1014.

[9] E. Harboure, C. Segovia, J.L. Torrea and B. Viviani. "Power weighted L^p-inequalities for Laguerre Riesz Transforms". Preprint.

[10] Y. Kanjin. "A transplantation theorem for Laguerre series". Tôhoku Math. J. **43** (1991), 537-555.

[11] J. Lindenstrauss and L. Tzafriri. *Classical Banach function spaces.* Springer-Verlag, Berlin, 1979.

[12] R. Macías, C. Segovia and J.L. Torrea. "Weighted norm estimates for the maximal operator of the Laguerre functions heat diffusion semigroup". Studia Math. **172** (2) (2006), 149–167.

[13] A. Nowak and K. Stempak. "Weighted estimates for the Hankel transform transplantation operator". Preprint.

[14] J.L. Rubio de Francia, F. Ruiz and J.L. Torrea. "Calderón-Zygmund theory for operator-valued kernels". Adv. in Math. **62** (1) (1986), 7–48.

[15] E. Stein, *Singular integrals and differentiability properties of functions.* Princeton University Press, Princeton N.J. 1970.

[16] E. Stein. *Topics in Harmonic Analysis Related to the Littlewood-Paley Theory.* Annals of Mathematical Studies **63**. Princeton University Press, 1970.

[17] K. Stempak and J.L. Torrea , "Poisson integrals and Riesz transforms for Hermite function expansions with weights". Jour. Funct. Anal. J. Funct. Anal. **202** (2003), no. 2, 443–472.

[18] K. Stempak and W. Trebels. "On weighted transplantation and multipliers for Laguerre expansions". Math. Ann. **300** (1994), 203-219.

[19] G. Szegő *Orthogonal Polynomials*, Amer. Math. Soc. Colloquium Publ. **23**, New York, 1939.

[20] S. Thangavelu. "Transplantation summability and multipliers for multiple Laguerre expansions". Tôhoku Math. J. **44** (1992), 279-298.

[21] S. Thangavelu. *Lecture notes on Hermite and Laguerre expansions.* Princeton Univ. Press. Princeton, New Jersey 1993.

G. Garrigós and J.L. Torrea
Dep. Matemáticas C-XV
Universidad Autónoma de Madrid
28049 Madrid, Spain
gustavo.garrigos@uam.es
joseluis.torrea@uam.es

E. Harboure and B. Viviani
IMAL FIQ
CONICET, Universidad Nacional del Litoral
Güemes 3450, 3000 Santa Fe, Argentina
harbour@ceride.gov.ar
viviani@ceride.gov.ar

T. Signes
Dep. Mat. Aplicada. Fac. Informática
Universidad de Murcia
30100 Espinardo (Murcia), Spain
tmsignes@um.es