Potential operators and their commutators acting between variable Lebesgue spaces with different weights

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ABSTRACT
We prove that a generalized Fefferman-Phong type conditions on a pair of weights u and v is sufficient for the boundedness of the potential type operator from $L_v^{p(\cdot)}$ into $L_u^{q(\cdot)}$. We also obtain an analogous estimates for their commutators with BMO symbols.

We include some estimates for a generalized maximal operator in the variable context $M_{s(\cdot)}$, and its fractional version, $M_{\beta(\cdot),s(\cdot)}$, between variable versions of $L\log L$ type spaces, where $s(\cdot)$ and $\beta(\cdot)$ are exponents belonging to certain classes.

KEYWORDS
Potential operators, Variable Lebesgue spaces, Commutators, Weights

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1. Introduction and main results

Let Φ be a nonnegative and locally integrable function. We shall consider potential type operators T_Φ defined by

$$T_\Phi f(x) = \int_{\mathbb{R}^n} \Phi(x-y)f(y)\,dy,$$

whenever this integral is finite where the kernel Φ satisfies certain weak growth condition.

In [1], E. Sawyer and R. Wheeden obtained Fefferman-Phong type conditions on the weights and proved weighted boundedness results for the fractional integral operators I_α between Lebesgue spaces. Motivated by this paper, in [2], C. Pérez considered weaker norms than those involved in the Fefferman-Phong type conditions in [1] and obtained weighted boundedness result for the potenctial operator T_Φ. This article was the motivation for a great variety of subsequent papers related to this kind of operator. For example, in [3] and [4], the authors obtained weighted L^p inequalities of Fefferman-Stein type for T_Φ and for the higher order commutators associated to this operator, respectively, whenever $1 < p < \infty$. For this commutators two weighted norm

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inequalities in the spirit of those given in [2] were proved in [5], and similar results were obtained in [6] and [7] in the general framework as spaces of homogeneous type.

Multilinear version oh the results described above can be found in [8] and extrapolation results involving these operator were given in [9] and [10].

On the other hand, it is well known that many applications in partial differential equations and quantum mechanics find in this type of operators relevant tool. We shall refer the reader to [1] and [11] for further information.

The aim of this paper is to describe the behavior of the operators mentioned above when they act between variable exponent Lebesgue spaces with different weights. Concretely, we proved that a generalized Fefferman-Phong type conditions on a pair of weights u and v is sufficient for the boundedness of the potential type operator from $L_v^{p(\cdot)}$ into $L_u^{q(\cdot)}$. We also obtain an analogous estimates for their commutators with BMO symbols.

We also include some interesting estimates for certain generalizations, in the variable context, of the Hardy-Litthewood maximal operator between variable version of the $L\log L$ type spaces. Fractional version of these results are also considered.

In the definition of T_Φ, the function Φ belongs to a certain class of kernels that satisfy that there exists positive constants δ, c and $0 \leq \varepsilon < 1$, with the property that

$$\sup_{2^k<|x|\leq 2^{k+1}} \Phi(x) \leq \frac{c}{2^{kn}} \int_{\delta(1-\varepsilon)2^k<|y|\leq 2\delta(1+\varepsilon)2^k} \Phi(y)\, dy,$$

for all $k \in \mathbb{Z}$. We shall denote this class by $\mathfrak{D}$.

For example, if Φ is radial an nonincreasing, then $\Phi \in \mathfrak{D}$. A basic example of potential operator with radial and nonincreasing kernel Φ is given by the fractional integral operator I_α, which is the convolution with the kernel $\Phi(t) = |t|^{\alpha-n}$, $0 < \alpha < n$. There are other important examples such as the Bessel potential $J_{\beta,\lambda}$, $\beta\,, \lambda > 0$ with kernels $K_{\beta,\lambda}$ best defined by means of its Fourier transform by $\widehat{K_{\beta,\lambda}}(\xi) = (\lambda^2+|\xi|^2)^{-\beta/2}$ and $K_{\beta,\lambda}$ is also radial and nonincreasing.

Nevertheless, condition $\mathfrak{D}$ involves other type of kernels Φ such that radial and non-decreasing functions. Moreover, if Φ is essentially constant on annuli, that is, $\Phi(y) \leq C\Phi(x)$ for $|y|/2 \leq |x| \leq 2|y|$, then $\Phi \in \mathfrak{D}$.

We now introduce the general context where we shall be working with.

The expression $A \lesssim B$ means that there exists a positive constant C such that $A \leq CB$. With $A \simeq B$ we mean $A \lesssim B$ and $B \lesssim A$.

Let $p(\cdot) : \mathbb{R}^n \to [1, \infty)$ be a measurable function. For $\mathrm{A} \subset \mathbb{R}^n$ we define $p_{\mathrm{A}}^- = ess\inf_{x\in \mathrm{A}} p(x)$ and $p_{\mathrm{A}}^+ = ess\sup_{x\in\mathrm{A}} p(x)$. For simplicity we denote $p^- = p_{\mathbb{R}^n}^-$ and $p^+ = p_{\mathbb{R}^n}^+$.

With $p'(\cdot)$ we denote the conjugate exponent of $p(\cdot)$ given by $p'(\cdot) = p(\cdot)/(p(\cdot) - 1)$. It is not hard to prove that $(p')^- = (p^+)'$ and $(p')^+ = (p^-)'$.

We say that $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ if $1 < p^- \leq p^+ < \infty$ and we denote by $\mathcal{P}^{log}(\mathbb{R}^n)$ the set of the exponents $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ that satisfy the following inequalities

$$|p(x) - p(y)| \lesssim 1/\log(e + 1/|x - y|),\ x, y \in \mathbb{R}^n$$

and

$$|p(x) - p_\infty| \lesssim 1/\log(e + |x|),\ x \in \mathbb{R}^n \tag{1}$$

for some positive constant p_∞. It is easy to see that the inequality (1) implies that $\lim_{|x|\to\infty} p(x) = p_\infty$. The conditions on $p(\cdot)$ above are known as local and global log-Hölder conditions, respectively.

If $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$, the variable exponent Lebesgue space $L^{p(\cdot)}(\mathbb{R}^n)$ is the set of the measurable functions f defined on $\mathbb{R}^n$ such that, for some positive λ, the convex functional modular

$$\varrho_{p(\cdot)}(f/\lambda) = \int_{\mathbb{R}^n} |f(x)/\lambda|^{p(x)}\, dx$$

is finite. A Luxemburg norm can be defined in $L^{p(\cdot)}(\mathbb{R}^n)$ by taking

$$\|f\|_{p(\cdot)} = \inf\left\{\lambda > 0 : \varrho_{p(\cdot)}(f/\lambda) \leq 1\right\}.$$

By $L^{p(\cdot)}_{\text{loc}}(\mathbb{R}^n)$ we denote the space of the functions f such that $f \in L^{p(\cdot)}(\mathbb{R}^n)(U)$ for every compact set $U \subset \mathbb{R}^n$.

A locally integrable function w defined in $\mathbb{R}^n$ which is positive almost everywhere is called a weight. For $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ we define the weighted variable Lebesgue space $L^{p(\cdot)}_w(\mathbb{R}^n)$ as the set of the measurable functions f defined on $\mathbb{R}^n$ such that $fw \in L^{p(\cdot)}(\mathbb{R}^n)$.

By a cube $Q \subset \mathbb{R}^n$ we shall understand a cube with sides parallel to the coordinate axes. The sidelength of Q is denoted by $\ell(Q)$ and aQ, $a > 0$, denotes the cube concentric with Q and with sidelength $a\ell(Q)$. By $\mathcal{X}_Q$ we denote the characteristic function of Q.

We are now in position to state our main results.

The next theorem gives a two weighted boundedness result for T_Φ between variable Lebesgue spaces with different exponents. In the classical Lebesgue space a proof can be found in [2]. The function $\widetilde{\Phi}$ involved in the condition on the weights is given by

$$\widetilde{\Phi}(t) = \int_{|z|\leq t} \Phi(z)\, dz.$$

Theorem 1.1. *Let $p(\cdot), q(\cdot) \in \mathcal{P}^{log}(\mathbb{R}^n)$ such that $p(\cdot) \leq q(\cdot)$ and $\Phi \in \mathfrak{D}$. Suppose that (v, w) is any couple of weights such that $v \in L^{p(\cdot)}_{\text{loc}}(\mathbb{R}^n)$ and for some constants r, $a > 1$,*

$$\sup_Q \widetilde{\Phi}(\ell(Q)) \frac{\|\mathcal{X}_Q\|_{q(\cdot)}}{\|\mathcal{X}_Q\|_{p(\cdot)}} \frac{\left\|\mathcal{X}_Q v^{-1}\right\|_{r(p^-)'}}{\|\mathcal{X}_Q\|_{r(p^-)'}} \frac{\|\mathcal{X}_Q w\|_{aq^+}}{\|\mathcal{X}_Q\|_{aq^+}} < \infty, \tag{2}$$

where the supremum is taken over all cubes $Q \subset \mathbb{R}^n$. Then $T_\Phi : L^{p(\cdot)}_v(\mathbb{R}^n) \hookrightarrow L^{q(\cdot)}_w(\mathbb{R}^n)$.

The maximal operator associated to T_Φ is defined by

$$M_{\widetilde{\Phi}} f(x) = \sup_{Q \ni x} \frac{\widetilde{\Phi}(\ell(Q))}{|Q|} \int_Q |f(y)|\, dy.$$

Note that the weights $w = u^{1/ap^+}$ and $v = (M_{\widetilde{\Phi}^{r(p^-)'}} u)^{1/r(p^-)'}$ where u a weight, $r > \max\{1, p^+/(p^-)'\}$ and $a = r(p^-)'/p^+$ satisfies condition (2) with $p = q$. In fact, since

$$M_{\widetilde{\Phi}^{r(p^-)'}} u = \sup_{Q \ni x} \frac{\widetilde{\Phi}(\ell(Q))^{r(p^-)'}}{|Q|} \int_Q u(y)\, dy,$$

then

$$\widetilde{\Phi}(\ell(Q))\frac{\left\|\mathcal{X}_Q v^{-1}\right\|_{r(p^-)'}}{\|\mathcal{X}_Q\|_{r(p^-)'}}\frac{\|\mathcal{X}_Q w\|_{ap^+}}{\|\mathcal{X}_Q\|_{ap^+}} \leq \left(\frac{1}{|Q|}\int_Q u(y)\,dy\right)^{-1/(r(p^-)')+1/(ap^+)} = 1.$$

In the theorem above the bump-conditions on the weights involve constant exponents. However, we can give a variable version by assuming certain adittional condition on the constants r and a, which restrict the range of them with respect to the previous theorem.

Theorem 1.2. *Let $p(\cdot), q(\cdot) \in \mathcal{P}^{log}(\mathbb{R}^n)$ such that $p(\cdot) \leq q(\cdot)$ and $\Phi \in \mathfrak{D}$. Suppose that (v, w) is any couple of weights such that $v \in L^{p(\cdot)}_{\text{loc}}(\mathbb{R}^n)$ and for some constants $r > (p')^+/(p')^-$ and $a > q^+/q^-$,*

$$\sup_Q \widetilde{\Phi}(\ell(Q))\frac{\|\mathcal{X}_Q\|_{q(\cdot)}}{\|\mathcal{X}_Q\|_{p(\cdot)}}\frac{\left\|\mathcal{X}_Q v^{-1}\right\|_{rp'(\cdot)}}{\|\mathcal{X}_Q\|_{rp'(\cdot)}}\frac{\|\mathcal{X}_Q w\|_{aq(\cdot)}}{\|\mathcal{X}_Q\|_{aq(\cdot)}} < \infty. \tag{3}$$

Then $T_\Phi : L^{p(\cdot)}_v(\mathbb{R}^n) \hookrightarrow L^{q(\cdot)}_w(\mathbb{R}^n)$.

It is easy to check that the pair $(w, \sup_Q \widetilde{\Phi}(\ell(Q))\|\mathcal{X}_Q w\|_{aq(\cdot)}\|\mathcal{X}_Q\|^{-1}_{aq(\cdot)})$, where $a > q^+/q^-$ satisfies condition (3) with $p = q$.

Remark 1. When p and q are constant exponents, Theorems 1.1 and 1.2 coincide. Moreover, they was proved in [2].

When T_Φ is the fractional integral operator, defined for $0 < \alpha < n$ by $I_\alpha f(x) = \int_{\mathbb{R}^n} f(y)|x-y|^{\alpha-n}dy$, condition (3) can be written as follows

$$\sup_Q |Q|^{\frac{\alpha}{n}}\frac{\|\mathcal{X}_Q\|_{q(\cdot)}}{\|\mathcal{X}_Q\|_{p(\cdot)}}\frac{\left\|\mathcal{X}_Q v^{-1}\right\|_{rp'(\cdot)}}{\|\mathcal{X}_Q\|_{rp'(\cdot)}}\frac{\|\mathcal{X}_Q w\|_{aq(\cdot)}}{\|\mathcal{X}_Q\|_{aq(\cdot)}} < \infty. \tag{4}$$

However, as in the classical case (see [1]), it can be seen that a necessary condition for the boundedness

$$I_\alpha : L^{p(\cdot)}_v \hookrightarrow L^{q(\cdot)}_w \tag{5}$$

is given by

$$A^\alpha_{p(\cdot),q(\cdot)}: \qquad \sup_Q |Q|^{\frac{\alpha}{n}}\frac{\|\mathcal{X}_Q\|_{q(\cdot)}}{\|\mathcal{X}_Q\|_{p(\cdot)}}\frac{\left\|\mathcal{X}_Q v^{-1}\right\|_{p'(\cdot)}}{\|\mathcal{X}_Q\|_{p'(\cdot)}}\frac{\|\mathcal{X}_Q w\|_{q(\cdot)}}{\|\mathcal{X}_Q\|_{q(\cdot)}} < \infty. \tag{6}$$

In fact, fix a cube $Q \subset \mathbb{R}^n$ and let $\tilde{f}(y) = \theta\, \mathcal{X}_Q(y) s_Q(y)^{p'(y)/p(y)} v(y)^{-p'(y)}$ where $s_Q(y) = (|Q|^{1/n} + |x_Q - y|)^{\alpha-n}$, x_Q is the center of Q and θ is a constant such that $\left\|\tilde{f}v\right\|_{p(\cdot)} = 1$. Since

$$|Q|^{\frac{1}{n}}|x-y| \leq |Q|^{\frac{1}{n}}|x_Q - x| + |Q|^{\frac{1}{n}}|x_Q - y| \leq (|Q|^{\frac{1}{n}} + |x_Q - x|)(|Q|^{\frac{1}{n}} + |x_Q - y|),$$

we have

$$|x-y|^{\alpha-n} \geq |Q|^{1-\alpha/n} s_Q(x) s_Q(y), \tag{7}$$

for every $x, y \in \mathbb{R}^n$. On the other hand, since $\left\|\tilde{f}v\right\|_{p(\cdot)} = 1$, we have

$$1 = \int_{\mathbb{R}^n} \tilde{f}(x)^{p(x)} v(x)^{p(x)}\,dx \simeq \int_Q s_Q(x)^{p'(x)} v(x)^{-p'(x)}\,dx, \tag{8}$$

and this implies that

$$\left\|s_Q v^{-1}\right\|_{p'(\cdot)} \simeq 1. \tag{9}$$

Then, by (7) and (8) we have

$$I_\alpha \tilde{f}(x) \gtrsim |Q|^{1-\alpha/n} s_Q(x) \int_Q s_Q(y)^{p'(y)} v(y)^{-p'(y)}\, dy \gtrsim |Q|^{1-\alpha/n} s_Q(x).$$

Thus by (5), the last inequality and (9) we obtain

$$\begin{aligned} 1 &\gtrsim \left\|I_\alpha \tilde{f} w\right\|_{q(\cdot)} \gtrsim |Q|^{1-\alpha/n} \left\|s_Q w\right\|_{q(\cdot)} \gtrsim |Q|^{1-\alpha/n} \left\|\mathcal{X}_Q s_Q v^{-1}\right\|_{p'(\cdot)} \left\|\mathcal{X}_Q s_Q w\right\|_{q(\cdot)} \\ &\gtrsim |Q|^{\alpha/n-1} \left\|\mathcal{X}_Q v^{-1}\right\|_{p'(\cdot)} \left\|\mathcal{X}_Q w\right\|_{q(\cdot)} \end{aligned}$$

since $s_Q(x) \geq |Q|^{\alpha/n-1}$. The last expression is equivalent to condition (6) since Lemma 2.2 (see section §2).

Therefore, as in the classical case, the sufficient condition given by (4) is stronger than condition $A^\alpha_{p(\cdot),q(\cdot)}$ given in (6). In fact, by Lemma 2.7 and Hölder's inequality (13) (see sections §3 and §2), since $1/p'(\cdot) = 1/(rp'(\cdot)) + 1/(r'p'(\cdot))$ and $1/q(\cdot) = 1/(aq(\cdot)) + 1/(a'q(\cdot))$, we have

$$|Q|^{\frac{\alpha}{n}} \frac{\|\mathcal{X}_Q\|_{q(\cdot)}}{\|\mathcal{X}_Q\|_{p(\cdot)}} \frac{\left\|\mathcal{X}_Q v^{-1}\right\|_{p'(\cdot)}}{\|\mathcal{X}_Q\|_{p'(\cdot)}} \frac{\|\mathcal{X}_Q w\|_{q(\cdot)}}{\|\mathcal{X}_Q\|_{q(\cdot)}} \lesssim |Q|^{\frac{\alpha}{n}} \frac{\|\mathcal{X}_Q\|_{q(\cdot)}}{\|\mathcal{X}_Q\|_{p(\cdot)}} \frac{\left\|\mathcal{X}_Q v^{-1}\right\|_{rp'(\cdot)}}{\|\mathcal{X}_Q\|_{rp'(\cdot)}} \frac{\|\mathcal{X}_Q w\|_{aq(\cdot)}}{\|\mathcal{X}_Q\|_{aq(\cdot)}}.$$

In this article we shall also deal with the commutators of T_Φ with BMO symbols. For a nonnegative, locally integrable function Φ, a function $b \in L^1_{\text{loc}}(\mathbb{R}^n)$ and a nonnegative integer m, the conmmutator of order m of T_Φ is formaly defined by

$$T_\Phi^{b,m} f(x) = \int_{\mathbb{R}^n} (b(x) - b(y))^m \Phi(x-y) f(y) dy.$$

In the classical Lebesgue context, these operators were study in [5] and in [8] in the multilinear framework.

The next result envolving commutators is the corresponding version of Theorem 1.1 for this case. When $m = 0$ both theorems are the same.

Theorem 1.3. *Let $p(\cdot), q(\cdot) \in \mathcal{P}^{log}(\mathbb{R}^n)$ such that $p(\cdot) \leq q(\cdot)$, $\Phi \in \mathfrak{D}$, $b \in BMO$ and $m \in \mathbb{N}$. Let two constants r, $a > 1$. Suppose that (v, w) is any couple of weights such that $v \in L^{p(\cdot)}_{\text{loc}}(\mathbb{R}^n)$, $w \in L^{aq(\cdot)}_{\text{loc}}(\mathbb{R}^n)$ and*

$$\sup_Q \widetilde{\Phi}(\ell(Q)) \frac{\|\mathcal{X}_Q\|_{q(\cdot)}}{\|\mathcal{X}_Q\|_{p(\cdot)}} \frac{\left\|\mathcal{X}_Q v^{-1}\right\|_{r(p^-)'}}{\|\mathcal{X}_Q\|_{r(p^-)'}} \frac{\|\mathcal{X}_Q w\|_{aq^+}}{\|\mathcal{X}_Q\|_{aq^+}} < \infty.$$

Then $T_\Phi^{b,m} : L_v^{p(\cdot)}(\mathbb{R}^n) \hookrightarrow L_w^{q(\cdot)}(\mathbb{R}^n)$.

As in the case of the operator T_Φ we can give a variable bump condition of the theorem above.

Theorem 1.4. *Let $p(\cdot), q(\cdot) \in \mathcal{P}^{log}(\mathbb{R}^n)$ such that $p(\cdot) \leq q(\cdot)$, $\Phi \in \mathfrak{D}$, $b \in BMO$ and $m \in \mathbb{N}$. Let two constants r, a such that $r > (p')^+/(p')^-$ and $a > q^+/q^-$. Suppose that (v, w) is any couple of weights such that $v \in L^{p(\cdot)}_{\text{loc}}(\mathbb{R}^n)$, $w \in L^{aq(\cdot)}_{\text{loc}}(\mathbb{R}^n)$ and*

$$\sup_Q \widetilde{\Phi}(\ell(Q)) \frac{\|\mathcal{X}_Q\|_{q(\cdot)}}{\|\mathcal{X}_Q\|_{p(\cdot)}} \frac{\left\|\mathcal{X}_Q v^{-1}\right\|_{rp'(\cdot)}}{\|\mathcal{X}_Q\|_{rp'(\cdot)}} \frac{\|\mathcal{X}_Q w\|_{aq(\cdot)}}{\|\mathcal{X}_Q\|_{aq(\cdot)}} < \infty.$$

Then $T_\Phi^{b,m} : L_v^{p(\cdot)}(\mathbb{R}^n) \hookrightarrow L_w^{q(\cdot)}(\mathbb{R}^n)$.

Let $s(\cdot) \geq 1$ and g be a locally integrable function. We shall consider the maximal operator $M_{s(\cdot)}$ introduced in [12] and defined by

$$M_{s(\cdot)}g(x) \doteq \sup_{Q \ni x} \|\mathcal{X}_Q g\|_{s(\cdot)} \|\mathcal{X}_Q\|_{s(\cdot)}^{-1}. \tag{10}$$

It was proved in [[12],Theorem 7.3.27] that, if $p(\cdot), s(\cdot), l(\cdot) \in \mathcal{P}^{log}(\mathbb{R}^n)$ then, $M_{s(\cdot)} : L^{p(\cdot)}(\mathbb{R}^n) \hookrightarrow L^{p(\cdot)}(\mathbb{R}^n)$ provided that $p(\cdot) = s(\cdot)l(\cdot)$. On the other hand in [13] the authors proved that the Hardy-Littlewood maximal operator M is bounded in $L^{p(\cdot)}(\log L)^{r(\cdot)}(\mathbb{R}^n)$ when $p(\cdot) \in \mathcal{P}^{log}(\mathbb{R}^n)$ and $r(\cdot) \in \mathcal{P}^{loglog}(\mathbb{R}^n)$.

We say that $r(\cdot) \in \mathcal{P}^{loglog}(\mathbb{R}^n)$ if $r(\cdot) : \mathbb{R}^n \to \mathbb{R}$ is bounded on $\mathbb{R}^n$ and it satisfies the following inequality

$$|r(x) - r(y)| \lesssim 1/\log(e + \log(e + 1/|x-y|)), \text{ for every } x, y \in \mathbb{R}^n.$$

Let $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ and $r(\cdot) \geq 0$, we denote with $\vartheta_{p(\cdot)}^{r(\cdot)}$ the function defined by

$$\vartheta_{p(\cdot)}^{r(\cdot)}(x,t) = t^{p(x)}(\log(e+t))^{r(x)}, \tag{11}$$

for $t > 0$ and $x \in \mathbb{R}^n$.

The space $L^{p(\cdot)}(\log L)^{r(\cdot)}(\mathbb{R}^n)$ is the set of the measurable functions f defined on $\mathbb{R}^n$ such that, for some positive λ, the convex functional modular

$$\varrho_{p(\cdot)}^{r(\cdot)}(f/\lambda) = \int_{\mathbb{R}^n} \vartheta_{p(\cdot)}^{r(\cdot)}\left(x, \frac{|f(x)|}{\lambda}\right) dx$$

is finite. A Luxemburg norm can be defined in $L^{p(\cdot)}(\log L)^{r(\cdot)}(\mathbb{R}^n)$ by taking

$$\|f\|_{p(\cdot),r(\cdot)} \doteq \|f\|_{L^{p(\cdot)}(\log L)^{r(\cdot)}} = \inf\left\{\lambda > 0 : \varrho_{p(\cdot)}^{r(\cdot)}(f/\lambda) \leq 1\right\}.$$

In the following result we extend both results mentioned above.

Theorem 1.5. *Let* $p(\cdot), s(\cdot), l(\cdot) \in \mathcal{P}^{log}(\mathbb{R}^n)$ *such that* $p(\cdot) = s(\cdot)l(\cdot)$. *Let* $r(\cdot) \in \mathcal{P}^{loglog}(\mathbb{R}^n)$ *such that* $r(\cdot) \geq 0$. *Then* $M_{s(\cdot)} : L^{p(\cdot)}(\log L)^{r(\cdot)}(\mathbb{R}^n) \hookrightarrow L^{p(\cdot)}(\log L)^{r(\cdot)}(\mathbb{R}^n)$.

For $\beta(\cdot) \geq 1$, we define the following fractional type version of the maximal operator $M_{s(\cdot)}$ given in (10), as follows

$$M_{\beta(\cdot),s(\cdot)}g(x) \doteq \sup_{Q \ni x} \|\mathcal{X}_Q\|_{\beta(\cdot)} \|\mathcal{X}_Q g\|_{s(\cdot)} \|\mathcal{X}_Q\|_{s(\cdot)}^{-1}. \tag{12}$$

A key result relating both operators above is given by the following lemma (see the proof in section §4).

Lemma 1.6. *Let* $p(\cdot), q(\cdot) \in \mathcal{P}^{log}(\mathbb{R}^n)$ *such that* $p(\cdot) \leq q(\cdot)$. *Suppose that* $1/\beta(\cdot) = 1/p(\cdot) - 1/q(\cdot)$ *and let* $s(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ *be such that* $s^+ < \beta^-$. *Then the following inequality*

$$M_{\beta(\cdot),s(\cdot)}(g)(x) \lesssim M_{(\beta(\cdot)/s(\cdot))' s(\cdot)}(g^{p(\cdot)/q(\cdot)})(x) \left\|g^{p(\cdot)/\beta(\cdot)}\right\|_{\beta(\cdot)}$$

holds for every function g.

By Lemma 1.6 and Theorem 1.5 we obtain the following boundedness result for the fractional type maximal operator $M_{\beta(\cdot),s(\cdot)}$, which is relevant in the proofs of our main results.

Theorem 1.7. *Let $p(\cdot),q(\cdot)\in\mathcal{P}^{log}(\mathbb{R}^n)$ such that $p(\cdot)\le q(\cdot)$ and $r(\cdot)\in\mathcal{P}^{loglog}(\mathbb{R}^n)$ such that $r(\cdot)\ge 0$. Let $\beta(\cdot)$ and $s(\cdot)$ be two functions such that $1/\beta(\cdot)=1/p(\cdot)-1/q(\cdot)$ and $s^+<p^-$. Then $M_{\beta(\cdot),s(\cdot)}:L^{p(\cdot)}(\log L)^{r(\cdot)}(\mathbb{R}^n)\hookrightarrow L^{q(\cdot)}(\log L)^{r(\cdot)}(\mathbb{R}^n)$.*

Particularly, if $r(\cdot)\equiv 0$ we obtain that $M_{\beta(\cdot),s(\cdot)}:L^{p(\cdot)}(\mathbb{R}^n)\hookrightarrow L^{q(\cdot)}(\mathbb{R}^n)$.

Let us return to the condition on the weights (3). For a given exponent $s(\cdot)$, a weight w and a cube Q, that inequality involves averages of the form $a^{s(\cdot)}(w,Q)=\|\mathcal{X}_Q w\|_{s(\cdot)}\|\mathcal{X}_Q\|_{s(\cdot)}^{-1}$.

We now consider the following Luxemburg type averages associated to a given N-function ϕ (see [12] for more information about N-function),

$$\|f\|_{\phi,Q}=\inf\left\{\lambda>0:\frac{1}{|Q|}\int_Q\phi\left(x,\frac{|f(x)|}{\lambda}\right)dx\le 1\right\},$$

and the corresponding maximal operator $\mathbf{M}_\phi f(x)=\sup_{Q\ni x}\|f\|_{\phi,Q}$.

For $\beta(\cdot)\in\mathcal{P}(\mathbb{R}^n)$, we define the following fractional type version of the maximal operator above $\mathbf{M}_{\beta(\cdot),\phi}f(x)=\sup_{Q\ni x}\|\mathcal{X}_Q\|_{\beta(\cdot)}\|f\|_{\phi,Q}$. When $\phi(\cdot,L)=L^{p(\cdot)}(\log L)^{r(\cdot)}$, it is easy to see the following relation between both averages given above.

Lemma 1.8. *Let $p(\cdot)\in\mathcal{P}^{log}(\mathbb{R}^n)$ such that $p^+\le p_\infty$ and let $r(\cdot)$ be a function such that $0\le r^-\le r^+<\infty$. If $\phi(x,t)=t^{p(x)}(\log(e+t))^{r(x)}$, then the inequality $\|f\|_{\phi,Q}\le C_\epsilon\, a^{p(\cdot)+\epsilon}(f,Q)$ holds for every cube Q, for every function $f\in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ and $\epsilon>0$.*

As a consequence of this lemma we have the following version of the Theorem 1.4 by considering conditions that involves this type of Luxemburg averages. The proof is similar to the proof of theorem 1.4.

Theorem 1.9. *Let $p(\cdot),q(\cdot)\in\mathcal{P}^{log}(\mathbb{R}^n)$ such that $p(\cdot)\le q(\cdot)$. Let $\Phi\in\mathfrak{D}$, $b\in BMO$ and $m\in\mathbb{N}\cup\{0\}$. Let $A_m,B_m,C_m,D_m\in N(\mathbb{R}^n)$ be funtions that verify*

(i) $\mathbf{M}_{A_m}:L^{p(\cdot)}(\mathbb{R}^n)\hookrightarrow L^{p(\cdot)}(\mathbb{R}^n)$,
(ii) $\mathbf{M}_{\beta(\cdot),C_m}:L^{q'(\cdot)}(\mathbb{R}^n)\hookrightarrow L^{p'(\cdot)}(\mathbb{R}^n)$ *where $\beta(\cdot)$ is defined as in Lemma 2.7,*
(iii) $A_m^{-1}(x,t)B_m^{-1}(x,t)\le t/(\log(e+t))^m$ *and*
(iv) $C_m^{-1}(x,t)D_m^{-1}(x,t)\le t/(\log(e+t))^m$.

If (w,v) is any couple of weights such that $v\in L^{p(\cdot)}_{\mathrm{loc}}(\mathbb{R}^n)$, $w\in L^{q(\cdot)}_{\mathrm{loc}}(\mathbb{R}^n)$ and

$$\sup_Q\widetilde{\Phi}(\ell(Q))\left\|v^{-1}\right\|_{B_m,Q}\|w\|_{D_m,Q}\|\mathcal{X}_Q\|_{q(\cdot)}\|\mathcal{X}_Q\|_{p(\cdot)}^{-1}<\infty.$$

Then $T_\Phi^{b,m}:L_v^{p(\cdot)}\to L_w^{q(\cdot)}$.

2. Preliminaries

When we deal with variable Lebesgue spaces, we have the following known results that we shall be using along this paper.

Lemma 2.1. ([12]) *Let $s(\cdot), p(\cdot), q(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ be such that $1/s(\cdot) = 1/p(\cdot) + 1/q(\cdot)$. Then*

$$\|fg\|_{s(\cdot)} \leq 2\,\|f\|_{p(\cdot)}\,\|g\|_{q(\cdot)}\,. \tag{13}$$

Moreover, if $s(\cdot) \equiv 1$, the inequality above gives

$$\int_{\mathbb{R}^n} |f(y)g(y)|\,dy \leq 2\,\|f\|_{p(\cdot)}\,\|g\|_{p'(\cdot)} \tag{14}$$

which is an extension of the classical Hölder inequality.

Lemma 2.2. ([12]) *Let $p(\cdot) \in \mathcal{P}^{log}(\mathbb{R}^n)$. Then, for every cubes $Q \subset \mathbb{R}^n$, $\|\mathcal{X}_Q\|_{p(\cdot)}\,\|\mathcal{X}_Q\|_{p'(\cdot)} \simeq |Q|$.*

Lemma 2.3. ([12]) *Let $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ and $s > 0$ such that $sp^- \geq 1$. Then $\||f|^s\|_{p(\cdot)} = \|f\|_{sp(\cdot)}^s$.*

Lemma 2.4. ([14]) *Let $p(\cdot) \in \mathcal{P}^{log}(\mathbb{R}^n)$ such that $p^+ \leq p_\infty$. Then $\|\mathcal{X}_Q\|_{p(\cdot)} \lesssim |Q|^{1/p(x)}$, holds for every cube $Q \subset \mathbb{R}^n$ and a.e. $x \in Q$.*

Lemma 2.5. ([12]) *Let $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$. Then, for every cubes $Q \subset \mathbb{R}^n$, $\min\{1, |Q|\} \leq \|\mathcal{X}_Q\|_{p(\cdot)} \leq \max\{1, |Q|\}$.*

Lemma 2.6. ([12]) *Let $p(\cdot) \in \mathcal{P}^{\log}(\mathbb{R}^n)$ and $Q \subset \mathbb{R}^n$ a cube such that $|Q| \leq 2^n$. Then, if and $x \in Q$, $\|\mathcal{X}_Q\|_{p(\cdot)} \simeq |Q|^{1/p(x)}$. Moreover, for every cube $Q \subset \mathbb{R}^n$, $\|\mathcal{X}_Q\|_{p(\cdot)} \simeq |Q|^{(1/p)_Q}$.*

The result above allow us to obtain the following lemma.

Lemma 2.7. *Let $p(\cdot), q(\cdot) \in \mathcal{P}^{log}(\mathbb{R}^n)$ such that $p(\cdot) \leq q(\cdot)$. Suppose that $1/\beta(\cdot) = 1/p(\cdot) - 1/q(\cdot)$, then for every cubes $Q \subset \mathbb{R}^n$, $\|\mathcal{X}_Q\|_{p(\cdot)}\,\|\mathcal{X}_Q\|_{q(\cdot)}^{-1} \simeq \|\mathcal{X}_Q\|_{\beta(\cdot)}\,.$*

Lemma 2.8. ([15]) *Let $p(\cdot) \in \mathcal{P}^{log}(\mathbb{R}^n)$, $b \in BMO(\mathbb{R}^n)$ and k be a positive integer. Then*

$$\sup_Q \frac{\left\|\mathcal{X}_Q|b - b_Q|^k\right\|_{p(\cdot)}}{\|\mathcal{X}_Q\|_{p(\cdot)}} \lesssim \|b\|_{BMO}^k \quad \textit{and} \quad \sup_Q \frac{\left\|\mathcal{X}_{jQ}|b - b_{iQ}|^k\right\|_{p(\cdot)}}{\|\mathcal{X}_{jQ}\|_{p(\cdot)}} \lesssim \|b\|_{BMO}^k$$

$\forall j, i \in \mathbb{Z}$ with $j > i$.

Lemma 2.9. ([14]) *If $p(\cdot) \in \mathcal{P}^{log}(\mathbb{R}^n)$ there exists a positive constant C_p such that, for every cube $Q \subset \mathbb{R}^n$, $\|\mathcal{X}_{2Q}\|_{p(\cdot)} \leq C_p\,\|\mathcal{X}_Q\|_{p(\cdot)}\,.$*

3. Key lemmas

In order to state the following Lemma, let us recall that $\|\cdot\|_{p(\cdot),r(\cdot)} \doteq \|\cdot\|_{L^{p(\cdot)}(\log L)^{r(\cdot)}}$. Given $f \in L^1_{\text{loc}}(\mathbb{R}^n)$, $s(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ and Q a cube, with $M_{s(\cdot),Q}f$ we denote the average

$$M_{s(\cdot),Q}f = \|\mathcal{X}_Q f\|_{s(\cdot)}\|\mathcal{X}_Q\|_{s(\cdot)}^{-1}.$$

Lemma 3.1. *Let $p(\cdot), s(\cdot) \in \mathcal{P}^{log}(\mathbb{R}^n)$ such that $p(\cdot) \geq s(\cdot)$. Let $r(\cdot) \in \mathcal{P}^{loglog}(\mathbb{R}^n)$ such that $r(\cdot) \geq 0$. Let f be a nonnegative measurable function on $\mathbb{R}^n$ with $\|f\|_{p(\cdot),r(\cdot)} \leq 1$*

such that $f(x) \geq 1$ or $f(x) = 0$ for each $x \in \mathbb{R}^n$. If $Q \subset \mathbb{R}^n$ is a cube with $|Q| \leq 1$ such that $M_{s(\cdot),Q} f \geq 1$ and $x \in Q$, then

$$M_{s(\cdot),Q} f \leq \left(\frac{1}{|Q|} \int_Q \vartheta_{p(\cdot)}^{r(\cdot)}(y, f(y))\, dy \right)^{\frac{1}{p(x)}} \left(\log \left(e + \frac{1}{|Q|} \int_Q \vartheta_{p(\cdot)}^{r(\cdot)}(y, f(y))\, dy \right) \right)^{\frac{-r(x)}{p(x)}}.$$

In order to prove the Lemma 3.1 we previously show two useful estimates contained in the next lemma. For simplicity we introduce de following notation. If f is a function defined on $\mathbb{R}^n$, for each $x \in \mathbb{R}^n$ and $Q \subset \mathbb{R}^n$ a cube we define $I = I(f, Q) = M_{s(\cdot),Q} f$ and $J = J(f, Q) = \fint_Q \vartheta_{p(\cdot)}^{r(\cdot)}(y, f(y))\, dy$.

Lemma 3.2. *Let $f, p(\cdot), s(\cdot)$ and Q as in the hypotheses of Lemma 3.1. Then there exists two positive constants C_1 and C_2 such that*
1. *If $I \geq 1$ then $J \geq C_1$. and* ***2.*** *If $J \geq C_1$ then $I \leq C_2 J$.*

Proof. 1. By hypothesis we have that $\varrho_{s(\cdot)}(\mathcal{X}_Q f / \|\mathcal{X}_Q\|_{s(\cdot)}) \geq 1$. Since $|Q| \leq 1$, by Lemma 2.6, the hypotheses on f and the exponents we get

$$1 \lesssim |Q|^{-1} \varrho_{s(\cdot)}(\mathcal{X}_Q f) \lesssim |Q|^{-1} \varrho_{p(\cdot)}(\mathcal{X}_{Q \cap \{y | f(y) \geq 1\}} f) \lesssim J.$$

2. Since $C_1/J \leq 1$ as above we obtain

$$\begin{aligned} \int_Q \left(\frac{C_1 f(y)}{J \|\mathcal{X}_Q\|_{s(\cdot)}} \right)^{s(y)} dy &\leq \frac{C_1}{J} \int_Q \left(\frac{f(y)}{\|\mathcal{X}_Q\|_{s(\cdot)}} \right)^{s(y)} dy \\ &= \frac{C_1}{J} \int_{Q \cap \{y | f(y) \geq 1\}} \left(\frac{f(y)}{\|\mathcal{X}_Q\|_{s(\cdot)}} \right)^{s(y)} dy \lesssim \frac{1}{J|Q|} \int_Q \vartheta_{p(\cdot)}^{r(\cdot)}(y, f(y))\, dy \lesssim 1. \end{aligned}$$

□

In [16] the authors proved the following estimate that we shall also use in the proof of Lemma 3.1. In order to state it, if Q is a cube and $x \in Q$, we define

$$K \doteq J^{1/p(x)} (\log(e + J))^{-r(x)/p(x)}. \tag{15}$$

Lemma 3.3. *Let $p(\cdot) \in \mathcal{P}^{log}(\mathbb{R}^n)$ and $r(\cdot) \in \mathcal{P}^{loglog}(\mathbb{R}^n)$ such that $r(\cdot) \geq 0$. Let f be a nonnegative measurable function on $\mathbb{R}^n$ with $\|f\|_{p(\cdot),r(\cdot)} \leq 1$ and $Q \subset \mathbb{R}^n$ a cube. If $1 \lesssim J$, then for every $x, y \in Q$*

$$K^{-p(y)} \lesssim J^{-1} (\log(e + J))^{r(x)} \ \text{ and } \ (\log(e + K))^{-r(y)} \lesssim (\log(e + J))^{-r(x)}.$$

We can now proceed with the proof ol Lemma 3.1.

Proof of Lemma 3.1. Since $M_{s(\cdot),Q} f \geq 1$, by Lemma 3.2, we have $1 \lesssim J$. Let $x \in Q$ and let K be defined as in (15) then

$$\begin{aligned} I/K &= \|\mathcal{X}_Q f / K\|_{s(\cdot)} \|\mathcal{X}_Q\|_{s(\cdot)}^{-1} \\ &\leq \left(\left\|\mathcal{X}_{Q \cap \{y | f(y) \leq K\}} f / K\right\|_{s(\cdot)} + \left\|\mathcal{X}_{Q \cap \{y | f(y) \geq K\}} f / K\right\|_{s(\cdot)} \right) \|\mathcal{X}_Q\|_{s(\cdot)}^{-1} \\ &\leq 1 + \left\| \mathcal{X}_{Q \cap \{y | f(y) \geq K\}} \left(\frac{f}{K} \right)^{p(\cdot)/s^+} \left(\frac{\log(e + f)}{\log(e + K)} \right)^{r(\cdot)/s^+} \|\mathcal{X}_Q\|_{s(\cdot)}^{-1} \right\|_{s(\cdot)} \doteq 1 + \ \mathrm{A}\ . \end{aligned} \tag{16}$$

Let us estimate A. Since $|Q| \le 1$, by Lemma 2.6, we have

$$\int_{Q\cap\{y|f(y)\ge K\}} \left[\left(\frac{f(y)}{K}\right)^{p(y)/s^+}\left(\frac{\log(e+f(y))}{\log(e+K)}\right)^{r(y)/s^+}\|\mathcal{X}_Q\|_{s(\cdot)}^{-1}\right]^{s(y)} dy$$
$$\lesssim \frac{1}{|Q|}\int_{Q\cap\{y|f(y)\ge K\}} \left[\left(\frac{f(y)}{K}\right)^{p(y)}\left(\frac{\log(e+f(y))}{\log(e+K)}\right)^{r(y)}\right]^{s(y)/s^+} dy$$
$$\lesssim \frac{1}{|Q|}\int_{Q\cap\{y|f(y)\ge K\}} \left(\frac{f(y)}{K}\right)^{p(y)}\left(\frac{\log(e+f(y))}{\log(e+K)}\right)^{r(y)} dy.$$

By Lemma 3.3 and the estimate above we obtain that A $\lesssim 1$ and, by (16), it follows that $I \lesssim K$ as required. □

The following lemma was proved in [12] and we shall use it in the proof of Theorem 1.7.

Lemma 3.4. [[12],Theorem 7.3.27] *Let $p(\cdot), s(\cdot), l(\cdot) \in \mathcal{P}^{log}(\mathbb{R}^n)$ such that $p(\cdot) = s(\cdot)l(\cdot)$ and $l^- > 1$. Then for any $m > n$ there exists $\theta \in (0,1)$ such that*

$$(\theta M_{s(\cdot),Q} f)^{p(x)} \lesssim M(f^{p(\cdot)/l^-})(x))^{l^-} + (M(H_m^{l(\cdot)/l^-})(x))^{l^-} + 2H_m(x)^{l^-}$$

holds for every cube $Q \subset \mathbb{R}^n$, $x \in Q$ and every function f such that $\|f\|_{p(\cdot)} \le 1/2$, where $H_m(x) = (e+|x|)^{-m}$.

The following lemma is useful to prove the Theorem 1.4.

Lemma 3.5. *Let $p(\cdot) \in \mathcal{P}^{log}(\mathbb{R}^n)$, $\nu \in \mathbb{Z}$ and Q_0 a dyadic cube. If for $i \in \mathbb{N}$ we define $\mathcal{O}_i = \{iQ$ dyadic cube $: Q \subset Q_0$ and $\ell(Q) = 2^{-\nu}\}$, then*

$$\sum_{Q\in\mathcal{O}_1} \|f\mathcal{X}_{3Q}\|_{p(\cdot)} \|g\mathcal{X}_{3Q}\|_{p'(\cdot)} \lesssim \|f\mathcal{X}_{3Q_0}\|_{p(\cdot)} \|g\mathcal{X}_{3Q_0}\|_{p'(\cdot)} \tag{17}$$

for every $f \in L^{p(\cdot)}_{\text{loc}}(\mathbb{R}^n)$ and $g \in L^{p'(\cdot)}_{\text{loc}}(\mathbb{R}^n)$.

Proof. Let $f \in L^{p(\cdot)}_{\text{loc}}(\mathbb{R}^n)$ and $g \in L^{p'(\cdot)}_{\text{loc}}(\mathbb{R}^n)$. By Lemma 2.2 and Hölder's inequality (14) we have

$$\sum_{Q\in\mathcal{O}_1} \|f\mathcal{X}_{3Q}\|_{p(\cdot)} \|g\mathcal{X}_{3Q}\|_{p'(\cdot)} \lesssim \int_{\mathbb{R}^n} \sum_{Q\in\mathcal{O}_1} \mathcal{X}_{3Q}(x)\frac{\|f\mathcal{X}_{3Q}\|_{p(\cdot)}}{\|\mathcal{X}_{3Q}\|_{p(\cdot)}}\frac{\|g\mathcal{X}_{3Q}\|_{p'(\cdot)}}{\|\mathcal{X}_{3Q}\|_{p'(\cdot)}}\, dx$$
$$\lesssim \int_{\mathbb{R}^n}\left(\sum_{Q\in\mathcal{O}_1} \mathcal{X}_{3Q}(x)\frac{\|f\mathcal{X}_{3Q}\|_{p(\cdot)}}{\|\mathcal{X}_{3Q}\|_{p(\cdot)}}\right)\left(\sum_{Q\in\mathcal{O}_1} \mathcal{X}_{3Q}(x)\frac{\|g\mathcal{X}_{3Q}\|_{p'(\cdot)}}{\|\mathcal{X}_{3Q}\|_{p'(\cdot)}}\right) dx$$
$$= \int_{\mathbb{R}^n}\left(\sum_{Q\in\mathcal{O}_3} \mathcal{X}_{3Q}(x)\frac{\|f\mathcal{X}_{3Q}\|_{p(\cdot)}}{\|\mathcal{X}_{3Q}\|_{p(\cdot)}}\right)\left(\sum_{Q\in\mathcal{O}_3} \mathcal{X}_{3Q}(x)\frac{\|g\mathcal{X}_{3Q}\|_{p'(\cdot)}}{\|\mathcal{X}_{3Q}\|_{p'(\cdot)}}\right) dx$$
$$\lesssim \left\|\sum_{Q\in\mathcal{O}_3} \mathcal{X}_{3Q}\frac{\|f\mathcal{X}_{3Q_0}\mathcal{X}_{3Q}\|_{p(\cdot)}}{\|\mathcal{X}_{3Q}\|_{p(\cdot)}}\right\|_{p(\cdot)} \left\|\sum_{Q\in\mathcal{O}_3} \mathcal{X}_{3Q}\frac{\|g\mathcal{X}_{3Q_0}\mathcal{X}_{3Q}\|_{p'(\cdot)}}{\|\mathcal{X}_{3Q}\|_{p'(\cdot)}}\right\|_{p'(\cdot)}.$$

Then, by [[12], Corollary 7.3.21] we obtain (17). □

4. Proof of main results

Proof of Theorem 1.1. Since $v \in L^{p(\cdot)}_{\text{loc}}(\mathbb{R}^n)$ implies that the set of bounded functions with compact support is dense in $L^{p(\cdot)}_v(\mathbb{R}^n)$ and T_Φ is a positive operator, it is enough to show that $\|T_\Phi f\|_{L^{q(\cdot)}_w} \lesssim \|f\|_{L^{p(\cdot)}_v}$ for each nonnegative bounded function with compact support f. This is in turn equivalent by duality to $\int_{\mathbb{R}^n} g(x)w(x)T_\Phi f(x)\,dx \lesssim \|f\|_{L^{p(\cdot)}_v}$ for all nonnegative bounded functions with compact support f, g such that $\|g\|_{q'(\cdot)} \leq 1$.

It was proved in [2] that, if $\Phi \in \mathfrak{D}$, there exists a family of maximal nonoverlaping dyadic cubes $\{Q_{k,j}\}$, the Calderón-Zygmund cubes, such that, if we denote $R_{k,j} = \gamma Q_{k,j}$ where $\gamma = \max\{3, \delta(1+\varepsilon)\}$ with ε, δ the numbers provided by condition $\mathfrak{D}$, we have the following estimation

$$\int_{\mathbb{R}^n} gwT_\Phi f \lesssim \sum_{k,j} |Q_{k,j}|\widetilde{\Phi}(\ell(R_{k,j}))\frac{1}{|R_{k,j}|}\int_{R_{k,j}} f(z)\,dz\,\frac{1}{|R_{k,j}|}\int_{R_{k,j}} g(z)w(z)\,dz.$$

Let us denote $s = r(p^-)'$. By Hölder's inequality, the hypotheses on the weights and Lemma 2.7, the last sum can be estimated by a multiple of

$$\begin{aligned}\sum_{k,j}&|Q_{k,j}|\widetilde{\Phi}(\ell(R_{k,j}))M_{s',R_{k,j}}(fv)\,M_{s,R_{k,j}}(v^{-1})\,M_{(aq^+)',R_{k,j}}(g)\,M_{aq^+,R_{k,j}}(w)\\ &\simeq \sum_{k,j}|Q_{k,j}|M_{s',R_{k,j}}(fv)M_{\beta(\cdot),(aq^+)',R_{k,j}}(g). \qquad (18)\end{aligned}$$

We shall use the following properties of Calderón-Zygmund cubes. For each $k, j \in \mathbb{Z}$ we can consider the sets $D_k = \bigcup_j Q_{k,j}$ and $F_{k,j} = Q_{k,j}\backslash(Q_{k,j}\cap D_{k+1})$. Then $\{F_{k,j}\}$ is a disjoint family of sets which satisfy $|Q_{k,j}| \lesssim |F_{k,j}|$. Then we can estimate (18) by a multiple of

$$\begin{aligned}\sum_{k,j}|F_{k,j}|M_{s',R_{k,j}}(fv)M_{\beta(\cdot),(aq^+)',R_{k,j}}(g) &\leq \int_{\mathbb{R}^n} M_{s'}(fv)(y)M_{\beta(\cdot),(aq^+)'}(g)(y)dy\\ &\lesssim \|M_{s'}(fv)\|_{p(\cdot)}\left\|M_{\beta(\cdot),(aq^+)'}(g)\right\|_{p'(\cdot)} \lesssim \|fv\|_{p(\cdot)}\end{aligned}$$

where we have used that, by [[12],Theorem 7.3.27], $M_{s'} : L^{p(\cdot)} \hookrightarrow L^{p(\cdot)}$ since $p^- > s'$ and, by Theorem 1.7, $M_{\beta(\cdot),(aq^+)'} : L^{q'(\cdot)} \hookrightarrow L^{p'(\cdot)}$. □

Proof of Theorem 1.2. The proof follow as in the proof of Theorem 1.1 replacing the exponents s ans l by $s(\cdot) = rp'(\cdot)$ and $l(\cdot) = aq(\cdot)$ respectively to obtain

$$\int_{\mathbb{R}^n} gwT_\Phi f \lesssim \left\|M_{s'(\cdot)}(fv)\right\|_{p(\cdot)}\left\|M_{\beta(\cdot),l'(\cdot)}g\right\|_{p'(\cdot)} \lesssim \|fv\|_{p(\cdot)}$$

where we have used that, by [[12],Theorem 7.3.27], $M_{s'(\cdot)} : L^{p(\cdot)} \hookrightarrow L^{p(\cdot)}$ since $p^- > (s')^+$ and, by Theorem 1.7, $M_{\beta(\cdot),l'(\cdot)} : L^{q'(\cdot)} \hookrightarrow L^{p'(\cdot)}$, since $(q')^- > (l')^+$. □

We first prove Theorem 1.4 since Theorem 1.3 follows similar arguments with minor changes.

Proof of Theorem 1.4. Since $v \in L^{p(\cdot)}_{\text{loc}}(\mathbb{R}^n)$ implies that the set of bounded functions with compact support is dense in $L^{p(\cdot)}_v(\mathbb{R}^n)$ and $|T^{b,m}_\Phi|$ is a positive operator, it is enough to show that $\left\|T^{b,m}_\Phi f\right\|_{L^{q(\cdot)}_w} \lesssim \|f\|_{L^{p(\cdot)}_v}$ for each nonnegative bounded function with compact support f. Moreover, by duality this is equivalent to prove that $\int_{\mathbb{R}^n} g(x)w(x)|T^{b,m}_\Phi f(x)|\,dx, \lesssim \|f\|_{L^{p(\cdot)}_v}$ for all nonnegative bounded functions with compact support f, g such that $\|g\|_{q'(\cdot)} \leq 1$.

Let $\overline{\Phi}$ be the function defined by $\overline{\Phi}(t) = \sup_{t<|x|\leq 2t}\Phi(x)$, for every $t > 0$. It was proved in [5] that, if $\Phi \in \mathfrak{D}$, we can estimate

$$\int_{\mathbb{R}^n} gw|T^{b,m}_\Phi f| \lesssim \sum_Q \overline{\Phi}\left(\frac{\ell(Q)}{2}\right)\sum_{j=0}^m \int_{3Q}|b-b_Q|^j f\int_Q |b-b_Q|^{m-j}g, \tag{19}$$

where $b_Q = |Q|^{-1}\int_Q b(z)dz$ and the sum is taking over all dyadic cubes of $\mathbb{R}^n$.

Let us denote $s(\cdot) \doteq rp'(\cdot)$ and $l \doteq aq(\cdot)$. Since $(p')^+ < r(p')^-$ and $q^+ < aq^-$ then $(s')^+ < p^-$ y $(l')^+ < (q^+)'$. Let ϖ, u two constants such that $(s')^+ < \varpi < p^-$ and $(l')^+ < u < (q^+)'$, and $\omega(\cdot), \tau(\cdot)$ defined by $1/\omega(\cdot) = 1/s(\cdot) + 1/\varpi$ and $1/\tau(\cdot) = 1/l(\cdot) + 1/u$. Observe that $\omega(\cdot), \tau(\cdot) \in \mathcal{P}^{log}(\mathbb{R}^n)$ since $s(\cdot), l(\cdot) \in \mathcal{P}^{log}(\mathbb{R}^n)$. By Hölder's inequality (14), Lemmas 2.2, 2.3 and 2.8 we can estimate (19) by a multiple of

$$\begin{aligned}
&\sum_Q \overline{\Phi}\left(\frac{\ell(Q)}{2}\right)\sum_{j=0}^m |3Q|\frac{\left\|\mathcal{X}_{3Q}|b-b_Q|^j\right\|_{\omega'(\cdot)}}{\|\mathcal{X}_{3Q}\|_{\omega'(\cdot)}}\frac{\|\mathcal{X}_{3Q}f\|_{\omega(\cdot)}}{\|\mathcal{X}_{3Q}\|_{\omega(\cdot)}}\\
&\qquad\times |Q|\frac{\left\|\mathcal{X}_Q|b-b_Q|^{m-j}\right\|_{\tau'(\cdot)}}{\|\mathcal{X}_Q\|_{\tau'(\cdot)}}\frac{\|\mathcal{X}_Q gw\|_{\tau(\cdot)}}{\|\mathcal{X}_Q\|_{\tau(\cdot)}}\\
&\lesssim \|b\|^m_{BMO}\sum_Q \overline{\Phi}\left(\frac{\ell(Q)}{2}\right)|3Q|\frac{\|\mathcal{X}_{3Q}f\|_{\omega(\cdot)}}{\|\mathcal{X}_{3Q}\|_{\omega(\cdot)}}|Q|\frac{\|\mathcal{X}_Q gw\|_{\tau(\cdot)}}{\|\mathcal{X}_Q\|_{\tau(\cdot)}}.
\end{aligned} \tag{20}$$

Since g has compact support and $w \in L^{aq(\cdot)}_{\text{loc}}(\mathbb{R}^n)$, $\lim_{\ell(Q)\to\infty} M_{\tau(\cdot),Q}(gw) = 0$. Let C_τ be the constant provided by Lemma 2.9. If $\alpha \geq C_\tau$ and $k \in \mathbb{Z}$, it follows that, if for some dyadic cube Q,

$$\alpha^k < M_{\tau(\cdot),Q}(gw), \tag{21}$$

then Q is contained in dyadic cubes satisfying this condition, which are maximal with respect to the inclusion. Thus, for each integer k there is a family of maximal nonoverlapping dyadic cubes $\{Q_{k,j}\}$ satisfying (21). Let $Q'_{k,j}$ be the dyadic cube containing $Q_{k,j}$ with sidelength $2\ell(Q_{k,j})$. Then, by maximality and Lemma 2.9, we have

$$\alpha^k < M_{\tau(\cdot),Q_{k,j}}(gw) \leq \left\|\mathcal{X}_{Q'_{k,j}}\right\|_{\tau(\cdot)}\|\mathcal{X}_{Q_{k,j}}\|^{-1}_{\tau(\cdot)}M_{\tau(\cdot),Q'_{k,j}}(gw) \leq C_\tau\,\alpha^k \leq \alpha^{k+1}.$$

For $k \in \mathbb{Z}$ we define the set $\mathcal{C}_k = \{Q \text{ dyadic} : \alpha^k < M_{\tau(\cdot),Q}(gw) \leq \alpha^{k+1}\}$. Then every dyadic cube Q for which $M_{\tau(\cdot),Q}(gw) \neq 0$ belongs to exactly one $\mathcal{C}_k$. Furthermore, if $Q \in \mathcal{C}_k$, it follows that $Q \subset Q_{k,j}$ for some j. Then from (20) we obtain that

$$\int_{\mathbb{R}^n} gw|T_\Phi^{b,m}| \lesssim \|b\|_{BMO}^m \sum_k \sum_{Q\in\mathcal{C}_k} \overline{\Phi}\left(\ell(Q)/2\right)|3Q|M_{\omega(\cdot),3Q}(f)|Q|M_{\tau(\cdot),Q}(gw)$$
$$\lesssim \|b\|_{BMO}^m \sum_{k,j} \alpha^k \sum_{Q\in\mathcal{C}_k\,:\,Q\subset Q_{k,j}} \overline{\Phi}\left(\ell(Q)/2\right)|3Q||Q|M_{\omega(\cdot),3Q}(f)$$
$$\lesssim \|b\|_{BMO}^m \sum_{k,j} M_{\tau(\cdot),Q_{k,j}}(gw) \sum_{Q\in\mathcal{C}_k\,:\,Q\subset Q_{k,j}} \overline{\Phi}\left(\ell(Q)/2\right)|3Q||Q|M_{\omega(\cdot),3Q}(f). \quad (22)$$

If we show that there is a constant C_Φ such that, for any dyadic cube Q_0,

$$\sum_{Q\,:\,Q\subset Q_0} \overline{\Phi}\left(\ell(Q)/2\right)|3Q||Q|M_{\omega(\cdot),3Q}(f) \leq C_\Phi\tilde{\Phi}(\delta(1+\varepsilon)\ell(Q_0))|3Q_0|M_{\omega(\cdot),3Q_0}(f), \quad (23)$$

with ε, δ the numbers provided by condition $\mathfrak{D}$, from (4.6) we obtain that

$$\int_{\mathbb{R}^n} gw|T_\Phi^{b,m}f| \lesssim \|b\|_{BMO}^m \sum_{k,j} \tilde{\Phi}(\delta(1+\varepsilon)\ell(Q_{k,j}))|3Q_{k,j}|M_{\omega(\cdot),3Q_{k,j}}(f)M_{\tau(\cdot),Q_{k,j}}(gw).$$

Let $\gamma = \max\{3, \delta(1+\varepsilon)\}$, we denote $R_{k,j} = \gamma Q_{k,j}$. Then, by Lemmas 2.9, 2.2, Hölder's inequality (13), the hypothesis on the weights and Lemma 2.7 we have

$$\int_{\mathbb{R}^n} gw|T_\Phi^{b,m}f| \lesssim \|b\|_{BMO}^m \sum_{k,j} \tilde{\Phi}(\ell(R_{k,j}))|R_{k,j}|M_{\omega(\cdot),R_{k,j}}(f)M_{\tau(\cdot),R_{k,j}}(gw)$$
$$\lesssim \|b\|_{BMO}^m \sum_{k,j} \tilde{\Phi}(\ell(R_{k,j}))|R_{k,j}|M_{\varpi,R_{k,j}}(fv)M_{s(\cdot),R_{k,j}}(v^{-1})M_{u,R_{k,j}}(g)M_{l(\cdot),R_{k,j}}(w)$$
$$\simeq \|b\|_{BMO}^m \sum_{k,j} |Q_{k,j}|M_{\varpi,R_{k,j}}(fv)M_{\beta(\cdot),u,R_{k,j}}(g).$$

As before, we take a disjoint family of sets $\{F_{k,j}\}$ such that $|Q_{k,j}| \lesssim |F_{k,j}|$. Then

$$\int_{\mathbb{R}^n} gw|T_\Phi^{b,m}f| \lesssim \|b\|_{BMO}^m \sum_{k,j} |F_{k,j}|M_{\varpi,R_{k,j}}(fv)M_{\beta(\cdot),u,R_{k,j}}(g)$$
$$\lesssim \|b\|_{BMO}^m \int_{\mathbb{R}^n} M_\varpi(fv)M_{\beta(\cdot),u}g$$
$$\lesssim \|b\|_{BMO}^m \|M_\varpi(fv)\|_{p(\cdot)} \left\|M_{\beta(\cdot),u}(g)\right\|_{p'(\cdot)} \lesssim \|b\|_{BMO}^m \|fv\|_{p(\cdot)},$$

where we have used that by [[12],Theorem 7.3.27], $M_\varpi : L^{p(\cdot)} \hookrightarrow L^{p(\cdot)}$ since $p^- > \varpi$, and by Theorem 1.7, $M_{\beta(\cdot),u} : L^{q'(\cdot)} \hookrightarrow L^{p'(\cdot)}$ since $(q')^- > u$.

In order to complete the proof let us prove (23). In fact, if $\ell(Q_0) = 2^{-\nu_0}$ with $\nu_0 \in \mathbb{Z}$, by Lemmas 2.2 and 3.5 we have

$$\sum_{Q\,:\,Q\subset Q_0} \overline{\Phi}\left(\ell(Q)/2\right)|Q||3Q|\,M_{\omega(\cdot),3Q}(f)$$
$$\lesssim \sum_{\nu\geq\nu_0} \overline{\Phi}(2^{-\nu-1})2^{-\nu n} \sum_{Q\subset Q_0\,:\,\ell(Q)=2^{-\nu}} \|f\mathcal{X}_{3Q}\|_{\omega(\cdot)} \|\mathcal{X}_{3Q}\|_{\omega'(\cdot)}$$
$$\lesssim \|f\mathcal{X}_{3Q_0}\|_{\omega(\cdot)} \|\mathcal{X}_{3Q_0}\|_{\omega'(\cdot)} \sum_{\nu\geq\nu_0} \overline{\Phi}(2^{-\nu-1})2^{-\nu n}$$
$$\lesssim \|f\mathcal{X}_{3Q_0}\|_{\omega(\cdot)} \|\mathcal{X}_{3Q_0}\|_{\omega'(\cdot)} \tilde{\Phi}(\delta(1+\varepsilon)\ell(Q_0)),$$

where the last estimation was proved in [2]. This conclude (23). $\square$

Proof of Theorem 1.3. The proof follow as in the proof of Theorem 1.4 replacing the exponents $s(\cdot)$ and $l(\cdot)$ by $s \doteq r(p^-)'$ and $l \doteq aq^+$ respectively. In fact, we have $s' < p^-$ and $l' < (q^+)'$ and we can take ϖ, u, ω, τ constants such that $s' < \varpi < p^-$, $l' < u < (q^+)'$, $1/\omega = 1/s + 1/\varpi$ and $1/\tau = 1/l + 1/u$. $\square$

Proof of Theorem 1.5. Let $f \in L^{p(\cdot)}(\log L)^{r(\cdot)}(\mathbb{R}^n)$ with $\|f\|_{p(\cdot),r(\cdot)} \leq 1$. It is enough to prove that $\int_{\mathbb{R}^n} \vartheta_{p(\cdot)}^{r(\cdot)}(x, M_{s(\cdot)}f(x))\, dx \lesssim 1$, where $\vartheta_{p(\cdot)}^{r(\cdot)}$ is defined as in (11).

Without lost of generality we can suppose that f is nonnegative. Define $f_1 = f\mathcal{X}_{\{x:f(x)\leq 1\}}$ and $f_2 = f - f_1$. Since, for each fixed x, the function $\vartheta_{p(\cdot)}^{r(\cdot)}(x, \cdot)$ is convex we have

$$\begin{aligned}\int_{\mathbb{R}^n} \vartheta_{p(\cdot)}^{r(\cdot)}(x, M_{s(\cdot)}f(x))dx &\lesssim \int_{\mathbb{R}^n} \vartheta_{p(\cdot)}^{r(\cdot)}(x, M_{s(\cdot)}f_1(x))dx + \int_{\mathbb{R}^n} \vartheta_{p(\cdot)}^{r(\cdot)}(x, M_{s(\cdot)}f_2(x))dx \\ &\doteq \mathrm{I} + \mathrm{II}.\end{aligned}$$

Fix $x \in \mathbb{R}^n$ and let $Q \subset \mathbb{R}^n$ a cube such that $x \in Q$. Then $M_{s(\cdot),Q}f_1 \leq 1$ which implies that $\log(e + M_{s(\cdot),Q}f_1) \lesssim 1$. Then, for a fix contant $m > n$, by Lemma 3.4, there exist $\theta \in (0, 1)$ such that

$$\begin{aligned}(M_{s(\cdot),Q}f_1)^{p(x)}(\log(e + M_{s(\cdot),Q}f_1))^{r(x)} &\lesssim (\theta M_{s(\cdot),Q}f_1)^{p(x)} \\ &\lesssim (M(f^{p(\cdot)/l^-})(x))^{l^-} + (M(H_m^{l(\cdot)/l^-})(x))^{l^-} + 2H_m(x)^{l^-}\end{aligned}$$

where $H_m(x) = (e + |x|)^{-m}$. By taking supremum over all cubes Q containing x and using that $l^- > 1$, we get that

$$\begin{aligned}\mathrm{I} &\lesssim \int_{\mathbb{R}^n} (M(f^{p(\cdot)/l^-})(x))^{l^-} dx + \int_{\mathbb{R}^n} (M(H_m^{l(\cdot)/l^-})(x))^{l^-} dx + 2\int_{\mathbb{R}^n} H_m(x)^{l^-} dx \\ &\lesssim \int_{\mathbb{R}^n} f(x)^{p(x)}\, dx + \int_{\mathbb{R}^n} H_m(x)^{l^-}\, dx \lesssim 1.\end{aligned}$$

Fix $x \in \mathbb{R}^n$ and let $\mathcal{A}, \mathcal{B}$ and $\mathcal{C}$ the sets defined by

$$\begin{aligned}\mathcal{A} &= \{Q \subset \mathbb{R}^n : x \in Q \ \text{ and } \ M_{s(\cdot),Q}f_2 < 1\}, \\ \mathcal{B} &= \{Q \subset \mathbb{R}^n : x \in Q, \ M_{s(\cdot),Q}f_2 \geq 1 \ \text{ and } \ |Q| \geq 1\}, \\ \mathcal{C} &= \{Q \subset \mathbb{R}^n : x \in Q, \ M_{s(\cdot),Q}f_2 \geq 1 \ \text{ and } \ |Q| < 1\}.\end{aligned}$$

In order to estimate II, we first observe that

$$\begin{aligned}M_{s(\cdot)}f_2(x) &\leq \sup_{\mathcal{A}} M_{s(\cdot),Q}f_2 + \sup_{\mathcal{B}} M_{s(\cdot),Q}f_2 + \sup_{\mathcal{C}} M_{s(\cdot),Q}f_2 \\ &\doteq M^1 f_2(x) + M^2 f_2(x) + M^3 f_2(x).\end{aligned}$$

The estimation of $M^1 f_2(x)$ follows similar arguments as in I. If $Q \in \mathcal{B}$, from Lemma 2.5, we have that $M_{s(\cdot),Q}f_2 = 1$. In fact, $\|\mathcal{X}_Q\|_{s(\cdot)} \geq 1$ implies that $M_{s(\cdot),Q}f_2 \leq \|\mathcal{X}_Q f_2\|_{s(\cdot)}$. But, since $f_2 \geq 1$ or $f_2 = 0$, we have

$$\begin{aligned}\int_{\mathbb{R}^n} f_2(y)^{s(y)}\, dy &= \int_{\mathbb{R}^n \cap \{y | f_2(y) \geq 1\}} f_2(y)^{s(y)}\, dy + \int_{\mathbb{R}^n \cap \{y | f_2(y) = 0\}} f_2(y)^{s(y)}\, dy \\ &\leq \int_{\mathbb{R}^n \cap \{y | f_2(y) \geq 1\}} f_2(y)^{p(y)}\, dy \leq \int_{\mathbb{R}^n} \vartheta_{p(\cdot)}^{r(\cdot)}(y, f_2(y))\, dy \leq 1.\end{aligned}$$

Then $\|\mathcal{X}_Q f_2\|_{s(\cdot)} \le 1$ and we can deduce $M_{s(\cdot),Q} f_2 = 1$. Then we can proceed as in the previous case. Let $Q \in \mathcal{C}$. Given $1 < p_1 < l^-$, by Lemma 3.1 with $p(\cdot)$ and $r(\cdot)$ replaced by $p(\cdot)/p_1$ and $r(\cdot)/p_1$ respectively, we get that

$$M_{s(\cdot),Q} f_2 \lesssim \left(\frac{1}{|Q|}\int_Q \vartheta^{r(\cdot)/p_1}_{p(\cdot)/p_1}(y, f_2(y))\,dy\right)^{p_1/p(x)} (\log(e+J))^{-r(x)/p(x)}$$

where J is defined as in Lemma 3.1. Then, by Lemma 3.2, we obtain

$$\vartheta^{r(\cdot)}_{p(\cdot)}(x, M_{s(\cdot),Q} f_2) \lesssim \left(M(\vartheta^{r(\cdot)/p_1}_{p(\cdot)/p_1}(\cdot, f_2(\cdot)))(x)\right)^{p_1}.$$

By taking supremum over all cubes $Q \in \mathcal{C}$ and noting that $p_1 > 1$ we get that

$$\begin{aligned}\int_{\mathbb{R}^n} \vartheta^{r(\cdot)}_{p(\cdot)}(x, M_{s(\cdot)} f_2(x))\,dx &\lesssim \int_{\mathbb{R}^n} \left(M(\vartheta^{r(\cdot)/p_1}_{p(\cdot)/p_1}(\cdot, f_2(\cdot)))(x)\right)^{p_1}\,dx \\ &\lesssim \int_{\mathbb{R}^n} \vartheta^{r(\cdot)}_{p(\cdot)}(x, f_2(x))\,dx \lesssim 1.\end{aligned}$$

We are done. □

Proof of Lemma 1.6. Let $x \in \mathbb{R}^n$ and $Q \subset \mathbb{R}^n$ be a cube such that $x \in Q$. Let $s(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ which satisfies $s^+ < \beta^-$, by taking into account that $1/\beta(\cdot) = 1/s(\cdot) - 1/[(\beta(\cdot)/s(\cdot))' s(\cdot)]$, by Lemma 2.7, $\|\mathcal{X}_Q\|_{\beta(\cdot)} \|\mathcal{X}_Q\|_{(\beta(\cdot)/s(\cdot))' s(\cdot)} \simeq \|\mathcal{X}_Q\|_{s(\cdot)}$. Noting that $p(\cdot)/q(\cdot) + p(\cdot)/\beta(\cdot) = 1$, by Hölder's inequality (13) we have

$$\begin{aligned}\|\mathcal{X}_Q\|_{\beta(\cdot)} M_{s(\cdot),Q}(g) &= \|\mathcal{X}_Q\|_{\beta(\cdot)} M_{s(\cdot),Q}(g^{p(\cdot)/q(\cdot)+p(\cdot)/\beta(\cdot)}) \\ &\lesssim M_{(\beta(\cdot)/s(\cdot))' s(\cdot)}(g^{p(\cdot)/q(\cdot)})(x) \left\|g^{p(\cdot)/\beta(\cdot)}\right\|_{\beta(\cdot)}.\end{aligned}$$

Thus the desired inequality follows inmediately. □

This following remark will be usefull in the proof of the Theorem 1.7.

Remark 2. Let $p(\cdot), q(\cdot), \beta(\cdot)$ as in the hypothesis of Lemma 2.7 and $s(\cdot)$ be a function such that $1 \le s^- \le s^+ < p^-$. Suppose that $l(\cdot) = q(\cdot)/[(\beta(\cdot)/s(\cdot))' s(\cdot)]$. Then $l^- > 1$. In fact, since $s^+ < p^-$, there exists $\varepsilon > 0$ such that $s^+(1+\varepsilon) < p^-$. Then, for $x \in \mathbb{R}^n$, $s(x) < s(x)(1+\varepsilon) < p(x)$ and $s(x)/[q(x) - s(x)(1+\varepsilon] < p(x)(q(x) - p(x))$, which implies that $q(x)/s(x) > (1+\varepsilon)\,(\beta(x)/s(x))'$ and then $l(x) > 1+\varepsilon > 1$ which proves that $l^- > 1$.

Proof of Theorem 1.7. Let $g \in L^{p(\cdot)}(\log L)^{r(\cdot)}(\mathbb{R}^n)$ such that $\|g\|_{p(\cdot),r(\cdot)} \le 1$. By Lemma 1.6 we have

$$\left\|M_{\beta(\cdot),s(\cdot)} g\right\|_{q(\cdot),r(\cdot)} \lesssim \left\|M_{(\beta(\cdot)/s(\cdot))' s(\cdot)}(g^{p(\cdot)/q(\cdot)})\right\|_{q(\cdot),r(\cdot)} \left\|g^{p(\cdot)/\beta(\cdot)}\right\|_{\beta(\cdot)}$$

Note that, if $g \in L^{p(\cdot)}(\log L)^{r(\cdot)}(\mathbb{R}^n)$ then $g \in L^{p(\cdot)}(\mathbb{R}^n)$. On other hand, by Theorem 1.5 with p and s replaced by q and $(\beta/s)'s$ respectively and Remark 2 we obtain that $M_{(\beta(\cdot)/s(\cdot))' s(\cdot)}$ is bounded in $L^{q(\cdot)}(\log L)^{r(\cdot)}(\mathbb{R}^n)$. Then we have that

$$\left\|M_{\beta(\cdot),s(\cdot)} g\right\|_{q(\cdot),r(\cdot)} \lesssim \left\|g^{p(\cdot)/q(\cdot)}\right\|_{q(\cdot),r(\cdot)} \lesssim \|g\|_{p(\cdot),r(\cdot)}$$

and we conclude de proof. □

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