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Presentations of Trivial Extensions of Finite Dimensional Algebras and a Theorem of Sheila Brenner

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Let Λ be a finite dimensional algebra over an algebraically closed field such that any oriented cycle in the ordinary quiver of Λ is zero in Λ. We describe the ordinary quiver and relations for $T(\Lambda) = \Lambda \ltimes D(\Lambda)$, the trivial extension of Λ by its minimal injective cogenerator $D(\Lambda)$, and also for the repetitive algebra $\widehat{\Lambda}$ of Λ. Associated with this description we give an application of a theorem of Sheila Brenner.

INTRODUCTION

A finite dimensional k-algebra Λ (associative, with identity) over an algebraically closed field k is called self-injective if all projective Λ-modules are injective. An important class of self-injective algebras is formed by the

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symmetric algebras. Recall that an algebra Λ is said to be symmetric if $\Lambda \simeq D(\Lambda)$ as two sided Λ-modules.

Important and interesting examples of symmetric algebras are provided by group algebras of finite groups over fields and trivial extensions of finite dimensional algebras. Denote by $T(\Lambda) = \Lambda \ltimes D(\Lambda)$ the trivial extension of Λ by its minimal injective cogenerator $D(\Lambda)$. We assume that Λ is basic and indecomposable, so Λ is given as the path algebra of a connected finite quiver Q modulo an ideal of relations I.

In this work we will describe the ordinary quiver and relations for $T(\Lambda)$, under the assumption that any oriented cycle in the ordinary quiver of Λ is zero in Λ. From this we deduce a description of the ordinary quiver and relations for $\widehat{\Lambda}$, the repetitive algebra of Λ.

The key point in the study of the ordinary quiver of $T(\Lambda)$ is showing that $Q_{T(\Lambda)}$ is obtained from Q_Λ by adding t arrows, where the number t is equal to the dimension of soc(Λ), considered as module over the enveloping algebra Λ^e. We illustrate here the situation when Λ is schurian. In this case we can choose a k-basis for the Λ^e-socle of Λ consisting of maximal nonzero paths. For each of these we add an arrow to Q_Λ, and in the opposite direction. In this way, all arrows are in oriented cycles. The relations are particularly interesting in the case when the ordinary quiver of Λ has no oriented cycles and parallel paths in Q_Λ are equal in Λ, because they can be formulated directly in terms of the cycles in $kQ_{T(\Lambda)}$, independently of the relations for Λ. More precisely, the ideal $I_{T(\Lambda)}$ of relations for $T(\Lambda)$ is generated by

(i) the paths consisting of $n+1$ arrows in a cycle of length n,

(ii) the paths whose arrows do not belong to a single cycle, and

(iii) the difference $q - q'$ of paths q, q' with the same origin and endpoint and having a common supplement in cycles of $Q_{T(\Lambda)}$. By a supplement of q in the cycle C we mean the path consisting of the remaining arrows of C.

This particular case has been crucial for the classification of all trivial extensions of finite representation type, and consequently for the study of iterated tilted algebras of Dynkin type, as done in the first author's doctoral dissertation [F], which will be published elsewhere.

The general case considered in this paper is stated in Theorem 3.9. Though technically more complicated, the essential ideas are contained in the above description.

Finally, we give an application of a theorem of Sheila Brenner. More precisely, Brenner shows in [B] how to determine the number of indecomposable direct summands of the middle term of an almost split sequence starting with a simple module. As a consequence of this result she obtains, for a self-injective artin algebra, the number of indecomposable direct summands of $\mathfrak{r}P/\operatorname{soc} P$, where P is indecomposable projective. In general, it is not easy to compute these numbers for a given algebra. We give here a very simple interpretation of them in the particular case of the trivial extension $T(\Lambda) = \Lambda \ltimes D\Lambda$, where Λ is an algebra such that any oriented cycle in Q_Λ is zero in Λ. Our description is given in terms of oriented cycles in the quiver $kQ_{T(\Lambda)}$.

1. PRELIMINARIES

In this section we fix some notation and recall some relevant definitions and results which will be needed in the next sections. For a general reference in representation theory, we refer the reader to [ARS].

Throughout this paper k will denote an algebraically closed field. By an algebra we mean a finite dimensional k-algebra which we will also assume to be basic and indecomposable. Thus $\Lambda \simeq kQ_\Lambda/I$, where Q_Λ is a finite connected quiver and the ideal I is admissible.

We denote by $\operatorname{mod}\Lambda$ the category of finitely generated left Λ-modules, and by D: $\operatorname{mod}\Lambda \to \operatorname{mod}\Lambda^{op}$ the standard duality $\operatorname{Hom}_k(\cdot, k)$. Also, we denote the Jacobson radical of Λ by $\operatorname{rad}\Lambda$ or simply by $\mathfrak{r}$.

For a given quiver Q, we will denote by Q_0 the set of vertices, and by Q_1 the set of arrows between vertices. For each arrow α, $s(\alpha)$ and $e(\alpha)$ will denote the start and end vertices of α, respectively.

For each i in Q_0, S_i will be the simple Λ-module associated to i, and P_i and I_i will denote the projective cover and injective envelope of S_i, respectively. Thus, if e_i is the idempotent element of Λ corresponding to the vertex i of Q, then $P_i = \Lambda e_i$.

Recall that the trivial extension $T(\Lambda) = \Lambda \ltimes D(\Lambda)$ of Λ by $D(\Lambda)$ is the algebra with underlying vector space $\Lambda \oplus D(\Lambda)$, and the product is defined by $(\lambda, f)(\mu, g) = (\lambda\mu, \lambda g + f\mu)$ for any $\lambda, \mu \in \Lambda$.

We will need the following known facts, whose proof is straightforward.

PROPOSITION 1.1. *Let Λ be an algebra. Then*

(i) $\operatorname{rad} T(\Lambda) = (\mathfrak{r}, D(\Lambda))$, *where $\mathfrak{r}$ denotes the radical of Λ.*

(ii) $\operatorname{rad}^2 T(\Lambda) = (\mathfrak{r}^2, \mathfrak{r}D(\Lambda) + D(\Lambda)\mathfrak{r})$.

(iii) $\operatorname{rad} T(\Lambda)/\operatorname{rad}^2 T(\Lambda)$ *and* $(\mathfrak{r}/\mathfrak{r}^2, D(\Lambda)/(\mathfrak{r}D(\Lambda) + D(\Lambda)\mathfrak{r}))$ *are isomorphic vector spaces.*

2. THE ORDINARY QUIVER OF $T(\Lambda)$

In this section we describe the ordinary quiver of $T(\Lambda)$ for any finite dimensional k-algebra Λ.

Let $\Lambda = kQ_\Lambda/I_\Lambda$ be a fixed presentation for Λ. Given an element x in kQ_Λ, we will denote by $\bar{x}$ the corresponding element in kQ_Λ/I_Λ.

For our purpose the following preliminary result will be useful.

LEMMA 2.1. *There is a short exact sequence of Λ-bimodules*

$$0 \to D(\Lambda)\mathfrak{r} + \mathfrak{r}D(\Lambda) \to D(\Lambda) \to D(\operatorname{soc}_{\Lambda^e} \Lambda) \to 0,$$

where Λ^e is the enveloping algebra of Λ.

Proof. Since $D(\operatorname{soc}_{\Lambda^e} \Lambda) = \operatorname{top}_{\Lambda^e}(D(\Lambda))$, there is a short exact sequence

$$0 \to \operatorname{rad}_{\Lambda^e} D(\Lambda) \to D(\Lambda) \to D(\operatorname{soc}_{\Lambda^e} \Lambda) \to 0.$$

We only have to describe $\operatorname{rad}_{\Lambda^e} D(\Lambda)$. Since

$$\operatorname{rad}(\Lambda^e) = \mathfrak{r} \otimes \Lambda^{op} + \Lambda \otimes \mathfrak{r}^{op},$$

we have $\operatorname{rad}_{\Lambda^e} D(\Lambda) \simeq \operatorname{rad} \Lambda^e \cdot D(\Lambda) \simeq \mathfrak{r}D(\Lambda) + D(\Lambda)\mathfrak{r}$. ∎

PROPOSITION 2.2. *If Λ is an algebra with ordinary quiver Q_Λ, then the ordinary quiver of $T(\Lambda)$ is given by*

(i) $(Q_{T(\Lambda)})_0 = (Q_\Lambda)_0$,

(ii) $(Q_{T(\Lambda)})_1 = (Q_\Lambda)_1 \cup \{\beta_{p_1}, \ldots, \beta_{p_t}\}$, *where* $\{\overline{p_1}, \ldots, \overline{p_t}\}$ *is a k-basis for* $\operatorname{soc}_{\Lambda^e} \Lambda$, *and for each* i, β_{p_i} *is an arrow from* $e(p_i)$ *to* $s(p_i)$.

Proof. (i) Let $Q_0 = \{1, 2, \ldots, n\}$ be the set of vertices of Q_Λ and let $\{e_1, \ldots, e_n\}$ be the set of trivial paths in kQ_Λ. It is easy to verify that $\{(\overline{e_1}, 0), \ldots, (\overline{e_n}, 0)\}$ is a complete set of primitive orthogonal idempotents in $T(\Lambda)$. So, $Q_{T(\Lambda)}$ has n vertices in one-to-one correspondence with $(\overline{e_1}, 0), \ldots, (\overline{e_n}, 0)$.

(ii) For each pair of integers i, j, with $1 \leq i, j \leq n$, the number of arrows from i to j is equal to $\dim_k((\overline{e_j}, 0) \operatorname{rad} T(\Lambda)/\operatorname{rad}^2 T(\Lambda)(\overline{e_i}, 0))$. By Proposition 1.1, we have

$$\begin{aligned}\dim_k&((\overline{e_j}, 0) \operatorname{rad} T(\Lambda)/\operatorname{rad}^2 T(\Lambda)(\overline{e_i}, 0)) \\ &= \dim_k(\overline{e_j}\mathfrak{r}/\mathfrak{r}^2\overline{e_i}) + \dim_k(\overline{e_j}(D(\Lambda)/(\mathfrak{r}D(\Lambda) + D(\Lambda)\mathfrak{r}))\overline{e_i}).\end{aligned}$$

The first summand is equal to the number of arrows from i to j in Q_Λ.

According to Lemma 2.1, the second summand is equal to the dimension of the subspace of $\operatorname{soc}_{\Lambda^e} \Lambda$ generated by all elements $\bar{p}_k$, where p_k is a k-linear combination of paths starting at j and ending at i. ∎

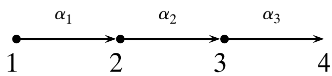
Remark 2.3. We say that a path q in Q_Λ is **maximal** if $\bar{q} \neq 0$ and $\overline{\alpha q} = 0 = \overline{q\alpha}$, for any arrow α in $(Q_\Lambda)_1$. Note that if q is a maximal path, then $\bar{q} \in \operatorname{soc}_{\Lambda^e} \Lambda$.

When Λ is a **schurian** algebra, which means $\dim_k \operatorname{Hom}_\Lambda(P, P') \leq 1$ for every pair of indecomposable projective Λ-modules P, P', we can also give another description of $D(\Lambda)/(\mathfrak{r}D(\Lambda) + D(\Lambda)\mathfrak{r}) \simeq D(\operatorname{soc}_{\Lambda^e} \Lambda)$. In this case we have the following result:

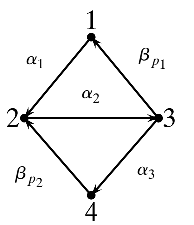
LEMMA 2.4. *Let Λ be a schurian algebra. Then $\operatorname{soc}_{\Lambda^e} \Lambda$ is the subspace of Λ generated by all $\bar{q} \in \Lambda$, with q a maximal path in Q_Λ.*

We now give some examples to illustrate the construction of $Q_{T(\Lambda)}$.

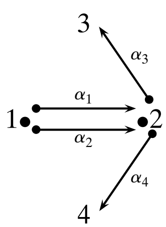
EXAMPLE 2.5. Let Λ be given by the quiver



with the relation $\alpha_3\alpha_2\alpha_1 = 0$. Then $\{\bar{p}_1 = \overline{\alpha_2\alpha_1}, \bar{p}_2 = \overline{\alpha_3\alpha_2}\}$ is a k-basis for $\operatorname{soc} \Lambda$. So, according to Proposition 2.2, $Q_{T(\Lambda)}$ is the quiver

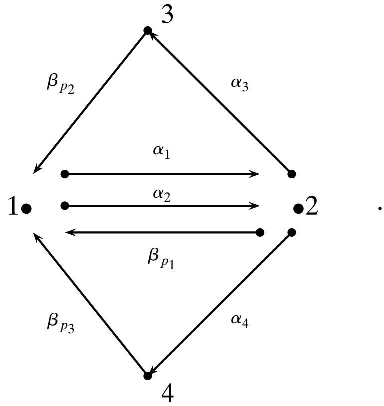


EXAMPLE 2.6. Let Λ be given by the quiver



with relations $\alpha_3\alpha_1 = \alpha_3\alpha_2$, $\alpha_4\alpha_1 = \alpha_4\alpha_2$.

In this case, $\{\bar{p}_1 = \overline{\alpha_1 - \alpha_2}, \bar{p}_2 = \overline{\alpha_3\alpha_1}, \bar{p}_3 = \overline{\alpha_4\alpha_1}\}$ is a k-basis for soc Λ. Then $Q_{T(\Lambda)}$ is the following quiver:



3. THE RELATIONS FOR $T(\Lambda)$

From now on we will assume that any oriented cycle in Q_Λ is zero in Λ. In particular, the class of schurian algebras has this property.

Notation. In all that follows, we fix a set $\mathbb{M} = \{p_1, \ldots, p_t\}$ of elements in kQ_Λ such that $\{\overline{p_1}, \ldots, \overline{p_t}\}$ is a basis for $\mathrm{soc}_{\Lambda^e} \Lambda$. Moreover, let $\{\overline{p_1}, \ldots, \overline{p_t}, \ldots, \overline{p_d}\}$ be a basis of Λ. We will denote by $\{\overline{p_1}^*, \ldots, \overline{p_d}^*\}$ the dual basis in $D(\Lambda)$.

Our next goal is to describe the ideal $I_{T(\Lambda)}$ of relations for $T(\Lambda)$. This will require some preliminary definitions and remarks.

DEFINITION 3.1. Let $\mathbb{C}$ be an oriented cycle in $Q_{T(\Lambda)}$. We say that $\mathbb{C}$ is **elementary** if $\mathbb{C} = \alpha_j \cdots \alpha_1 \beta_p \alpha_n \cdots \alpha_{j+1}$, with $\alpha_1, \ldots, \alpha_n \in (Q_\Lambda)_1$ and $p \in \mathbb{M}$ and $\bar{p}^*(\overline{\alpha_n \cdots \alpha_1}) \neq 0$. In this case, the **weight** of $\mathbb{C}$ is $w(\mathbb{C}) = \bar{p}^*(\overline{\alpha_n \cdots \alpha_1}) \in k^*$.

EXAMPLE 3.2. Let Λ be the algebra given in Example 2.6 and consider the following k-basis for Λ:

$$\{\overline{p_1} = \overline{\alpha_1 - \alpha_2},\ \overline{p_2} = \overline{\alpha_3\alpha_1},\ \overline{p_3} = \overline{\alpha_4\alpha_1},\ \overline{\alpha_1},\ \overline{\alpha_3},\ \overline{\alpha_4},\ \overline{e_1},\ \overline{e_2},\ \overline{e_3},\ \overline{e_4}\}.$$

Since $\overline{p_1}^*(\overline{\alpha_1}) = \overline{\alpha_1 - \alpha_2}^*(\overline{\alpha_1}) = 0$, the oriented cycle $\beta_{p_1}\alpha_1$ is not an elementary cycle. Now, it is not difficult to check that the oriented cycles

$$\beta_{p_2}\alpha_3\alpha_1,\ \beta_{p_3}\alpha_4\alpha_1,\ \beta_{p_2}\alpha_3\alpha_2,\ \beta_{p_3}\alpha_4\alpha_2,\ \text{and } \beta_{p_1}\alpha_2$$

are all the elementary cycles in $Q_{T(\Lambda)}$, up to cyclic permutations.

In all that follows, when we say that a path q is **contained** in the path q', it will be understood that $q' = \gamma_2 q \gamma_1$, where γ_1, γ_2 are paths with $e(\gamma_1) = s(q)$ and $s(\gamma_2) = e(q)$.

Remark 3.3. If $0 \neq \bar{v} \in \Lambda$, then there are paths δ_1, δ_2 in kQ_Λ and $p_j \in \mathbb{M}$ such that $\overline{p_j}^*(\overline{\delta_1 v \delta_2}) \neq 0$, and in particular, any nonzero path in Λ is contained in an elementary cycle.

In fact, let δ_1, δ_2 be paths in kQ_Λ of maximal length such that $\bar{z} = \overline{\delta_1 v \delta_2} \neq 0$. Then $\bar{z} \in \operatorname{soc} \Lambda$, so $\bar{z} = \sum_{i=1}^{t} b_i \overline{p_i}$, with $b_i \in k$ and $p_i \in \mathbb{M}$ for all $i = 1, \dots, t$. Since $\bar{z} \neq 0$, there exists j such that $b_j \neq 0$. Thus $\overline{p_j}^*(\bar{z}) \neq 0$. From this it follows that if v is a path in Q_Λ such that $\bar{v} \neq 0$, then $\beta_{p_j} \delta_1 v \delta_2$ is an elementary cycle containing v.

DEFINITION 3.4. Let q be a path in an elementary cycle $\mathbb{C}$ of length less than or equal to the length of $\mathbb{C}$. If $s(q) = e(q)$, the **supplement** of q in $\mathbb{C}$ is the trivial path $e_{s(q)}$; otherwise, it is the path formed by the remaining arrows of $\mathbb{C}$.

Consider the morphism of k-algebras $\Phi\colon kQ_{T(\Lambda)} \to T(\Lambda)$ defined on the trivial paths and the arrows as follows:

$$\Phi(e_i) = (\overline{e_i}, 0), \qquad \text{for } i = 1, \dots, n,$$

$$\Phi(\alpha) = (\bar{\alpha}, 0), \quad \Phi(\beta_p) = (0, \bar{p}^*), \qquad \text{for } \alpha \in (Q_\Lambda)_1 \text{ and } p \in \mathbb{M}.$$

Then Φ is surjective. Associated with Φ are the morphisms

$$\varphi_1 = \pi_1 \Phi\colon kQ_{T(\Lambda)} \to \Lambda \quad \text{and} \quad \varphi_2 = \pi_2 \Phi\colon kQ_{T(\Lambda)} \to D(\Lambda),$$

where π_1, π_2 are the projections induced by the decomposition $T(\Lambda) = \Lambda \oplus D(\Lambda)$.

We now make some simple but important observations.

Throughout this section, $(\beta_p)_{p \in \mathbb{M}}$ denotes the ideal generated by the elements β_p in $kQ_{T(\Lambda)}$.

LEMMA 3.5. *Let $\Lambda = kQ_\Lambda / I$ be an algebra such that any oriented cycle in kQ_Λ is zero in Λ and q, u be paths in KQ_T.*

(a) *If $v = v_1 + v_2$, with $v_1 \in kQ_\Lambda$, $v_2 \in (\beta_p)_{p \in \mathbb{M}}$, then $\Phi(v) = (\varphi_1(v_1), \varphi_2(v_2))$.*

(b) *$\varphi_2(q) \neq 0$ implies that $q \in (\beta_p)_{p \in \mathbb{M}}$.*

(c) *$\varphi_2(q) = 0$ if q contains two or more arrows β_p, $p \in \mathbb{M}$.*

(d) *$\varphi_2(q)(\bar{u}) \neq 0$ implies that u is a supplement of q.*

(e) *$\varphi_2(v)(\bar{u}) = \varphi_2(vu)(\overline{e_i}) = \varphi_2(uv)(\overline{e_j})$ if u is a path from i to j in kQ_Λ.*

(f) *If* $v = \sum_{s=1}^{l} a_s q_s$, *with* q_s *different paths and* $\varphi_2(v) \neq 0$, *then there exists a supplement* u *of one of the* q_i*s such that* $\varphi_2(vu) \neq 0$ *and* $\varphi_2(uv) \neq 0$.

(g) *Let* $\mathbb{C}$ *be an elementary cycle with origin* e. *Then* $\varphi_2(\mathbb{C})(\bar{e}) = w(\mathbb{C})$ *and* $\varphi_2(\mathbb{C})(\bar{u}) = 0$ *for any path* u *in* kQ_Λ, $u \neq e$.

(h) *If* q *has a supplement, then* $\Phi(q) \neq 0$.

(i) *Let* $v \in e_j kQ_{T(\Lambda)} e_i$ *and* γ *be a path from* j *to* i. *Then* $v\gamma \in \operatorname{Ker}\Phi$ *if and only if* $\gamma v \in \operatorname{Ker}\Phi$.

Proof. (a), (b), and (e) follow directly from the definitions, while (g) follows from the definitions together with the hypothesis over Λ.

We get (c) using that $D(\Lambda)^2 = 0$ in $T(\Lambda)$. Assume $\varphi_2(q)(\bar{u}) \neq 0$. To prove (d), we know by (b) and (c) that $q = \gamma\beta_p\delta$, with γ, δ paths in kQ_Λ. Then $0 \neq \varphi_2(q)(\bar{u}) = \bar{p}^*(\overline{\delta u \gamma})$, and therefore u is a supplement of q in the elementary cycle $\gamma\beta_p\delta u$.

Assume now v as in (f) and let u be such that $\varphi_2(v)(\bar{u}) \neq 0$. Hence $\varphi_2(q_s)(\bar{u}) \neq 0$ for some s; thus u is a supplement of q_s, by (d). Now (f) follows from (e).

Suppose now that u is a supplement for q in $\mathbb{C}$, $\mathbb{C} = qu$. By (g), $\varphi_2(\mathbb{C}) \neq 0$; thus $\Phi(q) \neq 0$.

Finally, let $v \in e_j kQ_{T(\Lambda)} e_i$ and let γ be a path from j to i. Then γv and $v\gamma$ are linear combinations of cycles with origin i and j, respectively, which we may assume are in (β_p) because Q_Λ has no nonzero oriented cycles. Thus $\varphi_2(\gamma v)$, $\varphi_2(v\gamma)$ vanish on all paths different from e_i and e_j, respectively. By (e), we have $\varphi_2(\gamma v)(\overline{e_i}) = \varphi_2(v\gamma)(\overline{e_j})$, and the statement follows now from (a), using the fact that γv and $v\gamma$ are in (β_p). ∎

In order to describe the relations for $T(\Lambda)$, we have to find generators for $\operatorname{Ker}\Phi$.

PROPOSITION 3.6. *Let* Φ *be as above. For each* $j \in (Q_{T(\Lambda)})_0$, *let* I'_j *be the ideal in* $kQ_{T(\Lambda)}$ *generated by*

(i) *oriented cycles from* j *to* j *which are not elementary, and*

(ii) *elements* $w(\mathbb{C}')\mathbb{C} - w(\mathbb{C})\mathbb{C}'$, *where* $\mathbb{C}, \mathbb{C}'$ *are elementary cycles with origin* j.

Then $\operatorname{Ker}\Phi \cap e_j kQ_{T(\Lambda)} e_j$ *generates* I'_j.

Proof. That oriented cycles in kQ_Λ lie in $\operatorname{Ker}\Phi$ is a direct consequence of the definition of Φ and the hypothesis on Λ. So let $\mathbb{C}$ be a cycle from j to j in $kQ_{T(\Lambda)}$, that is, in (β_p). Then $\Phi(\mathbb{C}) = (0, \varphi_2(\mathbb{C}))$, by Lemma 3.5(a). If $\mathbb{C}$ contains two or more arrows β_p, $p \in \mathbb{M}$, then $\varphi_2(\mathbb{C}) = 0$ by (c) of the same lemma, and therefore $\mathbb{C}$ lies in $\operatorname{Ker}\Phi$. Suppose now that $\mathbb{C}$ contains exactly one arrow β_p, for some $p \in \mathbb{M}$. Then $\mathbb{C} = \gamma\beta_p\delta$, with γ, δ paths in kQ_Λ. If $\varphi_2(\mathbb{C}) \neq 0$, there exists a path u in kQ_Λ such that

$\varphi_2(\mathbb{C})(u) = \bar{p}^*(\overline{\delta u \gamma}) \neq 0$. Hence u is a path from j to j and $\bar{u} \neq 0$. It follows from the hypothesis on Λ that $u = e_j$, so $\mathbb{C}$ is elementary. Thus nonelementary cycles are in $\operatorname{Ker}\Phi$.

Let now $z = w(\mathbb{C}')\mathbb{C} - w(\mathbb{C})\mathbb{C}'$ be an element as defined in (ii). Then $\Phi(z) = (0, \varphi_2(z))$, and $\varphi_2(z) = w(\mathbb{C}')\varphi_2(\mathbb{C}) - w(\mathbb{C})\varphi_2(\mathbb{C}')$.

Let u be a path in kQ_Λ. By Lemma 3.5(g), we have that $\varphi_2(z)(\bar{u}) = 0$ if $u \neq e_j$, and $\varphi_2(\mathbb{C})(\overline{e_j}) = w(\mathbb{C})$, $\varphi_2(\mathbb{C}')(\overline{e_j}) = w(\mathbb{C}')$, so $\varphi_2(z)(\overline{e_j}) = 0$.

Thus I'_j is contained in the ideal generated by $\operatorname{Ker}\Phi \cap e_j kQ_{T(\Lambda)} e_j$. To prove the other inclusion, let $z \in \operatorname{Ker}\Phi \cap e_j kQ_{T(\Lambda)} e_j$. We write $z = z_1 + z_2$, with $z_1 = \sum_{i=1}^{r} a_i \mathbb{C}_i$, $z_2 = \sum_{i=r+1}^{t} a_i \mathbb{C}_i$, where $\mathbb{C}_1, \ldots, \mathbb{C}_r$ are elementary cycles and $\mathbb{C}_{r+1}, \ldots, \mathbb{C}_t$ are cycles which are not elementary.

Then $z_2 \in I'_j$, and we just proved that $I'_j \subseteq \operatorname{Ker}\Phi$. So $0 = \Phi(z) = \Phi(z_1)$. Since $\mathbb{C}_i$ is elementary, $\mathbb{C}_i \in (\beta_p)$, so $\Phi(\mathbb{C}_i) = (0, \varphi_2(\mathbb{C}_i))$, $i = 1, \ldots, r$. Thus $0 = \Phi(z_1) = (0, \sum_{i=1}^{r} a_i \varphi_2(\mathbb{C}_i))$, and we get from Lemma 3.5(g)

$$0 = \sum_{i=1}^{r} a_i \varphi_2(\mathbb{C}_i)(\bar{e}_j) = \sum_{i=1}^{r} a_i w(\mathbb{C}_i).$$

So $a_1 = -\sum_{i=2}^{r} a_i w(\mathbb{C}_i)/w(\mathbb{C}_1)$, and $z_1 = \sum_{i=2}^{r}(a_i \mathbb{C}_i - a_i(w(\mathbb{C}_i)/w(\mathbb{C}_1))\mathbb{C}_1) = \sum_{i=2}^{r}(a_i/w(\mathbb{C}_1))((w(\mathbb{C}_1)\mathbb{C}_i - w(\mathbb{C}_i)\mathbb{C}_1)) \in I'_j$, since it is a linear combination of elements of the type (ii). Thus the ideal generated by $\operatorname{Ker}\Phi \cap e_j kQ_{T(\Lambda)} e_j$ is contained in I'_j, and therefore coincides with I'_j. ∎

Remark 3.7. As a direct consequence of the preceding proposition, we know that the classes of oriented cycles with origin j in $kQ_{T(\Lambda)}$ generate a one-dimensional subspace of $kQ_{T(\Lambda)}/I'_j$.

We have now the following consequence of Proposition 3.6.

COROLLARY 3.8. *Let Λ be as in Lemma 3.5 and let $\mathbb{C}$ be an oriented cycle in $Q_{T(\Lambda)}$. Then the following conditions are equivalent*:

(i) *$\mathbb{C}$ is an elementary cycle.*

(ii) *$\mathbb{C}$ is nonzero in $T(\Lambda)$.*

Proof. It follows from Lemma 3.5(g) that (i) implies (ii).

We get directly from Proposition 3.6 that (ii) implies (i), since nonelementary cycles are in I'_j, which is contained in $\operatorname{Ker}\Phi$. ∎

We can now prove the main result of this section.

THEOREM 3.9. *Let $\Lambda = kQ_\Lambda/I$ be an algebra such that any oriented cycle in Q_Λ is zero in Λ. Let I' be the ideal in $kQ_{T(\Lambda)}$ generated by*

(i) *I,*

(ii) *the paths consisting of $n+1$ arrows of an elementary cycle of length n,*

(iii) *the paths whose arrows do not belong to a single elementary cycle, and*

(iv) *the elements* $\sum_{s=1}^{l} a_s\mu_s$, *where* $a_s \in k^*$ *and* μ_s *are different paths from* i *to* j *in* $(\beta_p)_{p\in\mathbb{M}}$, *for* $s = 1, \dots, l$, *and such that*

$$\gamma\left(\sum_{s=1}^{l} a_s\mu_s\right) \in I'_i \quad or \quad \left(\sum_{s=1}^{l} a_s\mu_s\right)\gamma \in I'_j,$$

where I'_t *is the ideal defined in Proposition* 3.6, *for each supplement* γ *of one of the* μ_s*s.*

Then I' *is admissible and* $I' = I_{T(\Lambda)}$. *That is,* $T(\Lambda) \simeq kQ_{T(\Lambda)}/I'$.

Proof. It is sufficient to prove that $I' = \operatorname{Ker}\Phi$, where Φ is the morphism defined above. First we show that $I' \subseteq \operatorname{Ker}\Phi$.

Let q be a path in $kQ_{T(\Lambda)}$, which is not in $\operatorname{Ker}\Phi$. If $\varphi_2(q) \neq 0$, then $q = \gamma\beta_p\delta$, with $p \in \mathbb{M}$ and γ, δ paths in kQ_Λ. Let u in kQ_Λ be a path such that $\varphi_2(q)(\bar{u}) \neq 0$. Then we have $0 \neq \bar{p}^*(\overline{\delta u\gamma}) = \varphi_2(q)(\bar{u})$.

Since Λ has no nonzero oriented cycles, δ and γ contain no common arrows. Thus q cannot consist of $n+1$ arrows in an elementary cycle of length n. Moreover, q is in the elementary cycle $\mathbb{C} = qu$.

If $\varphi_1(q) \neq 0$, then q is a path in kQ_Λ, and clearly $q \notin I$. We know that q is contained in an elementary cycle, by Remark 3.3. So, if I'' is the ideal generated by the classes (i)–(iii) of $kQ_{T(\Lambda)}$, then $q \notin I''$, and so $I'' \subseteq \operatorname{Ker}\Phi$.

Assume now that $v = \sum_{s=1}^{l} a_s q_s \notin \operatorname{Ker}\Phi$, where the q_ss are different paths in $(\beta_p)_{p\in\mathbb{M}}$ from i to j. Then $\Phi(v) = (0, \varphi_2(v)) \neq 0$.

Recall now that we know from Proposition 3.6 that $I'_j \subseteq \operatorname{Ker}\Phi$ for all j. Then, if $\varphi_2(v) \neq 0$, Lemma 3.5(f) states precisely that v is not an element in the class (iv).

Thus we have shown that $q \notin \operatorname{Ker}\Phi$ implies $q \notin I'$. This completes the proof that $I' \subseteq \operatorname{Ker}\Phi$.

Since Φ: $kQ_{T(\Lambda)} \to T(\Lambda)$ is surjective and $I' \subseteq \operatorname{Ker}\Phi$, to prove that the equality holds it is enough to prove that $\dim_k kQ_{T(\Lambda)}/I' = \dim_k T(\Lambda) = 2\dim_k \Lambda$.

The image of an element $y \in kQ_{T(\Lambda)}$ under the canonical epimorphism $kQ_{T(\Lambda)} \to kQ_{T(\Lambda)}/I'$ is denoted by $\tilde{y}$.

Since $I' \subseteq \operatorname{Ker}\Phi$, we have the canonical epimorphism $kQ_{T(\Lambda)}/I' \to kQ_{T(\Lambda)}/\operatorname{Ker}\Phi = T(\Lambda)$. The inclusion of Λ in $T(\Lambda)$ factors through $kQ_{T(\Lambda)}/I'$ because $I \subseteq I'$. Thus the map ι: $\Lambda \to kQ_{T(\Lambda)}/I'$ induced by the embedding of kQ_Λ in $kQ_{T(\Lambda)}$ is a monomorphism.

We have that $kQ_{T(\Lambda)} = kQ_\Lambda + (\beta_p)_{p\in\mathbb{M}}$. Therefore, $e_j kQ_{T(\Lambda)} e_i = e_j kQ_\Lambda e_i + e_j(\beta_p)_{p\in\mathbb{M}} e_i$, for each i and j in $(Q_{T(\Lambda)})_0$. Let π: $kQ_{T(\Lambda)} \to kQ_{T(\Lambda)}/I'$ be the canonical epimorphism.

We define in $kQ_{T(\Lambda)}/I'$ the subspaces $\mathscr{P}_{ij} = \pi(e_j kQ_\Lambda e_i)$ and $\mathscr{F}_{ij} = \pi(e_j(\beta_p)_{p\in\mathbb{M}} e_i)$. Then $\mathscr{P}_{ij} = \iota(e_j \Lambda e_i) \simeq e_j \Lambda e_i$, so $\sum_{i,j} \dim_k \mathscr{P}_{ij} = \dim_k \Lambda$.

We will prove that $\dim_k(\mathscr{P}_{ij}) \geq \dim_k(\mathscr{F}_{ji})$. Though these dimensions depend on i and j, to simplify notation we will denote them by n and m, respectively.

We observe first that a path q in $kQ_{T(\Lambda)}$ is not in I' if and only if there is a supplement for q in some elementary cycle $\mathbb{C}$. In fact, we know by Corollary 3.8 that elementary cycles are nonzero in $T(\Lambda)$ and therefore are not in I', because $I' \subseteq \operatorname{Ker} \Phi$. On the other hand, if $q \notin I'$, then all arrows of q belong to a single elementary cycle $\mathbb{C}$ since q does not belong to the class (iii) of I'. Moreover, using the fact that q is not in the class (ii) of I', we deduce that q is contained in $\mathbb{C}$.

As a first consequence, we find that $\mathscr{F}_{ji} \neq 0$ if and only if $\mathscr{P}_{ij} \neq 0$. So we assume that both are nonzero and choose paths $\gamma_1, \ldots, \gamma_n$ in kQ_Λ and $\mu_1, \ldots, \mu_m$ in $(\beta_p)_{p\in\mathbb{M}}$, so that $\{\widetilde{\gamma_1}, \ldots, \widetilde{\gamma_n}\}$ and $\{\widetilde{\mu_1}, \ldots, \widetilde{\mu_m}\}$ are bases for $\mathscr{P}_{ij}$ and $\mathscr{F}_{ji}$, respectively.

Now we will show that $m \leq n$. Suppose on the contrary that $m > n$. We will find a nonzero element w in $\mathscr{F}_{ji}$ such that $w\tilde{\gamma} = 0$ for each supplement γ of any of the μ_t. This will contradict that w does not belong to the class (iv) of I'. Clearly, it is enough to find $w \neq 0$ in $\mathscr{F}_{ji}$ such that $w\widetilde{\gamma_t} = 0$ for each t. So we start looking for relations among the $\mu_k\gamma_t$s.

Let $1 \leq t \leq n$. We prove first that not all $\widetilde{\mu_k\gamma_t}$, $k = 1, \ldots, m$, are zero. In fact, $\gamma_t \notin I'$, so, as we observed above, it has a supplement, say δ_t, in an elementary cycle $\mathbb{C}_t$. Using again the above observation, we get that $\delta_t\gamma_t \notin I'$. Now, $\tilde{\delta}_t \in \mathscr{F}_{ji}$ because $\tilde{\gamma}_t \in \mathscr{P}_{ij}$, so $\tilde{\delta}_t$ is a linear combination of $\widetilde{\mu_1}, \ldots, \widetilde{\mu_m}$, and therefore, for every t with $1 \leq t \leq n$, there exists an index r_t such that $\widetilde{\mu_{r_t}\gamma_t} \neq 0$ and $1 \leq r_t \leq m$. Now we know from Remark 3.7 that the cycles $\widetilde{\mu_k\gamma_t}$, $k = 1, \ldots, m$, generate a subspace of $kQ_{T(\Lambda)}/I'$ of dimension 1, because all these cycles have origin i. We conclude that $\widetilde{\mu_k\gamma_t} = a_{kt}\widetilde{\mu_{r_t}\gamma_t}$ for some $a_{kt} \in k$ and for all $k = 1, \ldots, m$.

Now, $(\sum_{k=1}^m x_k\widetilde{\mu_k})\widetilde{\gamma}_t = \sum_{k=1}^m x_k a_{kt}\widetilde{\mu_{r_t}\gamma_t}$. The system $\sum_{k=1}^m a_{kt}X_k = 0$, with $t = 1, \ldots, n$, has a nontrivial solution because $m > n$, say $(x_1, \ldots, x_m)$. Then the element $w = \sum_{k=1}^m x_k\widetilde{\mu_k}$ satisfies the required conditions: $w \neq 0$ and $w\widetilde{\gamma}_t = 0$ for all t, ending the proof that $m \leq n$.

We are now in a position to prove that $\dim_k kQ_{T(\Lambda)}/I' = \dim_k T(\Lambda)$. Since $kQ_{T(\Lambda)}/I'$ maps onto $T(\Lambda)$, we get that $\dim_k kQ_{T(\Lambda)}/I' \geq \dim_k T(\Lambda)$. On the other hand, $\dim_k kQ_{T(\Lambda)}/I' \leq \sum_{i,j}(\dim_k \mathscr{P}_{ij} + \dim_k \mathscr{F}_{ij}) \leq \sum_{i,j}(\dim_k \mathscr{P}_{ij} + \dim_k \mathscr{P}_{ji}) = 2\dim_k \Lambda = \dim T(\Lambda)$. This ends the proof of the theorem. ∎

As an immediate consequence, we get that any nonzero path in $T(\Lambda)$ is in a nonzero oriented cycle.

We will see next that the hypothesis that $\mu_s \in (\beta_p)_{p\in\mathbb{M}}$ in (iv) of the preceding theorem can be omitted.

COROLLARY 3.10. *Let $v = \sum_{s=1}^{l} a_s q_s$, where $a_s \in k^*$ and the q_s are pairwise different paths from i to j in $kQ_{T(\Lambda)}$. Suppose that either $\overline{\gamma v} = 0$ or $\overline{v\gamma} = 0$ in $T(\Lambda)$ for every supplement γ of any of the q_ss. Then $\bar{v} = 0$.*

Proof. Suppose first that $v = \sum_{s=1}^{l} a_s q_s$ satisfies $\overline{\gamma v} = 0$ for a supplement γ of some q_ss. Then, for some $0 \le r \le l$, we may suppose that $q_1, \dots, q_r \in kQ_\Lambda$, $q_{r+1}, \dots, q_l \in (\beta_p)_{p\in\mathbb{M}}$. Let $v_1 = \sum_{s=1}^{r} a_s q_s$, $v_2 = \sum_{s=r+1}^{l} a_s q_s$.

Assume that γ is a supplement of q_j, with $j > r$. Then $\gamma \in kQ_\Lambda$, and therefore $\overline{\gamma v_1} = 0$ because oriented cycles are zero in Λ. Thus $\overline{\gamma v_2} = 0$ and we conclude from the preceding theorem that $\overline{v_2} = 0$.

So $v = v_1 \in kQ_\Lambda$. If $\bar{v}_1 \ne 0$, then we know by Remark 3.3 that there exist $\delta_1, \delta_2 \in kQ_\Lambda$ and $p \in \mathbb{M}$ so that $\bar{p}^*(\overline{\delta_1 v \delta_2}) \ne 0$, and thus, for some s, $\bar{p}^*(\overline{\delta_1 q_s \delta_2}) \ne 0$, so $\mathbb{C} = \beta_p \delta_1 q_s \delta_2$ is an elementary cycle. Then $\gamma = \delta_2 \beta_p \delta_1$ is a supplement of q_s. Moreover,

$$\varphi_2(\gamma)(\bar{v}) = (\overline{\delta_2}\,\bar{p}^*\,\overline{\delta_1})(\bar{v}) = \bar{p}^*(\overline{\delta_1 v \delta_2}) \ne 0.$$

We conclude from Lemma 3.5(e) that $\varphi_2(\gamma v) \ne 0$. So $\Phi(\gamma v) \ne 0$, contradicting the assumption that $\overline{\gamma v} = 0$ in $kQ_{T(\Lambda)}$. Thus we have $\bar{v} = 0$, as desired. This ends the proof of the corollary, since we know by Lemma 3.5(i) that $\overline{v\gamma} = 0$ if and only if $\overline{\gamma v} = 0$. ∎

When the algebra Λ is schurian, the description of the ideal I' is easier, as we state in the following corollary.

COROLLARY 3.11. *Let $\Lambda = kQ_\Lambda/I$ be a schurian algebra. The ideal I'' generated by*

(a) *the paths consisting of $n+1$ arrows of an elementary cycle of length n,*

(b) *the paths whose arrows do not belong to a single elementary cycle, and*

(c) *the elements $a'q - aq'$, where q, q' are paths from i to j admitting a common supplement $\gamma \in kQ_\Lambda$ in elementary cycles,*

$$\beta_p \alpha_n \cdots \alpha_1 \quad \textit{and} \quad \beta_{p'} \alpha'_m \cdots \alpha'_1,$$

respectively, with $\overline{\alpha_n \cdots \alpha_1} = a\bar{p}$ and $\overline{\alpha'_m \cdots \alpha'_1} = a'\overline{p'}$, $a, a' \in k^*$,

is admissible and $I'' = I_{T(\Lambda)}$. That is, $T(\Lambda) \simeq kQ_{T(\Lambda)}/I''$.

Proof. First we observe, for a path $\gamma \in kQ_\Lambda$ and $p \in \mathbb{M}$, that $\bar{\gamma} = a\bar{p}$ with $a \in k$ if and only if $w(\beta_p \gamma) = a$. So the elements $\omega(\mathbb{C}')\mathbb{C} - \omega(\mathbb{C})\mathbb{C}'$ with $\mathbb{C}, \mathbb{C}'$ cycles with the same origin are in the class (c) of I''.

Assume now that $v = \sum_{s=1}^{l} a_s \mu_s$ is an element in the class (iv) of $I' = I_{T(\Lambda)}$, where $a_s \in k^*$ and μ_s are different paths from i to j in $(\beta_p)_{p \in \mathbb{M}}$, for $s = 1, \dots, l$. That is, $\overline{\gamma v} \in I'_i$ for any supplement γ of one of the μ_ss. We may assume that $\mu_1, \dots, \mu_l$ are not in the class (b) of I''. Since Λ is schurian, they have the same supplements. Let γ be one of them.

Then $\overline{\gamma\mu_i} = (\omega(\overline{\gamma\mu_i})/\omega(\overline{\gamma\mu_1}))\overline{\gamma\mu_1}$ by Proposition 3.6. So

$$0 = \overline{\gamma v} = \left(a_1 + \sum_{i=2}^{l} a_i \omega(\overline{\gamma\mu_i})/\omega(\overline{\gamma\mu_1})\right)\overline{\gamma\mu_1}.$$

Thus $a_1 = -\sum_{i=2}^{l} a_i \omega(\overline{\gamma\mu_i})/\omega(\overline{\gamma\mu_1})$. By replacing this expression in v, we obtain that

$$v = \sum_{i=2}^{l} a_i(\overline{\mu_i} - (\omega(\overline{\gamma\mu_i})/\omega(\overline{\gamma\mu_1}))\overline{\mu_1})$$

belongs to the class (c) of I''.

Now we will prove that $I \subseteq I''$. If a path γ is in I, then γ is in the class (b) of I''. If γ, γ' are paths in kQ_Λ such that $\bar{\gamma}, \overline{\gamma'} \neq 0$ in Λ and $\gamma - a\gamma' \in I$, then they have the same supplements, and we know by Corollary 3.8 that they have at least one. Let μ be one of them, and $\mathbb{C} = \mu\gamma$ and $\mathbb{C}' = \mu\gamma'$. Then $\Phi(\mathbb{C}) = a\Phi(\mathbb{C}')$, so $w(\mathbb{C}) = aw(\mathbb{C}')$ and thus $\gamma - a\gamma'$ is in the class (iv) of I' and therefore in I''. So $I \subseteq I''$.

Since the classes (a), (b) of I'' coincide with the classes (ii), (iii) of I', we have that $I' \subseteq I''$.

On the other hand, the previous corollary shows that the class (c) of I'' is contained in I', proving that $I'' \subseteq I'$. ∎

Suppose Λ is a schurian algebra such that the ordinary quiver of Λ has no oriented cycles. Then all oriented cycles in $kQ_{T(\Lambda)}$ not passing through any vertex more than once are nonzero in $T(\Lambda)$. So we can replace "elementary cycle" by "cycle" in the statement of Corollary 3.11.

Schurian algebras have the property that if γ and η are paths in kQ_Λ from i to j and $\bar{\gamma}, \bar{\eta} \neq 0$ in Λ, then $\bar{\gamma} = a\bar{\eta}$, with $a \in k$. Sometimes the presentation for Λ can be chosen so that always $a = 1$, and we say then that parallel paths are equal in Λ. In this case, the class (c) in Corollary 3.11 can be described in a simpler way, and if we further assume that the ordinary quiver of Λ has no oriented cycles, then the situation is particularly nice, because the relations can be formulated directly in terms of the cycles in $kQ_{T(\Lambda)}$, independently of the relations for Λ, as we state in the following corollary.

COROLLARY 3.12. *Let $\Lambda = kQ_\Lambda/I$ be a schurian algebra such that the ordinary quiver of Λ has no oriented cycles and parallel paths in Q_Λ are equal in Λ. Then the ideal $I_{T(\Lambda)}$ of relations for $T(\Lambda)$ is generated by*

(i) *the paths consisting of $n+1$ arrows in a cycle of length n,*

(ii) *the paths whose arrows do not belong to a single cycle*, *and*

(iii) *the difference* $q - q'$ *of paths* q, q' *with the same origin and endpoint and having a common supplement in cycles of* $Q_{T(\Lambda)}$.

We observe here that the hypothesis of Corollary 3.12 holds for any algebra Λ such that $T(\Lambda)$ is of finite representation type [Y].

Now we give two examples.

EXAMPLE 3.13. Let Λ be as in Example 2.5. Using the description of $kQ_{T(\Lambda)}$ given there and Corollary 3.12, we find that $T(\Lambda)$ is the algebra given by $kQ_{T(\Lambda)}$ with the relations

$$\mathfrak{r}^4 = 0,$$
$$\alpha_1\beta_{p_1} = \beta_{p_2}\alpha_3,$$
$$\alpha_3\alpha_2\alpha_1 = \beta_{p_1}\alpha_2\beta_{p_2} = 0.$$

EXAMPLE 3.14. Let Λ be as in Example 2.6. This algebra is not schurian. In this case we find that $T(\Lambda)$ is the algebra given by $Q_{T(\Lambda)}$ with the relations

$$\alpha_3\alpha_1 = \alpha_3\alpha_2; \quad \alpha_4\alpha_1 = \alpha_4\alpha_2,$$
$$\alpha_1\beta_{p_2}\alpha_3\alpha_1 = \alpha_3\alpha_1\beta_{p_2}\alpha_3 = \beta p_2\alpha_3\alpha_1\beta p_2 = 0,$$
$$\alpha_4\alpha_1\beta_{p_3}\alpha_4 = \beta_{p_3}\alpha_4\alpha_1\beta_{p_3} = 0,$$
$$\alpha_2\beta_{p_1}\alpha_2 = \beta_{p_1}\alpha_2\beta_{p_1} = 0,$$
$$\alpha_3\alpha_1\beta_{p_3} = 0; \quad \beta_{p_1}\alpha_1 = \alpha_1\beta_{p_1} = 0; \quad \beta_{p_1}\alpha_1\beta_{p_3} = 0,$$
$$\alpha_4\alpha_1\beta_{p_2} = 0; \quad \beta_{p_1}\alpha_2\beta_{p_2} = 0,$$
$$\beta_{p_2}\alpha_3 = \beta_{p_3}\alpha_4; \quad \alpha_1\beta_{p_2} = \alpha_2\beta_{p_2}; \quad \alpha_1\beta_{p_3} = \alpha_2\beta_{p_3}; \quad \beta_{p_1}\alpha_2 = \beta_{p_2}\alpha_3\alpha_1.$$

We go on now to study the ordinary quiver and relations for the repetitive algebra $\widehat{\Lambda}$ of Λ, as defined by Hughes and Waschbusch in [HW],

$$\widehat{\Lambda} = \begin{pmatrix} \ddots & \ddots & & & 0 \\ & \Lambda_{m-1} & Q_{m-1} & & \\ & & \Lambda_m & Q_m & \\ & & & \Lambda_{m+1} & \ddots \\ 0 & & & & \ddots \end{pmatrix},$$

where $\Lambda_m = \Lambda$, $Q_m = D(\Lambda)$ for all $m \in Z$.

The vertices of $Q_{\widehat{\Lambda}}$ are pairs (m, i) with $m \in Z$, $i \in (Q_\Lambda)_0$, and the set of arrows is

$$(Q_{\widehat{\Lambda}})_1 = \{\alpha^m\colon (m,i) \to (m,j)\}_{\alpha:\ i\to j \in (Q_\Lambda)_1} \cup \{\beta_p^m\colon (m, e(p)) \to (m+1, s(p))\}_{p\in\mathbb{M}}$$

for $m \in Z$.

It is well known that $\widehat{\Lambda}$ is a Galois covering of $T(\Lambda)$. If $\gamma = \alpha_1 \cdots \alpha_n$ is a path in kQ_Λ, we will denote by γ^m the path $\alpha_1^m \cdots \alpha_n^m$ in $kQ_{\widehat{\Lambda}}$.

The relations are obtained by lifting the relations in $kQ_{T(\Lambda)}$. We have to replace the notions of elementary cycle, weight of an elementary cycle, and supplement of a path by the following notions.

DEFINITION 3.15. A path C in $kQ_{\widehat{\Lambda}}$ is called **elementary** if $C = \delta^{m+1}\beta_p^m\gamma^m$, with δ, γ paths in kQ_Λ, $p \in \mathbb{M}$, such that $\bar{p}^*(\overline{\gamma\delta}) \neq 0$. In this case the **weight** of C is $w(C) = p^*(\overline{\gamma\delta})$. A path q in $kQ_{\widehat{\Lambda}}$ has a **supplement** if there is a path q' in $kQ_{\widehat{\Lambda}}$ such that qq' is an elementary path.

We are now in a position to describe the relations in $kQ_{\widehat{\Lambda}}$ generalizing a result by Asashiba for triangular schurian algebras such that parallel paths in Q_Λ are equal in Λ [A]. We start with the ideal $I_{\widehat{\Lambda}(m,j)}$ of relations from (m, j) to $(m+1, j)$. This ideal is generated by

(a) paths from (m, j) to $(m+1, j)$ which are not elementary,

(b) elements $w(C')C - w(C)C'$, where C, C' are elementary paths from (m, j) to $(m+1, j)$.

THEOREM 3.16. *Let $\Lambda = kQ_\Lambda/I$ be an algebra such that any oriented cycle in Q_Λ is zero in Λ. Let I' be the ideal in $kQ_{\widehat{\Lambda}}$ generated by*

(a) *the sets $I^m = \{\sum a_i\gamma_i^m,$ such that $\sum a_i\gamma_i \in I\}, m \in Z$,*

(b) *the paths not contained in an elementary path, and*

(c) *the elements $x = \sum_{s=1}^{l} a_s q_s$, where $a_s \in k^*$ and q_s are different paths from (m, i) to (m', j) in $(\beta_p^m)_{m\in Z,\, p\in\mathbb{M}}$ for $s = 1, \ldots, l$, and such that*

$$\gamma x \in I_{\widehat{\Lambda}(m,i)} \quad \textit{and} \quad x\gamma \in I_{\widehat{\Lambda}(m',j)}$$

for each supplement γ of one of the q_ss and $I_{\widehat{\Lambda}(m,i)}$ the above defined ideal.

Then I' is admissible and $I' = I_{\widehat{\Lambda}}$. That is, $\widehat{\Lambda} \simeq kQ_{\widehat{\Lambda}}/I'$.

Remark 3.17. We observe that the elements corresponding to the class (ii) of Theorem 3.9 belong to the class (b).

Proof. Let 1_Λ^m be the element of $\widehat{\Lambda}$ which has 1_Λ as its mth diagonal entry and all other entries zero, and let $e_{(m,i)} = 1_\Lambda^m e_i$ be the idempotent of $\widehat{\Lambda}$ corresponding to the vertex (m, i) of $Q_{\widehat{\Lambda}}$. We define $\Psi: kQ_{\widehat{\Lambda}} \to \widehat{\Lambda}$ by $\Psi((m, j)) = e_{(m,j)}$, $\Psi(\alpha^m) = \bar{\alpha}.1_\Lambda^m$, and $\Psi(\beta_p^m) = \overline{p^*}_{(m,m+1)}$, the matrix with $\overline{p^*}$ in the $(m, m+1)$ entry and zero elsewhere, for $p \in \mathbb{M}$, $m \in Z$.

Then Ψ is surjective and $I' \subseteq \operatorname{Ker} \Psi$. This can be proven with a straightforward adaptation of the arguments used in the proof of Theorem 3.9 (see also [R]). The other inclusion follows from the fact that

$$\dim_k e_{(m,i)}(kQ_{\widehat{\Lambda}}/I')e_{(m,j)} + \dim_k e_{(m,i)}(kQ_{\widehat{\Lambda}}/I')e_{(m+1,j)}$$
$$= \dim_k e_i(kQ_{T(\Lambda)}/I_{T(\Lambda)})e_j.$$

Summing up over i and j, we see that

$$\dim_k 1_\Lambda^m.kQ_{\widehat{\Lambda}}/I' = \dim_k T(\Lambda) = \dim_k \Lambda + \dim_k D(\Lambda) = \dim 1_\Lambda^m.\widehat{\Lambda},$$

and this ends the proof of the theorem. ∎

4. AN APPLICATION OF A THEOREM OF S. BRENNER

This section is devoted to a very simple interpretation of the results established in [B] in the particular case of the trivial extension $T(\Lambda) = \Lambda \ltimes D(\Lambda)$, where Λ is an algebra such that any oriented cycle in Q_Λ is zero in Λ.

We start by recalling Brenner's results.

Let Λ be an artin algebra. An element of Λ of the form $\alpha = f\alpha g$, where f and g are primitive idempotents of Λ and $\alpha \in \mathfrak{r}\backslash\mathfrak{r}^2$, will be called an **arrow**.

Let e be a primitive idempotent of Λ. A set A of arrows will be called a **complete set of arrows** for $\mathfrak{r}e$ if

(1) it generates $\mathfrak{r}e$ (as a Λ-module).

(2) no proper subset of A generates $\mathfrak{r}e$.

A complete set of arrows for $e\mathfrak{r}$ is defined similarly.

Let e be a primitive idempotent and let $\mathcal{N}$ be the set of pairs (N, n) of integers such that there exist sets of arrows A_i and B_i, $0 \le i \le n$, of which only A_0 and B_0 can be empty, satisfying the following conditions:

(1) $i \neq j$ implies $A_i \cap A_j = \varnothing = B_i \cap B_j$.

(2) $\bigcup_{i=0}^n A_i$ is a complete set of arrows for $e\mathfrak{r}$.

(3) $\bigcup_{i=0}^n B_i$ is a complete set of arrows for $\mathfrak{r}e$.

(4) If $i \neq j$, or $i = 0$, or $j = 0$, then $\alpha \in A_i$ and $\beta \in B_j$ implies $\beta\alpha = 0$.

(5) $N = n + \operatorname{card} A_0$.

Let $N_e = \max\{N\colon \text{there exists } n \text{ with } (N, n) \in \mathscr{N}\}$ and $n_e = \min\{n\colon (N_e, n) \in \mathscr{N}\}$.

THEOREM 4.1 (Brenner). *Let S be a noninjective simple Λ-module, and let e be a primitive idempotent of Λ such that $S \simeq \Lambda e/\mathfrak{r}e$. The middle term of the almost split sequence starting at S has exactly N_e indecomposable direct summands. Furthermore, the number of indecomposable projective direct summands is equal to $N_e - n_e$.*

COROLLARY 4.2 (Brenner). *If Λ is self-injective, then the number of indecomposable direct summands of $\mathfrak{r}P/\operatorname{soc} P$, where $P = \Lambda e$, is equal to n_e.*

In all that follows, let Λ be an algebra such that any oriented cycle in Q_Λ is zero in Λ.

For each $h \in (Q_{T(\Lambda)})_0$, let $\mathscr{C}_h$ be the set of oriented cycles $\mathbb{C}$ such that $\mathbb{C} \neq 0$ in $T(\Lambda)$, and $s(\mathbb{C}) = e(\mathbb{C}) = h$. By Corollary 3.10, we have $\mathscr{C}_h \neq \varnothing$.

We begin with the following definitions:

DEFINITION 4.3. Let $\mathbb{C}, \mathbb{C}'$ be in $\mathscr{C}_h$. We say that $\mathbb{C}$ and $\mathbb{C}'$ are related, and write $\mathbb{C}\mathfrak{R}C'$, if there exists an arrow α belonging to $\mathbb{C}$ and $\mathbb{C}'$ with $s(\alpha) = h$ or $e(\alpha) = h$.

On the other hand, we will define a relation $\mathfrak{R}'$ on the set

$$A_h = \{\alpha \in (Q_{T(\Lambda)})_1\colon e(\alpha) = h\}.$$

DEFINITION 4.4. Let α, α' be in A_h. We say that α and α' are related and write $\alpha\mathfrak{R}'\alpha'$ if there exists an arrow $\beta \in (Q_{T(\Lambda)})_1$ such that $\beta\alpha \neq 0$ and $\beta\alpha' \neq 0$ in $T(\Lambda)$.

From now on, we denote by "$\equiv$" and "$\approx$" the equivalence relations generated by $\mathfrak{R}$ in $\mathscr{C}_h$ and by $\mathfrak{R}'$ in A_h, respectively.

We next want to give the precise connection between these equivalence relations. For this purpose, the following results will be useful.

LEMMA 4.5. *Let $\alpha_1, \ldots, \alpha_m$ be arrows in A_h, such that $\alpha_1\mathfrak{R}'\alpha_2\mathfrak{R}'\cdots\mathfrak{R}'\alpha_m$. Then there exist cycles $\mathbb{C}_1, \ldots, \mathbb{C}_m$ in $\mathscr{C}_h$, with $\alpha_i \in \mathbb{C}_i$, for all $i = 1, \ldots, m$ and $\mathbb{C}_1 \equiv \mathbb{C}_m$.*

Proof. We prove this by induction on m. Our claim clearly holds if $m = 1$. Assume now that $\alpha_1\mathfrak{R}'\alpha_2\mathfrak{R}'\cdots\mathfrak{R}'\alpha_m$, where $m \geq 2$. Then there exists $\beta \in (Q_{T(\Lambda)})_1$, such that $\beta\alpha_{m-1} \neq 0$ and $\beta\alpha_m \neq 0$. This implies that the paths $\beta\alpha_{m-1}$ and $\beta\alpha_m$ belong to cycles $\mathbb{C}'_{m-1}$ and $\mathbb{C}_m$, respectively. Therefore, $\mathbb{C}'_{m-1}\mathfrak{R}\mathbb{C}_m$ and $\alpha_{m-1} \in \mathbb{C}'_{m-1}$, $\alpha_m \in \mathbb{C}_m$.

By the induction assumption, we have cycles $\mathbb{C}_1, \ldots, \mathbb{C}_{m-1}$, with $\alpha_i \in \mathbb{C}_i$ for $i = 1, \ldots, m-1$, and $\mathbb{C}_1 \equiv \mathbb{C}_{m-1}$. Since α_{m-1} is an arrow belonging to $\mathbb{C}'_{m-1}$ and $\mathbb{C}_{m-1}$, we have that $\mathbb{C}_{m-1}\mathfrak{R}\mathbb{C}'_{m-1}$. Then $\mathbb{C}_1 \equiv \mathbb{C}_m$, and this ends the proof of the lemma. ∎

LEMMA 4.6. *Let* $\mathbb{C}_1, \ldots, \mathbb{C}_m$ *in* $\mathscr{C}_h$ *such that* $\mathbb{C}_1 \mathfrak{R} \mathbb{C}_2 \mathfrak{R} \cdots \mathfrak{R} \mathbb{C}_m$. *Then there exist arrows* $\alpha_1, \ldots, \alpha_m$ *in* A_h, *with* $\alpha_i \in \mathbb{C}_i$, *for* $i = 1, \ldots, m$ *and* $\alpha_1 \approx \alpha_m$.

Proof. This follows by induction on m, applying arguments similar to those used in the proof of Lemma 4.5. ∎

We now give the desired connection between the equivalence relations "≡" and "≈."

PROPOSITION 4.7. *Let* $\alpha, \alpha' \in A_h, \mathbb{C}, \mathbb{C}' \in \mathscr{C}_h$ *be such that* $\alpha \in \mathbb{C}$ *and* $\alpha' \in \mathbb{C}'$. *Then we have* $\alpha \approx \alpha'$ *if and only if* $\mathbb{C} \equiv \mathbb{C}'$.

Proof. The result is an immediate consequence of the previous lemmas. ∎

This readily gives the following

COROLLARY 4.8. $\operatorname{card}(\mathscr{C}_h/\equiv) = \operatorname{card}(A_h/\approx)$.

We are now ready to describe the numbers N_{e_h} and n_{e_h}.

PROPOSITION 4.9. *Let* h *be a vertex in* $Q_{T(\Lambda)}$, *and let* e_h *be the idempotent element corresponding to* h. *If* $\dim_k \Lambda > 1$, *then* $N_{e_h} = \operatorname{card}(\mathscr{C}_h/\equiv) = n_{e_h}$.

Proof. Consider the partition $A_1, \ldots, A_t$ induced by the relation "≈" on the set A_h. Define, for $1 \leq i \leq t, B_i = \{\beta\colon$ there exists $\alpha \in A_i$ with $\beta\alpha \neq 0\}$ and $A_0 = B_0 = \emptyset$. Since $\dim_k \Lambda > 1$, it follows that $B_i \neq \emptyset$, for $i = 1, \ldots, t$. By construction, the pair (t, t) is in $\mathscr{N}$.

We shall prove that if $(N, n) \in \mathscr{N}$, then $N = n \leq t$. In fact, let A'_i, B'_i be sets of arrows satisfying Brenner's condition, for $0 \leq i \leq n$.

Using the above conditions and Corollary 3.10, we conclude that

(1) $B'_i = \{\beta\colon$ there exists $\alpha \in A'_i$ with $\beta\alpha \neq 0\}$ for $i = 1, \ldots, n$.

(2) $B'_0 = A'_0 = \emptyset$.

It follows from (2) that $N = n$. On the other hand, it is easy to see that if $\alpha \approx \alpha'$, then there exists j such that $\alpha, \alpha' \in A'_j$. Then, for $1 \leq i \leq t$, $A_i \subset A'_{j_i}$, which implies $n \leq t$. Therefore, $N_{e_h} = n_{e_h} = \operatorname{card}(A_h/\approx)$, and the proof is finished by applying Corollary 4.8. ∎

Remark 4.10. If $\dim_k \Lambda = 1$, then $\dim_k T(\Lambda) = 2$ and $\mathscr{N} = \{(1, 0), (1, 1)\}$, so $N_{e_1} = 1$ and $n_{e_1} = 0$.

We are now in a position to restate Brenner's results in the particular case of a trivial extension $T(\Lambda)$.

THEOREM 4.11. *Let S_h be a simple $T(\Lambda)$-module corresponding to the vertex h. The number of indecomposable direct summands of the middle term of the almost split sequence*

$$0 \to S_h \to E \to \operatorname{Tr} \operatorname{D} S_h \to 0$$

is equal to the number of equivalence classes in $\mathscr{C}_h$. Furthermore, the number of indecomposable projective summands of E is equal to zero, except if $\dim_k \Lambda = 1$.

COROLLARY 4.12. *Let P_h be the indecomposable projective $T(\Lambda)$-module corresponding to the vertex h. If* $\dim_k \Lambda > 1$, *then the number of indecomposable direct summands of $\mathfrak{r}P_h/\operatorname{soc} P_h$ is equal to the number of equivalence classes in $\mathscr{C}_h$.*

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