

# CONSTRUCTING A COMPLETELY INTEGRABLE SYSTEM VIA ALGEBRO-GEOMETRIC DATA

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ABSTRACT. We use the algebro-geometric data given by a genus 2 Jacobian, a curve and a line bundle on the Jacobian, and the action of a group of translates on the theta sections of this line bundle, to reconstruct an integrable system: the geodesic motion on  $SO(4)$ , metric II (so termed after Adler and van Moerbeke).

## 1. INTRODUCTION

Since the early days of Mechanics, finite dimensional integrable systems have been related to algebraic geometry. That is shown in examples like the rigid body cases or Jacobi's geodesic motion on the ellipsoid.

Most of the known examples are a particular class of integrable systems, whose solutions, expressible in terms of theta functions, are associated to abelian varieties (i.e. complex tori in projective space) with divisors (codimension one subvarieties) on them, and the Hamiltonian flows are linear on these abelian varieties. Roughly speaking, such systems are called algebraic completely integrable (a.c.i.).

Starting from an a.c.i. system we can produce algebro-geometric data like a divisor on an abelian variety (the divisor at infinity), its polarization, the linear system associated with this divisor, and a finite group of translations leaving invariant the divisor and the holomorphic vector fields.

We can ask whether it is possible to go in the backward direction and view the integrable system as deformation of a suitable geometric data.

In this paper we show how to recover an algebraic completely integrable system from algebro-geometric data. The regarded system is a geodesic motion on  $SO(4)$  (the metric II case studied by Adler and van Moerbeke in [1],[2]). Here, the commuting complexified flows linearize on Jacobians  $A_\alpha$  of genus 2 curves, upon adding to the complexified invariant manifolds, divisors  $D_\alpha$  (curves called  $SO(4)$  divisors) at infinity, with a precise pattern (they consist of four translates of the theta divisor intersecting at triple points, which are half-periods, like the figure below). One considers the action of the group of translations leaving invariant  $D_\alpha$  and the sections of the linear system of functions blowing up once at  $D_\alpha$  and vanishing at least twice at the triple points. Surprisingly, this gives the sections for the right phase space. In the projective closure of the complexified phase space  $\mathbb{C}^6$  (i.e.  $\mathbb{P}^6$ ) the invariant manifolds compactify to set theoretical complete intersections, into which the Jacobians map birationally, so that a Jacobian minus its divisor at infinity is isomorphic to the respective (complexified) affine invariant manifold.

The question arises whether it is possible to reconstruct such a system by providing its Jacobian  $A_\alpha$ , its configuration divisor at infinity  $\mathcal{D}_\alpha$  (for instance an  $SO(4)$  divisor as explained in Theorem 1), a group  $G = \mathbb{Z}_2 \times \mathbb{Z}_2$  of symmetries (which essentially are translates by half periods in the Jacobi variety), and a line bundle  $L_\alpha \rightarrow A_\alpha$ , whose sections are projective coordinates of the ambient space  $\mathbb{P}^6$ . We provide such a construction by finding a convenient basis of theta functions for the above data, with the property that the same theta functions (up to permissible change of basis) are sections of the line bundle

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1991 *Mathematics Subject Classification.* 58F07, 14K25.

*Key words and phrases.* integrable systems,

$L_\alpha \rightarrow A_\alpha$ , and in which  $G$  has a “nice” representation for all Jacobians. We deduce quadratic equations for the image of the Jacobians in  $\mathbb{P}(H^0(A_\alpha, L_\alpha)^*) = \mathbb{P}^6$ , in terms of certain parameters. Also, we find the quadratic equations for the holomorphic vector fields in terms of this basis.

The quadratic equations that describe the image of  $A_\alpha$  in  $\mathbb{P}^6$  contain natural parameters  $\alpha$  which serve as the moduli data. Now, one of the theta sections, say  $\Theta$ , will cut out on each  $A_\alpha$  the divisor at infinity, and in the affine variables  $(Z_i/\Theta)$  in  $\mathbb{C}^6$  we obtain a smooth piece for each generic  $\alpha$  (the affine piece). The question is whether such a family of affine surfaces put together in  $\mathbb{C}^6$  has a Poisson structure so that they are the complexified invariant manifolds for a Hamiltonian structure. Indeed, such a Poisson structure, polynomial in the affine variables, is uniquely determined up to a Poisson transformation and choice of Casimirs.

The above considerations lead us to the following theorem that will be shown along the paper.

**Theorem 1.** *Consider the family  $\{A_\alpha\}$  of genus 2 Jacobians and divisors  $\{\mathcal{D}_\alpha = \Theta_0 + \Theta_1 + \Theta_2 + \Theta_3\}$  on them, such that  $\Theta_i$  is a translate by a half period  $e_i$  of the theta divisor, and the  $\Theta_i$ 's intersect into four triple points  $\{e_0, \dots, e_3\}$ . This family posses a group of translations  $G = \mathbb{Z}_2 \times \mathbb{Z}_2$  leaving invariant each  $\mathcal{D}_\alpha$  and  $A_\alpha \setminus \mathcal{D}_\alpha$ . Let  $H^0(A_\alpha, \mathcal{D}_\alpha - 2e_0 - 2e_1 - 2e_2 - 2e_3)$  be the space of sections linearly equivalent to  $\mathcal{D}_\alpha$  that vanish at least twice at the  $e_i$ 's. Then, this space has dimension 7 and decomposes into odd and even parts, with respect to the  $(-1)$ -involution, of dimensions 1 and 6 respectively; the odd section vanishing at  $\mathcal{D}_\alpha$ .*

One can find a basis  $\{v_i = \frac{Z_i}{\Theta}\}$ , with  $\Theta$  odd and the  $Z_i$ 's even, such that  $G$  acts as in the following table, for all generic Jacobians  $A_\alpha$ :

Table I						
	$v_1$	$v_2$	$v_3$	$v_4$	$v_5$	$v_6$
$\sigma$	$-v_2$	$-v_1$	$-v_3$	$-v_4$	$-v_6$	$-v_5$
$\tau$	$-v_1$	$-v_2$	$v_4$	$v_3$	$v_6$	$v_5$

The image of  $A_\alpha$  in  $\mathbb{P}^6 = \mathbb{P}(H^0(A_\alpha, \mathcal{D}_\alpha - 2e_0 - 2e_1 - 2e_2 - 2e_3)^*)$  is a set theoretical complete intersection of four quadrics  $q_1 = v_1v_2 = \alpha_1$ ,  $q_2 = v_3v_4 = \alpha_2$ ,  $q_3 = v_5v_6 = \alpha_3$  and another quadric  $q_4 = \alpha_4$ . If the quadrics  $q_2$  and  $q_3$  are chosen as Casimirs for a polynomial Poisson structure in the affine variables  $v_i$ 's, then, there is an integrable system with nontrivial hamiltonians  $X_{q_1}$ ,  $X_{q_4}$ , and linear Poisson matrix. This system is, up to Poisson isomorphism, the metric II case of the geodesic motion on  $SO(4)$  studied by Adler and Van Moerbeke [1], [9].

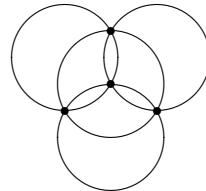


Figure I

We use the normalized action of  $G$  and the tangency condition  $X(q_i) = 0$  for any holomorphic vector field  $X$  and equation  $q_i$  to find all the equations of the image of the Jacobian in  $\mathbb{P}^6$ . The moduli parameters appear in the freedom we have to choose different basis of sections for  $L_\alpha \rightarrow A_\alpha$  so that the action of  $G$  on the equations of the variety and holomorphic vector fields are in a “normal form” (i.e. roughly speaking, this means nice expressions without parameters).

The theta functions that would do the trick are products of genus 2 half integer characteristic theta functions belonging to  $H^0(A_\alpha, \mathcal{D}_\alpha - 2e_0 - 2e_1 - 2e_2 - 2e_3)$  (i.e. the system of theta functions whose zero locus is linearly equivalent to  $4\theta$  and vanish at least twice at the points  $e_0, e_1, e_2, e_3$ , which are half periods and triple points of the configuration divisor at infinity  $\mathcal{D}_\alpha$ ). Such linear system was suggested in [3]. An explicit computation of the dimension of this space and higher powers of it did not come easily

until a procedure by Bauer [4] was available. He considers the pull back of linear systems on  $A$  to the surface  $A_\Omega$  (the blow up of the abelian surface  $A$  at the 16 half periods). The computation can be done by using a bijection between symmetric curves on  $A_\Omega$  and curves on the K3 surface  $K_\Omega = A_\Omega/\{1, -1\}$  and then Riemann-Roch formula and a theorem by Kodaira as explained in [10]. Some of the algebraic geometric assertions were already done by Szemberg [12] in his thesis. However, the step to reconstruct the integrable system is new and uses techniques already present in [9]. We are grateful to W. Barth, Th. Bauer and T. Szemberg for showing us their approach to quadrics in  $\mathbb{P}^6$  and to Pol Vanhaecke for useful discussions about this problem.

## 2. PRELIMINARIES

Let  $p : \tilde{A} \rightarrow A$  be the blow up of the abelian surface  $A$  at the 16 half periods  $\{e_0, \dots, e_{15}\}$ , and  $\{E_i = p^{-1}e_i, i = 0, \dots, 15\}$  the 16  $(-1)$ -curves. Let us denote by  $(-1)_A$  the reflexion with respect to the origin in  $A$ . This reflexion induces an involution  $(-1)_{\tilde{A}}$  in  $\tilde{A}$ . The quotient by the action of this involution  $\tilde{K}_A = \tilde{A}/\langle (-1)_{\tilde{A}} \rangle$  is a smooth K3 surface and the projection  $\pi : \tilde{A} \rightarrow \tilde{K}_A$  has the disjoint union of the 16  $(-1)$ -curves  $E_i$  as ramification divisor, and the disjoint union of the 16  $(-2)$ -curves  $B_i = \pi(E_i)$  as branch locus.

Let  $\mathcal{D}$  be a curve in  $A$  with multiplicities  $\mu_i$ 's at the half periods  $e_i$ 's and let  $\nu_i$ 's be given nonnegative integers for each  $i$ . We start from a symmetric divisor  $\mathcal{D}$  (given by an even or odd section in  $H^0(A, [\mathcal{D}])$ ) and consider the line bundle  $\mathcal{L}_\nu$  on  $\tilde{A}$  generated by  $p^*(\mathcal{D}) - \sum \nu_i E_i = \hat{\mathcal{D}} + \sum (\mu_i - \nu_i) E_i$  ( $\hat{\mathcal{D}}$  = strict transform of  $\mathcal{D}$ ). Then  $\mathcal{L}_\nu$  is symmetric with respect to  $(-1)_{\tilde{A}}$  (i.e.  $\mathcal{L}_\nu \simeq (-1)_{\tilde{A}}^* \mathcal{L}_\nu$ ) if the  $\nu_i$ 's have the same parity (prop. 3.1 [10]). The space  $H^0(\tilde{A}, \mathcal{L}_\nu)$  is identified under  $p_*$  with the sections in  $H^0(A, [\mathcal{D}])$  that vanish to order  $\geq \nu_i$  at the  $e_i$ 's. Also, if  $\mathcal{D}'$  and  $\nu_i'$ 's are another divisor and positive integers, one has the intersection formula  $(p^*(\mathcal{D}) - \sum \nu_i E_i) \cdot (p^*(\mathcal{D}') - \sum \nu_i' E_i) = \mathcal{D} \cdot \mathcal{D}' - \sum \nu_i \nu_i'$ .

Let us consider a  $\mathcal{L}_\nu$  symmetric and let  $(-1)_{\mathcal{L}_\nu}$  be the involution of  $\mathcal{L}_\nu$  over  $(-1)_{\tilde{A}}$  induced by the corresponding involution  $(-1)_{[\mathcal{D}]}$  of  $[\mathcal{D}]$  over  $(-1)_A$ . The action of  $(-1)_{[\mathcal{D}]}$  on the fibers over the half periods  $e_i$ 's is multiplication by  $s_i = +1$  (in whose case the half period  $e_i$  is called even) or by  $s_i = -1$  (where  $e_i$  is called odd).

There is an involution on sections  $\varphi : H^0(\mathcal{L}_\nu) \rightarrow H^0(\mathcal{L}_\nu)$  defined by  $\varphi(s) = (-1)_{\mathcal{L}_\nu} s (-1)_{\tilde{A}}$ . This involution splits  $H^0(\mathcal{L}_\nu)$  into  $(+1)$  and  $(-1)$  eigenspaces :  $H^0(\mathcal{L}_\nu)^\pm$ . Moreover,  $\pi_* \mathcal{L}_\nu = \mathcal{M}^+ \oplus \mathcal{M}^-$  is a rank-2 bundle over  $\tilde{K}_A$  which decomposes, with regard to  $s \mapsto (-1)_{\mathcal{L}_\nu} s (-1)_{\tilde{A}}$ , into  $(+1)$  and  $(-1)$  line bundles  $\mathcal{M}^\pm$ , and there are isomorphisms  $H^0(\tilde{K}_A, \mathcal{M}^\pm) \simeq H^0(\tilde{A}, \mathcal{L}_\nu)^\pm$  [4].

Let  $\hat{\mathcal{D}} = \hat{\mathcal{D}} + \sum (\mu_i - \nu_i) E_i$  be a symmetric effective curve in the linear system  $|\mathcal{L}_\nu|$ , then, one can associate a curve in  $\tilde{K}_A$  as follows:  $\tilde{\mathcal{C}} = \pi(\hat{\mathcal{D}}) + \sum [\frac{\mu_i - \nu_i}{2}] B_i$ , where the square brackets is the integer part,  $B_i = \pi_* E_i = \pi(E_i)$  and  $\pi(\hat{\mathcal{D}})$  is the image of  $\hat{\mathcal{D}}$ . Starting from an odd or even curve  $\mathcal{D}$  in the linear system  $|\mathcal{D}|$  on  $A$ , we construct the curve  $\tilde{\mathcal{C}}$  on  $\tilde{K}_A$  by this way associated to the divisor  $p^*(\mathcal{D}) - \sum \nu_i E_i$ . Then, by proposition 3.1 in [10] we have  $\mathcal{M}^+ = \mathcal{O}_{\tilde{K}_A}(\tilde{\mathcal{C}})$  if  $\nu$  and  $\mathcal{D}$  have the same parity, and  $\mathcal{M}^- = \mathcal{O}_{\tilde{K}_A}(\tilde{\mathcal{C}})$  otherwise.

The Riemann-Roch formula for a K3 surface  $\tilde{K}_A$  and an effective curve  $\mathcal{C}$  on it goes as follows:

$$(1) \quad h^0(\mathcal{C}) = \frac{\mathcal{C}^2}{2} + 2 + h^1(\mathcal{C}),$$

where  $h^1(\mathcal{C}) = m - 1$  and  $m$  is the number of connected components of  $\mathcal{C}$  (see [10]). This gives an effective way of computing the dimensions of  $H^0(\mathcal{L}_\nu^{\otimes n})$  since we have an isomorphism  $H^0(\mathcal{L}_\nu) \simeq H^0(\mathcal{M}^+) \oplus H^0(\mathcal{M}^-)$ . Also, by Bauer [4], we have the formulas

$$(2) \quad \pi_* \mathcal{M}^\pm = \mathcal{L}_\nu \otimes [Z^\mp]^{-1},$$

where  $Z^\pm = \sum_{s_i = \pm 1} E_i$  and  $s_i$  the parity of the half period  $e_i$ .

**Example 1.** Let  $\mathcal{D}$  be the divisor of Figure I on a genus two Jacobian. This curve contains all 16 half periods. The ones that are triple points are labeled  $\{e_0, e_1, e_2, e_3\}$ .  $\mathcal{D}$  is an odd divisor in the linear system

$|4\Theta|$  with respect to the  $(-1)$  involution that fixes the half periods (Lemma 7.7.1 [6]). Now, consider the bundle  $\mathcal{L}_\nu = [p^*(\mathcal{D}) - \sum_{i=0}^3 2E_i]$  on  $\tilde{A}$ . We want to compute the dimensions of  $H^0(\tilde{A}, \mathcal{L}_\nu)^\pm$ .

- (1) Let us find  $h^0(\mathcal{L}_\nu)^-$ . We have that  $p^*(\mathcal{D}) = \sum_{i=0}^3 D_i + \sum_{i=0}^3 3E_i + \sum_{i=4}^{15} E_i$ , where the  $D_i$ 's represent the genus two curves. Then, the curve  $\mathcal{D}^- = \sum_{i=0}^3 D_i + \sum_{i=0}^{15} E_i$  belongs to  $|\mathcal{L}_\nu|^-$ , and the associated curve in  $\tilde{K}_A$  is  $\mathcal{C}^- = \sum_{i=0}^3 \pi(D_i)$ . One obtains  $\pi^*\mathcal{C}^- = p^*\mathcal{D} - (\sum_{i=0}^3 3E_i + \sum_{i=4}^{15} E_i)$ . Therefore, by formula (1),  $h^0(\mathcal{L}_\nu)^- = h^0([\mathcal{C}^-]) = \frac{(\pi^*\mathcal{C}^-)^2}{4} + 2 + h^1(\mathcal{C}^-) = -4 + 2 + 3 = 1$ .
- (2) Let  $\mathcal{D}^+ \in |p^*\mathcal{D} - \sum_{i=0}^3 2E_i|^+$ . First, one constructs the divisors  $Z^\pm$  as above. Taking into account that  $\mathcal{D}$  is a totally symmetric divisor and that the parity of the origin is even, all periods turn out to be even. So, we obtain  $Z^+ = \sum_{i=0}^{15} E_i$  and  $Z^- = 0$ . Let us denote by  $B^\pm$  the direct image of  $Z^\pm$  respectively. By applying  $\pi_*$  at the level of curves in formulas (2) we get the linear equivalence  $2\mathcal{C}^+ + B^- \sim 2\mathcal{C}^* + B^+ - \sum_{i=0}^3 2B_i$ , where  $\mathcal{C}^+$  and  $\mathcal{C}^*$  are the associated curves to  $\mathcal{D}^+$  and  $p^*\mathcal{D}$  respectively. This leads to the equivalence  $2\mathcal{C}^+ \sim 2\sum_{i=0}^3 \pi(D_i) + \sum_{i=0}^{15} B_i$ , or by pulling back to  $\tilde{A}$ :  $\pi^*\mathcal{C}^+ \sim \sum_{i=0}^3 D_i + \sum_{i=0}^{15} E_i = p^*\mathcal{D} - \sum_{i=0}^3 2E_i$ . Therefore, we calculate  $(\mathcal{C}^+)^2 = \frac{(\pi^*\mathcal{C}^+)^2}{2} = 8$ . It follows that such a curve on a K3 surface has  $h^1(\mathcal{C}^+) = 0$  [11]. Then, by (1)  $h^0(\mathcal{L}_\nu)^+ = h^0(\mathcal{C}^+) = 6$ .
- (3) From 1 and 2 we conclude that the space  $H^0(\tilde{A}, \mathcal{L}_\nu)$  splits into a 1-dimensional odd piece and a 6-dimensional even part.

**Example 2.** Compute the dimensions of the spaces  $H^0(\mathcal{L}_\nu^{\otimes 2})^\pm$ . We write  $\pi_*\mathcal{L}_\nu^{\otimes 2} = \mathcal{M}_2^+ \oplus \mathcal{M}_2^-$  for the decomposition into  $\pm 1$  bundles of  $\pi_*\mathcal{L}_\nu^{\otimes 2}$ . By 1. in the above example, we have that  $\pi^*\mathcal{M}^+ \simeq \mathcal{L}_\nu$ . Therefore,  $\pi_*(\mathcal{L}_\nu^{\otimes 2}) \simeq \pi_*(\mathcal{L}_\nu \otimes \pi^*\mathcal{M}^+) \simeq \pi_*(\mathcal{L}_\nu) \otimes \mathcal{M}^+ \simeq (\mathcal{M}^+ \oplus \mathcal{M}^-) \otimes \mathcal{M}^+ \simeq (\mathcal{M}^+)^{\otimes 2} \oplus (\mathcal{M}^- \otimes \mathcal{M}^+)$ . It follows that  $\mathcal{M}_2^+ \simeq (\mathcal{M}^+)^{\otimes 2} = [2\mathcal{C}^+]$ , and  $\mathcal{M}_2^- \simeq \mathcal{M}^- \otimes \mathcal{M}^+ = [\mathcal{C}^- + \mathcal{C}^+]$ , because  $(\mathcal{M}^+)^{\otimes 2}$  and  $\mathcal{M}^- \otimes \mathcal{M}^+$  are eigenspaces under the action of the involution. The selfintersection numbers of the divisors representing  $\mathcal{M}^\pm$  are bigger than 8. So, in both cases  $h^1(\mathcal{M}^\pm) = 0$  [11]. Thus, by the Riemann-Roch formula we get  $h^0(\mathcal{M}_2^+) = 4\frac{(\mathcal{C}^+)^2}{2} + 2 = 18$ , and  $h^0(\mathcal{M}_2^-) = \frac{(\mathcal{C}^+)^2 + (\mathcal{C}^-)^2 + 2\mathcal{C}^+ \cdot \mathcal{C}^-}{2} + 2 = 6$ . In this case, the dimension of  $H^0(\tilde{A}, \mathcal{L}_\nu^{\otimes 2})$  turns out to be  $h^0(\mathcal{M}_2^+) + h^0(\mathcal{M}_2^-) = 18 + 6 = 24$ .

### 3. GENUS TWO THETA FUNCTIONS.

Let  $\tau$  be the  $2 \times 2$  Riemann matrix of a (generic and principally polarized) genus 2 Jacobian. A pair of real vectors  $(m, m^*)$  is associated univocally with the point  $m^* + m\tau$  of  $\mathbb{C}^2$ .

For the pair of row vectors  $(m, m^*)$  (called characteristics) we define the classical theta functions [6, §8.5] as (1) below, where  $e(z) = \exp(2\pi iz)$ ,  $z \in \mathbb{C}$ . They have the properties (2),(3),(3'),(4).

- (1)  $\vartheta_{m, m^*}(\tau, \zeta) = \sum_{\psi \in \mathbb{Z}^n} e(\frac{1}{2}(\psi + m)\tau^t(\psi + m) + (\psi + m)^t(\zeta + m^*))$
- (2)  $\vartheta_{m, m^*}(\tau, -\zeta) = \vartheta_{-m, -m^*}(\tau, \zeta)$
- (3)  $\vartheta_{m+\psi, m^*+\psi^*}(\tau, \zeta) = e(m^t\psi^*)\vartheta_{m, m^*}(\tau, \zeta)$ , for  $\psi, \psi^* \in \mathbb{Z}^n$
- (3')  $\vartheta_{m, m^*}(\tau, \zeta + u\tau + u^*) = e(-\frac{1}{2}u\tau^t u - u^t(\zeta + u^*))e(-u^t m^*)\vartheta_{m+u, m^*+u^*}(\tau, \zeta)$ .

We also use the customary notation  $\vartheta_{m, m^*}(\tau, \zeta) = \vartheta \left[ \begin{matrix} m \\ m^* \end{matrix} \right] (\tau, \zeta)$ , and agree to represent the point  $m^* + m\tau$  either by  $\left[ \begin{matrix} m \\ m^* \end{matrix} \right]$  or  $\left\{ \begin{matrix} m \\ m^* \end{matrix} \right\}$ , when  $\tau$  is fixed.

If  $\left\{ \begin{matrix} m \\ m^* \end{matrix} \right\} \in \frac{1}{2}\mathbb{Z}^{2g}/\mathbb{Z}^{2g}$  is a half period, then we have the formula (Prop. 3.14 - Ch II. p. 167 [8]).

- (4)  $\vartheta_{m, m^*}(\tau, -\zeta) = e(2m^t m^*)\vartheta_{m, m^*}(\tau, \zeta) = e_*(m^* + m\tau)\vartheta_{m, m^*}(\tau, \zeta)$

There are  $2^{2g}$  half periods on an abelian variety of dimension  $g$ . We say that a half period of characteristic  $\begin{Bmatrix} m \\ m^* \end{Bmatrix}$  is odd (even) if the factor  $e_* \left( \begin{Bmatrix} m \\ m^* \end{Bmatrix} \right)$  is negative (positive).

For a genus 2 Jacobian the even half period characteristics are given by

$$\begin{aligned} e_{35} &= \begin{Bmatrix} 0 & 0 \\ 0 & 0 \end{Bmatrix}, e_{23} = \begin{Bmatrix} 0 & 0 \\ 1/2 & 0 \end{Bmatrix}, e_{45} = \begin{Bmatrix} 0 & 0 \\ 0 & 1/2 \end{Bmatrix}, e_{13} = \begin{Bmatrix} 1/2 & 0 \\ 0 & 0 \end{Bmatrix} \\ e_{12} &= \begin{Bmatrix} 0 & 1/2 \\ 0 & 0 \end{Bmatrix}, e_{25} = \begin{Bmatrix} 1/2 & 1/2 \\ 0 & 0 \end{Bmatrix}, e_{14} = \begin{Bmatrix} 1/2 & 0 \\ 0 & 1/2 \end{Bmatrix}, e_{15} = \begin{Bmatrix} 0 & 1/2 \\ 1/2 & 0 \end{Bmatrix} \\ e_{24} &= \begin{Bmatrix} 0 & 0 \\ 1/2 & 1/2 \end{Bmatrix}, e_{34} = \begin{Bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{Bmatrix}. \end{aligned}$$

While the odd characteristics are the following

$$\begin{aligned} e_0 &= \begin{Bmatrix} 1/2 & 1/2 \\ 1/2 & 0 \end{Bmatrix}, e_1 = \begin{Bmatrix} 1/2 & 1/2 \\ 0 & 1/2 \end{Bmatrix}, e_2 = \begin{Bmatrix} 0 & 1/2 \\ 1/2 & 1/2 \end{Bmatrix}, e_3 = \begin{Bmatrix} 1/2 & 0 \\ 1/2 & 1/2 \end{Bmatrix} \\ e_4 &= \begin{Bmatrix} 1/2 & 0 \\ 1/2 & 0 \end{Bmatrix}, e_5 = \begin{Bmatrix} 0 & 1/2 \\ 0 & 1/2 \end{Bmatrix} \end{aligned}$$

It follows that the theta functions  $\vartheta[e_i]$  are odd functions with respect to the involution, while the  $\vartheta[e_{ij}]$ 's are even. One has the relations  $\bar{e}_i + \bar{e}_j = \bar{e}_{ij} + \bar{e}_0$  and  $\sum_{i=0}^5 \bar{e}_i = 0$  on the Jacobian.

The odd half periods are the Weierstrass points of the theta divisor  $\{\zeta : \vartheta_{0,0}(\tau, \zeta) = 0\} = \Gamma$  and  $\Gamma$  is also the genus 2 curve into its Jacobian.

#### 4. THE ACTION OF A GROUP.

Any  $\text{SO}(4)$  divisor  $\mathcal{D}$  can be constructed from one of the 16 symmetric curves  $\{\Gamma + \bar{e}_i, \Gamma + \bar{e}_{ij}\}$  by acting on it with a particular group of translates  $G = \mathbf{Z}_2 \times \mathbf{Z}_2 = \{1, t_1, t_2, t_3 = t_1 + t_2\}$ , in such a way that  $G$  fixes the triple points. Actually, the four triple points define translates by half periods that coincide with  $G$ , once one of the these points is chosen as origin. It is easy to see that the same divisor  $\mathcal{D}$  is obtained from a single symmetric curve, say  $\Gamma$ , by letting  $G$  act on it. Moreover,  $\Gamma$  contains three half periods that together define an origin and  $G$ . Therefore, there are  $80 = \binom{6}{3} \cdot 4$  ways of giving an origin and a group.

Let  $\Gamma$  be defined as in the previous section. Call  $\Theta_3 = t_3(\Gamma)$ ,  $\Theta_2 = t_2(\Gamma)$ ,  $\Theta_1 = t_1(\Gamma)$ ,  $\Theta_0 = \Gamma$ , and  $e_0, e_1, e_2, e_3$  the triple points of  $\mathcal{D} = \Theta_0 + \Theta_1 + \Theta_2 + \Theta_3$ . Then,

**Proposition 2.** [12]. *A basis for  $H^0(\tilde{A}, [p^*(\mathcal{D}) - \sum_{i=0}^3 2E_i])$  is given by the odd section  $s_0 s_1 s_2 s_3$  and the even sections  $s_0^2 s_1^2, s_0^2 s_2^2, s_1^2 s_2^2, s_1^2 s_3^2, s_2^2 s_3^2$ , where  $s_0, s_1, s_2, s_3$  are theta functions vanishing on  $\Theta_0, \Theta_1, \Theta_2, \Theta_3$  respectively.*

We pick three points  $e_0, e_1, e_2$  in  $\Gamma = \overline{\{\zeta : \vartheta(\tau, \zeta) = 0\}}$ ,  $e_0$  as origin and consider the group  $G$  generated by  $e_1 - e_0, e_2 - e_0$ . This has an extra element  $e_{12} - e_0$ . We write

$$\begin{aligned} t_1 &= e_1 - e_0 = \begin{Bmatrix} 0 & 0 \\ -1/2 & 1/2 \end{Bmatrix} = e_{24} + \begin{Bmatrix} 0 & 0 \\ -1 & 0 \end{Bmatrix} \\ t_2 &= e_2 - e_0 = \begin{Bmatrix} -1/2 & 0 \\ 0 & 1/2 \end{Bmatrix} = e_{14} + \begin{Bmatrix} -1 & 0 \\ 0 & 0 \end{Bmatrix} \\ t_3 &= e_{12} - e_0 = \begin{Bmatrix} -1/2 & 0 \\ -1/2 & 0 \end{Bmatrix} = e_4 + \begin{Bmatrix} -1 & 0 \\ -1 & 0 \end{Bmatrix} \end{aligned}$$

and consider the translates of the  $\vartheta$ -divisor  $\Gamma$  by these elements of  $\mathbf{Z}_2 \times \mathbf{Z}_2$ . These translates are given by the sections

$$s_o = \vartheta[e_{35}](\tau, \zeta) = \vartheta(\tau, \zeta), s_1 = \vartheta[e_{24}](\tau, \zeta), s_2 = \vartheta[e_{14}](\tau, \zeta), \text{ and } s_3 = \vartheta[e_4](\tau, \zeta).$$

Thus, the zero locus of  $\Theta(\tau, \zeta) = \vartheta[e_{35}]\vartheta[e_{24}]\vartheta[e_{14}]\vartheta[e_4]$  gives a typical  $\text{SO}(4)$  divisor. As  $\Theta$  is the product of 3 even sections and one odd section,  $\Theta$  is odd.

We will make a table with the action of  $t_x$  defined by  $t_x\vartheta(\zeta) = \vartheta(\zeta + x)$ . This action is associated with the Schrodinger representation of the Theta group (see [6] and [9] for those matters).

	$s_o = \vartheta[e_{35}](\tau, \zeta)$	$s_1 = \vartheta[e_{24}]$	$s_2 = \vartheta[e_{14}]$	$s_3 = \vartheta[e_4]$
(3) $t_1$	$\vartheta[e_{35}](\tau, \zeta + e_1 - e_0) = \vartheta[e_{24}]$	$\vartheta[e_{35}]$	$-\vartheta[e_4]$	$\vartheta[e_{14}]$
$t_2$	$\vartheta[e_{35}](\tau, \zeta + e_2 - e_0) = f(\zeta)\vartheta[e_{14}]$	$f(\zeta)i\vartheta[e_4]$	$f(\zeta)\vartheta[e_{35}]$	$f(\zeta)i\vartheta[e_{24}]$
$t_3$	$\vartheta[e_{35}](\tau, \zeta + e_{12} - e_0) = -g(\zeta)\vartheta[e_4]$	$g(\zeta)i\vartheta[e_{14}]$	$g(\zeta)\vartheta[e_{24}]$	$g(\zeta)i\vartheta[e_{35}]$

Let  $u_{ij} = s_i^2 s_j^2$ ,  $i < j$ , be the even sections of  $[p^*(\mathcal{D}) - \sum_{i=0}^3 2E_i]$ . Then  $G$  acts as follows,

	$u_{01}$	$u_{02}$	$u_{03}$	$u_{12}$	$u_{13}$	$u_{23}$	$\Theta$
(4) $t_1$	$u_{01}$	$u_{13}$	$u_{12}$	$u_{03}$	$u_{02}$	$u_{23}$	$-\Theta$
$t_2$	$-f^4 u_{23}$	$f^4 u_{02}$	$-f^4 u_{12}$	$-f^4 u_{03}$	$f^4 u_{13}$	$-f^4 u_{01}$	$-f^4 \Theta$
$t_3$	$-g^4 u_{23}$	$g^4 u_{13}$	$-g^4 u_{03}$	$-g^4 u_{12}$	$g^4 u_{02}$	$-g^4 u_{01}$	$g^4 \Theta$

This action is similar to the one described in [12] for a different basis. We needed to go down to the classical theta functions to make the action on  $\Theta$  explicit. If we put  $v_3 = \frac{u_{01}}{\Theta}$ ,  $v_4 = \frac{u_{23}}{\Theta}$ ,  $v_1 = \frac{u_{02}}{\Theta}$ ,  $v_2 = \frac{u_{13}}{\Theta}$ ,  $v_5 = \frac{u_{03}}{\Theta}$ ,  $v_6 = \frac{u_{12}}{\Theta}$ , we obtain Table I of the theorem, with  $\sigma = t_1$  and  $\tau = t_2$ .

Since this action admits a separable rescaling, that is, a rescaling in each set of variables:  $\{v_1, v_2\}$ ,  $\{v_3, v_4\}$ , and  $\{v_5, v_6\}$ . Then, by the very definition of the  $v_i$ 's, we have relations  $v_1 v_2 = c_1$ ,  $v_3 v_4 = c_2$ ,  $v_5 v_6 = c_3$ , where now, new  $v_i$ 's are substituted in place of the old ones rescaled. These equations are three independent integrals with three free parameters for the would be system.

## 5. THE QUADRATIC VECTOR FIELDS INVARIANTS UNDER THE GROUP.

The above sections  $\{u_{01}, u_{02}, u_{03}, u_{12}, u_{13}, u_{23}, \Theta\}$  of  $H^0(p^*(\mathcal{D}) - 2E_0 - 2E_1 - 2E_2 - 2E_3)$ , yield a rational map  $A \rightarrow \tilde{A} \rightarrow \mathbb{P}^6$  (which is not defined at  $e_0, e_1, e_2$  and  $e_3$ ). The degree of the image is  $(p^*\mathcal{D} - \sum 2E_i)^2 = \mathcal{D}^2 + \sum 4E_i^2 = 32 - 4 \cdot 4 = 16$ .

Now, let  $\mathcal{L}$  be the line bundle associated to the divisor  $p^*\Theta - \sum_{i=0}^3 2E_i$ . As said before  $h^0(\mathcal{L}) = 7$  and  $h^0(\mathcal{L})^+ = 6$ ,  $h^0(\mathcal{L})^- = 1$ . The vector fields in the affine variables  $v_i$ 's are given by Wronskians of an even section with the odd section, and these define sections in  $H^0(\mathcal{L}^{\otimes 2})^+$ . This follows from properties of wronskians [9]:  $W_Y : H^0(\mathcal{L})^+ \otimes H^0(\mathcal{L})^- \rightarrow H^0(\mathcal{L}^{\otimes 2})^+$ . Thus, the vector fields can be written quadratically in terms of the even sections, which follows from the proposition:

**Proposition 3.** *The map  $S^2 H^0(\mathcal{L})^+ \rightarrow H^0(\mathcal{L}^{\otimes 2})^+$  is surjective, and  $H^0(\mathcal{L})^+ \otimes H^0(\mathcal{L})^- \rightarrow H^0(\mathcal{L}^{\otimes 2})^-$  is an isomorphism.*

*Proof:* We have canonical isomorphisms  $H^0(\tilde{A}, \mathcal{L})^\pm \cong H^0(\tilde{K}_A, \mathcal{M}^\pm)$  and  $H^0(\tilde{A}, \mathcal{L}^2)^\pm \cong H^0(\tilde{K}_A, \mathcal{M}_2^\pm)$  and consider the conclusions of example 2.

The hypothesis of Theorem 1 by Saint-Donat [11] is checked out in [12]. Therefore we conclude that there is a surjective morphism  $S^2 H^0(\tilde{K}_A, \mathcal{M}^+) \rightarrow H^0(\tilde{K}_A, (\mathcal{M}^+)^{\otimes 2}) \cong H^0(\tilde{K}_A, \mathcal{M}_2^+) \cong H^0(\mathcal{L}^{\otimes 2})^+$  from which follows the first part of the proposition.

For the second part, notice that the sections  $u_{ij} \otimes \Theta$ ,  $i < j$ , of  $H^0(\mathcal{L}^{\otimes 2})^-$  are linearly independent on  $A$ .  $\square$

**Remark 1.** The map  $S^2H^0(\mathcal{L}) = S^2(H^0(\mathcal{L})^+ \oplus H^0(\mathcal{L})^-) = S^2H^0(\mathcal{L})^+ \oplus (H^0(\mathcal{L})^+ \otimes H^0(\mathcal{L})^-) \oplus S^2H^0(\mathcal{L})^- \rightarrow H^0(\mathcal{L}^{\otimes 2})^+ \oplus H^0(\mathcal{L}^{\otimes 2})^-$  is surjective, and this means there are  $4 = s^2h^0(\mathcal{L}) - h^0(\mathcal{L}^{\otimes 2}) = 28 - 24$  quadratic equations defining the image of  $\tilde{A}$  in  $\mathbb{P}^6$ .

Since the nontrivial holomorphic vector fields on the Jacobians have to be tangent to the affine variety defined by the quadrics  $q_1 = v_1v_2 = c_1$ ,  $q_2 = v_3v_4 = c_2$ ,  $q_3 = v_5v_6 = c_3$ , and invariant under the translations  $\sigma$ ,  $\tau$ , we get

$$(5) \quad \begin{aligned} \dot{v}_1 &= v_1(\alpha_3(v_3 + v_4) + \alpha_5(v_5 + v_6)) = v_1f_1 \\ \dot{v}_2 &= -v_2(\alpha_3(v_3 + v_4) + \alpha_5(v_5 + v_6)) = -v_2f_1 \\ \dot{v}_3 &= v_3(\beta_1(v_1 - v_2) + \beta_5(v_5 - v_6)) = v_3f_2 \\ \dot{v}_4 &= -v_4(\beta_1(v_1 - v_2) + \beta_5(v_5 - v_6)) = -v_4f_2 \\ \dot{v}_5 &= v_5(\gamma_1(v_1 + v_2) + \gamma_3(v_3 - v_4)) = v_5f_3 \\ \dot{v}_6 &= -v_6(\gamma_1(v_1 + v_2) + \gamma_3(v_3 - v_4)) = -v_6f_3 \end{aligned}$$

which gives several two dimensional families of vector fields nonvanishing on each variable.

## 6. THE EXTRA QUADRATIC INVARIANT.

We want to find the remaining invariant under the group  $G$  which is killed by the quadratic vector fields. Such an invariant must be of the form

$$(6) \quad \begin{aligned} q_4 &= \alpha(v_1^2 + v_2^2) + \beta(v_3^2 + v_4^2) + \gamma(v_5^2 + v_6^2) \\ &+ \delta(v_1 + v_2)(v_3 - v_4) + \epsilon(v_1 - v_2)(v_5 - v_6) + \eta(v_3 + v_4)(v_5 + v_6). \end{aligned}$$

This has to satisfy the equation  $\dot{q}_4 = 0$  under all vector fields, which leads to the linear system

$$\begin{aligned} 2\alpha_3\alpha + \beta_1\delta &= 0 & \alpha_3\delta + 2\beta_1\beta &= 0 \\ 2\alpha_5\alpha + \gamma_1\epsilon &= 0 & \alpha_5\delta + \beta_1\eta + \gamma_3\epsilon &= 0 \\ \alpha_3\epsilon + \beta_5\delta + \gamma_1\eta &= 0 & 2\beta_5\beta + \gamma_3\eta &= 0 \\ \alpha_5\epsilon + 2\gamma_1\gamma &= 0 & \beta_5\eta + 2\gamma_3\gamma &= 0 \end{aligned}$$

with the following rank 5 matrix

$$(7) \quad \begin{bmatrix} 2\alpha_3 & 0 & 0 & \beta_1 & 0 & 0 \\ 2\alpha_5 & 0 & 0 & 0 & \gamma_1 & 0 \\ 0 & 0 & 0 & \beta_5 & \alpha_3 & \gamma_1 \\ 0 & 0 & 2\gamma_1 & 0 & \alpha_5 & 0 \\ 0 & 2\beta_1 & 0 & \alpha_3 & 0 & 0 \\ 0 & 0 & 0 & \alpha_5 & \gamma_3 & \beta_1 \\ 0 & 2\beta_5 & 0 & 0 & 0 & \gamma_3 \\ 0 & 0 & 2\gamma_3 & 0 & 0 & \beta_5 \end{bmatrix}.$$

**Lemma 4.** Besides the invariants  $v_1v_2, v_3v_4, v_5v_6$ , an extra quadratic invariant of the form (6) killed by two independent quadratic vector fields exists, if it has one of the following coefficients  $(\alpha, \beta, \gamma, \delta, \epsilon, \eta)$

- (a)  $(0, -\frac{a}{2}, -\frac{1}{2a}, 0, 0, 1)$
- (b)  $(-\frac{a}{2}, -\frac{1}{2a}, 0, 1, 0, 0)$
- (c)  $(-\frac{1}{2}ab, -\frac{1}{2}\frac{a}{b}, -\frac{1}{2}\frac{b}{a}, \pm a, b, 1)$
- (d)  $(-\frac{a}{2}, 0, -\frac{1}{2a}, 0, 1, 0)$

*Proof:* The conditions for the existence of an extra quadratic invariant of the form (6) are those that make matrix (7) to have rank 5. This is the vanishing of 28  $6 \times 6$  minors. We do not write these cumbersome expressions and leave it for the reader wanting to check computations.

Next, we write a table with the solution of these equations and a basis of the kernel of (7) (that is the coefficients of (6)) under the condition that there must be two linearly independent vector fields which

do not vanish on the variables  $v_i$ 's. Also, this is left to the reader: first we assumed  $\alpha_5\beta_1\gamma_3 = \alpha_3\beta_5\gamma_1$ , and then  $\alpha_5\beta_1\gamma_3 = -\alpha_3\beta_5\gamma_1$ .  $\square$

**Remark 2.** Any of the quadrics (a), (b), (d) together with  $q_1, q_2$ , and  $q_3$  lead to a reducible affine variety. This is not desirable since we want the images of irreducible jacobians. Therefore, we discard those cases.

**Remark 3.** Let us consider the set theoretical complete intersection  $CA$  in  $\mathbb{P}^6$  determined by the quadrics  $q_i = c_i, i = 1, \dots, 4$ . This surface contains the image of  $A$  in  $\mathbb{P}^6$ , call it  $\bar{A}$ . Moreover, the degree of  $CA$  is  $16 = 2 \cdot 2 \cdot 2 \cdot 2$ , the same as the degree of  $\bar{A}$ . Therefore, the rational map  $\bar{A} \rightarrow CA$  induced by  $A \rightarrow \mathbb{P}^6$  is generically one to one, and this means  $CA$  coincide with  $\bar{A}$  on an open set. The image of  $A - \mathcal{D}$  in  $\bar{A}$  is smooth and one to one because it corresponds to lifting from  $\tilde{K}_A$  to  $\tilde{A}$ , away from the branch locus, the smooth embedding  $\tilde{K}_A \rightarrow \mathbb{P}^5$  given by the even sections. Also,  $A - \mathcal{D}$  maps onto the affine piece  $\{v \in \mathbb{C}^6 : q_i(v) = c_i, i = 1, \dots, 4\}$ , and since this piece is smooth by construction and contains the image of  $A - \mathcal{D}$  in  $\mathbb{P}^6$ , it coincides with it. Thus,  $A - \mathcal{D}$  is isomorphic to the affine piece determined by the four quadrics.

## 7. THE POISSON MATRIX.

Assume the vector fields we are looking at have the special following form with matrix  $J$  polynomial in the affine variables

$$(8) \quad \dot{w}_j = X_H(w_j) = (J(w) \cdot \text{grad } H(w))_j = \sum_{k=1}^6 J_{jk}(w) \cdot \frac{\partial H}{\partial w_k}.$$

We want to find a Poisson structure of the form  $\{f, g\}(w) = \langle \text{grad } f(w), J(w) \cdot \text{grad } g(w) \rangle$  so that the Hamiltonian vector fields correspond to the holomorphic vector fields already found.

Let  $T_x$  be a translation  $T_x : A \rightarrow A$  on the abelian variety  $A$ , then, we have the equivariance relation  $dT_x \cdot X_H = X_H \circ T_x$ . The action of  $T_x \in G$  on generating functions is linear  $T_x^*(w_i) = \sum_{j=1}^6 \lambda_{ij} w_j$ . Then  $X_H(y)(T_x^*(w_i)) = \sum \lambda_{ij} \sum \mathcal{J}_j(w) \frac{\partial H}{\partial w_k}$ . But  $X_H(y)(T_x^*(w_i)) = d(w_i \circ T_x) \circ X(y) = dw_i \cdot dT_x X = dw_i \cdot X(T_x(y))(w_i) = X(T_x(y))(w_i) = T_x^*(X(y)(w_i))$  where  $X(y)(w_i)$  is a polynomial in the variables  $w_1, \dots, w_6$ .

Now, for a globally defined polynomial  $H$  invariant under  $G$ , we have  $T_x^*H(w_1, \dots, w_6) = H(T_x^*w_1, \dots, T_x^*w_6) = H(w_1, \dots, w_6)$ , and for the set of functions  $T_x^*(w_i)$ , the exchange of differentials  $dT_x^*(w_i) = \sum_{j=1}^6 \lambda_{ij} dw_j$  occurs. This induces on derivations the transformation formula  $\frac{\partial}{\partial(T_x^*w_i)} = \sum_{j=1}^6 \mu_{ij} \frac{\partial}{\partial w_j}$ , where  $(\mu_{ij}) = ((\lambda_{ij})^t)^{-1}$ . Moreover  $T_x^*\left(\frac{\partial H}{\partial w_i}\right) = T_x^*\left(\left(\frac{\partial}{\partial w_i}\right)(H)\right) = \left(\left(\frac{\partial}{\partial w_i}\right) \circ T_x\right)(H \circ T_x) = \frac{\partial}{\partial(T_x^*w_i)}(T_x^*H) = \sum \mu_{ij} \frac{\partial}{\partial w_j} T_x^*H = \sum \mu_{ij} \frac{\partial}{\partial w_j} H$ .

Therefore, we obtain the relation

$$\sum \lambda_{ij} J_{jk}(w) \frac{\partial H}{\partial w_k} = \sum T_x^* J_{ik}(w) \cdot \mu_{kj} \frac{\partial H}{\partial w_j}.$$

So, for any integral invariant  $H$

$$(9) \quad \sum_k \left( \sum_j \lambda_{ij} J_{jk}(w) - \sum_l T_x^* J_{il}(w) \cdot \mu_{lj} \right) \frac{\partial H}{\partial w_k} = 0.$$

This means that the functions  $\epsilon_{ik}(w) = \sum_j \lambda_{ij} J_{jk}(w) - J_{ij}(T_x^*w) \cdot \mu_{jk}$  are killed by the gradients  $\text{grad } H$ , for  $H$  an (integral of the motion) invariant by  $G$ . Namely, the vectors  $\epsilon_i(w) = (\epsilon_{i1}(w), \epsilon_{i2}(w), \dots, \epsilon_{i6}(w))$  belong to the tangent space of the affine variety  $\{w : q_i(w) = c_i \quad i = 1, \dots, 4\}$ . Thus, there must be a function  $H$ , linear combination of the nontrivial invariants  $H_1, H_2$ , such that  $\epsilon_i(w) = J(w) \cdot \text{grad } H$ .

Assuming that there are no nontrivial Hamiltonians which are linear in the affine coordinates, we conclude that  $\epsilon_i(w) = 0$  if the matrix  $J$  has linear entries.



Indeed, for a nontrivial Hamiltonian  $H$ , the polynomials in  $w_1, \dots, w_6, (J(w) \cdot \text{grad}H)$ ,  $i = 1, \dots, 6$ , have at least degree 2. So, it follows:

**Lemma 5.** *If the matrix  $J$  has linear entries, then it is equivariant by the action of translations in  $G$ , and skewequivariant by the action of  $(-1)$ -involution. Namely, if  $\Lambda(\sigma)$  is the matrix of the translate  $\sigma$ ,  $J(\sigma w) = \Lambda(\sigma)J(w)\Lambda(\sigma)^t$ ,  $\sigma \in G$ , and if  $\Lambda(\iota)$  is the matrix of the  $(-1)$ -involution  $\iota$ ,  $J(\iota w) = -\Lambda(\iota)J(w)\Lambda(\iota)^t$ .*

Let us assume that the Poisson matrix of this would be system have linear entries. Being the invariants quadratic, we deduce that the matrix  $J$  satisfies the relations described by the lemma 5.

Let us write a general linear  $6 \times 6$  matrix in  $2 \times 2$  blocks

$$(10) \quad J = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}$$

Let  $I$  be the  $2 \times 2$  identity matrix and

$$D = \begin{Bmatrix} 0 & 1 \\ 1 & 0 \end{Bmatrix}$$

the  $2 \times 2$  transposition. In matrix notation, the operators  $\sigma, \tau$ , and the  $(-1)$ -involution  $\iota$  are written as follows:

$$\sigma = \begin{bmatrix} -D & 0 & 0 \\ 0 & -I & 0 \\ 0 & 0 & -D \end{bmatrix} \quad \tau = \begin{bmatrix} -I & 0 & 0 \\ 0 & D & 0 \\ 0 & 0 & D \end{bmatrix} \quad \iota = \begin{bmatrix} -I & 0 & 0 \\ 0 & -I & 0 \\ 0 & 0 & -I \end{bmatrix}$$

The invariance property of  $J$  is described by the following two relations

$$\sigma \cdot J = \begin{bmatrix} DA_{11}D & DA_{12} & DA_{13}D \\ A_{21}D & A_{22} & A_{23}D \\ DA_{31}D & DA_{32} & DA_{33}D \end{bmatrix}$$

$$\tau \cdot J = \begin{bmatrix} A_{11} & -A_{12}D & -A_{13}D \\ -DA_{21} & DA_{22}D & DA_{23}D \\ -DA_{31} & DA_{32}D & DA_{33}D \end{bmatrix}$$

We also use the fact that  $J$  is skew symmetric. Thus, it is enough to solve the following equations:

$$(11) \quad \begin{cases} \sigma \cdot A_{11} = DA_{11}D \\ \tau \cdot A_{11} = A_{11} \end{cases} \quad \begin{cases} \sigma \cdot A_{22} = A_{22} \\ \tau \cdot A_{22} = DA_{22}D \end{cases} \quad \begin{cases} \sigma \cdot A_{33} = DA_{33}D \\ \tau \cdot A_{33} = DA_{33}D \end{cases}$$

$$\begin{cases} \sigma \cdot A_{12} = DA_{12} \\ \tau \cdot A_{12} = -A_{12}D \end{cases} \quad \begin{cases} \sigma \cdot A_{23} = A_{23} \\ \tau \cdot A_{23} = DA_{23}D \end{cases} \quad \begin{cases} \sigma \cdot A_{13} = DA_{13}D \\ \tau \cdot A_{13} = -A_{13}D \end{cases}$$

The equations for  $A_{11} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$  can be written as  $\begin{bmatrix} \sigma a & \sigma b \\ \sigma c & \sigma d \end{bmatrix} = \begin{bmatrix} d & c \\ b & a \end{bmatrix}$ , and  $\begin{bmatrix} \tau a & \tau b \\ \tau c & \tau d \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ . This means  $A_{11} = \begin{bmatrix} a & b \\ \sigma a & \sigma b \end{bmatrix}$ , with  $a = \tau a, b = \tau b$ .

Analogously, for the remaining  $A_{ij}$  we get  $A_{22} = \begin{bmatrix} a & b \\ \tau b & \tau a \end{bmatrix}$ , with  $a = \sigma a, b = \sigma b$ .  $A_{33} = \begin{bmatrix} a & b \\ \sigma b & \sigma a \end{bmatrix}$  with  $\tau \sigma b = b, \tau \sigma a = a$ .

$$(12) \quad A_{13} = \begin{bmatrix} a' & -\tau a' \\ -\sigma \tau a' & \sigma a' \end{bmatrix}, \quad A_{23} = \begin{bmatrix} a'' & \sigma a'' \\ \tau \sigma a'' & \tau a'' \end{bmatrix}, \quad A_{12} = \begin{bmatrix} a & -\tau a \\ \sigma a & -\sigma \tau a \end{bmatrix}$$

Since  ${}^t A_{ii} = -A_{ii}$ , we obtain

$$(13) \quad \begin{aligned} A_{11} &= \begin{bmatrix} 0 & b \\ \sigma b & 0 \end{bmatrix}, \text{ with } b = \tau b = -\sigma b \\ A_{22} &= \begin{bmatrix} 0 & b' \\ \tau b' & 0 \end{bmatrix}, \text{ with } b' = -\tau b' = \sigma b' \\ A_{33} &= \begin{bmatrix} 0 & b'' \\ \sigma b'' & 0 \end{bmatrix}, \text{ with } b'' = -\sigma b'' = \tau \sigma b'' \end{aligned}$$

Using table I, we finally get

$$(14) \quad \begin{aligned} A_{11} &= \begin{bmatrix} 0 & f_1 \\ -f_1 & 0 \end{bmatrix}, \quad f_1 = \alpha_3(v_3 + v_4) + \alpha_5(v_5 + v_6) \\ A_{22} &= \begin{bmatrix} 0 & f_2 \\ -f_2 & 0 \end{bmatrix}, \quad f_2 = \beta_1(v_1 - v_2) + \beta_5(v_5 - v_6) \\ A_{33} &= \begin{bmatrix} 0 & f_3 \\ -f_3 & 0 \end{bmatrix}, \quad f_3 = \gamma_1(v_1 + v_2) + \gamma_3(v_3 - v_4) \end{aligned}$$

**Proposition 6.** *There is a system with Poisson matrix (10) and entries (12) and (14) that has the functions  $q_2 = v_3 v_4, q_3 = v_5 v_6$  as Casimirs. Up to a change of basis, it is the  $SO(4)$  case of Adler and Van Moerbeke.*

*Proof:* Since  $J \cdot \text{grad } q_2 = 0, J \cdot \text{grad } q_3 = 0$ , namely  $J \cdot [0, 0, v_4, v_3, 0, 0]^t = 0, J \cdot [0, 0, 0, 0, v_6, v_5]^t = 0$ . These impose conditions on  $A_{13}, A_{23}, A_{12}, A_{22}$ , and  $A_{33}$ . Clearly  $f_2 = 0$  and  $f_3 = 0$ . Also  $av_4 - \tau av_3 = 0, a'v_6 - \tau a'v_5 = 0, -\sigma \tau a'v_6 + \sigma a'v_5 = 0, a''v_6 + \tau a''v_5 = 0, \tau \sigma a''v_6 + \tau a''v_5 = 0, a''v_4 + \tau \sigma a''v_3 = 0, \sigma a''v_4 + \tau a''v_3 = 0, av_4 - \tau av_3 = 0 \Rightarrow a = \alpha'_3 v_3, a'v_6 - \tau a'v_5 = 0 \Rightarrow a' = \beta'_5 v_5, a''v_6 + \sigma a''v_5 = 0 \Rightarrow a'' = \gamma'_5 v_5$ .

Thus

$$(15) \quad A_{12} = \begin{bmatrix} \alpha'_3 v_3 & -\alpha_3 v_4 \\ -\alpha'_3 v_3 & \alpha'_3 v_4 \end{bmatrix}, \quad A_{13} = \begin{bmatrix} \beta'_5 v_5 & -\beta'_5 v_6 \\ \beta'_5 v_5 & -\beta'_5 v_6 \end{bmatrix}, \quad A_{23} = \begin{bmatrix} \gamma'_5 v_5 & -\gamma'_5 v_6 \\ -\gamma'_5 v_5 & \gamma'_5 v_6 \end{bmatrix}$$

but the equations imply  $A_{23} = 0$ .

The  $SO(4)$  system for the metric II is the system of differential equations [2],[9].

$$\begin{aligned} \dot{\tau}_1 &= \tau_2 \tau_6 & \dot{\tau}_4 &= \tau_3 \tau_5 \\ \dot{\tau}_2 &= \frac{1}{2} \tau_3 (\tau_1 + \tau_4) & \dot{\tau}_5 &= \tau_3 \tau_4 \\ \dot{\tau}_3 &= \frac{1}{2} \tau_3 (\tau_1 + \tau_4) & \dot{\tau}_6 &= \tau_1 \tau_2. \end{aligned}$$

One can pick a Poisson matrix for this system

$$J_{SO(4)} = \begin{bmatrix} 0 & \tau_3 & \tau_2 & 0 & 0 & (2\tau_2 - \tau_5) \\ -\tau_5 & 0 & 0 & 0 & 0 & 0 \\ -\tau_2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \tau_5 \\ 0 & 0 & 0 & 0 & 0 & \tau_4 \\ -(\tau_4 - \tau_5) & 0 & 0 & -\tau_5 & -\tau_4 & 0 \end{bmatrix}$$

with Poisson bracket  $\{f, g\} = \left\langle \frac{\partial f}{\partial \tau}, J_{SO(4)} \cdot \frac{\partial g}{\partial \tau} \right\rangle$ .

By making the change of variables

$$v_1 = \tau_1 + \tau_0, \quad v_2 = \tau_6 - \tau_1, \quad v_3 = \tau_2 + \tau_3, \quad v_4 = \tau_2 - \tau_3, \quad v_5 = \tau_4 + \tau_5, \quad v_6 = \tau_5 - \tau_4$$

the Poisson matrix takes the form

$$J_{SO(4)} = \begin{bmatrix} 0 & v_3 + v_4 - 1/2(v_5 + v_6) & v_3 & -v_4 & -v_5 & v_6 \\ -(v_3 + v_4) + 1/2(v_5 + v_6) & 0 & -v_3 & v_4 & -v_5 & v_6 \\ -v_3 & v_3 & 0 & 0 & 0 & 0 \\ v_4 & -v_4 & 0 & 0 & 0 & 0 \\ v_5 & v_5 & 0 & 0 & 0 & 0 \\ v_6 & -v_6 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} A & B & C \\ -{}^tB & 0 & 0 \\ -{}^tC & 0 & 0 \end{bmatrix}$$

and the invariants are  $Q_1 = v_1v_2$ ,  $Q_2 = v_3v_4$ ,  $Q_3 = v_5v_6$ ;  $Q_4 = \frac{1}{2}(v_4 + v_3 - v_5 - v_6)^2 + \frac{1}{2}(v_3 - v_4 - v_1 - v_2)^2 - \frac{1}{4}(v_1 - v_2 - v_5 + v_6)^2$  with  $Q_2$  and  $Q_3$  as the Casimirs.

Another linear change of variables  $w_i = f_i(v_1, \dots, v_6)$  with matrix

$$\left( \frac{\partial w_i}{\partial v_j} \right) = \begin{bmatrix} B_{11} & B_{12} & B_{13} \\ B_{21} & B_{22} & B_{23} \\ B_{31} & B_{32} & B_{33} \end{bmatrix}$$

brings the  $J_{SO(4)}$  to the desired matrix. Indeed,

$$J_{SO(4)} = \begin{bmatrix} B_{11} & B_{12} & B_{13} \\ B_{21} & B_{22} & B_{23} \\ B_{31} & B_{32} & B_{33} \end{bmatrix} \begin{bmatrix} A & B & C \\ -{}^tB & 0 & 0 \\ -{}^tC & 0 & 0 \end{bmatrix} \begin{bmatrix} {}^tB_{11} & {}^tB_{21} & {}^tB_{31} \\ {}^tB_{12} & {}^tB_{22} & {}^tB_{32} \\ {}^tB_{13} & {}^tB_{23} & {}^tB_{33} \end{bmatrix}$$

$$= \begin{bmatrix} B_{11} & B_{12} & B_{13} \\ B_{21} & B_{22} & B_{23} \\ B_{31} & B_{32} & B_{33} \end{bmatrix} \cdot \begin{bmatrix} A{}^tB_{11} + B{}^tB_{12} + C{}^tB_{13} & A{}^tB_{21} + B{}^tB_{22} + C{}^tB_{23} & A{}^tB_{31} + B{}^tB_{32} + C{}^tB_{33} \\ -{}^tB{}^tB_{11} & -{}^tB{}^tB_{21} & -{}^tB{}^tB_{31} \\ -{}^tC{}^tB_{11} & -{}^tC{}^tB_{21} & -{}^tC{}^tB_{31} \end{bmatrix}$$

pick  $B_{21} = B_{31} = 0$ ,  $B_{11} = I$ ,  $B_{22} = \alpha'_3 I$ ,  $B_{23} = 0$ ,  $B_{32} = 0$ ,  $B_{33} = -\beta'_5 I$ .

The upper-left corner of the product matrix is  $A' = A + B{}^tB_{12} - (B{}^tB_{12})^t + C{}^tB_{13} - (C{}^tB_{13})^t$ , and if

$${}^tB_{12} = a \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix} \quad {}^tB_{13} = b \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

for convenient scalars  $a$  and  $b$ , we obtain  $A' = A_{11}$  as in (14). The other entries are treated similarly.  $\square$

## 8. CONCLUSIONS

As seen in this paper, the procedure of assigning the algebro-geometric data  $(A_\alpha, \mathcal{D}_\alpha, \mathcal{L}_\alpha, G)$  ( $A_\alpha$  an abelian variety,  $\mathcal{D}_\alpha$  a divisor on it,  $\mathcal{L}_\alpha$  a line bundle on  $A_\alpha$  and  $G$  a group (of translates) leaving invariant  $\mathcal{D}_\alpha$  and  $\mathcal{L}_\alpha$ ) to an a.c.i. system, can be accomplished in some cases in a successful way. There is a question of how unique a system is obtained from a given data. The question for the  $SO(4)$  case would be settled if the 80 different configurations referred to in §4 would be shown to be isomorphic to the  $SO(4)$  system. A more invariant way of looking at these problems is needed. If such a problem is carried out successfully one can ask if it is possible to characterize such systems as the three body periodic Toda Lattice, Kowalevski's Top or other a.c.i. systems with two degrees of freedom. A more interesting question is whether these procedures allow to find new "mathematical" integrable systems. This would be a nice outcome for these techniques.

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