NEW MAXIMAL FUNCTIONS AND MULTIPLE WEIGHTS FOR THE MULTILINEAR CALDERÓN-ZYGMUND THEORY

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ABSTRACT. A multi(sub)linear maximal operator that acts on the product of m Lebesgue spaces and is smaller that the m-fold product of the Hardy-Littlewood maximal function is studied. The operator is used to obtain a precise control on multilinear singular integral operators of Calderón-Zygmund type and to build a theory of weights adapted to the multilinear setting. A natural variant of the operator which is useful to control certain commutators of multilinear Calderón-Zygmund operators with BMO functions is then considered. The optimal range of strong type estimates, a sharp end-point estimate, and weighted norm inequalities involving both the classical Muckenhoupt weights and the new multilinear ones are also obtained for the commutators.

1. INTRODUCTION

The groundbreaking work of Calderón and Zygmund in the 50's [3] is the basis for what is today named after them Calderón-Zygmund theory. Their initial work on operators given by convolution with singular kernels was motivated by connections with potential theory and elliptic partial differential equations, and by the need to study operators which are higher-dimension analogs of the classical Hilbert transform. The tools developed over the years to deal with these and related problem in $\mathbb{R}^n$ form the core of what are nowadays called real-variable techniques.

The theory has had quite a success in the solution of many problems in both real and complex analysis, operator theory, approximation theory, and partial differential equations. This success is, in part, a consequence of the broad extension of the methods employed to different geometrical and multivariable contexts, which include homogeneous and non-homogeneous spaces, and multiparameter, non-linear, and multiliear settings. We refer to Coifman-Meyer [12], Christ [6], Fefferman [20], Stein [49],

Grafakos-Torres [27] and Volberg [50] for surveys and historical details about these different aspects of the subject.

Adapting the methods of the Calderón-Zygmund theory to each different context is, however, not always immediate. The theory provides a blueprint for the kind of results to be expected but, typically, the general approach needs to be complemented with the development of tools intrinsic to each particular new situation being faced. In particular, it is of relevance in each application to identify appropriate maximal functions that control in various ways many operators and functionals quantities that need to be estimated. As we will describe in this article, this is also the case for the multilinear Calderón-Zygmund theory. A collection of maximal functions that we will introduce will give us a way to obtain several sharp bounds for multilinear Calderón-Zygmund operators and their commutators.

The multilinear version of the Calderón-Zygmund theory originated in the works of Coifman and Meyer in the 70's, see e.g. [10], [11], and it was oriented towards the study of the Calderón commutator. Later on the topic was retaken by several authors; including Christ and Journé [8], Kenig and Stein [33], and Grafakos and Torres [25]. This last work provides a comprehensive approach to general multilinear Calderón-Zygmund operators that we will follow in this article.

As it is well-known, linear Calderón-Zygmund operators map L^p into itself for $1 < p < \infty$, with an $L^1 \to L^{1,\infty}$ estimate as one end-point and an $L^\infty \to BMO$ estimate as the other. It is then natural that the first results obtained for multilinear Calderón-Zygmund operators (see Section 2 below for technical definitions) were of the form $L^p \times L^q \to L^r$, with $1 < p, q, r < \infty$ satisfying the Hölder relation $1/p + 1/q = 1/r$. The fact that positive results also hold for $r > 1/2$ was somehow overlooked until Lacey and Thiele obtained their boundedness results for the bilinear Hilbert transform [36], [37]. The bilinear Hilbert transform is an operator far more singular than the bilinear Calderón-Zygmund operators and yet it satisfies bounds for $r > 2/3$ (it is not known yet whether the bounds are also true for $r > 1/2$). It was then shown in [33] and [25] that the full range $r > 1/2$ is achieved for bilinear Calderón-Zygmund operators, with an end-point estimate of the form $L^1 \times L^1 \to L^{1/2,\infty}$. (An m-linear version also holds; see (2.5) below.)

Weighted estimates and commutators in this multilinear setting were then studied in [26] and [46]. These works opened up some new problems that we resolve in this article.

The first set of problems that we consider relates directly to multilinear singular integrals. It was shown in [26] that if T is an m-linear Calderón-Zygmund operator, then $T(f_1, \cdots, f_m)$ is controlled in terms of L^p-norms by $\prod_{j=1}^m Mf_j$, where M is the usual Hardy-Littlewood maximal operator. As a consequence, it was deduced that if $\frac{1}{p_1} + \cdots + \frac{1}{p_m} = \frac{1}{p}$ and $p_0 = \min\{p_j\} > 1$, then T is a bounded from $L^{p_1}(w) \times \cdots \times L^{p_m}(w)$ into $L^p(w)$, provided that the weight w is in the class A_{p_0}. It is a simple observation that the same approach shows that

$$(1.1) \qquad\qquad T : L^{p_1}(w_1) \times \cdots \times L^{p_m}(w_m) \to L^p(\nu),$$

where $\nu = \prod_{j=1}^{m} w_j^{p/p_j}$ and w_j is in A_{p_j}. Such weights ν were used in [24] to obtain multilinear extrapolation results.

Nevertheless, the question of the existence of a multiple weight theory was posed in [27], and it has been since then an open problem whether the control of T by $\prod_{j=1}^{m} Mf_j$ is optimal and whether the conditions on w_j for which (1.1) holds cannot be improved. In this article we answer these questions by studying a multi(sub)linear maximal function $\mathcal{M}$ defined by

$$\mathcal{M}(\vec{f})(x) = \sup_{x \in Q} \prod_{i=1}^{m} \frac{1}{|Q|} \int_Q |f_i(y_i)| dy_i.$$

This operator is strictly smaller than the m-fold product of M. We develop the corresponding theory of weights for this new maximal function which, in turn, gives the right class of multiple weights for m-linear Calderón-Zygmund operators.

We use some analogous tools to study a second set of problems related now to multilinear versions of the commutators of Coifman, Rochberg and Weiss [13]. We recall that the operators introduced in [13] are defined by $[b, T]f = b\,T(f) - T(bf)$, where b is a locally integrable function in $\mathbb{R}^n$, usually called the symbol, and T is a Calderón-Zygmund singular integral. The original interest in the study of such operators was related to generalizations of the classical factorization theorem for Hardy spaces. Further applications have then been found in partial differential equations [4] [5] [17] [28]. Recently multiparameter versions have also received renewed attention; see e.g. [21] and [35].

The main result from [13] states that if b is in BMO, then $[b, T]$ is a bounded operator on $L^p(\mathbb{R}^n)$, $1 < p < \infty$. In fact, the BMO membership of b is also a necessary condition for the L^p-boundedness of the commutator when, for example, $T = H$, the Hilbert transform. An interesting fact is that, unlike what it is done with singular integral operators, the proof of the L^p-boundedness of the commutator does not rely on a weak type $(1, 1)$ inequality. In fact, simple examples show that in general $[b, T]$ fails to be of weak type $(1, 1)$ when $b \in BMO$. Instead, it was proved by Pérez [42] that a weak-$L(\log L)$ type estimate holds (see (3.15) below).

Given a collection of locally integrable functions $\vec{b} = (b_1, \ldots, b_m)$, we define the m-linear commutator of $\vec{b}$ and the m-linear Calderón-Zygmund operator T to be

$$(1.2) \qquad T_{\vec{b}}(f_1, \cdots, f_m) = \sum_{j=1}^{m} T_{\vec{b}}^j(\vec{f}),$$

where each term is the commutator of b_j and T in the j-th entry of T, that is,

$$T_{\vec{b}}^j(\vec{f}) = b_j T(f_1, \cdots, f_j, \cdots, f_m) - T(f_1, \cdots, b_j f_j, \cdots, f_m).^1$$

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[1] We chose this definition to follow [46] and for symmetry and simplicity in some statements, but for most estimates it will be enough to consider only one term $T_{\vec{b}}^j(\vec{f})$. On the other hand, we shall see that in the end-point result the presence of just one symbol produce, surprisingly, the appearance of non-linear functional estimates in all the entries of T.

This definition coincides with the linear commutator $[b, T]$ when $m = 1$. The m-linear commutators were considered by Pérez and Torres in [46]. They proved that if $\vec{b} \in (BMO)^m$, $1 < p < \infty$, and $p_1, p_2, \cdots, p_m$ are such that $\frac{1}{p_1} + \cdots + \frac{1}{p_m} = \frac{1}{p}$, then

$$T_{\vec{b}} : L^{p_1} \times \cdots \times L^{p_m} \longrightarrow L^p.$$

Observe that a crucial condition $p > 1$ was assumed in this result. The restriction arose in [46] because of the method used which as in the linear case [13], rely on strong A_p estimates (hence limiting the approach to $p > 1$). The experience with the linear commutators and multilinear Calderón-Zygmund operators, however, suggests that the optimal range should be $1/m < p < \infty$. We will see in this article that this is in fact the case. Moreover, the question of the existence of an end-point result along the lines of the work [42] was also stated in [46], and we find an answer involving an appropriate weak-$L(\log L)$ estimate when $p = 1/m$. The bounds that we obtain hold also for the new multiple weights and we achieve them by using yet other new maximal functions, $\mathcal{M}_{L(\log L)}^i$, $i = 1, \ldots, m$, and $\mathcal{M}_{L(\log L)}$, defined by the expressions

$$\mathcal{M}_{L(\log L)}^i(\vec{f})(x) = \sup_{Q \ni x} \|f_i\|_{L(\log L), Q} \prod_{j \neq i} \frac{1}{|Q|} \int_Q f_j \, dx$$

and

$$\mathcal{M}_{L(\log L)}(\vec{f})(x) = \sup_{Q \ni x} \prod_{j=1}^m \|f_j\|_{L(\log L), Q}.$$

(See Section 2 below for more details about the norm $\| \cdot \|_{L(\log L), Q}$.)

Observe that $\mathcal{M}_{L(\log L)}$ is bigger than $\mathcal{M}$ reflecting the presence of the BMO symbols. One can see that $\mathcal{M}_{L(\log L)}(\vec{f})$ is pointwise controlled by a multiple of $\prod_{j=1}^m M^2 f_j(x)$, but this product is too big to derive the sharp weighted estimate for commutators that we are interested in. That is why we use instead $\mathcal{M}_{L(\log L)}$.

The article is organized as follows. Section 2 contains some basic definitions and facts concerning multilinear singular integrals, weights, sharp maximal functions, and Orlicz spaces needed throughout the rest of this work. The reader familiar with the subject, however, may skip directly to Section 3 where all the theorems are stated. The proofs of the results involving $\mathcal{M}$ and multilinear Calderón-Zygmund operators are presented in Section 4, while Section 5 contains the pointwise and strong-type estimates for $\mathcal{M}_{L(\log L)}^i$ and the commutators. The proof of the end-point estimate for $\mathcal{M}_{L(\log L)}^i$ and the commutators are postponed until Section 6. Various examples (and counterexamples) are collected in Section 7.

2. Preliminaries

2.1. **Multilinear Calderón-Zygmund operators.** Let T be a multilinear operator initially defined on the m-fold product of Schwartz spaces and taking values into the space of tempered distributions,

$$T : S(\mathbb{R}^n) \times \cdots \times S(\mathbb{R}^n) \to S'(\mathbb{R}^n).$$

Following [25], we say that T is an m-linear Calderón-Zygmund operator if, for some $1 \leq q_j < \infty$, it extends to a bounded multilinear operator from $L^{q_1} \times \cdots \times L^{q_m}$ to L^q, where $\frac{1}{q} = \frac{1}{q_1} + \cdots + \frac{1}{q_m}$, and if there exists a function K, defined off the diagonal $x = y_1 = \cdots = y_m$ in $(\mathbb{R}^n)^{m+1}$, satisfying

$$T(f_1, \ldots, f_m)(x) = \int\limits_{(\mathbb{R}^n)^m} K(x, y_1, \ldots, y_m) f_1(y_1) \ldots f_m(y_m) \, dy_1 \ldots dy_m$$

for all $x \notin \cap_{j=1}^m f_j$;

$$(2.1) \qquad |K(y_0, y_1, \ldots, y_m)| \leq \frac{A}{\left(\sum\limits_{k,l=0}^m |y_k - y_l| \right)^{mn}};$$

and

$$(2.2) \qquad |K(y_0, \ldots, y_j, \ldots, y_m) - K(y_0, \ldots, y_j', \ldots, y_m)| \leq \frac{A|y_j - y_j'|^\varepsilon}{\left(\sum\limits_{k,l=0}^m |y_k - y_l| \right)^{mn+\varepsilon}},$$

for some $\varepsilon > 0$ and all $0 \leq j \leq m$, whenever $|y_j - y_j'| \leq \frac{1}{2} \max_{0 \leq k \leq m} |y_j - y_k|$.

It was shown in [25] that if $\frac{1}{r_1} + \cdots + \frac{1}{r_m} = \frac{1}{r}$, then an m-linear Calderón-Zygmund operator satisfies

$$(2.3) \qquad\qquad T : L^{r_1} \times \cdots \times L^{r_m} \to L^r$$

when $1 < r_j < \infty$ for all $j = 1, \cdots, m$; and

$$(2.4) \qquad\qquad T : L^{r_1} \times \cdots \times L^{r_m} \to L^{r,\infty},$$

when $1 \leq r_j < \infty$ for all $j = 1, \cdots, m$, and at least one $r_j = 1$. In particular,

$$(2.5) \qquad\qquad T : L^1 \times \cdots \times L^1 \to L^{1/m,\infty}$$

2.2. **Weights.** By a weight we mean a non-negative measurable function. We recall that a weight w belongs to the class A_p, $1 < p < \infty$, if

$$\sup_Q \left(\frac{1}{|Q|} \int_Q w(y) \, dy \right) \left(\frac{1}{|Q|} \int_Q w(y)^{1-p'} \, dy \right)^{p-1} < \infty.$$

This number is called the A_p constant of w. A weight w belongs to the class A_1 if there is a constant C such that

$$\frac{1}{|Q|} \int_Q w(y) \, dy \leq C \inf_Q w,$$

and the infimum of these constants C is called the A_1 constant of w. Since the A_p classes are increasing with respect to p, the A_∞ class of weights is defined in a natural way by $A_\infty = \cup_{p>1} A_p$ and the A_∞ constant of $w \in A_\infty$ is the smallest of the infimum of the A_p constant such that $w \in A_p$.

A well-known result obtained by Muckenhoupt [41] is that the Hardy-Littlewood maximal function,

$$M f(x) = \sup_{Q \ni x} \frac{1}{Q} \int_Q |f(y)| dy,$$

satisfies $M : L^p(w) \to L^p(w)$ if and only if w is in A_p (see [39] for a new simple proof which yields the sharp A_p constant). He also obtained a characterization of the weak-type inequalities for M. Namely, $M : L^p(w) \to L^{p,\infty}(\nu)$ if and only if

$$(2.6) \qquad \sup_Q \left(\frac{1}{|Q|} \int_Q \nu(y)\, dy \right) \left(\frac{1}{|Q|} \int_Q w(y)^{1-p'}\, dy \right)^{p-1} < \infty.$$

We will also need some results about one-sided weights. Given an interval $I = [a, b]$, we denote $I^+ = [b, 2b - a]$. A weight w is said to belong to the A_p^+ condition if

$$\sup_I \left(\frac{1}{|I|} \int_I w(x) dx \right) \left(\frac{1}{|I|} \int_{I^+} w(x)^{-1/(p-1)} dx \right)^{p-1} < \infty.$$

It is a known fact in the theory of one-sided weights (see, e.g., [40]) that if w satisfies the A_p^+ condition, then there exists a constant c such that for any interval I,

$$(2.7) \qquad w(I) \le c w(I^+).$$

2.3. **Sharp maximal operators.** For $\delta > 0$, let M_δ be the maximal function

$$M_\delta f(x) = M(|f|^\delta)^{1/\delta}(x) = \left(\sup_{Q \ni x} \frac{1}{|Q|} \int_Q |f(y)|^\delta\, dy \right)^{1/\delta}.$$

Also, let $M^\#$ be the usual sharp maximal function of Fefferman and Stein [19],

$$M^\#(f)(x) = \sup_{Q \ni x} \inf_c \frac{1}{|Q|} \int_Q |f(y) - c|\, dy \approx \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |f(y) - f_Q|\, dy,$$

where as usual $f_Q = \frac{1}{|Q|} \int_Q f(y)\, dy$ denotes the average of f over Q.

We will use the following form of the classical result of Fefferman and Stein [19]. See also [32].

Let $0 < p, \delta < \infty$ and let w be a weight in A_∞. Then, there exists $C > 0$ (depending on the A_∞ constant of w), such that

$$(2.8) \qquad \int_{\mathbb{R}^n} (M_\delta f(x))^p\, w(x) dx \le C \int_{\mathbb{R}^n} (M_\delta^\# f(x))^p\, w(x) dx,$$

for all function f for which the left hand side is finite.

Similarly, if $\varphi : (0, \infty) \to (0, \infty)$ is doubling, then there exists a constant c (depending on the A_∞ constant of w and the doubling condition of φ) such that

$$(2.9) \quad \sup_{\lambda > 0} \varphi(\lambda)\, w(\{y \in \mathbb{R}^n : M_\delta f(y) > \lambda\}) \le c \sup_{\lambda > 0} \varphi(\lambda)\, w(\{y \in \mathbb{R}^n : M_\delta^\# f(y) > \lambda\})$$

for every function f such that the left hand side is finite. Extension of these estimates for a large class of spaces can be found in [16].

2.4. **Orlicz spaces and normalized measures.** We need some basic facts from the theory of Orlicz spaces that we will state without proof. For more information and a lively exposition about these spaces the reader may consult the upcoming book by Wilson [51] or the classical one [47].

Let $\Phi : [0, \infty) \to [0, \infty)$ be a Young function. That is, a continuous, convex, increasing function with $\Phi(0) = 0$ and such that $\Phi(t) \to \infty$ as $t \to \infty$. The Orlicz space with respect to the measure μ, $L_\Phi(\mu)$, is defined to be the set of measurable functions f such that for some $\lambda > 0$,

$$\int_{\mathbb{R}^n} \Phi\left(\frac{|f(x)|}{\lambda}\right) d\mu < \infty.$$

The space L_Φ is a Banach space when endowed with the Luxemburg norm

$$\|f\|_\Phi = \|f\|_{L_\Phi} = \inf\left\{\lambda > 0 : \int_{\mathbb{R}^n} \Phi\left(\frac{|f(x)|}{\lambda}\right) d\mu \leq 1\right\}.$$

The Φ-average of a function f over a cube Q is defined to be $L_\Phi(\mu)$ with μ the normalized measure of the cube Q and it is denoted by $\|f\|_{\Phi,Q}$. That is,

$$\|f\|_{\Phi,Q} = \inf\{\lambda > 0 : \frac{1}{|Q|} \int_Q \Phi\left(\frac{|f(x)|}{\lambda}\right) dx \leq 1\}.$$

It is a simple but important observation that

$$\|f\|_{\Phi,Q} > 1 \quad \text{if and only if} \quad \frac{1}{|Q|} \int_Q \Phi\left(|f(x)|\right) dx > 1.$$

Another useful observation is that if Φ_1 and Φ_2 are two Young functions with $\Phi_1(t) \leq \Phi_2(t)$, for $t \geq t_0 > 0$, then

$$(2.10) \qquad \|f\|_{\Phi_1,Q} \leq C \|f\|_{\Phi_2,Q},$$

which can be seen as a *generalized Jensen's inequality*.

Associated to each Young function Φ, one can defined a complementary function

$$(2.11) \qquad \bar{\Phi}(s) = \sup_{t>0}\{st - \Phi(t)\}.$$

Such $\bar{\Phi}$ is also a Young function and the $\bar{\Phi}$-averages it defines are related to the L_Φ-averages via a *the generalized Hölder's inequality*. Namely,

$$(2.12) \qquad \frac{1}{|Q|} \int_Q |f(x)\,g(x)|\,dx \leq 2 \|f\|_{\Phi,Q} \|g\|_{\bar{\Phi},Q}.$$

A particular case of interest, an especially in this paper, are the Young functions

$$\Phi(t) = t\,(1 + \log^+ t) \text{ and } \Psi(t) = e^t - 1,$$

defining the classical Zygmund spaces $L(\log L)$, and $\exp L$ respectively. The corresponding averages will be denoted by

$$\|\cdot\|_{\Phi,Q} = \|\cdot\|_{L(\log L),Q} \text{ and } \|\cdot\|_{\Psi,Q} = \|\cdot\|_{\exp L,Q}.$$

Observe that the above function Φ is submultiplicative. That is, for $s, t > 0$

$$\Phi(st) \leq \Phi(s)\,\Phi(t).$$

This submultiplicative property will be used several times in this article. A computation shows that the complementary function of Ψ defined by (2.11) satisfies

$$\bar{\Psi}(t) \leq \Phi(t),$$

and so from the generalized Hölder inequality (2.12) and (2.10) we also get

$$(2.13) \qquad \frac{1}{|Q|}\int_Q |f(x)\,g(x)|\,dx \leq C\,\|f\|_{\exp L, Q}\,\|g\|_{L(\log L), Q}.$$

This inequality allows to write the following formula that will be used in this article:

$$(2.14) \qquad \frac{1}{|Q|}\int_Q |b(y) - b_Q|\,f(y)\,dy \leq C\|b\|_{BMO}\|f\|_{L(\log L), Q}.$$

for any function $b \in BMO$ and any non negative function f. This inequality follows from (2.13) and the John-Nirenberg inequality [31] for BMO functions: there are dimensional positive constants $c_1 < 1$ and $c_2 > 2$ such that

$$\frac{1}{|Q|}\int_Q \exp(\frac{c_1|b(y) - b_Q|}{\|b\|_{BMO}})\,dy \leq c_2$$

which easily implies that for appropriate constant $c > 0$

$$\|b - b_Q\|_{\exp L, Q} \leq c\,\|b\|_{BMO}.$$

In view of this result and its applications it is natural to define as in [43] a maximal operator

$$M_{L(\log L)}f(x) = \sup_{Q \ni x} \|f\|_{L(\log L), Q},$$

where the supremum is taken over all the cubes containing x. (Other equivalent definitions can be found in the literature.) We will also use the pointwise equivalence

$$(2.15) \qquad M_{L(\log L)}f(x) \approx M^2 f(x).$$

This equivalence was obtained in [44] using Stein's lemma [48] (see [16] for a different argument) and it is shown in [42] the relationship with linear commutators.

Finally, we will employ several times the following simple Kolmogorov inequality. Let $0 < p < q < \infty$, then there is a constant $C = C_{p,q}$ such that for any measurable function f

$$(2.16) \qquad \|f\|_{L^p(Q, \frac{dx}{|Q|})} \leq C\,\|f\|_{L^{q,\infty}(Q, \frac{dx}{|Q|})}.$$

3. Main results

3.1. The key pointwise estimate.

Definition 3.1. *Given $\vec{f} = (f_1, \ldots, f_m)$, we define the maximal operator $\mathcal{M}$ by*

$$\mathcal{M}(\vec{f})(x) = \sup_{Q \ni x} \prod_{i=1}^{m} \frac{1}{|Q|} \int_Q |f_i(y_i)| dy_i,$$

where the supremum is taken over all cubes Q containing x.

With some abuse, will refer to $\mathcal{M}$ as a *multilinear maximal function*, even though it is obviously only sublinear in each entry. The main result connecting multilinear Calderón-Zygmund operators and this multilinear maximal function is the following.

Theorem 3.2. *Let T be an m-linear Calderón-Zygmund operator and let $\delta > 0$ such that $\delta < 1/m$. Then for all $\vec{f}$ in any product of $L^{q_j}(\mathbb{R}^n)$ spaces, with $1 \le q_j < \infty$,*

$$(3.1) \qquad\qquad M_\delta^\#(T(\vec{f}))(x) \le C\mathcal{M}(\vec{f})(x).$$

The linear version of this estimate can be found in [1] (see also [30] for a earlier result related to (3.1)).

We note that (3.1) improves the inequality

$$(3.2) \qquad\qquad M_\delta^\#(T(\vec{f}))(x) \le C \prod_{j=1}^{m} Mf_j(x)$$

obtained in [46]. Since $\mathcal{M}$ is trivially controlled by the m-fold product of M, (3.1) can be used to recover all the weighted estimates results of [26] and [46] that follow from (3.2). The point here, however, is that (3.1) opens up the possibility of considering more general weights. We exploit this possibility in our next result.

3.2. Weighted estimates for the multilinear maximal function.
We investigate the boundedness properties of $\mathcal{M}$ on various weighted spaces.

Theorem 3.3. *Let $1 \le p_j < \infty, j = 1, \ldots, m$ and $\frac{1}{p} = \frac{1}{p_1} + \cdots + \frac{1}{p_m}$. Let ν and w_j be weights. Then the inequality*

$$(3.3) \qquad\qquad \|\mathcal{M}(\vec{f})\|_{L^{p,\infty}(\nu)} \le c \prod_{j=1}^{m} \|f_j\|_{L^{p_j}(w_j)}$$

holds for any $\vec{f}$ if and only if

$$(3.4) \qquad\qquad \sup_Q \Big(\frac{1}{|Q|} \int_Q \nu\Big)^{1/p} \prod_{j=1}^{m} \Big(\frac{1}{|Q|} \int_Q w_j^{1-p_j'}\Big)^{1/p_j'} < \infty,$$

where $\Big(\frac{1}{|Q|} \int_Q w_j^{1-p_j'}\Big)^{1/p_j'}$ in the case $p_j = 1$ is understood as $(\inf_Q w_j)^{-1}$.

Note that this result for $\mathcal{M}$ is a natural extension to the multilinear setting of Muckenhoupt's weak-type characterization for M, since we recover (2.6) when $m = 1$.

Observe also that condition (3.4) combined with the Lebesgue differentiation theorem implies that $\nu(x) \leq c \prod_{j=1}^{m} w_j(x)^{p/p_j}$ a.e. This suggests a way to define an analogue of the Muckenhoupt A_p classes for multiple weights.

Definition 3.4. *For m exponents $p_1, \cdots, p_m$, we will often write p for the number given by $\frac{1}{p} = \frac{1}{p_1} + \cdots + \frac{1}{p_m}$, and $\vec{P}$ for the vector $\vec{P} = (p_1, \cdots, p_m)$.*

Definition 3.5. *Let $1 \leq p_1, \ldots, p_m < \infty$. Given $\vec{w} = (w_1, \ldots, w_m)$, set*

$$\nu_{\vec{w}} = \prod_{j=1}^{m} w_j^{p/p_j}.$$

We say that $\vec{w}$ satisfies the $A_{\vec{P}}$ condition if

$$(3.5) \qquad \sup_Q \Big(\frac{1}{|Q|} \int_Q \nu_{\vec{w}}\Big)^{1/p} \prod_{j=1}^{m} \Big(\frac{1}{|Q|} \int_Q w_j^{1-p_j'}\Big)^{1/p_j'} < \infty.$$

When $p_j = 1$, $\Big(\frac{1}{|Q|} \int_Q w_j^{1-p_j'}\Big)^{1/p_j'}$ is understood as $(\inf_Q w_j)^{-1}$.

We will refer to (3.5) as the *multilinear $A_{\vec{P}}$ condition*. Observe that $A_{(1,\cdots,1)}$ is contained in $A_{\vec{P}}$ for each $\vec{P}$, however the classes $A_{\vec{P}}$ are not increasing with the natural partial order. See the example in Remark 7.3.

Observe that if each w_j is in A_{p_j}, then by Hölder's inequality

$$\sup_Q \Big(\frac{1}{|Q|} \int_Q \nu_{\vec{w}}\Big)^{1/p} \Big(\frac{1}{|Q|} \int_Q w_j^{1-p_j'}\Big)^{1/p_j'}$$

$$\leq \sup_Q \prod_{j=1}^{m} \Big(\frac{1}{|Q|} \int_Q w_j\Big)^{1/p_j} \Big(\frac{1}{|Q|} \int_Q w_j^{1-p_j'}\Big)^{1/p_j'} < \infty,$$

so we have

$$\prod_{j=1}^{m} A_{p_j} \subset A_{\vec{P}}.$$

However it is shown in Remark 7.2 that this inclusion is strict, in fact $\vec{w} \in A_{\vec{P}}$ does not imply in general $w_j \in L_{\text{loc}}^1$ for any j.

Note also that, again using Hölder's inequality

$$\sup_Q \Big(\frac{1}{|Q|} \int_Q \nu_{\vec{w}}\Big)^{1/mp} \Big(\frac{1}{|Q|} \int_Q \nu_{\vec{w}}^{-\frac{1}{mp-1}}\Big)^{(mp-1)/mp}$$

$$\leq \sup_Q \prod_{j=1}^{m} \Big(\frac{1}{|Q|} \int_Q w_j\Big)^{1/mp_j} \Big(\frac{1}{|Q|} \int_Q w_j^{-\frac{1}{p_j-1}}\Big)^{(p_j-1)/mp_j} < \infty,$$

where we have used that $m - 1/p = \sum (p_j - 1)/p_j$. It follows that $\nu_{\vec{w}}$ is in A_{mp}.

It turns out that something more general happens and the multilinear $A_{\vec{P}}$ condition has the following interesting characterization in terms of the linear A_p classes.

Theorem 3.6. *Let $\vec{w} = (w_1, \cdots, w_m)$ and $1 \leq p_1, \ldots, p_m < \infty$.*
   Then $\vec{w} \in A_{\vec{P}}$ if and only if

(3.6)
$$\begin{cases} w_j^{1-p_j'} \in A_{mp_j'}, \ j = 1, \ldots, m \\ \nu_{\vec{w}} \in A_{mp}, \end{cases}$$

where the condition $w_j^{1-p_j'} \in A_{mp_j'}$ in the case $p_j = 1$ is understood as $w_j^{1/m} \in A_1$.

Observe that in the linear case ($m = 1$) both conditions included in (3.6) represent the same A_p condition. However, when $m \geq 2$ none of the two conditions in (3.6) implies the other. See Remark 7.1 in Section 7 below.

The theorem also shows that as the index m increases the $A_{\vec{P}}$ condition gets weaker.

Theorem 3.6 plays an important role in the following characterization of the strong-type inequalities for $\mathcal{M}$ with one weight.

Theorem 3.7. *Let $1 < p_j < \infty, j = 1, \ldots, m$ and $\frac{1}{p} = \frac{1}{p_1} + \cdots + \frac{1}{p_m}$. Then the inequality*

(3.7)
$$\|\mathcal{M}(\vec{f})\|_{L^p(\nu_{\vec{w}})} \leq C \prod_{j=1}^{m} \|f_j\|_{L^{p_j}(w_j)}$$

holds for every $\vec{f}$ if and only if $\vec{w}$ satisfies the $A_{\vec{P}}$ condition.

The counterexample in Remark 7.4 shows that the assumption $p_j > 1$ for all j is essential in Theorem 3.7 even in the unweighted case. We want to emphasize that (3.7) does not hold with $\mathcal{M}(\vec{f})$ replaced by $\prod_{j=1}^{m} Mf_j$. See Remark 7.5 in Section 7. We also note that in order to prove Theorem 3.7 one cannot apply some basic technique used to work with M in the linear case. For example, it is well-known [22, p. 137] that the distribution functions of M and the dyadic maximal function M^d are comparable, but in the multilinear case $\mathcal{M}$ and its dyadic version $\mathcal{M}^d$ are not. In fact, take for instance $m = 2$ and $n = 1$ and set $f_1 = \chi_{(-1,0)}$ and $f_2 = \chi_{(0,1)}$. Then, $\mathcal{M}^d(\vec{f}) \equiv 0$ but $\mathcal{M}(\vec{f})(x) > 0$ everywhere. Nevertheless, the dyadic analysis is still useful; see the second proof of Theorem 3.7 in Section 4 below.

3.3. **Weighted estimates for multilinear Calderón-Zygmund operators.** We show that the multilinear classes $A_{\vec{P}}$ are also the appropriate ones for multilinear Calderón-Zygmund operators. First using Theorem 3.2 we will show the following result which can be viewed as an extension of the Coifman-Fefferman theorem [9] to the multilinear case.

Corollary 3.8. *Let T be an m-linear Calderón-Zygmund operator, let w be a weight in A_∞ and let $p > 0$. There exists $C > 0$ (depending on the A_∞ constant of w) so that*

the inequalities

$$(3.8) \qquad \|T(\vec{f})\|_{L^p(w)} \le C\|\mathcal{M}(\vec{f})\|_{L^p(w)}$$

and

$$(3.9) \qquad \|T(\vec{f})\|_{L^{p,\infty}(w)} \le C\|\mathcal{M}(\vec{f})\|_{L^{p,\infty}(w)}$$

hold for all bounded function $\vec{f}$ with compact support.

From Theorems 3.3, 3.6, 3.7, and the above Corollary 3.8 we obtain the following weighted estimates.

Corollary 3.9. *Let T be an m-linear Calderón-Zygmund operator, $\frac{1}{p} = \frac{1}{p_1} + \cdots + \frac{1}{p_m}$, and $\vec{w}$ satisfy the $A_{\vec{P}}$ condition.*

  (i) *If $1 < p_j < \infty$, $j = 1, \ldots, m$, then*

$$(3.10) \qquad \|T(\vec{f})\|_{L^p(\nu_{\vec{w}})} \le C\prod_{j=1}^{m} \|f_j\|_{L^{p_j}(w_j)}.$$

  (ii) *If $1 \le p_j < \infty$, $j = 1, \ldots, m$, and at least one of the $p_j = 1$, then*

$$(3.11) \qquad \|T(\vec{f})\|_{L^{p,\infty}(\nu_{\vec{w}})} \le C\prod_{j=1}^{m} \|f_j\|_{L^{p_j}(w_j)}.$$

As we already mentioned, $\mathcal{M}$ cannot be replaced by the m-fold product of M in (3.7) and, hence, Corollary 3.9 cannot be obtained from the previously known estimates from [26], where $T(\vec{f})$ is controlled by $\prod_{j=1}^{m} Mf_j$.

It turns out that the classes $A_{\vec{P}}$ are also characterized by the boundedness of certain multilinear singular integral operators.

Definition 3.10. *For $i = 1, \cdots, n$, the m-linear i-th Riesz transform is defined by*

$$R_i(\vec{f})(x) = p.v. \int_{(\mathbb{R}^n)^m} \frac{\sum_{j=1}^{m}(x_i - (y_j)_i)}{\left(\sum_{j=1}^{m} |x - y_j|^2\right)^{\frac{nm+1}{2}}} f_1(y_1) \ldots f_m(y_m) dy_1 \ldots dy_m,$$

where $(y_j)_i$ denotes the i-th coordinate of y_j.

Theorem 3.11.
If (3.11) or (3.10) holds for each of the m-linear Riesz transforms $R_i(\vec{f})$, then $\vec{w}$ is in the class $A_{\vec{P}}$.

3.4. **Mixed weak-type inequalities.** The multilinear operator defined by $\prod_{j=1}^{m} Mf_j$ is too big to obtain the weighted estimates obtained in Section 3.3. Nevertheless, we show in this section that it does satisfy sharp weighted weak-type estimates by means of the mixed weak-type inequalities derived in [14].

It follows from the classical Fefferman-Stein inequality

$$\|Mf\|_{L^p(w)} \le C\|f\|_{L^p(Mw)} \quad (1 < p < \infty)$$

that if $p_j > 1$ for all j and $\frac{1}{p} = \frac{1}{p_1} + \cdots + \frac{1}{p_m}$, then

$$(3.12) \qquad \|\prod_{j=1}^{m} M f_j\|_{L^p(\nu_{\vec{w}})} \leq C \prod_{j=1}^{m} \|f_j\|_{L^{p_j}(M w_j)}.$$

However, if at least one of $p_j = 1$, then even a weak-type analogue of (3.12) for arbitrary weights w_j is not true; see Remark 7.6 in Section 7. On the other hand, assuming that all $p_j = 1$ and all weights w_j in A_1, we have the following.

Theorem 3.12. *Assume that w_i is a weight in A_1 for all $i = 1, 2, ..., m$, and set $\nu = \left(\prod_{j=1}^{m} w_j \right)^{1/m}$. Then*

$$(3.13) \qquad \|\prod_{j=1}^{m} M f_j\|_{L^{\frac{1}{m},\infty}(\nu)} \leq C \prod_{j=1}^{m} \|f_i\|_{L^1(w_i)}.$$

To prove this theorem, we will use a recent result proved by Cruz-Uribe, Martell and Pérez [14] related to mixed weak-type inequalities.

Finally, the following simple proposition shows that a weak-type analogue of (3.12) can be obtained by taking $\mathcal{M}(\vec{f})$ instead of $\prod_{j=1}^{m} M f_j$.

Proposition 3.13. *Let $\frac{1}{p} = \frac{1}{p_1} + \cdots + \frac{1}{p_m}$. If $1 \leq p_j < \infty$, then*

$$(3.14) \qquad \|\mathcal{M}(\vec{f})\|_{L^{p,\infty}(\nu_{\vec{w}})} \leq C \prod_{j=1}^{m} \|f_j\|_{L^{p_j}(M w_j)}.$$

Indeed, inequality (3.14) follows from Theorem 3.3 if we put there $\nu = \nu_{\vec{w}}$ and $w_j = M w_j$. It is easy to see that condition (3.4) holds as a consequence of Hölder's inequality.

3.5. **Results for multilinear commutators.** The following end-point estimate for the linear case was obtained in [42],

$$(3.15) \qquad |\{y \in \mathbb{R}^n : |[b,T]f(y)| > \lambda\}| \leq C \int_{\mathbb{R}^n} \frac{|f(y)|}{\lambda} \left(1 + \log^+ \left(\frac{|f(y)|}{\lambda}\right)\right) dy$$

for all $\lambda > 0$, where C depends on the BMO norm of b. One of the main results in this article is Theorem 3.16 which is a multilinear version of (3.15).

We first prove a pointwise estimate relating multilinear commutators and the following maximal operators already mentioned in the introduction.

Definition 3.14. *Given $\vec{f} = (f_1, \ldots, f_m)$, we define the maximal operators*

$$\mathcal{M}_{L(\log L)}^i(\vec{f})(x) = \sup_{Q \ni x} \|f_i\|_{L(\log L),Q} \prod_{j \neq i} \frac{1}{|Q|} \int_Q f_j \, dx$$

and

$$\mathcal{M}_{L(\log L)}(\vec{f})(x) = \sup_{Q \ni x} \prod_{j=1}^{m} \|f_j\|_{L(\log L), Q},$$

where the supremum is taken over all cubes Q containing x.

In the linear case, the idea of relating commutators to sharp maximal operators goes back to Strömberg (cf. [29]), who used it to derive strong type estimates. To derive the end-point estimate (3.15), however, different methods were used in [42] and, later, in [45]. In the multilinear case we obtain the following estimate involving $M_\delta^\#$.

We will use the following notation, if $\vec{b} = (b_1, \cdots, b_m)$ in BMO^m, then we denote the norm $\|\vec{b}\|_{BMO^m} = \sup_{i=1,\cdots,m} \|b_i\|_{BMO}$.

Theorem 3.15. *Let $T_{\vec{b}}$ a multilinear commutator with $\vec{b} \in BMO^m$ and let $0 < \delta < \varepsilon$, with $0 < \delta < 1/m$. Then, there exists a constant $C > 0$, depending on δ and ε, such that*

(3.16) $$M_\delta^\#(T_{\vec{b}}(\vec{f}))(x) \leq C\|\vec{b}\|_{BMO^m} \left(\mathcal{M}_{L(\log L)}(\vec{f})(x) + M_\varepsilon(T(\vec{f}))(x) \right)$$

for all m-tuples $\vec{f} = (f_1, .., f_m)$ of bounded measurable functions with compact support.

We remark that the proof of this theorem actually shows that we can replace $\mathcal{M}_{L(\log L)}(\vec{f})$ in the right-hand side of (3.16) by the slightly smaller operator

$$\sum_{i=1}^{m} \mathcal{M}_{L(logL)}^i(\vec{f}).$$

Of course, estimate (3.16) also holds for $\mathcal{M}_{L(\log L)}(\vec{f})$ replaced by $\prod_{j=1}^{m} M^2(f_j)$. But, again, the smaller operator $\mathcal{M}_{L(\log L)}$ allows us to obtain more general and sharper results.

Recall that the multiple weight $\vec{w}$ satisfies the $A_{(1,\cdots,1)}$ condition, if there is a constant C such that for any cube Q,

$$\frac{1}{|Q|} \int_Q \nu_{\vec{w}} \leq A \prod_{j=1}^{m} \inf_Q w_j^{1/m},$$

where $\nu_{\vec{w}} = \prod_{j=1}^{m} w_j^{1/m}$.

Theorem 3.16. *Let $\vec{w} \in A_{(1,\cdots,1)}$ and $\vec{b} \in BMO^m$. Then there exists a constant C depending on $\|\vec{b}\|_{BMO}$ such that*

(3.17) $$\nu_{\vec{w}}\{x \in \mathbb{R}^n : |T_{\vec{b}}(\vec{f})(x)| > t^m\} \leq C \prod_{j=1}^{m} \left(\int_{\mathbb{R}^n} \Phi(\frac{|f_j(x)|}{t}) w_j(x) dx \right)^{1/m}.$$

Furthermore, this weak-type estimate is sharp in a very general sense. In fact, if we replace the right-hand side of (3.17) by a product of m functionals, one of with is a norm or is homogeneous in λ, then the resulting estimate does not even hold for characteristic functions of intervals. See the counterexample in Remark 7.7. In particular $T_{\vec{b}}$ cannot be a bounded map from any product of Banach spaces that contain characteristic functions of intervals into $L^{1/m,\infty}$. All of this still applies if we just consider any of the $T_{\vec{b}}^j$ involving only one symbol.

A simple homogeneity argument using that Φ submultiplicative shows that the constant C can be taken to be a multiple of $\Phi(\|\vec{b}\|_{BMO})^{1/m}$.

The proof of the Theorem 3.16 will be based on the following result.

Theorem 3.17. *Let $\vec{w} \in A_{(1,\cdots,1)}$. Then there exists a constant C such that*

$$\nu_{\vec{w}}\big\{x \in \mathbb{R}^n : \mathcal{M}_{L(logL)}^i(\vec{f})(x)| > t^m\big\} \leq C \prod_{j=1}^m \left(\int_{\mathbb{R}^n} \Phi(\frac{|f_j(x)|}{t})\, w_j(x)dx\right)^{1/m}.$$

In the linear case, it is possible to interpolate between (3.15) and, say, an strong L^{p_0} estimate to obtain strong-type results for all L^p with $1 < p < p_0$. One approach to obtain strong-type results for $1/m < p \leq 1$ in the m-linear case could be then to try to interpolate between the above end-point results and the results for $p > 1$ in [46]. We have been unable to find a reference for such form of multilinear interpolation and we do not know if the approach is really feasible. Nevertheless, we are able to obtain the strong-type estimates directly. We will derive them again from the pointwise result. This approach has the advantage that can be also used in the weighted context.

Theorem 3.18. *Let $\vec{w} \in A_{\vec{P}}$ and $\vec{b} \in BMO^m$ with $\frac{1}{p} = \frac{1}{p_1} + \cdots + \frac{1}{p_m}$ with $1 < p_j < \infty, j = 1, \ldots, m,$. Then there exists a constant C such that*

$$\tag{3.18} \|T_{\vec{b}}(\vec{f})\|_{L^p(\nu_{\vec{w}})} \leq C\|\vec{b}\|_{BMO^m} \prod_{j=1}^m \|f_j\|_{L^{p_j}(w_j)};$$

These results will be a consequence of the analog of (3.8) for commutators, with $\mathcal{M}$ replaced by $\mathcal{M}_{L(\log L)}$.

Theorem 3.19. *Let $p > 0$ and let w be a weight in A_∞. Suppose that $\vec{b} \in BMO^m$ with $\|\vec{b}\|_{BMO^m} = 1$. Then, there exists a constant $C > 0$, depending on the A_∞ constant of w, such that*

$$\tag{3.19} \int_{\mathbb{R}^n} |T_{\vec{b}}(\vec{f})(x)|^p w(x)dx \leq C \int_{\mathbb{R}^n} \mathcal{M}_{L(\log L)}(\vec{f})(x)^p w(x)dx,$$

and

$$\sup_{t>0} \frac{1}{\Phi(\frac{1}{t})} w(\{y \in \mathbb{R}^n : |T_{\vec{b}}(\vec{f})(y)| > t^m\})$$

$$\tag{3.20} \leq C \sup_{t>0} \frac{1}{\Phi(\frac{1}{t})} w(\{y \in \mathbb{R}^n : \mathcal{M}_{L(logL)}(\vec{f})(y) > t^m\}),$$

for all $\vec{f} = (f_1, .., f_m)$ bounded with compact support.

4. Proofs of the results for $\mathcal{M}$ and Calderón-Zygmund operators

Proof of Theorem 3.2. We will use ideas from [26], [38], and [46], although with some modifications. Fix a point x and a cube Q containing x. As it is well-known, to obtain (3.1) it suffices to prove for $0 < \delta < \frac{1}{m}$

$$(4.1) \qquad \left(\frac{1}{|Q|} \int_Q \left| |T(\vec{f})(z)|^\delta - |c_Q|^\delta \right| dz \right)^{1/\delta} \leq C\mathcal{M}(\vec{f})(x),$$

for some constant c_Q to be determined. In fact we will show, using $||\alpha|^r - |\beta|^r| \leq |\alpha - \beta|^r$, $0 < r < 1$, that

$$(4.2) \qquad \left(\frac{1}{|Q|} \int_Q |T(\vec{f})(z) - c_Q|^\delta \, dz \right)^{1/\delta} \leq C\mathcal{M}(\vec{f})(x).$$

Let $f_j = f_j^0 + f_j^\infty$, where $f_j^0 = f_j \chi_{Q^*}, j = 1, \ldots, m$, where $Q^* = 3Q$. Then

$$
\begin{aligned}
\prod_{j=1}^m f_j(y_j) &= \prod_{j=1}^m \left(f_j^0(y_j) + f_j^\infty(y_j) \right) \\
&= \sum_{\alpha_1, \ldots, \alpha_m \in \{0, \infty\}} f_1^{\alpha_1}(y_1) \ldots f_m^{\alpha_m}(y_m) \\
&= \prod_{j=1}^m f_j^0 + \sideset{}{'}\sum f_1^{\alpha_1}(y_1) \ldots f_m^{\alpha_m}(y_m),
\end{aligned}
$$

where each term of $\sum'$ contains at least one $\alpha_j \neq 0$. Write then

$$(4.3) \qquad T(\vec{f})(z) = T(\vec{f^0})(z) + \sideset{}{'}\sum T(f_1^{\alpha_1}, \cdots, f_m^{\alpha_m})(z).$$

Applying Kolmogorov's inequality (2.16) to the term

$$T(\vec{f^0}(z)) = T(f_1^0, \ldots, f_m^0)(z)$$

with $p = \delta$ and $q = 1/m$, we derive

$$
\begin{aligned}
\left(\frac{1}{|Q|} \int_Q |T(\vec{f^0}(z)|^\delta \, dz \right)^{1/\delta} &\leq C_{m,\delta} \|T(\vec{f^0}(z)\|_{L^{1/m,\infty}(Q, \frac{dx}{|Q|})} \\
&\leq C \prod_{j=1}^m \frac{1}{|3Q|} \int_{3Q} |f_j(z)| \, dz \\
&\leq C \, \mathcal{M}(\vec{f})(x),
\end{aligned}
$$

since $T : L^1 \times \cdots \times L^1 \to L^{1/m}$.

In order to study the other terms in (4.3), we set now

$$c = \sideset{}{'}\sum T(f_1^{\alpha_1}, \ldots, f_m^{\alpha_m})(x),$$

and we will show that, for any $z \in Q$, we also get an estimate of the form

$$(4.4) \qquad \sideset{}{'}\sum |T(f_1^{\alpha_1}, \cdots, f_m^{\alpha_m})(z) - T(f_1^{\alpha_1}, \cdots, f_m^{\alpha_m})(x)| \leq C\mathcal{M}(\vec{f})(x).$$

Consider first the case when $\alpha_1 = \cdots = \alpha_m = \infty$ and define

$$T(\vec{f}^\infty) = T(f_1^\infty, \cdots, f_m^\infty).$$

By the regularity condition (2.2), for any $z \in Q$ we obtain

$$|T(\vec{f}^\infty)(z) - T(\vec{f}^\infty)(x)|$$

$$\leq C \int_{(\mathbb{R}^n \backslash 3Q)^m} \frac{|x-z|^\varepsilon}{(|z-y_1| + \cdots + |z-y_m|)^{nm+\varepsilon}} \prod_{i=1}^m |f_i(y_i)| d\vec{y}$$

$$\leq C \sum_{k=1}^\infty \int_{(3^{k+1}Q)^m \backslash (3^k Q)^m} \frac{|x-z|^\varepsilon}{(|z-y_1| + \cdots + |z-y_m|)^{nm+\varepsilon}} \prod_{i=1}^m |f_i(y_i)| d\vec{y}$$

$$\leq C \sum_{k=1}^\infty \frac{|Q|^{\varepsilon/n}}{(3^k |Q|^{1/n})^{nm+\varepsilon}} \int_{(3^{k+1}Q)^m} \prod_{i=1}^m |f_i(y_i)| d\vec{y}$$

$$\leq C \sum_{k=1}^\infty \frac{1}{3^{k\varepsilon}} \prod_{i=1}^m |f_i|_{3^{k+1}Q} \leq C\mathcal{M}(\vec{f})(x)$$

(here we have used the notation $E^m = E \times \cdots \times E$ and $d\vec{y} = dy_1 \ldots dy_m$).

What remains to be considered are the terms in (4.4) such that $\alpha_{j_1} = \cdots = \alpha_{j_l} = 0$ for some $\{j_1, \ldots, j_l\} \subset \{1, \ldots, m\}$ and $1 \leq l < m$. By (2.2),

$$|T(f_1^{\alpha_1}, \ldots, f_m^{\alpha_m})(z) - T(f_1^{\alpha_1}, \ldots, f_m^{\alpha_m})(x)|$$

$$\leq \prod_{j \in \{j_1, \ldots, j_l\}} \int_{3Q} |f_j| dy_j \int_{(\mathbb{R}^n \backslash 3Q)^{m-l}} \frac{|x-z|^\varepsilon \prod_{j \notin \{j_1, \ldots, j_l\}} |f_j| dy_j}{(|z-y_1| + \cdots + |z-y_m|)^{nm+\varepsilon}}$$

$$\leq \prod_{j \in \{j_1, \ldots, j_l\}} \int_{3Q} |f_j| dy_j \sum_{k=1}^\infty \frac{|Q|^{\varepsilon/n}}{(3^k |Q|^{1/n})^{nm+\varepsilon}} \int_{(3^{k+1}Q)^{m-l}} \prod_{j \notin \{j_1, \ldots, j_l\}} |f_j| dy_j$$

$$\leq C \sum_{k=1}^\infty \frac{|Q|^{\varepsilon/n}}{(3^k |Q|^{1/n})^{nm+\varepsilon}} \int_{(3^{k+1}Q)^m} \prod_{i=1}^m |f_i(y_i)| d\vec{y},$$

and we arrived to the expression considered in the previous case. This gives (4.4) and concludes the proof of the theorem. $\square$

Proof of Theorem 3.3. The proof is very similar to the one in the linear situation (see [41]). We consider only the case when $p_j > 1$ for all $j = 1, \ldots, m$. Minor modifications for the case of some $p_j = 1$ can be done exactly as in the linear situation.

Suppose that (3.3) holds. Then for any $\vec{f}$ we clearly get

$$(4.5) \qquad \left(\int_Q \nu \right)^{1/p} \prod_{j=1}^m |f_j|_Q \le C \prod_{j=1}^m \|f_j \chi_Q\|_{L^{p_j}(w_j)}.$$

Setting here $f_j = w_j^{-1/(p_j-1)}$, we obtain (3.4).

Assume now that (3.4) holds. Then, by Hölder's inequality we obtain (4.5). It follows easily from (4.5) that

$$\mathcal{M}(\vec{f})(x) \le C \prod_{j=1}^m M_\nu^c(|f_j|^{p_j} w_j/\nu)(x)^{1/p_j},$$

where M_ν^c denotes the weighted centered maximal function. From this, using the well-known fact (based on the Besicovitch covering theorem) that M_ν^c is of weak type $(1,1)$ with respect to ν, and the Hölder inequality for weak spaces (see, [23, p.15]), we obtain

$$
\begin{aligned}
\|\mathcal{M}(\vec{f})\|_{L^{p,\infty}(\nu)} \ &\le\ C \| \prod_{j=1}^m M_\nu^c(|f_j|^{p_j} w_j/\nu)^{1/p_j} \|_{L^{p,\infty}(\nu)} \\
&\le\ C \prod_{j=1}^m \| M_\nu^c(|f_j|^{p_j} w_j/\nu)^{1/p_j} \|_{L^{p_j,\infty}(\nu)} \\
&=\ C \prod_{j=1}^m \| M_\nu^c(|f_j|^{p_j} w_j/\nu) \|_{L^{1,\infty}(\nu)}^{1/p_j} \\
&\le\ C \prod_{j=1}^m \| f_j \|_{L^{p_j}(w_j)}.
\end{aligned}
$$

The theorem is proved. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Proof of Theorem 3.6. Consider first the case when there exists at least one $p_j > 1$. Without loss of generality we can assume that $p_1, \dots, p_l = 1, 0 \le l < m$, and $p_j > 1$ for $j = l+1, \dots, m$.

Suppose that $\vec{w}$ satisfies the multilinear $A_{\vec{P}}$ condition.

Fix $j \ge l+1$ and define the numbers

$$q_j = p\Big(m - 1 + \frac{1}{p_j} \Big) \quad \text{and} \quad q_i = \frac{p_i}{p_i - 1} \frac{q_j}{p}, i \ne j, i \ge l+1.$$

We first prove that $w_j^{1-p_j'} \in A_{mp_j'}$ for $j \ge l+1$, i.e.,

$$(4.6) \qquad \left(\int_Q w_j^{-1/(p_j-1)} \right) \left(\int_Q w_j^{\frac{p}{p_j q_j}} \right)^{\frac{q_j p_j}{p(p_j-1)}} \le c|Q|^{\frac{mp_j}{p_j-1}}.$$

Since

$$\sum_{i=l+1}^{m} \frac{1}{q_i} = \frac{1}{m-1+1/p_j}\left(\frac{1}{p} + \sum_{i=l+1, i\neq j}^{m}(1-1/p_i)\right) = 1,$$

applying the Hölder inequality, we obtain

$$\int_Q w_j^{\frac{p}{p_j q_j}} = \int_Q \Big(\prod_{i=l+1}^{m} w_i^{\frac{p}{p_i q_j}}\Big)\Big(\prod_{i=l+1, i\neq j}^{m} w_i^{-\frac{p}{p_i q_j}}\Big)$$

$$\leq \Big(\int_Q \prod_{i=l+1}^{m} w_i^{p/p_i}\Big)^{1/q_j} \prod_{i=l+1, i\neq j}^{m}\Big(\int_Q w_i^{-1/(p_i-1)}\Big)^{1/q_i}.$$

From this inequality and the $A_{\vec{P}}$ condition we easily get (4.6).

Next we show that $\nu_{\vec{w}} \in A_{mp}$. Setting $s_j = (m-1/p)p'_j$, $j \geq l+1$, we have $\sum_{j=l+1}^{m} \frac{1}{s_j} = 1$ and, therefore, by Hölder's inequality,

$$(4.7) \qquad \int_Q \prod_{j=l+1}^{m} w_j^{-\frac{p}{p_j(pm-1)}} \leq \prod_{j=l+1}^{m}\Big(\int_Q w_j^{-1/(p_j-1)}\Big)^{1/s_j}.$$

Hence,

$$\int_Q (\nu_{\vec{w}})^{-\frac{1}{pm-1}} \leq \prod_{j=1}^{l}(\inf_Q w_j)^{-\frac{p}{pm-1}} \prod_{j=l+1}^{m}\Big(\int_Q w_j^{-1/(p_j-1)}\Big)^{1/s_j}.$$

Combining this inequality with the $A_{\vec{P}}$ condition gives $\nu_{\vec{w}} \in A_{mp}$.

Suppose now that $l > 0$, and let us show that $w_j^{1/m} \in A_1, j = 1, \dots, l$. Fix $1 \leq i_0 \leq l$. By Hölder's inequality and (4.7),

$$\int_Q w_{i_0}^{1/m} \leq \Big(\int_Q w_{i_0}^p \prod_{j=l+1}^{m} w_j^{p/p_j}\Big)^{1/pm}\Big(\int_Q \prod_{j=l+1}^{m} w_j^{-\frac{p}{p_j(pm-1)}}\Big)^{1-1/pm}$$

$$\leq \Big(\int_Q w_{i_0}^p \prod_{j=l+1}^{m} w_j^{p/p_j}\Big)^{1/pm} \prod_{j=l+1}^{m}\Big(\int_Q w_j^{1-p'_j}\Big)^{\frac{1}{mp'_j}}$$

This inequality combined with the $A_{\vec{P}}$ condition proves $w_{i_0}^{1/m} \in A_1$. Thus we have proved that $\vec{w} \in A_{\vec{P}} \Rightarrow (3.6)$.

To prove that (3.6) is sufficient for $\vec{w} \in A_{\vec{P}}$, we first observe that for any weight w_j,

$$(4.8) \qquad 1 \leq \Big(\frac{1}{|Q|}\int_Q \nu_{\vec{w}}^{-\frac{1}{pm-1}}\Big)^{m-1/p} \prod_{j=1}^{m}\Big(\frac{1}{|Q|}\int_Q w_j^{\frac{1}{p_j(m-1)+1}}\Big)^{m-1+1/p_j}.$$

Indeed, let $\alpha = \frac{1}{1+pm(m-1)}$ and $\alpha_j = \frac{1/p+m(m-1)}{1/p_j+m-1}$. Then $\sum_{j=1}^{m} 1/\alpha_j = 1$, and by Hölder's inequality,

$$\int_Q \nu_{\vec{w}}^{\alpha} \leq \prod_{j=1}^{m}\Big(\int_Q w_j^{\frac{\alpha p \alpha_j}{p_j}}\Big)^{1/\alpha_j} = \prod_{j=1}^{m}\Big(\int_Q w_j^{\frac{1}{p_j(m-1)+1}}\Big)^{\alpha p(m-1+1/p_j)}.$$

Using again the Hölder inequality, we have

$$1 \le \Big(\frac{1}{|Q|}\int_Q \nu_{\vec{w}}^\alpha\Big)\Big(\frac{1}{|Q|}\int_Q \nu_{\vec{w}}^{-\frac{1}{pm-1}}\Big)^{\alpha(pm-1)}.$$

This inequality along with the previous one yields (4.8). Finally, (4.8) combined with (3.6) easily gives that $\vec{w} \in A_{\vec{P}}$.

It remains to consider the case when $p_j = 1$ for all $j = 1, \cdots, m$. Assume that $\vec{w} \in A_{(1,\cdots,1)}$, i.e.,

$$(4.9) \qquad \Big(\frac{1}{|Q|}\int_Q \Big(\prod_{j=1}^m w_j\Big)^{1/m}\Big)^m \le c\prod_{j=1}^m \inf_Q w_j.$$

It is clear that (4.9) implies that $w_j^{1/m} \in A_1$, $j = 1, \ldots, m$ and $\nu_{\vec{w}} \in A_1$. Conversely, combining these last conditions with Hölder's inequality we obtain

$$\begin{aligned}
\Big(\frac{1}{|Q|}\int_Q \Big(\prod_{j=1}^m w_j\Big)^{1/m}\Big)^m &\le c\inf_Q\Big(\prod_{j=1}^m w_j\Big) \le c\Big(\frac{1}{|Q|}\int_Q \Big(\prod_{j=1}^m w_j\Big)^{1/m^2}\Big)^{m^2}.\\
&\le c\prod_{j=1}^m \Big(\frac{1}{|Q|}\int_Q w_j^{1/m}\Big)^m \le c\prod_{j=1}^m \inf_Q w_j.
\end{aligned}$$

This proves that $\vec{w} \in A_{(1,\cdots,1)}$ is equivalent to $w_j^{1/m} \in A_1$, $j = 1, \ldots, m$ and $\nu_{\vec{w}} \in A_1$. The theorem is proved. $\qquad\square$

Proof of Theorem 3.7. The necessity follows immediately from Theorem 3.3, so we only have to prove the sufficiency. We give two proofs based on different ideas. They parallel to some extend the different proofs given in the linear situation in [9], and [7].

1st Proof. Assume that $\vec{w} \in A_{\vec{p}}$. Then by Theorem 3.6 each $w_j^{-\frac{1}{p_j-1}}$ satisfies the reverse Hölder inequality, i.e., there exist $r_j > 1$ and $c > 0$ such that for all $1 \le r \le r_j$ and for any cube Q,

$$(4.10) \qquad \Big(\frac{1}{|Q|}\int_Q w_j^{-\frac{r}{p_j-1}}\Big)^{1/r} \le c\frac{1}{|Q|}\int_Q w_j^{-\frac{1}{p_j-1}}.$$

Let

$$\xi = \min_{1\le j\le m} r_j \quad\text{and}\quad q = \max_{1\le j\le m}\frac{pm}{pm+(1-1/\xi)(p_j-1)},$$

and observe that $qp_j > 1$ for any j.

We claim that the following pointwise inequality holds:

$$(4.11) \qquad \mathcal{M}(\vec{f})(x) \le c\prod_{j=1}^m M_{\nu_{\vec{w}}}^c\big((|f_j|^{p_j}w_j/\nu_{\vec{w}})^q\big)(x)^{1/qp_j}.$$

Then the proof of the theorem follows from Hölder's inequality and the boundedness of the centered maximal operator.

To verify the claim we first observe that, by Hölder's inequality,

$$(4.12) \qquad \int_Q |f_j| \leq \Big(\int_Q |f_j|^{p_j q} w_j^q \nu_{\vec{w}}^{1-q} \Big)^{\frac{1}{qp_j}} \Big(\int_Q \big(w_j^q \nu_{\vec{w}}^{1-q} \big)^{-\frac{1}{qp_j-1}} \Big)^{1-\frac{1}{qp_j}}.$$

Set $\gamma_j = \frac{qp_j-1}{(1-q)(pm-1)}$. By the definition of q, $\gamma_j > 1$ for any j. Applying again Hölder's inequality, we get

$$(4.13) \qquad \int_Q \big(w_j^q \nu_{\vec{w}}^{1-q} \big)^{-\frac{1}{qp_j-1}} \leq \Big(\int_Q w_j^{-\frac{q\gamma_j'}{qp_j-1}} \Big)^{1/\gamma_j'} \Big(\int_Q \nu_{\vec{w}}^{-\frac{1}{pm-1}} \Big)^{1/\gamma_j}.$$

Note now that for any j,

$$\frac{q(p_j-1)\gamma_j'}{qp_j-1} = \frac{q(p_j-1)}{q(p_j-1)-(1-q)pm} \leq \xi.$$

Therefore, by (4.10),

$$(4.14) \qquad \int_Q w_j^{-\frac{q\gamma_j'}{qp_j-1}} = \int_Q w_j^{-\frac{1}{p_j-1}\frac{q(p_j-1)\gamma_j'}{qp_j-1}}$$

$$\leq c|Q|^{1-\frac{q(p_j-1)\gamma_j'}{qp_j-1}} \Big(\int_Q w_j^{-\frac{1}{p_j-1}} \Big)^{\frac{q(p_j-1)\gamma_j'}{qp_j-1}}.$$

Applying (4.13), (4.14) and the fact that $\nu_{\vec{w}} \in A_{pm}$ (see Theorem 3.6), we obtain

$$\Big(\int_Q \big(w_j^q \nu_{\vec{w}}^{1-q} \big)^{-\frac{1}{qp_j-1}} \Big)^{1-\frac{1}{qp_j}}$$

$$\leq c|Q|^{-\frac{pm(1-q)}{qp_j}} \Big(\int_Q w_j^{-\frac{1}{p_j-1}} \Big)^{1-1/p_j} \Big(\int_Q \nu_{\vec{w}}^{-\frac{1}{pm-1}} \Big)^{\frac{(1-q)(pm-1)}{qp_j}}$$

$$\leq \frac{c}{\nu_{\vec{w}}(Q)^{\frac{1-q}{qp_j}}} \Big(\int_Q w_j^{-\frac{1}{p_j-1}} \Big)^{1-1/p_j}.$$

Finally, combining this inequality with (4.12) and the $A_{\vec{P}}$ condition we can estimate

$$\prod_{j=1}^m |f_j|_Q \leq c \prod_{j=1}^m \Big(\frac{1}{\nu_{\vec{w}}(Q)} \int_Q (|f_j|^{p_j} w_j / \nu_{\vec{w}})^q \nu_{\vec{w}} \Big)^{1/qp_j}.$$

This yields (4.11) and hence the theorem is proved.

$\square$

2nd Proof. We first give the proof for the dyadic version of $\mathcal{M}$ defined by

$$\mathcal{M}^d(\vec{f})(x) = \sup_{x \in Q \in \mathcal{D}} \prod_{i=1}^m \frac{1}{|Q|} \int_Q |f_i(y_i)| dy_i,$$

where $\mathcal{D}$ is the family of all dyadic cubes in $\mathbb{R}^n$. Observe that

$$\|\mathcal{M}^d(\vec{f})\|_{L^p(\nu_{\vec{w}})} \le c \prod_{j=1}^m \|f_j\|_{L^{p_j}(w_j)}$$

is equivalent to

(4.15)
$$\|\mathcal{M}^d(\vec{f_\sigma})\|_{L^p(\nu_{\vec{w}})} \le c \prod_{j=1}^m \|f_j\|_{L^{p_j}(\sigma_j)},$$

where $\sigma_j = w_j^{-\frac{1}{p_j - 1}}$ and $\vec{f_\sigma} = (f_1 \sigma_1, \ldots, f_m \sigma_m)$.

Fix $a > 2^{mn}$. For each integer k let

$$\Omega_k = \{x \in \mathbb{R}^n : \mathcal{M}^d(\vec{f})(x) > a^k\}.$$

It is easy to see that a full analogue of the classical Calderón–Zygmund decomposition holds for $\mathcal{M}^d(\vec{f})$ and, therefore, there is a family of maximal non-overlapping dyadic cubes $\{Q_{k,j}\}$ for which $\Omega_k = \cup_j Q_{k,j}$ and

(4.16)
$$a^k < \prod_{i=1}^m \frac{1}{|Q_{k,j}|} \int_{Q_{k,j}} |f_i \sigma_i(y_i)| dy_i \le 2^{nm} a^k.$$

It follows that

$$\int_{\mathbb{R}^n} \mathcal{M}^d(\vec{f_\sigma})^p \, \nu_{\vec{w}} dx = \sum_k \int_{\Omega_k \setminus \Omega_{k+1}} \mathcal{M}^d(\vec{f_\sigma})^p \, \nu_{\vec{w}} dx$$

$$\le a^p \sum_k a^{kp} \nu_{\vec{w}}(\Omega_k) = a^p \sum_{k,j} a^{kp} \nu_{\vec{w}}(Q_{k,j})$$

$$\le a^p \sum_{k,j} \left(\prod_{i=1}^m \frac{1}{|Q_{k,j}|} \int_{Q_{k,j}} |f_i \sigma_i(y_i)| dy_i \right)^p \nu_{\vec{w}}(Q_{k,j})$$

$$= a^p \sum_{k,j} \left(\prod_{i=1}^m \frac{1}{\sigma_i(Q_{k,j})} \int_{Q_{k,j}} |f_i \sigma_i(y_i)| dy_i \right)^p \left(\prod_{i=1}^m \frac{\sigma_i(Q_{k,j})}{|Q_{k,j}|} \right)^p \nu_{\vec{w}}(Q_{k,j})$$

$$\le c \sum_{k,j} \left(\prod_{i=1}^m \frac{1}{\sigma_i(Q_{k,j})} \int_{Q_{k,j}} |f_i \sigma_i(y_i)| dy_i \right)^p \prod_{i=1}^m \sigma_i(Q_{k,j})^{p/p_i}$$

(where in the last estimate we have used the $A_{\vec{P}}$ condition).

Set now $E_{k,j} = Q_{k,j} \setminus Q_{k,j} \cap \Omega_{k+1}$. We claim that there exists a constant $\beta > 0$ such that

(4.17)
$$|Q_{k,j}| < \beta \, |E_{k,j}|$$

for each k, j. Indeed, by (4.16) and Hölder's inequality,

$$
\begin{aligned}
|Q_{k,j} \cap \Omega_{k+1}| &= \sum_{Q_{k+1,l} \subset Q_{k,j}} |Q_{k+1,l}| \\
&< \frac{1}{a^{(k+1)/m}} \sum_{Q_{k+1,l} \subset Q_{k,j}} \left(\prod_{i=1}^{m} \int_{Q_{k+1,l}} |f_i \sigma_i| \right)^{1/m} \\
&\leq \left(\frac{1}{a^{k+1}} \prod_{i=1}^{m} \int_{Q_{k,j}} |f_i \sigma_i| \right)^{1/m} \leq \frac{2^n}{a^{1/m}} |Q_{k,j}|,
\end{aligned}
$$

which proves (4.17) with $1/\beta = 1 - 2^n/a^{1/m}$.

By Theorem 3.6, each σ_i satisfies the A_{q_i} condition for appropriate $q_i > 1$. It follows from the definition of A_p and Hölder's inequality (see, e.g., [23, p. 693]) that there exists a constant c such that, for any cube Q and any measurable subset $E \subset Q$,

$$
\left(\frac{|E|}{|Q|} \right)^{q_i} \leq c \frac{\sigma_i(E)}{\sigma_i(Q)}.
$$

Combining this inequality with (4.17), we get

$$
\sigma_i(Q_{k,j}) \leq \gamma_i \sigma_i(E_{k,j})
$$

for each $i = 1, \ldots, m$ and each k, j. Hence, using Hölder's inequality and the fact that the sets $E_{k,j}$ are pairwise disjoint, we obtain (with $\gamma = \max_i \gamma_i$)

$$
\begin{aligned}
&\int_{\mathbb{R}^n} \mathcal{M}^d(\vec{f_\sigma})^p \nu_{\vec{w}} dx \\
&\leq c \sum_{k,j} \left(\prod_{i=1}^{m} \frac{1}{\sigma_i(Q_{k,j})} \int_{Q_{k,j}} |f_i \sigma_i(y_i)| dy_i \right)^p \prod_{i=1}^{m} \sigma_i(E_{k,j})^{p/p_i}. \\
&\leq c \prod_{i=1}^{m} \left(\sum_{k,j} \left(\frac{1}{\sigma_i(Q_{k,j})} \int_{Q_{k,j}} |f_i \sigma_i(y_i)| dy_i \right)^{p_i} \sigma_i(E_{k,j}) \right)^{p/p_i}. \\
&\leq c \prod_{i=1}^{m} \left(\sum_{k,j} \int_{E_{k,j}} M_{\sigma_i}(f_i)^{p_i} \sigma_i \right)^{p/p_i} \leq c\gamma \prod_{i=1}^{m} \left(\int_{\mathbb{R}^n} M_{\sigma_i}(f_i)^{p_i} \sigma_i \right)^{p/p_i} \\
&\leq c \prod_{i=1}^{m} \left(\int_{\mathbb{R}^n} |f_i|^{p_i} \sigma_i \right)^{p/p_i},
\end{aligned}
$$

where in the last inequality we have used the boundedness of M_{σ_i} on $L^{p_i}(\sigma_i)$. The proof is complete in the dyadic case. Observe that it is enough to assume that the weight $\vec{w}$ satisfies the $A_{\vec{P}}$ condition only for dyadic cubes.

To pass from the dyadic version to the general situation we use established tools to handle such passage. We need the following easy variant of a result of Fefferman-Stein [18] which can be also found in [22, p. 431].

Lemma 4.1. *For each integer k, each $\vec{f}$, all x in $\mathbb{R}^n$ and $p > 0$ there exists a constant c, depending only on n, m and p, so that*

$$\mathcal{M}^k(\vec{f})(x)^p \leq \frac{c}{|Q_k|} \int_{Q_k} \left(\tau_{-t} \circ \mathcal{M}^d \circ \tau_t \right)(\vec{f})(x)^p \, dt.$$

Here $\tau_t g(x) = g(x - t)$, Q_k is the cube centered at the origin with side length 2^{k+2}, and $\mathcal{M}^k$ is the operator defined as $\mathcal{M}$ but with cubes having sides of length smaller than 2^k.

Clearly, to estimate the left-hand side of (3.7) it suffices to estimate $\|\mathcal{M}^k(\vec{f})\|_{L^p(\nu_{\vec{w}})}$. It follows from the above pointwise inequality and Fubini's theorem that

$$\|\mathcal{M}^k(\vec{f})\|_{L^p(\nu_{\vec{w}})} \leq c \sup_t \|\tau_{-t} \circ \mathcal{M}^d \circ \tau_t\|_{L^p(\nu_{\vec{w}})}.$$

We have now to estimate $\|\tau_{-t} \circ \mathcal{M}^d \circ \tau_t\|_{L^p(\nu_{\vec{w}})}$ with constant independent of t. The bound

$$\tau_{-t} \circ \mathcal{M}^d \circ \tau_t : L^{p_1}(w_1) \times \cdots \times L^{p_m}(w_m) \to L^p(\vec{w})$$

with constant independent of t is equivalent to the bound

$$\mathcal{M}^d : L^{p_1}(\tau_t w_1) \times \cdots \times L^{p_m}(\tau_t w_m) \to L^p(\tau_t(\vec{w}))$$

with constant independent of t. But $\tau_t(\vec{w})$ satisfies the $A_{\vec{P}}$ with constant independent of t because $A_{\vec{p}}$ is invariant under translation and, hence, we can apply the first part of the proof (i.e, the one considered for the dyadic maximal case). $\square$

Proof of Corollary 3.8. It is enough to prove (3.8) when the right-hand side is finite (or there is nothing to prove). We have using (2.8) and (3.1)

$$\|T(\vec{f})\|_{L^p(w)} \leq \|M_\delta(T(\vec{f}))\|_{L^p(w)} \leq C\|M_\delta^\#(T(\vec{f}))\|_{L^p(w)} \leq C\|\mathcal{M}(\vec{f})\|_{L^p(w)},$$

which gives the desired result provided we can show that $\|M_\delta(T(\vec{f}))\|_{L^p(w)}$ is finite. Note that since w is in A_∞, w is also in A_{p_0} with $0 < \max(1, pm) < p_0 < \infty$. So with $\delta < p/p_0 < 1/m$ we have, in addition to the above inequalities,

$$\|M_\delta(T(\vec{f}))\|_{L^p(w)} \leq \|M_{p/p_0}(T(\vec{f}))\|_{L^p(w)} = C\|M(T(\vec{f})^{p/p_0})\|_{L^{p_0}(w)}^{p_0/p}$$

$$\leq C\|(T(\vec{f})^{p/p_0})\|_{L^{p_0}(w)}^{p_0/p} \leq C\|T(\vec{f})\|_{L^p(w)}.$$

It is enough then to prove that $\|T(\vec{f})\|_{L^p(w)}$ is finite for each family $\vec{f}$ of bounded functions with compact support for which $\|\mathcal{M}(\vec{f})\|_{L^p(w)}$ is finite. We will see that this is always the case. The standard arguments are as follows.

The weight w is also in L_{loc}^q for q sufficiently close to 1 so that its dual exponent q' satisfies $pq' > 1/m$. Then, for any ball B center at the origin $\|T(\vec{f})\|_{L^p(B,w)}$ is finite by Hölder's inequality and the unweighted theory for T. On the other hand, outside a sufficiently large ball B,

$$(4.18) \qquad \mathcal{M}(\vec{f})(x) \geq C_1|x|^{-mn} \geq C_2|Tf(x)|$$

(with constants depending on $\vec{f}$ of course). From the assumption $\|\mathcal{M}(\vec{f})\|_{L^p(w)}$ finite and (4.18), we conclude

$$\|T(\vec{f})\|_{L^p(R^n \setminus B, w)} \leq C\|\mathcal{M}(\vec{f})\|_{L^p(R^n \setminus B, w)} < \infty.$$

Similar arguments give the weak-type estimate (3.9). □

Proof of Corollary 3.9. Since $\nu_{\vec{w}}$ is in A_∞ and the intersection of the space of simple functions with $L^p(w)$ is dense in $L^p(w)$ for any weight w ([2, p. 211], the corollary immediately follows from the previous one and the boundedness properties of $\mathcal{M}$ on weighted spaces. □

Proof of Theorem 3.11. For simplicity we consider only the one-dimensional case. The higher dimensional one is only notationally more complicated. Recall that the sum of the m-linear Riesz transforms is

$$T(\vec{f})(x) = \text{p.v.} \int_{\mathbb{R}^m} \frac{\sum_{j=1}^m (x - y_j)}{\left(\sum_{j=1}^m |x - y_j|^2\right)^{\frac{m+1}{2}}} f_1(y_1) \dots f_m(y_m) dy_1 \dots dy_m.$$

Clearly, it suffices to show that if (3.11) holds, then $\vec{w} \in A_{\vec{P}}$. We follow similar a argument to the one used in the linear case (see [22, p. 417]) but with some modifications.

First, we suppose that all functions $f_i \geq 0$ and that $supp(f_j) \subset I$. Then, if $x \in I^+$ and $y_j \in I$ for all j, we get that

$$\frac{\sum_{j=1}^m (x - y_j)}{\left(\sum_{j=1}^m |x - y_j|\right)^{m+1}} = \frac{1}{\left(\sum_{j=1}^m |x - y_j|\right)^m} \geq \frac{c_m}{|I|^m}$$

Therefore, if $x \in I^+$ we have

$$T(\vec{f})(x) \geq c_m \prod_{j=1}^m |f_j|_I,$$

and hence

$$I^+ \subset \{x : |T(\vec{f})(x)| > \lambda\},$$

whenever $0 < \lambda < c_m \prod_{j=1}^m |f_j|_I$. Arguing exactly as in Theorem 3.3, we obtain from this that for any interval I,

$$(4.19) \qquad \left(\frac{1}{|I|} \int_{I^+} \nu_{\vec{w}}\right)^{1/p} \prod_{j=1}^m \left(\frac{1}{|I|} \int_I w_j^{-1/(p_j-1)}\right)^{1-1/p_j} \leq c.$$

In a similar way we can prove that (3.11) implies that for any interval I,

$$(4.20) \qquad \left(\frac{1}{|I|} \int_I \nu_{\vec{w}}\right)^{1/p} \prod_{j=1}^m \left(\frac{1}{|I|} \int_{I^+} w_j^{-1/(p_j-1)}\right)^{1-1/p_j} \leq c.$$

From (4.20), using the same argument of the proof of the second condition in (3.6), we get that for any interval I,

$$\left(\int_I \nu_{\vec{w}}\right)\left(\int_{I^+} \nu_{\vec{w}}^{-1/(pm-1)}\right)^{pm-1} \le c|I|^{pm}.$$

Hence, $\nu_{\vec{w}} \in A_{pm}^+$. Finnally, combining (2.7) and (4.19) we can see that $\vec{w} \in A_{\vec{P}}$, which completes the proof. □

Before proving Theorem 3.12, we state several auxiliary facts that will be needed. By RH_∞ we denote the class of weights w satisfying

$$\sup_Q w \le \frac{c}{|Q|}\int_Q w.$$

It was shown in [15, Th. 4.8] that if $f, g \in RH_\infty$, then $fg \in RH_\infty$. In particular, since $(Mf)^{-1} \in RH_\infty$ for any f, we have that

(4.21)
$$v = \frac{1}{Mf_1 Mf_2 \ldots Mf_s} \in RH_\infty.$$

In a recent article [14], it was proved that if $u \in A_1$ and $v \in RH_\infty$, then for every $f \in L_u^1$,

(4.22)
$$\|Mf/v\|_{L^{1,\infty}(uv)} \le c\|f\|_{L^1(u)}$$

(more precisely, this follows from Theorem 1.3 in [14]).

Proof of Theorem 3.12. Let

$$\mu_\nu(\lambda) = \nu\left\{x : \prod_{j=1}^m Mf_j(x) > \lambda\right\}.$$

Observe that if suffices to show that

(4.23)
$$\mu_\nu(\lambda) \le \frac{c}{\lambda^{\frac{1}{m}}}\left(\prod_{j=1}^m \|f_i\|_{L^1(w_i)}\right)^{\frac{1}{m}} + \mu_\nu(2\lambda).$$

Indeed, by iterating (4.23) the weak-estimate (3.13) follows easily.
Define

$$E = \left\{x : \lambda < \prod_{j=1}^m Mf_j \le 2\lambda\right\} \quad \text{and} \quad v_i = \prod_{j=1, j\neq i}^m (Mf_j)^{-1}.$$

Using Hölder's inequality along with (4.21) and (4.22), we obtain

$$
\begin{aligned}
\mu_\nu(\lambda) - \mu_\nu(2\lambda) = \nu(E) \ &\leq \ \lambda^{-\frac{1}{m}} \int_E \Big(\prod_{j=1}^m M f_j w_j \Big)^{\frac{1}{m}} \\
&\leq \ \lambda^{-\frac{1}{m}} \prod_{j=1}^m \Big(\int_E M f_j w_j \Big)^{\frac{1}{m}} \\
&\leq \ 2\lambda^{1-\frac{1}{m}} \prod_{j=1}^m \Big(\int_{\{M f_j > \lambda v_j\}} v_j w_j \Big)^{\frac{1}{m}} \\
&\leq \ c\lambda^{-\frac{1}{m}} \Big(\prod_{j=1}^m \|f_i\|_{L^1(w_i)} \Big)^{\frac{1}{m}} .
\end{aligned}
$$

This proves (4.23) and the theorem. $\qquad\square$

5. Proof of the pointwise and weighted results for $\mathcal{M}_{L(\log L)}$ and the multilinear commutators

Proof of Theorem 3.15. By linearity it is enough to consider the operator with only one symbol. Fix then $b \in BMO$ and consider the operator

$$(5.1) \qquad T_b(\vec{f})(x) = b(x) T(f_1, \cdots, f_m) - T(bf_1, \cdots, f_m).$$

Note that for any constant λ we also have

$$T_b(\vec{f})(x) = (b(x) - \lambda) T(\vec{f})(x) - T((b-\lambda)f_1, .., f_m)(x).$$

Fix $x \in \mathbb{R}^n$. Since $0 < \delta < 1$, for any number c and any cube Q centered at x, we can estimate

$$
\left(\frac{1}{|Q|} \int_Q \Big| |T_b(\vec{f})(z)|^\delta - |c|^\delta \Big| \, dz \right)^{1/\delta} \leq \left(\frac{1}{|Q|} \int_Q \Big| T_b(\vec{f})(z) - c \Big|^\delta dz \right)^{1/\delta}
$$

$$
\leq \left(\frac{C}{|Q|} \int_Q \Big| (b(z) - \lambda) T(\vec{f})(z) \Big|^\delta dz \right)^{1/\delta} + \left(\frac{C}{|Q|} \int_Q |T((b-\lambda)f_1, .., f_m)(z) - c|^\delta dz \right)^{1/\delta}
$$

$$
= I + II.
$$

We analyze each term separately. Recall that $Q^* = 3Q$ and let $\lambda = (b)_{Q^*}$ be the average of b on Q^*. For any $1 < q < \varepsilon/\delta$ we have by Hölder's and Jensen's inequalities,

$$
\begin{aligned}
I \ &\leq \ C \left(\frac{1}{|Q|} \int_Q |b(z) - \lambda|^{\delta q'} dz \right)^{1/\delta q'} \left(\frac{1}{|Q|} \int_Q \Big| T(\vec{f})(z) \Big|^{\delta q} dz \right)^{1/\delta q} \\
&\leq \ C\|b\|_{BMO} \, M_{\delta q}(T(\vec{f}))(x) \\
&\leq \ C\|b\|_{BMO} \, M_\varepsilon(T(\vec{f}))(x).
\end{aligned}
$$

To estimate II we split again each f_i as $f_i = f_i^0 + f_i^\infty$ where $f_i^0 = f\chi_{Q^*}$ and $f_i^\infty = f_i - f_i^0$.

This yields

$$
\begin{aligned}
\prod_{j=1}^m f_j(y_j) &= \sum_{\{\alpha_1,\cdots,\alpha_m\}\in\{0,\infty\}} f_1^{\alpha_1}(y_1)\cdots f_m^{\alpha_m}(y_m) \\
&= \prod_{j=1}^m f_j^0(y_j) + {\sum}' f_1^{\alpha_1}(y_1)\cdots f_m^{\alpha_m}(y_m),
\end{aligned}
$$

where each term in $\sum'$ contains at least one $\alpha_j \neq 0$.

We let $c = \sum' c_{\alpha_1,\cdots,\alpha_m}$ with $c_{\alpha_1,\cdots,\alpha_m} = T((b-\lambda)f_1^{\alpha_1}, f_2^{\alpha_2}, \ldots, f_m^{\alpha_m})(x)$ and obtain

$$
\begin{aligned}
II &\leq C\left(\left(\frac{1}{|Q|}\int_Q \left|T((b-\lambda)f_1^0,\cdots,f_m^0)(z)\right|^\delta dz\right)^{1/\delta}\right. \\
&\quad + \left.{\sum}'\left(\frac{1}{|Q|}\int_Q \left|T((b-\lambda)f_1^{\alpha_1},\ldots,f_m^{\alpha_m})(\xi) - c_{\alpha_1,\cdots,\alpha_m}\right|^\delta dz\right)^{1/\delta}\right] \\
&= II_0 + {\sum}' II_{\alpha_1,\cdots,\alpha_m}.
\end{aligned}
$$

Using again $\delta < 1/m$, it follows that

$$
\begin{aligned}
II_0 &= C\left(\frac{1}{|Q|}\int_Q \left|T((b-\lambda)f_1^0,\cdots,f_m^0)(z)\right|^\delta dz\right)^{1/\delta} \\
&\leq C\|T((b-\lambda)f_1^0,\cdots,f_m^0)\|_{L^{1/m,\infty}(Q,\frac{dx}{|Q|})} \\
&\leq C\frac{1}{|Q|}\int_Q \left|(b(z)-\lambda)f_1^0(z)\right| dz \prod_{j=2}^m \frac{1}{|Q|}\int_Q |f_j^0(z)|\, dz \\
&\leq C\|b\|_{BMO}\|f_1\|_{L(\log L),Q}\prod_{j=2}^m |f_j|_{Q^*} \\
&\leq C\|b\|_{BMO}\,\mathcal{M}_{L(\log L)}(f_1,\cdots,f_m)(x).
\end{aligned}
$$

Consider now the term $II_{\infty,\cdots,\infty}$. We have

$$
\left(\frac{1}{|Q|}\int_Q |T((b-\lambda)f_1^\infty,\ldots,f_m^\infty)(z) - T((b-\lambda)f_1^\infty,\ldots,f_m^\infty)(x)|^\delta dz\right)^{1/\delta}
$$

$$
\leq \frac{C}{|Q|}\int_Q |T((b-\lambda)f_1^\infty,\ldots,f_m^\infty)(z) - T((b-\lambda)f_1^\infty,\ldots,f_m^\infty)(x)|dz
$$

$$
\leq C\int_{(\mathbb{R}^n\setminus 3Q)^m} \frac{|(b(y_1)-\lambda)f_1(y_1)|\prod_{i=2}^m |f_i(y_i)||x-z|^\varepsilon}{(|z-y_1|+\cdots+|z-y_m|)^{nm+\varepsilon}}d\vec{y}dz
$$

$$
\leq \frac{C}{|Q|}\int_Q \sum_{k=1}^\infty \int_{(3^{k+1}Q)^m\setminus(3^kQ)^m} \frac{|(b(y_1)-\lambda)f_1(y_1)|\prod_{i=2}^m |f_i(y_i)||x-z|^\varepsilon}{(|z-y_1|+\cdots+|z-y_m|)^{nm+\varepsilon}}d\vec{y}dz
$$

$$
\leq C\sum_{k=1}^\infty \frac{|Q|^{\varepsilon/n}}{(3^k|Q|^{1/n})^{nm+\varepsilon}}\int_{(3^{k+1}Q)^m}|(b(y_1)-\lambda)f_1(y_1)|\prod_{i=2}^m |f_i(y_i)|d\vec{y}
$$

$$
\leq C\|b\|_{BMO}\sum_{k=1}^\infty \frac{k}{3^{k\varepsilon}}\|f_1\|_{L(\log L),3^{k+1}Q}\prod_{j=2}^m |f_j|_{3^{k+1}Q}
$$

$$
\leq C\|b\|_{BMO}\,\mathcal{M}_{L(\log L)}(f_1,...,f_m)(x).
$$

We are left now to consider the terms $II_{\alpha_1,\cdots,\alpha_m}$ such that $\alpha_{j_1} = \cdots = \alpha_{j_l} = 0$ for some $\{j_1,\ldots,j_l\}\subset\{1,\ldots,m\}$, where $1\leq l < m$. We consider only the case $\alpha_1 = \infty$ since the other ones follow in analogous way. By (2.2),

$$
\left(\frac{1}{|Q|}\int_Q |T((b-\lambda)f_1^{\alpha_1},\ldots,f_m^{\alpha_m})(z) - T((b-\lambda)f_1^{\alpha_1},\ldots,f_m^{\alpha_m})(x)|^\delta dz\right)^{1/\delta}
$$

$$
\leq \frac{C}{|Q|}\int_Q |T((b-\lambda)f_1^{\alpha_1},\ldots,f_m^{\alpha_m})(z) - T((b-\lambda)f_1^{\alpha_1},\ldots,f_m^{\alpha_m})(x)|dz
$$

$$
\leq \frac{C}{|Q|}\int_Q \prod_{j\in\{j_1,\ldots,j_l\}}\int_{3Q}|f_j|dy_j \int_{(\mathbb{R}^n\setminus 3Q)^{m-l}} \frac{|x-z|^\varepsilon|b(y_1)-c|\prod_{j\notin\{j_1,\ldots,j_l\}}|f_j|dy_j}{(|z-y_1|+\cdots+|z-y_m)^{nm+\varepsilon}}dz
$$

$$
\leq C\prod_{j\in\{j_1,\ldots,j_l\}}\int_{3Q}|f_j|dy_j \sum_{k=1}^\infty \frac{|Q|^{\varepsilon/n}}{(3^k|Q|^{1/n})^{nm+\varepsilon}}\int_{(3^{k+1}Q)^{m-l}}|b(y_1)-c|\prod_{j\notin\{j_1,\ldots,j_l\}}|f_j|dy_j
$$

$$
\leq C\sum_{k=1}^\infty \frac{|Q|^{\varepsilon/n}}{(3^k|Q|^{1/n})^{nm+\varepsilon}}\int_{(3^{k+1}Q)^m}|b(y_1)-c|\prod_{i=1}^m |f_i(y_i)|d\vec{y}
$$

$$
\leq C\|b\|_{BMO}\sum_{k=1}^\infty \frac{k}{3^{k\varepsilon}}\|f\|_{L(\log L),3^{k+1}Q}\prod_{j=2}^m |f_j|_{3^{k+1}Q}
$$

$$
\leq C\|b\|_{BMO}\,\mathcal{M}_{L(\log L)}(f_1,...,f_m)(x).
$$

The proof of the theorem is now complete. $\qquad\square$

Proof of Theorem 3.19. Recall that $\vec{b} \in BMO^m$ with $\|\vec{b}\|_{BMO^m} = 1$. Again, it is enough to prove the result for, say,

$$T_b(\vec{f}) = bT(f_1, \ldots, f_m) - T(bf_1, \ldots, f_m).$$

We may also assume that the right-hand side of (3.19) is finite, since otherwise there is nothing to be proved.

Using Theorem 3.15 and Theorem 3.2, with exponents $0 < \delta < \varepsilon < 1/m$, we have

$$
\begin{aligned}
\|T_b(\vec{f})\|_{L^p(w)} &\leq \|M_\delta(T_b(\vec{f}))\|_{L^p(w)} \\
&\leq C \|M_\delta^\#(T_b(\vec{f}))\|_{L^p(w)} \\
&\leq C \|b\|_{BMO} \left(\|\mathcal{M}_{L(\log L)}(\vec{f})\|_{L^p(w)} + \|M_\varepsilon(T(\vec{f}))\|_{L^p(w)} \right) \\
&\leq C \left(\|\mathcal{M}_{L(\log L)}(\vec{f})\|_{L^p(w)} + \|M_\varepsilon^\#(T(\vec{f}))\|_{L^p(w)} \right) \\
&\leq C \left(\|\mathcal{M}_{L(\log L)}(\vec{f})\|_{L^p(w)} + \|\mathcal{M}(\vec{f})\|_{L^p(w)} \right) \\
&\leq C \|\mathcal{M}_{L(\log L)}(\vec{f})\|_{L^p(w)}.
\end{aligned}
$$

To apply the inequality (2.8), in the above computations we need to check that $\|M_\delta(T_{\vec{b}}(f_1, \cdots, f_m))\|_{L^p(w)}$ is finite and that $\|M_\epsilon(T(f_1, \cdots, f_m))\|_{L^p(w)}$ is finite. The latter was already checked in the proof of Corollary 3.8, since under our current assumptions we still have that

$$\|\mathcal{M}(\vec{f})\|_{L^p(w)} \leq \|\mathcal{M}_{L(\log L)}(\vec{f})\|_{L^p(w)} < \infty.$$

To check the former condition we can also use similar arguments.

As in the proof of Corollary 3.8, but with T there replaced by T_b, we can reduce matters to show that $\|T_{\vec{b}}(f_1, \cdots, f_m)\|_{L^p(w)}$ is finite. If we assume that b is bounded, this last condition follows from the unweighted theory for T_b for $p > 1$ in [46]. Indeed, observe that in the local part of the arguments used in Corollary 3.8 we can take q' so that $pq' > 1$. At infinity, on the other hand, we have for b bounded and x outside a sufficiently large ball B

$$
\begin{aligned}
|T_{\vec{b}}f(x)| &\leq C \int \frac{|b_1(x) - b_1(y)|}{|x - y_1|^n \ldots |x - y_m|^n} |f_1(y_1)| \ldots |f_m(y_m)| \, dy_1 \ldots dy_m \\
&\leq C \frac{1}{|x|^n} \int_{B(0,|x|)} |f_1(y)| dy \cdots \frac{1}{|x|^n} \int_{B(0,|x|)} |f_m(y)| dy \\
(5.2) \qquad &\leq C \mathcal{M}(\vec{f})(x) \leq C\mathcal{M}_{L(\log L)}(\vec{f})(x),
\end{aligned}
$$

which is again an appropriate bound because we are assuming that $\|\mathcal{M}_{L(\log L)}(\vec{f})\|_{L^p(w)}$ is finite. This proves (3.19) provided b is bounded.

To obtain the result for a general b in BMO we use a limiting argument as in [42]. Consider the sequence of functions $\{b_j\}$ given by

$$b_j(x) = \begin{cases} j, & g(x) > j \\ b(x), & |b(x)| \leq j \\ -j, & g(x) < -j \end{cases}$$

Note that the sequence converges pointwise to b and $\|b_j\|_{BMO} \leq c\|b\|_{BMO} = c$.

Since the family $\vec{f}$ is bounded with compact support and T is bounded we have that $\{T(b_j f_1, \ldots, f_m)\}$ is convergent in L^p for every $1 < p < \infty$. It follows that for a subsequence $\{b_{j'}\}$, $T_{b_{j'}}(\vec{f})$ converges to $T_b(\vec{f})$ almost everywhere. The required estimate for T_b follows now form the ones for the $T_{b_{j'}}$ and Fatou's lemma.

We now prove (3.20). We may assume that w is bounded. Indeed, note that $w_r = \min\{w, r\}$ is bounded and that its A_p constant is bounded by the double of the A_p constant of w. The result for general w will follow then by applying the Motonone Convergence Theorem.

As usual, we can also assume that the right-hand side of (3.20) is finite since otherwise there is, again, nothing to be proved.

Now, by the Lebesgue differentiation theorem we have

$$\sup_{t>0} \frac{1}{\Phi(\frac{1}{t})} w(\{y \in \mathbb{R}^n : |T_b(\vec{f})(y)| > t^m\}) \leq \sup_{t>0} \frac{1}{\Phi(\frac{1}{t})} w(\{y \in \mathbb{R}^n : M_\delta(T_b(\vec{f}))(y) > t^m\}).$$

Then, if we assume for the moment that the last term is finite, we can estimate it using the generalized weak-type Fefferman-Stein inequality (2.9) (here we use that $\frac{1}{\Phi(\frac{1}{t})}$ is doubling and that $\|b\|_{BMO} = 1$) by

$$C \sup_{t>0} \frac{1}{\Phi(\frac{1}{t})} w(\{y \in \mathbb{R}^n : M_\delta^\#(T_b \vec{f})(y) > t^m\})$$

$$\leq C \sup_{t>0} \frac{1}{\Phi(\frac{1}{t})} w(\{y \in \mathbb{R}^n : C\left[\mathcal{M}_{L(logL)}^1(\vec{f})(y) + M_\varepsilon(T(\vec{f}))(y)\right] > t^m\})$$

$$\leq C \sup_{t>0} \frac{1}{\Phi(\frac{1}{t})} w(\{y \in \mathbb{R}^n : \mathcal{M}_{L(logL)}^1(\vec{f})(y) > t^m\})$$

$$+ C \sup_{t>0} \frac{1}{\Phi(\frac{1}{t})} w(\{y \in \mathbb{R}^n : M_\varepsilon(T(\vec{f}))(y) > t^m\})$$

Suppose, again, that the last quantity is finite, then using again (2.9) we can continue with

$$\sup_{t>0} \frac{1}{\Phi(\frac{1}{t})} w(\{y \in \mathbb{R}^n : \mathcal{M}_{L(logL)}^1(\vec{f})(y) > t^m\})$$

$$+ \sup_{t>0} \frac{1}{\Phi(\frac{1}{t})} w(\{y \in \mathbb{R}^n : M_\varepsilon^\#(T(\vec{f}))(y) > t^m\})$$

$$\leq \sup_{t>0} \frac{C}{\Phi(\frac{1}{t})} w(\{y \in \mathbb{R}^n : \mathcal{M}_{L(logL)}^1(\vec{f})(y) > t^m\})$$

$$+ \sup_{t>0} \frac{C}{\Phi(\frac{1}{t})} w(\{y \in \mathbb{R}^n : \mathcal{M}(\vec{f})(y) > t^m\})$$

$$\leq \sup_{t>0} \frac{C}{\Phi(\frac{1}{t})} w(\{y \in \mathbb{R}^n : \mathcal{M}^1_{L(logL)}(\vec{f})(y) > t^m\}).$$

We need to verify now that

$$(5.3) \qquad \sup_{t>0} \frac{1}{\Phi(\frac{1}{t})} w(\{y \in \mathbb{R}^n : M_\delta(T_b(\vec{f}))(y) > t^m\}) < \infty.$$

and

$$(5.4) \qquad \sup_{t>0} \frac{1}{\Phi(\frac{1}{t})} w(\{y \in \mathbb{R}^n : M_\varepsilon(T(\vec{f}))(y) > t^m\}) < \infty.$$

We will only show (5.3) because the proof of (5.4) is very similar but easier.

Recall that we are assuming that w is bounded, so

$$\sup_{t>0} \frac{1}{\Phi(\frac{1}{t})} w(\{y \in \mathbb{R}^n : M_\delta(T_b(\vec{f}))(y) > t^m\})$$

$$\leq \|w\|_{L^\infty} \sup_{t>0} \frac{1}{\Phi(\frac{1}{t})} |\{y \in \mathbb{R}^n : M_{m\delta}\left(|T_{\vec{b}}\vec{f}|^{1/m}\right)(y) > t\}|$$

Now, using $\Phi(t) \geq t$, $m\delta < 1$, and the fact

$$\eta < 1 \quad \implies \quad M_\eta : L^{1,\infty}(\mathbb{R}^n) \to L^{1,\infty}(\mathbb{R}^n)$$

(which is a consequence of $M : L^{r,\infty}(\mathbb{R}^n) \to L^{r,\infty}(\mathbb{R}^n)$, $r > 1$), we obtain

$$\sup_{t>0} \frac{1}{\Phi(\frac{1}{t})} |\{y \in \mathbb{R}^n : M_{m\delta}\left(|T_b\vec{f}|^{1/m}\right)(y) > t\}|$$

$$\leq \sup_{t>0} t|\{y \in \mathbb{R}^n : M_{m\delta}\left(|T_b\vec{f}|^{1/m}\right)(y) > t\}|$$

$$\leq C \sup_{t>0} t|\{y \in \mathbb{R}^n : |T_b\vec{f}(y)|^{1/m} > t\}|.$$

Recalling that $\vec{f}$ has compact support, we may assume that $\text{supp}\,\vec{f} \subset B(0, R)$ for some $R > 0$. Write then

$$\sup_{t>0} t|\{y \in \mathbb{R}^n : |T_b\vec{f}(y)|^{1/m} > t\}| \leq$$

$$\sup_{t>0} t|\{y \in B_{2R} : |T_b\vec{f}(y)|^{1/m} > t\}| + \sup_{t>0} t|\{y \notin B_{2R} : |T_b\vec{f}(y)|^{1/m} > t\}| = I + II.$$

For I we estimate the L^1 norm instead and then use Hölder's inequality to compute

$$I \leq \int_{B_{2R}} |T_b f(y)|^{1/m} dy \leq CR^{(1-1/p)n} \left(\int_{\mathbb{R}^n} |T_b\vec{f}|^{p/m} dy\right)^{1/p}.$$

This last term is finite by the strong case if we choose p sufficiently large.

For II, we can control as before $T_b(\vec{f})(x)$ by $\mathcal{M}(\vec{f})(x)$ is we assume that b is bounded. Then we have

$$II^m \leq C \sup_{t>0} t^m |\{y \in \mathbb{R}^n : \mathcal{M}(\vec{f})(y)^{1/m} > t\}|^m$$

$$= C\|\mathcal{M}(\vec{f})\|_{L^{1/m,\infty}} \leq C \prod_{i=1}^m \int_{\mathbb{R}^n} |f_i| dx < \infty.$$

Summarizing we have shown that

(5.5) $$\sup_{t>0} t|\{y \in \mathbb{R}^n : |T_b\vec{f}(y)|^{1/m} > t\}| < \infty,$$

which gives in turn (5.3), provided w is bounded and b is bounded. As already explained, we can pass to a general w in A_∞ using monotone convergence and in this way we obtain the result for arbitrary w in A_∞ and b in L^∞, and with the constant in (3.20) depending on the BMO norm of b.

We now eliminate the assumption b bounded. Observe first that it is enough to prove (3.20) with the level set $\{y \in \mathbb{R}^n : |T_{\vec{b}}(\vec{f})(y)| > t^m\}$ replaced by $\{y \in B(0,N) : |T_{\vec{b}}(\vec{f})(y)| > t^m\}$ for arbitrary $N > 0$ and with a constant on the right-hand side independent of N. Then, we can approximate b by $\{b_j\}$ as before and use now that, for each compact set, an appropriate subsequence $\{|T_{b_j}\vec{f}|\}$ also converges to $|T_b\vec{f}|$ in measure. Taking limit in j gives then the required estimate for arbitrary b in BMO with a constant independent of N. Finally taking the sup in N completes the proof of the theorem.

$\square$

6. Proof of the end-point estimate for the multilinear commutator

Proof of Theorem 3.18. We need the following preliminary lemma.

Lemma 6.1. *Assume that $\vec{w} = (w_1, \ldots, w_m)$ satisfies the $A_{\vec{P}}$ condition. Then there exists a finite constant $r > 1$ such that $\vec{w} \in A_{\vec{P}/r}$.*

Proof. By Theorem 3.6, each $\sigma_j = w_j^{-\frac{1}{p_j-1}}$ belongs to A_∞ and, hence, there are constants $c_j, t_j > 1$, depending on the A_∞ constant of σ_j, such that for any cube Q

$$\left(\frac{1}{|Q|} \int_Q w_j^{-\frac{t_j}{p_j-1}}\right)^{\frac{1}{t_j}} \leq \frac{c_j}{|Q|} \int_Q w_j^{-\frac{1}{p_j-1}}.$$

Let $r_j > 1$ be selected so that

$$\frac{t_j}{p_j-1} = \frac{1}{\frac{p_j}{r_j}-1}.$$

Then, if $r = \min\{r_1, \cdots, r_m\}$ and $c = \max\{c_1, \cdots, c_m\}$, we have

$$\left(\frac{1}{|Q|} \int_Q \nu_{\vec{w}}\right)^{1/p/r} \prod_{j=1}^m \left(\frac{1}{|Q|} \int_Q w_j^{-1/(\frac{p_j}{r}-1)}\right)^{1-\frac{1}{\frac{p_j}{r}}},$$

$$= \Big(\frac{1}{|Q|}\int_Q \nu_{\vec{w}}\Big)^{r/p} \prod_{j=1}^{m}\Big(\frac{1}{|Q|}\int_Q w_j^{-1/(\frac{p_j}{r}-1)}\Big)^{(\frac{p_j}{r}-1)\frac{r}{p_j}}$$

$$\leq \Big(\frac{1}{|Q|}\int_Q \nu_{\vec{w}}\Big)^{r/p} \prod_{j=1}^{m}\Big(\frac{1}{|Q|}\int_Q w_j^{-1/(\frac{p_j}{r_j}-1)}\Big)^{(\frac{p_j}{r_j}-1)\frac{r}{p_j}}$$

$$= \Big(\frac{1}{|Q|}\int_Q \nu_{\vec{w}}\Big)^{r/p} \prod_{j=1}^{m}\Big(\frac{1}{|Q|}\int_Q w_j^{-\frac{t_j}{p_j-1}}\Big)^{\frac{p_j-1}{t_j}\frac{r}{p_j}},$$

$$\leq c^{rm}\Big(\frac{1}{|Q|}\int_Q \nu_{\vec{w}}\Big)^{r/p} \prod_{j=1}^{m}\Big(\frac{1}{|Q|}\int_Q w_j^{-\frac{1}{p_j-1}}\Big)^{(p_j-1)\frac{r}{p_j}} \leq c^{rm}[w]_{A_{\vec{P}}}^r.$$

Since, $\vec{w}=(w_1,\ldots,w_m)$ satisfies the $A_{\vec{P}}$ condition the proof of the lemma is finished.
□

Now, by Theorem 3.19 and since $\nu_{\vec{w}}$ is also in A_∞,

$$\int_{\mathbb{R}^n}|T_{\vec{b}}(\vec{f})(x)|^p\,\nu_{\vec{w}}(x)dx \leq C\int_{\mathbb{R}^n}\mathcal{M}_{L(\log L)}(\vec{f})(x)^p\,\nu_{\vec{w}}(x)dx.$$

To finish the proof we use a bigger operator than $\mathcal{M}_{L(\log L)}$ that is enough for our purposes. Indeed, if $r>1$ and since $\Phi(t)=t(1+\log^+(t))\leq t^r$, $t>1$ we have by the generalized Jensen's inequality (2.10).

$$\|f\|_{L(\log L),Q} \leq c\Big(\frac{1}{|Q|}\int_Q |f(y)|^r dy\Big)^{1/r},$$

and we can therefore estimate the maximal operator $\mathcal{M}_{L(\log L)}$ by the larger one

$$\mathcal{M}_r(\vec{f})(x) = \sup_{Q\ni x}\prod_{j=1}^{m}\Big(\frac{1}{|Q|}\int_Q |f_j|^r\Big)^{1/r},$$

to obtain

$$\|T_{\vec{b}}(\vec{f})\|_{L^p(\nu_{\vec{w}})} \leq c\|\mathcal{M}_r(\vec{f})\|_{L^p(\nu_{\vec{w}})}.$$

Now, to prove

$$\|\mathcal{M}_r(\vec{f})\|_{L^p(\nu_{\vec{w}})} \leq c\prod_{j=1}^{m}\|f_j\|_{L^{p_j}(w_j)}$$

is equivalent to prove

$$\|\mathcal{M}(\vec{f})\|_{L^{p/r}(\nu_{\vec{w}})} \leq c\prod_{j=1}^{m}\|f_j\|_{L^{p_j/r}(w_j)}.$$

By Theorem 3.7, this is equivalent to show that $\vec{w}\in A_{\vec{P}/r}$ and we already know that this is true for some small $r>1$ because of Lemma 6.1.
□

Proof of Theorem 3.17. Without loss of generality we may assume $i = 1$. Also, by homogeneity, we may assume that $t = 1$. Finally, we may also assume that $\vec{f} \geq 0$. Define the set

$$\Omega = \{x \in \mathbb{R}^n : \mathcal{M}^1_{L(logL)}(\vec{f})(x) > 1\}$$

It is easy to see that Ω is open and we may assume that it is not empty (or there is nothing to prove). To estimate the size of Ω, it is enough to estimate the size of every compact set F contained in Ω. We can cover any such F by a finite family of cubes $\{Q_j\}$ for which

$$(6.1) \qquad\qquad 1 < \|f_1\|_{\Phi, Q_j} \prod_{j=2}^{m} (f_j)_{Q_j}$$

Using Vitali's covering lemma, we can extract a subfamily of disjoint cubes $\{Q_i\}$ such that

$$(6.2) \qquad\qquad F \subset \cup_i 3Q_i.$$

By homogeneity,

$$1 < \big\| \prod_{j=2}^{m} (f_j)_{Q_i} \big\|_{\Phi, Q_i}$$

and by the properties of the norm $\|\cdot\|_{\Phi, Q}$, this is the same as

$$1 < \frac{1}{|Q_i|} \int_{Q_i} \Phi \left(f_1(y) \prod_{j=2}^{m} (f_j)_{Q_i} \right) dy,$$

Using now that Φ is submultiplicative and Jensen's inequality

$$1 < \prod_{j=1}^{m} \frac{1}{|Q_i|} \int_{Q_i} \Phi(f_j(y)).$$

Finally by the condition on the weights and Hölder's inequality at discrete level,

$$\nu_{\vec{w}}(F)^m \approx \left(\sum_i \nu_{\vec{w}}(Q_i) \right)^m \leq \left(\sum_i \prod_{j=1}^{m} \inf_Q w_j^{1/m} |Q_i|^{1/m} \left(\frac{1}{|Q_i|} \int_{Q_i} \Phi(f_j(y))\, dy \right)^{1/m} \right)^m$$

$$\leq \left(\sum_i \prod_{j=1}^{m} \left(\int_{Q_i} \Phi(f_j(y)) w_j(y)\, dy \right)^{1/m} \right)^m \leq \prod_{j=1}^{m} \int_{\mathbb{R}^n} \Phi(|f_j(y)|)\, w_j(y) dy,$$

which concludes the proof. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Proof of Theorem 3.16. We have all the ingredients to prove Theorem 3.16.

As in the proof of Theorem 3.15, by linearity it is enough to consider the operator with only one symbol. By homogeneity it is enough to assume $t = 1$ and hence we

must prove

$$\nu_{\vec{w}}\big\{x \in \mathbb{R}^n \,:\, |T_b(\vec{f})(x)| > 1\big\}^m \leq C \prod_{j=1}^{m} \int_{\mathbb{R}^n} \Phi(|f_j(x)|)\, w_j(x)dx.$$

Now, since Φ is submultiplicative, we have by Theorem 3.19 and Theorem 3.17

$$
\begin{aligned}
\nu_{\vec{w}}\big\{x \in \mathbb{R}^n \,:\, |T_b(\vec{f})(x)| > 1\big\}^m
&\leq C \sup_{t>0} \frac{1}{\Phi(\frac{1}{t})^m} \nu_{\vec{w}}\big\{x \in \mathbb{R}^n \,:\, |T_b(\vec{f})(x)| > t^m\big\}^m \\
&\leq C \sup_{t>0} \frac{1}{\Phi(\frac{1}{t})^m} \nu_{\vec{w}}(\{y \in \mathbb{R}^n : \mathcal{M}^1_{L(logL)}(\vec{f})(x) > t^m\})^m \\
&\leq C \sup_{t>0} \frac{1}{\Phi(\frac{1}{t})^m} \prod_{j=1}^{m} \int_{\mathbb{R}^n} \Phi(\frac{|f_j(x)|}{t})\, w_j(x)dx \\
&\leq C \sup_{t>0} \frac{1}{\Phi(\frac{1}{t})^m} \prod_{j=1}^{m} \int_{\mathbb{R}^n} \Phi(|f_j(x)|)\, \Phi(\frac{1}{t}) w_j(x)dx \\
&\leq C \prod_{j=1}^{m} \int_{\mathbb{R}^n} \Phi(|f_j(x)|)\, w_j(x)dx
\end{aligned}
$$

as we wanted to prove. $\qquad\square$

7. Remarks, examples and counterexamples

In this section we provide the examples and counterexamples mentioned earlier in the article and which establish that several of the estimates obtained are, in appropriate senses, sharp.

Remark 7.1. *The two conditions in (3.6) are independent of each other.*

Set $\vec{w} = (w_1, w_1^{-p_2/p_1})$. We have then $\nu_{\vec{w}} = 1$ which trivially belongs to A_{2p} for any w_1. If we select w_1 so that $w_1^{-\frac{1}{p_1-1}} \notin L^1_{loc}$, we see that the first condition in (3.6) does not hold. Conversely, let $n = 1, m = 2$ and $p_1 = p_2 = 2$. Set $w_1 = w_2 = |x|^{-2}$. Then the first condition in (3.6) holds (because $w_j^{-1} = |x|^2 \in A_4$), while $\nu_{\vec{w}} = |x|^{-2} \notin L^1_{loc}$, and hence $\nu_{\vec{w}} \notin A_2$.

Remark 7.2. *The condition $\vec{w} \in A_{\vec{P}}$ does not imply in general $w_j \in L^1_{loc}$ for any j.*

Take, for instance,

$$w_1 = \frac{\chi_{[0,2]}(x)}{|x-1|} + \chi_{\mathbb{R}/[0,2]}(x)$$

and $w_j(x) = \frac{1}{|x|}$ for $j = 2, ..., m$. Then, using the definition, it is not difficult to check that $\nu_{\vec{w}} \in A_1$. We also have $\inf_Q \nu_{\vec{w}} \sim \prod_{j=1}^{m} \inf_Q w_j^{p/p_j}$. These last two facts together easily imply that $\vec{w} \in A_{\vec{P}}$.

Remark 7.3. *The classes $A_{\vec{P}}$ are not increasing*

Let us consider the partial order relation between vectors $\vec{P} = (p_1, \ldots, p_m)$ and $\vec{Q} = (q_1, \ldots, q_m)$ given by $\vec{P} \lesssim \vec{Q}$ if $p_j \leq q_j$ for all j. Then, for $\vec{P} \lesssim \vec{Q}$ we have

$$\prod_{j=1}^{m} A_{p_j} \subseteq \prod_{j=1}^{m} A_{q_j},$$

but $A_{\vec{P}}$ is not contained in $A_{\vec{Q}}$. To see this, consider $n = 1$, $m = 2$, $\vec{P} = (p_1, p_2) = (2, 2)$, and

$$\vec{w} = (w_1, w_2) = (|x|^{-5/3}, 1)$$

Then since $w_1^{1/2} \in A_1$ it is easy to see that $\vec{w} \in A_{\vec{P}}$. Also, since w raised to an appropriate large power becomes non-locally integrable, it is easy to show that $\vec{w} \notin A_{\vec{Q}}$ if, for instance, $\vec{Q} = (2, 6)$.

Remark 7.4. *The assumption $p_j > 1$ for all j is essential in Theorem 3.7 even in the unweighted case.*

Let $n = 1$, $m = 2$, and suppose that (3.7) holds with $w_j \equiv 1$ for $p_1 = 1$ and $p_2 > 1$. Then taking f_1 to be the Dirac mass at the origin and $f_2 = g$, where g is any non-increasing function on $(0, \infty)$, we get that

$$\frac{g(x)}{x} \leq \frac{1}{x^2} \int_0^x g(t)dt \leq \mathcal{M}(f_1, f_2)(x),$$

and hence (3.7) would imply

$$\left(\int_0^\infty (g(x)/x)^p dx \right)^{1/p} \leq c \left(\int_0^\infty g(x)^{p_2} \right)^{1/p_2},$$

where $1/p = 1 + 1/p_2$. The simple choice of

$$g(x) = x^{1-1/p}(\log(1/x))^{-1/p}\chi_{(0,1/2)}$$

shows that this inequality is not true.

Remark 7.5. *The estimate (3.7) does not hold if $\mathcal{M}(\vec{f})$ is replaced by $\prod_{j=1}^{m} M(f_j)$, and therefore Theorem 3.9 cannot be obtained from the known estimates in* [26].

Let $m = 2$ and let $p_1, p_2 \geq 1$ satisfy $1/p_1 + 1/p_2 = 1/p > 1$. Chose $\varepsilon > 0$ such that $p_2 < p_2/p - \varepsilon$. Set now $w_1 = 1$ and $w_2 = |x|^{\varepsilon - p_2/p}$. Then it is easy to check that $\vec{w} \in A_{\vec{P}}$ (this follows from the fact that $w_2^{p/p_2} \in A_1$). Nevertheless, the inequality

(7.1) $$\|Mf_1 Mf_2\|_{L^{p,\infty}(\nu_{\vec{w}})} \leq C \|f_1\|_{L^{p_1}(w_1)} \|f_2\|_{L^{p_2}(w_2)}$$

does not hold for all f_1, f_2. Indeed, set $f_1 = \chi_{[0,1]}$ and $f_1 = N\chi_{[N,N+1]}$, for N big enough. It is clear that

$$[0, 1] \subset \{x : Mf_1 Mf_2 > 1/2\},$$

so the left-hand side of (7.1) is bigger than some constant $c > 0$. Furthermore, $\|f_1\|_{L^{p_1}(w_1)} = 1$ and $\|f_2\|_{L^{p_2}(w_2)} \sim N^{1+\frac{\varepsilon}{p_2}-\frac{1}{p}}$. We see then that (7.1) would imply $c \leq N^{1+\frac{\varepsilon}{p_2}-\frac{1}{p}}$, which is a contradiction.

Remark 7.6. *A weak-type analogue of (3.12) is not true for arbitrary weights w_j if at least one $p_j = 1$.*

Let $n = 1$ and $m = 2$. Let $1 \leq p_1 < \infty$ and $p_2 = 1$. For $k \geq 4$ set $J_k = (k + \frac{1}{4k}, k + \frac{1}{2k})$. Let now $w_1(x) = \sum_{k=4}^{\infty} k\chi_{J_k}(x)$ and $w_2(x) = \sum_{k=4}^{\infty} \frac{1}{k}\chi_{J_k}(x)$. Suppose that the inequality

$$(7.2) \qquad \|Mf_1 Mf_2\|_{L^{p,\infty}(\nu_{\vec{w}})} \leq c\|f_1\|_{L^{p_1}(Mw_1)}\|f_2\|_{L^1(Mw_2)}$$

holds with a constant c independent of f_1 and f_2. Set $f_1 = \chi_{(0,1)}$ and $f_2 = \sum_{k=1}^{N} \delta_k$, where δ_k is the Dirac mass at the point k. Simple computations show that

$$\bigcup_{k=4}^{N} J_k \subset \{x : Mf_1(x)Mf_2(x) > 1\}.$$

On the other hand, $\|f_1\|_{L^{p_1}_{Mw_1}} \leq c$ and $Mw_2(k) \leq c/k$. Therefore, (7.2) would imply $\sum_{k=1}^{N} \frac{1}{k} \leq c$, which is a obviously a contradiction.

Remark 7.7. *An estimate of the form*

$$(7.3) \qquad |\{x : |T_b(\vec{f})| > \lambda^m\}| \leq C_{(\|b\|_{BMO})} \left(\|\frac{f_i}{\lambda}\| \prod_{j \neq i} \|\Phi(\frac{|f_j|}{\lambda})\|_{L^1} \right)^{1/m}$$

cannot hold for characteristics functions of intervals if $\| \cdot \|$ is finite on characteristic functions and satisfies $\|\lambda f\| = \lambda\|f\|$. In particular (a bounded) mapping property of the form

$$T_{\vec{b}} : L^1 \times \cdots \times L^1 \to L^{1/m,\infty}$$

does not hold.

For $m = 1$ this was already shown in [42]. We adapt the arguments to the multilinear case. For simplicity we consider the case $n = 1$, $m = 2$. Suppose that (7.3) holds for some $\| \cdot \|$ with the required properties, some Calderón-Zygmund operator T like the bilinear Riezs transforms, and $b(x) = \log|1 + x|$. Let $f_1 = f_2 = \chi_{(0,1)}$. If (7.3) were to hold, we would have by multilinearity and homogeneity

$$|\{x \in \mathbb{R} : |T_b(\vec{f})(x)| > \lambda^2\}| \leq C \left(\|\frac{f_1}{\lambda^2}\| \|\Phi(f_2)\|_{L^1} \right)^{1/2},$$

and hence

$$(7.4) \qquad \sup_{\lambda > 0} \lambda |\{x \in \mathbb{R} : |T_b(\vec{f})(x)| > \lambda\}|^2 \leq C \|f_1\| \|\Phi(f_2)\|_{L^1} \leq C.$$

However, the left-hand side of (7.4) is not smaller than a multiple of

$$(7.5) \qquad \sup_{\lambda > 0} \lambda |\{x > e : \frac{\log(x)}{x^2} > \lambda\}|^2 = \infty$$

arriving to a contradiction.

To see (7.5), let $\varphi(x) = \frac{\log x}{x^2}$ and simply observe that for, say, positive integers k

$$\sup_{\lambda > 0} \lambda |\{x > e : \varphi(x) > \lambda\}|^2 \geq \sup_k \varphi(e^k) |\{x > e : \varphi(x) > \varphi(e^k)\}|^2 \geq$$

$$\sup_k \frac{k}{e^{2k}} (e^k - e)^2 = \infty.$$

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