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Cooperation between fluctuations and spatial coupling for two symmetry breakings in a gradient system

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A zero-dimensional system that is affected by field-dependent fluctuations evolves toward the field's values in which the fluctuations' effect is minimized. For a high enough noise intensity, it causes an exchange of roles between the stable and unstable state. In this paper, we report symmetry breaking in two stable states, but one of them stabilized by the fluctuations while exchanging its role with a previously stable state.

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I. INTRODUCTION

Constructive effects of the noise have been, and even today are, a very active line of research. Some examples are stochastic resonance [1–10], noise-induced transport [11–18], and noise-induced transitions [19]. The extended systems are of particular interest, where noise, in cooperation with the spatial coupling, results in phase transitions and structure formation [20–39]. Some of these phenomena originate from a short time instability [20–24], while others originate from an entropic mechanism [25,28–39], which was first reported in Ref. [25] and used in many others investigations [28–39]. In said paper, a class of exactly soluble models was introduced, exhibiting an ordering noise-induced phase transition in which the order arises as a result of a balance between the relaxing deterministic dynamics and the randomizing character of the fluctuations. In this regard, we highlight a paragraph that seems significant: "A simple interpretation can be found in terms of a balance between the role of the deterministic monostable local potential, which tends to take the system towards the disordered phase, and the stochastic motion, which is due to the fact that fluctuations are more intense in the disordered phase, and thus *push the system* away from it. The presence of a spatial coupling K helps to break the symmetry[1] of the homogeneous state." The fact that a noise may push a system designates a kind of dynamic, an average "force" that acts on the homogeneous state of the system. We use this concept in a series of works in order to displace the homogeneous state[2] toward a constructive field value region, meaning a region where any inhomogeneity destabilizes the homogeneous state, resulting in the formation of structures [35–39]. However, we did a comprehensive study on the subject, in which we showed that a multiplicative noise that corresponds to a given dynamic is indeed able to push the homogeneous state (HS) toward predetermined field values by the noise factor [40]. Moreover, we found that the effect's strength increases with the negative slope of the multiplicative factor [40].

Nature displays many cases of physical interest where the role of fluctuating parameters can be considered by a multiplicative noise [32,41]. In this regard, we recently reported a study about the generalized Nagumo model's parameter fluctuation effects (the extinction option is replaced by one of low density homogeneous population) [42]. These fluctuations can be mimicked by a multiplicative noise with a factor that depends on the positive parabola-shaped field. First, we solved the problem analytically by mapping it as a nongradient relaxational dynamic in a free-energy function with a field-dependent kinetic coefficient, and then we checked the results numerically. We found that, for high enough noise intensities, the noise-affected state exchanges its role with the unstable state (called adversity); therefore, the previously unstable state becomes stable, and the one affected by the fluctuations plays the role of the unstable one. This transition is another example originated by an entropic mechanism and must be interpreted as mentioned above. The HS is determined by the balance between the deterministic force and the fluctuations which drive the system toward field values where the noise's effect is minimized. We note that the transition can occur not only in the Stratonovich interpretation but also in the Itô interpretation [28]. A more interesting situation can arise by introducing spatial coupling between nearest neighboring sites (diffusion term for a spatially continuous system). Therefore, in this work, we turned the zero-dimensional system into an extended one by adding this coupling, and, as a result, we found interesting phase transitions.

To sum up, in a previous work, and using a method inspired by Ibañes *et al.* [25], in which a generalization of Nagumo model (zero-dimensional) was studied, we found interesting state transitions which are driven by the parameter fluctuations that characterize the aforementioned model. In this paper, we reported effects of spatial coupling between nearest individuals in the already strongly modified dynamic by such fluctuations.

In the first section, we explain our proposal and results for the zero-dimensional case. A mean-field analysis is carried out in Sec. II, in which we obtain evidence of two symmetry breakings. In the following section, we show numerical results which confirm the result predicted by the mean-field approximation. Finally, in the last section, we analyze our results, and we expose the corresponding conclusions.

II. MODEL AND PREVIOUS RESULTS

By applying the Nagumo model, we consider that each individual requires a minimum vital space (a space that cannot be invaded by another individual). Thus, we define

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[1]The translational symmetry is broken resulting in a true ordering phase transition (a disordered state becomes unstable, and two more ordered stable states arise).

[2]State: stationary solution for a given system.

u as the covering of the space available (with u a dimensionless variable normalized to 1). As aforementioned, we also consider a low-density homogeneous population: $u = \beta b$ as an alternative proposal to extinction. Under these conditions, the modified Nagumo nonlinearity can be expressed as $F(u) = (u - \beta b)(u - \alpha b)(b - u)$. This also shows two uniform attractors ($u = \beta b$ and $u = b$) and an ejector (αb), where α is known as the *adversity factor*. The ejector $u = \alpha b$ marks a limit between domains (low and high population density). When u is lower than this value, the system evolves toward the low population density $u = \beta b$, and when u is higher, the system evolves toward a population with density $u = b$.

Natural systems are undeniably subject to random fluctuations, arising from either environmental variability or thermal effects. In this regard, we choose to consider the fluctuation effects of the background population density ($u = \beta b$), by turning the β parameter in a stochastic variable using a Gaussian white noise with zero mean and correlation:

$$\langle \eta(t)\eta(t')\rangle = 2\lambda^2\delta(t - t'). \tag{1}$$

Thus, by describing β fluctuations as $\beta = \beta_0 + \eta(t)$, the modified zero-dimensional dynamic that the Nagumo model imposes can be expressed as

$$\partial_t u = F_{\beta_o}(u) + \Gamma^{1/2}(u)\,\eta(t), \tag{2}$$

where $F_{\beta_o}(u) = (u - \beta_0 b)(u - \alpha b)(b - u)$ and $\Gamma^{1/2}(u) = -b(u - \alpha b)(b - u)$ [Eq. (2) is to be interpreted in the Stratonovich sense]. We note that $\Gamma^{1/2}(u)$ is a parabola with positive curvature and has a minimum value located right at the midpoint between the two roots which are not affected by fluctuations ($u_{\min} = b\frac{1+\alpha}{2}$).

According to our proposal (to introduce a spatial coupling between nearest individuals), we add a diffusion term to the dynamic. Therefore, the dynamic is modified as

$$\partial_t u = F_{\beta_o}(u) + D\partial_x^2 u + \Gamma^{1/2}(u)\,\eta(x,t), \tag{3}$$

where the correlation is now $\langle \eta(x,t)\eta(x,t)\rangle = 2\lambda^2\delta(x - x')\delta(t - t')$.[3]

In order to turn the system into a nongradient relaxational one, we first take Eq. (2) in the absence of fluctuations (nor spatial coupling) and multiply and divide $F_{\beta_o}(u)$ by $-\Gamma(u)$. Thus, we can force a dynamic nongradient relaxational in some free-energy function $\mathcal{F}(u)$ with a field-dependent kinetic coefficient $\Gamma(u)$. The fictitious relaxation function so defined is written as

$$\mathcal{F}(u) = -\int du \frac{F_{\beta_o}(u)}{\Gamma(u)} = \ln\left\{\frac{|u - b|^{\frac{1-\beta_0}{(1-\alpha)b^2}}}{|u - \alpha b|^{\frac{\alpha-\beta_0}{(1-\alpha)b^2}}}\right\}. \tag{4}$$

It is clear that the fictitious relaxation function does not include the solution to be affected by fluctuations. However, the real dynamic does not change because of the action of the relaxation coefficient which reintroduces the aforementioned solution.

Therefore, Eq. (2) written in terms of the fictitious relaxation function is

$$\partial_t u = -\Gamma(u)\frac{d\mathcal{F}(u)}{du} + \Gamma^{1/2}(u)\,\eta(t). \tag{5}$$

When raising this issue, we observe that the fluctuations fulfill the fluctuation-dissipation theorem [25,43,44].

Under these conditions, the stationary probability distribution function (SPDF) for the field $P_{\text{th}}(u)$ is of the Boltzmann's type and can be described by effective relaxation function [25,28]:

$$P_{\text{th}}(u) \propto \exp\left\{-\frac{\mathcal{F}_{\text{eff}}(u)}{\lambda^2}\right\} = \frac{|u - \alpha b|^{e^{\alpha b}}}{b^2|u - b|^{e^b}}, \tag{6}$$

where $\mathcal{F}_{\text{eff}}(u) = \mathcal{F}(u) + \frac{\lambda^2}{2}\ln[\Gamma(u)]$, $e^{\alpha b} = 2\frac{\alpha-\beta_0-(1-\alpha)b^2\lambda^2}{(1-\alpha)b^2\lambda^2}$, and $e^b = 2\frac{1-\beta_0+(1-\alpha)b^2\lambda^2}{(1-\alpha)b^2\lambda^2}$.

We have already reported these results as well as the results of the simulation of stochastic process described by Eq. (2) and the corresponding numeric calculation of the stationary probability density $P_{\text{th}}(u)$ [37]. All these confirm the analytic result of the expression (6). Specially, we observe that the fluctuations always "displace" the fluctuations-affected state toward the middle point between the two nonaffected states. Moreover, for high enough noise intensities, the affected state exchanges its role with the adversity (α); therefore, the previously unstable state becomes stable and the one affected by the fluctuations plays the role of the unstable one. In a normal situation, a larger (smaller) adversity value means to promote the low (high) density population state, but with the contribution of fluctuations, once λ_c[4] is overcome, the affected state (now in the adversity role) gives the previously unstable (and now stable) state more weight (by being displaced toward the aforementioned middle point), while the noise intensity increases, until both stable states reach an equilibrium for higher intensities. Moreover, the aforementioned middle point coincides with the value that minimizes the multiplicative factor of the noise, which corroborate the vision about the possible existence of an underlying average dynamic that is able to push the system toward such minimum, until balanced by the deterministic forces. This vision is enhanced when we consider that the forcefulness of this effect grows proportionally to the negative multiplicative noise factor's derivative with respect to the field [40]. Then, when the noise intensity is high enough, the stochastic average dynamic prevails over the deterministic dynamic, and, therefore, the system is localized in a state that minimizes the multiplicative factor of the noise. In this paper, we locate the system in a situation so that the two stable states are $u = \alpha b$ and $u = b$. That is to say, choosing the noise intensity values in such a way that the (driven by fluctuations) exchange of roles between $u = \beta b$ (fluctuations-affected state which is stable in its absence) and $u = \alpha b$ (unstable state in absence of fluctuations) is carried out. To clarify, Fig. 1 shows a temporal

[3]A typical reaction-diffusion equation which is affected by a multiplicative noise produced by generalized Nagumo model's β-parameter fluctuations, where the reaction term is defined by the nonlinearity corresponding to generalized Nagumo model.

[4]λ_c indicate the noise intensity in which the exchange of roles occurs.

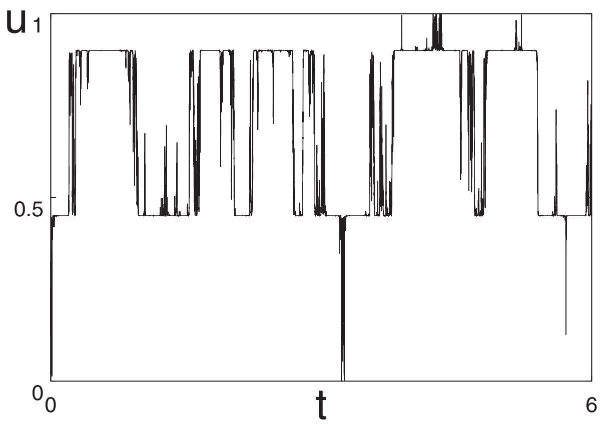

FIG. 1. Numerical simulation of the zero-dimensional system (fluctuations of β): u vs t (time unit: $10^7\delta t$). $\lambda = 10$, $\alpha = 0.5$, $\beta_0 b = 0.001$, and $b = 0.9$.

evolution (numerical simulation of the stochastic process) of field u revealing both stable states. It can be observed that the state $u = \alpha b$ ($u = 0.45$ for the illustrated case) has been stabilized by the fluctuations. In the absence of fluctuation, $u = \alpha b$ is an unstable state, being the stable state: $u = \beta b$ ($u = 0.001$ for the illustrated case). As aforementioned, our objective is to research the impact that can be produced on these previous results by adding spatial coupling between nearest neighbors. Results obtained by using a mean-field approximation are shown in the following section.

III. SPATIAL COUPLING AND FLUCTUATIONS: MEAN-FIELD APPROXIMATION

First, we use a mean-field approximation (MFA) to know what to expect of spatial coupling's effects on the states affected by fluctuations. Of course, this technique does not give accurate quantitative information, but it can give us guidance in this regard. Even more, taking into account that without spatial coupling but with fluctuations affecting the system, the stationary states are not disordered (two stable solutions, $u = \alpha b$ and $u = b$ when the fluctuating parameter is β). We are mainly interested in knowing how the spatial coupling could impact in the unstable state that is stabilized because of fluctuations ($u = \alpha b$).

Consequently and to simplify, starting from Eq. (3), we proceed to do a spatial discretization in a regular one-dimensional lattice with spacing δx, i.e., $u(x_i) \longrightarrow u_i$, with i the cell index. Considering that $\partial_x^2 u \longrightarrow \frac{1}{\delta x^2}\sum_j (u_j - u_i)$ (one dimension), where the sum is over nearest neighbors, the discretized version of Eq. (3) is written as

$$\partial_t u_i = F_{\beta_o}(u_i) + \frac{K}{2}\sum_j (u_j - u_i) + \Gamma^{1/2}(u_i)\,\eta_i(t), \qquad (7)$$

where $K = \frac{2D}{\delta x^2}$ and now the correlation is $\langle \eta(t)\eta(t')\rangle = 2\lambda^2\delta(t-t')\delta_{ij}$.

The mean-field approximation is tantamount to replacing the exact value of the neighbors by a mean-field common value $\langle u\rangle$. When starting from a disordered state, there are two typical ways to proceed: (1) finding the functional form corresponding to the deterministic dynamic including the diffusion term and then performing the MFA, and (2) performing the MFA first and, subsequently, finding the corresponding functional form. Given the context of our approach, the first way is unfeasible; therefore, we choose to use the second option. Therefore, the equation describing the stochastic dynamic in MFA is

$$\partial_t u = F_{\beta_o}(u) + K(\langle u\rangle - u) + \Gamma^{1/2}(u)\,\eta(t). \qquad (8)$$

Here we eliminate the subindex i, because it is irrelevant. Thus, by proceeding as in the zero-dimensional case, we define a free-energy function $F^c(u)$ as

$$F^c(u) = -\int du\,\frac{F_{\beta_o}(u) + K(\langle u\rangle - u)}{\Gamma(u)}.$$

And then we obtain the SPDF by calculating an effective free-energy function as

$$\begin{aligned} F^c_{\text{eff}}(u) &= -\int du\,\frac{F_{\beta_o}(u) + K(\langle u\rangle - u)}{\Gamma(u)} + \frac{\lambda^2}{2}\ln[\Gamma(u)] \\ &= \ln\left\{\frac{b^2|u-b|^{\frac{K\Theta}{\lambda^2}+e^b}}{|u-\alpha b|^{\frac{K\Theta}{\lambda^2}+e^{\alpha b}}}\right\} \\ &\quad + \frac{K\Psi}{b^4(1-\alpha)^2(u-b)(u-\alpha b)}, \end{aligned} \qquad (9)$$

where $\Theta = 2\frac{2\langle u\rangle(1-\alpha)-(1+\alpha)b}{b^4(1-\alpha)^3}$ and $\Psi = 2\langle u\rangle u - \langle u\rangle(1+\alpha)b + u(1-\alpha)b$.

As before, the corresponding SPDF is

$$P^c_{\text{th}}(u) \propto \exp\left\{-\frac{F^c_{\text{eff}}(u)}{\lambda^2}\right\},$$

and using (9), this latter is written as

$$\begin{aligned} P^c_{\text{th}}(u) \propto &\frac{|u-\alpha b|^{\frac{K\Theta}{\lambda^2}+e^{\alpha b}}}{b^2|u-b|^{\frac{K\Theta}{\lambda^2}+e^b}} \\ &\times \exp\left\{\frac{-K\Psi}{\lambda^2 b^4(1-\alpha)^2(u-b)(u-\alpha b)}\right\}. \end{aligned} \qquad (10)$$

Then the mean-field value $\langle u\rangle$ can be computed by using the consistency relation:

$$f(\langle u\rangle) = \frac{\int du\,P^c_{\text{th}}(u)u}{\int du\,P^c_{\text{th}}(u)} = \langle u\rangle. \qquad (11)$$

The expression (10) takes infinite values in $u = \alpha b$ and $u = b$, but it is relevant only in the exponent $\frac{-K\Psi}{\lambda^2 b^4(1-\alpha)^2(u-b)(u-\alpha b)}$, since this shows discontinuous jumps between $+\infty$ and $-\infty$, which indicate destabilizing effects driven by the cooperation between fluctuations and spatial coupling. In other words, the discontinuous jumps are produced in the noise multiplicative factor, but its effect is enabled by the spatial coupling.

Before we continue this exposition, we would like to emphasize that even without coupling, the stable states are already ordered states; this is why, we must introduce a variant in the usual *modus operandi* for the mean-field approximation. Since there are two stable states to consider, we did two mean field calculations, one around each state. We are particularly interested in obtaining qualitative evidences of the two symmetry breakings (one per each stable state) that could be caused by the cooperation between fluctuations and spatial

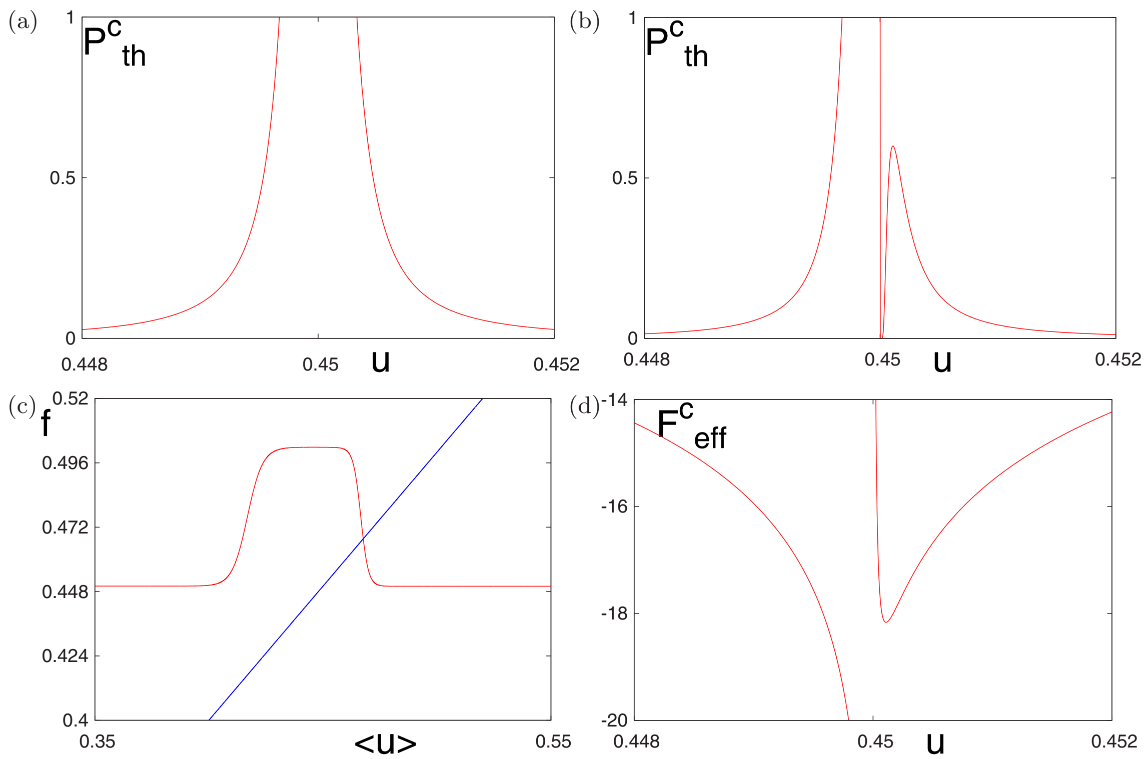

FIG. 2. (Color online) Results by using mean field in the state $u = \alpha b = 0.45$ ($b = 0.9$, $\alpha = 0.5$, and $\lambda = 10$). (a) $P^c_{\rm th}$ vs u for $K = 0$, (b) $P^c_{\rm th}$ vs u, (c) f vs $\langle u\rangle$, (d) $F^c_{\rm eff}$ vs u, for $K = 0.2$ ($P^c_{\rm th}$ is multiplied by the factor 10^4).

coupling. Consequently, we propose an integration range for each stable state, and therefore, we use Eq. (10) to find the $\langle u\rangle$.

Figures 2 and 3 highlight the joint effect of fluctuations and ($K = 0.005$) spatial coupling, calculated by MFA in

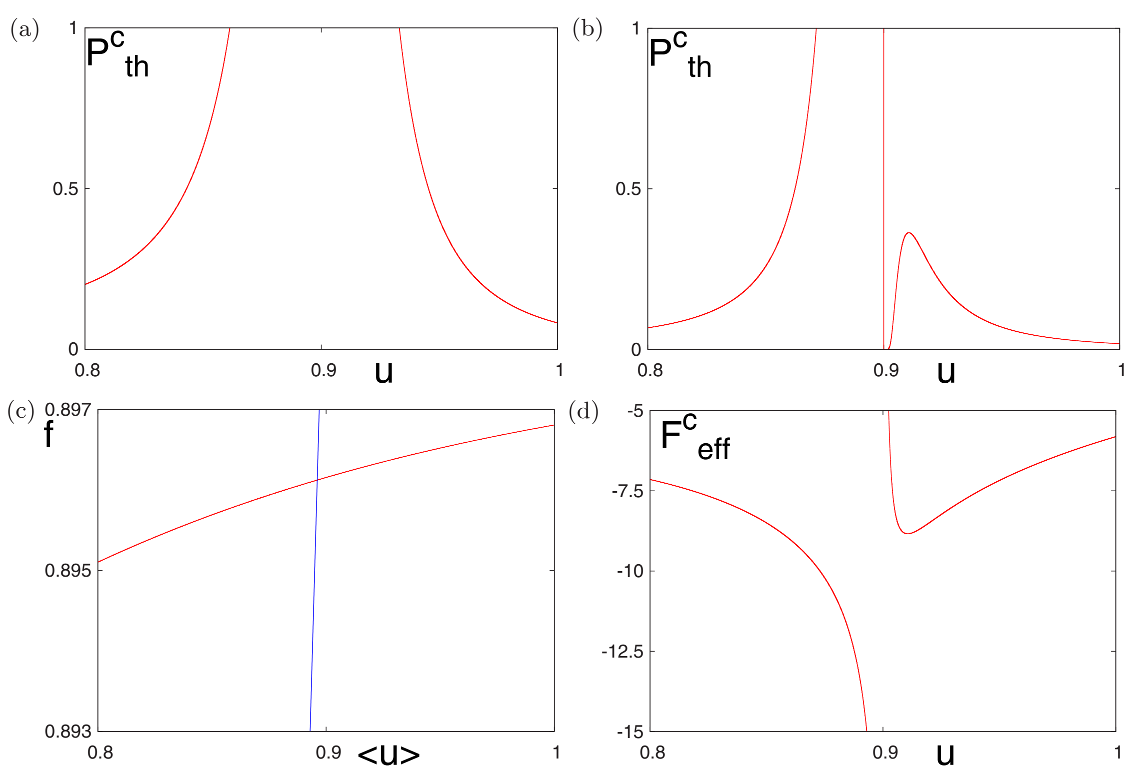

FIG. 3. (Color online) Results by using mean field in the state $u = b$ ($b = 0.9$, $\alpha = 0.5$, and $\lambda = 10$). (a) $P^c_{\rm th}$ vs u for $K = 0$, (b) $P^c_{\rm th}$ vs u, (c) f vs $\langle u\rangle$, (d) $F^c_{\rm eff}$ vs u, for $K = 0.2$ ($P^c_{\rm th}$ is multiplied by the factor 10^4).

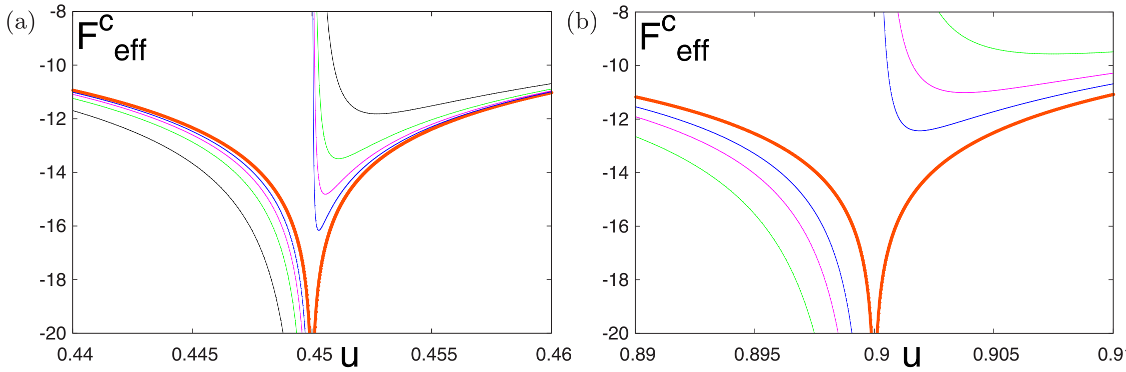

FIG. 4. (Color online) $F^c_{\rm eff}(u)$ vs u for different K values, increasing from top to bottom on the left side and from bottom to top on the right side of destabilized states ($u = 0.45$ or $u = 0.9$). (a) Close to $u = \alpha b = 0.45$, (thick line) $K = 0$, (thin line) $K = 0.04, 0.08, 0.4, 2$. (b) Close to $u = b$, (thick line) $K = 0$, (thin line) $K = 0.04, 0.08, 0.4$. The other parameters are $b = 0.9$, $\alpha = 0.5$, and $\lambda = 10$.

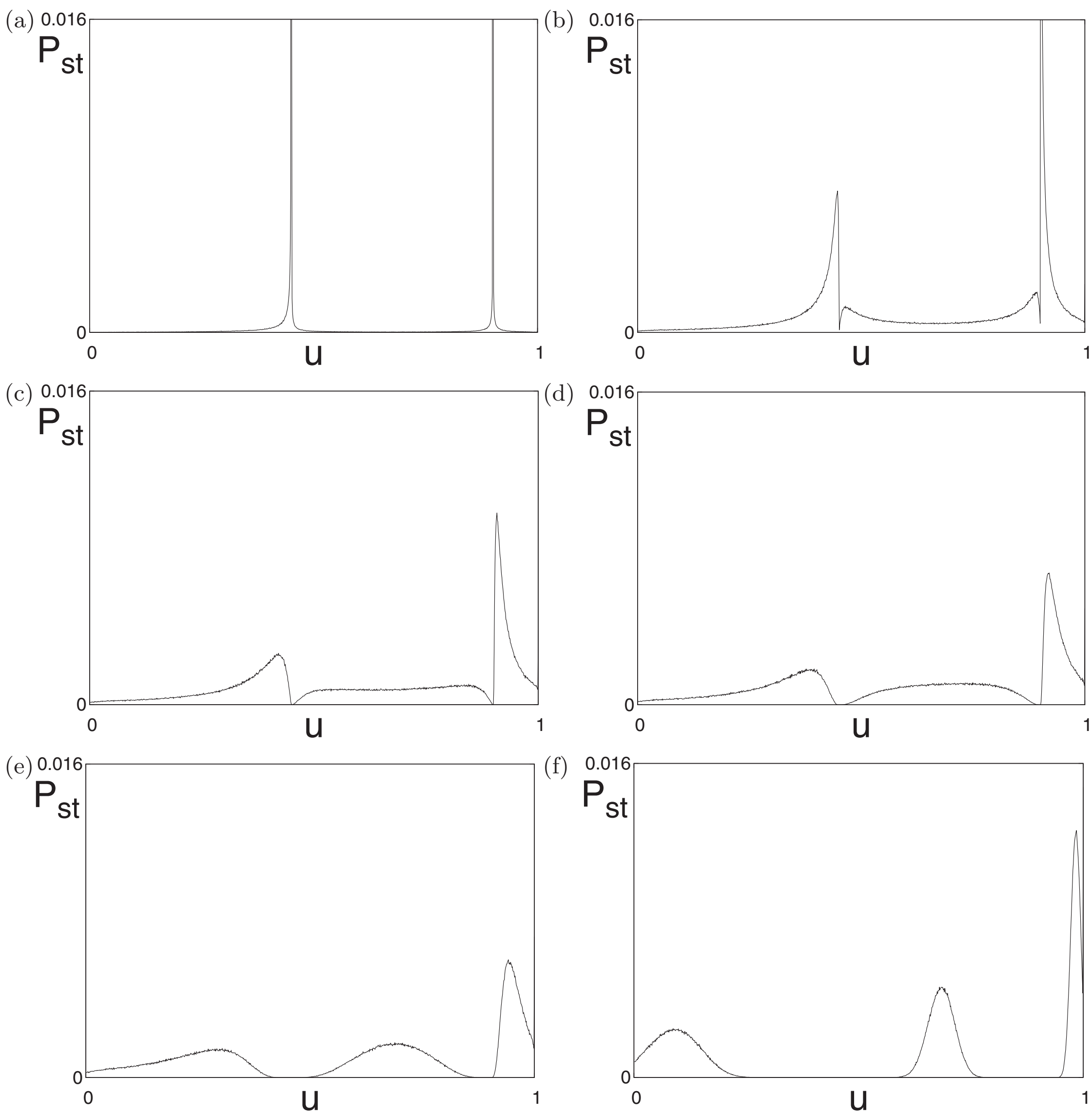

FIG. 5. Numerically computed SPDF: $P_{\rm st}$ vs u. (a) $K_a = 0$, (b) $K_a = 0.0032$, (c) $K_a = 0.0176$, (d)$K_a = 0.048$, (e) $K_a = 0.32$, and (f) $K_a = 16$ ($K_a = K/200$). Other parameters: $b = 0.9$, $\alpha = 0.5$, and $\lambda = 10$.

the states $u = \alpha b$ and $u = b$, respectively. These show the SPDFs (with and without spatial coupling), the mean-field solution and the effective free-energy function $F^c_{\rm eff}(u)$ for both states. While Figs. 2(a) and 3(a) [SPDF, case without spatial coupling ($K = 0$)] confirm the stability of $u = \alpha b$ and $u = b$, respectively, Figs. 2(b) and 3(b) [SPDF with spatial coupling ($K = 0.005$)] clearly denote a symmetry breaking of both states (the probability in $u = \alpha b$ and $u = b$ is zero; therefore, we expect four new state states to emerge). The curves of Figs. 2(d) and 3(d) show that the potential's walls that separate the new states are infinite; thus, we expect the phase transitions between these new states to be unlikely. Then we calculated curves of $F^c_{\rm eff}(u)$ versus u for different K values, which are shown in Fig. 4. These curves indicate that, by having added the spatial coupling, the new states move away from the already destabilized states as K increases.

We note that the discontinuous jumps can be observed for any $\langle u\rangle$ value. This means that, regardless of the used criteria to do the calculation, the evidence for the symmetry breakings is strong.

In short, we believe the only noteworthy result in this approximation is the possibility of two symmetry breakings. Of course, this is just a mean-field approximation as well as an atypical situation. Therefore, a numeric simulation is required.

IV. SPATIAL COUPLING AND FLUCTUATIONS: NUMERICAL RESULTS

In this section, we show the results of the equation's numerical resolution (7) by using the Heun's method [45,46]. We define three subregions within the $[0,1]$ field variability's range: $\mathrm{I} = [0,\alpha b]$, $\mathrm{II} = [\alpha b,b]$, and $\mathrm{III} = [b,1]$. When the simulations are initiated from a given subregion, the system always remains within such subregion, as long as $K \neq 0$ (transitions between subregions do not happen). We calculate the ($P_{\rm st}$) SPDF by taking samples while we let the system evolve for a T period of time (long time). More specifically, we divide the field variability's range in M cells and count the number of times that u takes values within each cell. We choose a (Δt) sampling time long enough to ensure independence. Since transitions between subregions do not happen, first, we take the same amount of samples per simulations initiated in each subregion (enabling the calculation of the SPDF by subregion), and then we obtain the SPDF corresponding to the region by means of addition and data normalization. We defined our lattice with size $L = 14\,400\delta x$ and spacing $\delta x = 0.025$, and for simulations: temporary step $\delta t = 10^{-4}$, $T = 600\,000\delta t$ (the system reaches the equilibrium in less than $T = 60\,000$), $\Delta t = 1000$ (corresponding to $L.T/\Delta t = 8\,640\,000$ samples), and $M = 1000$. Figure 5 shows the SPDF for different K values, highlighting the main qualitative aspects of our observations.

The first observation that should be made is that the curve of Fig. 5(b) is like the ones on Figs. 2(b) and 3(b). This result confirms the symmetry breaking that predicts the mean-field approximation: the two stable states are destabilized by the effect of the cooperation between fluctuations and spatial coupling causing the emergence of four stable states. We also note that the new states move away from their precursors as K increases, a phenomenon which was already predicted by the mean-field approximation. Other outcomes to highlight are the successive transformations produced as K varies. First, we note that the two states within region II approach each

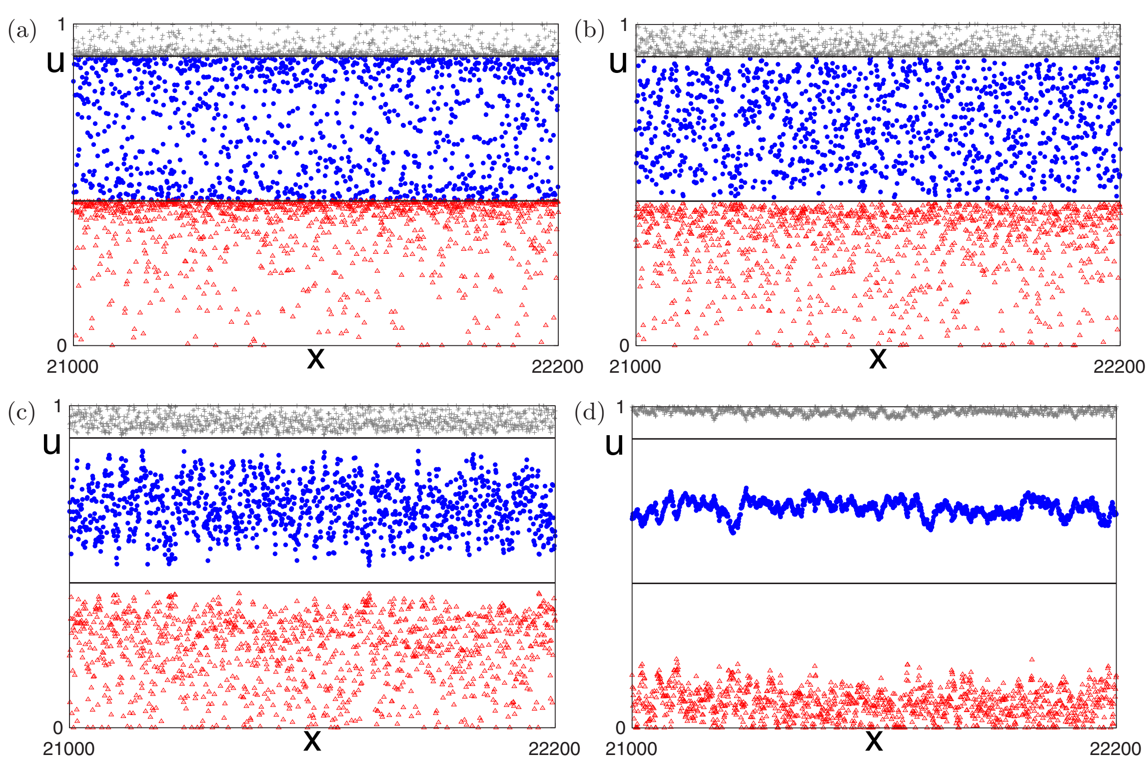

FIG. 6. (Color online) Numerically computed profiles: u vs x. (a) $K_a = 0.004$, (b) $K_a = 0.04$, (c) $K_a = 0.4$, and (d) $K_a = 16$ ($K_a = K/200$). Other parameters: $b = 0.9$, $\alpha = 0.5$, and $\lambda = 10$. Subregion I $=\triangle$, subregion II $= \bullet$, and subregion III $= +$.

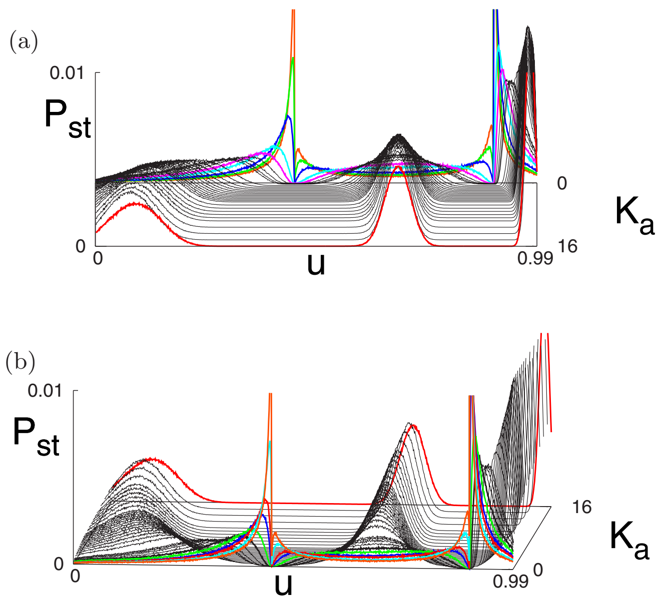

FIG. 7. (Color online) Numerically computed SPDF: P_{st} vs u and K_a. (a) Front view from $K_a = 16$–0. (b) Front view from $K_a = 0$–16 ($K_a = K/200$).

another until they overlap, resulting in only three stable states for large enough K values. We also observe modifications in the height, width, and location of the probability's peaks as K varies. Particularly, for intermediate K values, such peaks are widened; nevertheless, these later become narrow again for larger K values. In order to complete the depiction of these results, Fig. 6 shows profiles of u versus x for different K values which are consistent with the curves in Fig. 5. As aforementioned, once the system has been initiated in a given subregion, it always remains within this last subregion. Therefore we calculate the profiles for each subregion separately, and then we compose the graphics by superposing all of them. Since none of the points $u(x)$ corresponding to one subregion invade the neighboring subregion, all the points $u(x)$ indicated within a given subregion were initiated in such subregion. When Fig. 6 is interpreted under these considerations, we can visualize the curves of Figs. 5(b) and 5(c) in those of Figs. 6(a) and 6(b), as well as the curves of Figs. 5(d)–5(f) in those of Figs. 6(c) and 6(d). It is striking how the field variability's effective range decreases for larger K values. In addition, Fig. 7 shows a more detailed version of the SPDF changing with K.

V. ANALYSIS AND CONCLUSIONS

In this work, we have considered the study of the effects of spatial coupling (closest neighbors) in a system (a generalization of the Nagumo model) where, for a zero-dimensional case, the fluctuations can induce an exchange of roles between an unstable state and a stable one. Under this circumstance, there are two stable states before introducing the spatial coupling, but one of them is unstable in absence of fluctuations. By using a mean-field approximation, we found evidence of two spatial symmetry breakings as a result of cooperation between fluctuations and spatial coupling. We emphasize that one of the states that is destabilized by the symmetry breaking is the one stabilized by fluctuations. In other words, without spatial coupling (the zero-dimensional model), an unstable state is stabilized by fluctuations (by changing its role with another stable state), but with spatial coupling; it goes back to an unstable state, but, this time, it is also the precursor of the other two stable states. The MFA also predicts three regions separated by two infinite walls. This means that a system localized in one of the regions will remain inside of it. Moreover, the MFA shows us that the four new stable states move away from their precursors. The numerical simulations of corresponding stochastic process not only confirm the MFA's predictions, but also show other interesting phenomena. The two states, located within region II, approach one another as the spatial coupling constant increases, until they overlap. That is to say, for larger K values, there are only three stable states with height, width, and localization varying with K.

It is worth noting that the model used by us does not really involve a field-dependent relaxation coefficient (our model is not a nongradient relaxational system such as it is in Refs. [25,28–31]). We mapped the gradient model in another model with nongradient relaxational flow only to find the SPDF, a crucial link to be able to discover our results. This means that the relaxation coefficient has no impact on the results, since it does not exist in the studied model. What matters is the noise's multiplicative factor form (the noise pushes the system toward field values, minimizing its effect until it is balanced by the deterministic forces) and, of course, the spatial coupling as well.

We believe that our results go beyond the particular model here studied. Both by nature and human actions, the dynamic of a system can be changed when developing specific noises, with a suitable field-dependent multiplicative factor, and when adding a simple spatial coupling (closest neighbors). There is a competition between deterministic and stochastic forces. The deterministic ones drive the system toward the deterministic attractors, while the stochastic ones push the system toward situations in which the noise effects are minimized. The possible stationary states are the result of a balance between both forces. Due to the noise cooperation, the spatial coupling drives ordering noise-induced phase transitions.

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