Decoherence and the Loschmidt Echo

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Decoherence causes entropy increase that can be quantified using, e.g., the purity $\varsigma = \mathrm{Tr}\rho^2$. When the Hamiltonian of a quantum system is perturbed, its sensitivity to such perturbation can be measured by the Loschmidt echo $M(t)$. It is given by the squared overlap between the perturbed and unperturbed state. We describe the relation between the temporal behavior of $\varsigma(t)$ and the average $\bar{M}(t)$. In this way we show that the decay of the Loschmidt echo can be analyzed using tools developed in the study of decoherence. In particular, for systems with a classically chaotic Hamiltonian the decay of ς and $\bar{M}$ has a regime where it is dominated by the Lyapunov exponents.

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Decoherence is an essential ingredient of the quantum-classical transition [1,2]. Its implications for quantum versions of classically chaotic systems are especially intriguing, as they go beyond the restoration of the quantum-classical correspondence. Two of us discussed this issue in [3], presenting a surprising result that has since been amply corroborated [4–7]: For a quantum system with a classically chaotic Hamiltonian the rate at which the environment degrades information about the initial state becomes independent of the system-environment coupling strength. This rate (e.g., the von Neumann entropy production rate computed from the reduced density matrix of the system) is set by the classical Lyapunov exponents, provided that the coupling strength is within a certain (wide) range. This result has important implications and can be used to define quantum chaos [8]. A related but independent way to do this was considered by Peres [9]. He observed that quantum dynamics, insensitive to small differences in initial conditions, is highly sensitive to perturbations in the Hamiltonian [10]. More recently, Levstein, Pastawski, and collaborators [11,12] experimentally studied sensitivity to perturbations by measuring the Loschmidt echo in a many-body spin system. Their work motivated further analytical [13] and numerical [14] studies of the true nature of this sensitivity, which in turn triggered intense activity on the subject [15–17].

The measure of the echo signal is the overlap between two states that evolve from the same initial wave function Ψ_0 under the influence of two Hamiltonians (the unperturbed one H_0 and the perturbed one $H_\Delta = H_0 + \Delta$). More precisely, when U_0 and U_Δ denote the corresponding evolution operators, the echo is defined as

$$M_\Delta(t) = |\langle\Psi_0|U_\Delta^\dagger(t)U_0(t)|\Psi_0\rangle|^2. \qquad (1)$$

The quantity $\bar{M}(t)$, obtained by averaging M_Δ over an ensemble of perturbations, can be studied analytically and displays a rich temporal dependence. One interesting regime was analyzed by Jalabert and Pastawski [13] who showed, using a semiclassical approximation, that there is a window of values for the perturbation strength for which $\bar{M}(t)$ decays with a rate equal to the classical Lyapunov exponent. In spite of the simple discussion presented above, the physically relevant evolution will typically not be unitary: Environment-induced decoherence will suppress the echo even in the absence of the perturbation Δ. We shall, however, adhere to the usual assumption [11–17] that the evolutions are unitary, and show that even in that case of decoherence-free echo suppression it is possible to draw useful conclusions from the analogy with decoherence.

In this Letter we establish a direct connection between decoherence and the decay of the Loschmidt echo. We relate the evolution of $\bar{M}(t)$ and the linear entropy (or purity) of an open quantum system. The existence of a kinship between these two quantities was noted, for example, in Refs. [12,13], but never formally established. Such a relation is interesting not only from a fundamental point of view but also allows the use of results obtained in the theory of open quantum systems to understand better the behavior of the echo.

The key step in our demonstration is a simple observation: The average echo $\bar{M}(t)$ for an ensemble of perturbations with probability density $P(\Delta)$ is

$$\bar{M}(t) = \int D\Delta P(\Delta)|\langle\Psi_0|U_\Delta^\dagger(t)U_0(t)|\Psi_0\rangle|^2. \qquad (2)$$

This can be rewritten by defining the density matrix of the average perturbed state as

$$\bar{\rho}(t) = \int D\Delta P(\Delta)U_\Delta(t)|\Psi_0\rangle\langle\Psi_0|U_\Delta^\dagger(t). \qquad (3)$$

Thus, $\bar{M}(t)$ is simply the overlap between the average state $\bar{\rho}(t)$ and the unperturbed density matrix $\rho_0(t)$ evolved from the initial state with U_0:

$$\bar{M}(t) = \mathrm{Tr}(\bar{\rho}(t)\rho_0(t)). \quad (4)$$

Once we recognize this simple fact we can go one step further and notice that the state $\bar{\rho}$ generally evolves in time according to a master equation which is of the same kind as the ones arising in the study of decoherence. Hence, the evolution of the echo $\bar{M}(t)$ is directly placed in the context of open quantum systems and decoherence.

Equation (4) can be used to establish an inequality between the echo $\bar{M}(t)$ and the purity $\varsigma(t) = \mathrm{Tr}\bar{\rho}^2(t)$ (used to characterize decoherence). Using Schwartz inequality and assuming that the initial state is pure we find that $\bar{M}^2(t) \leq \varsigma(t)$. Related inequalities were noticed and used in a somewhat different context in [17,18]. This equation implies that when the purity $\varsigma(t)$ decays exponentially with a rate γ_D, then $\bar{M}(t)$ should also decay exponentially (or faster) with a rate at least $\gamma_D/2$. However, as we will see later (and as has been established in the literature [6,13,14]), there is an important regime (the so-called Lyapunov regime) where both quantities decay with the same rate set by the Lyapunov exponent.

Let us now analyze some generic features of the evolution of the average state $\bar{\rho}(t)$. In general, $\bar{\rho}$ obeys a master equation with nonunitary terms, which arise because averaging of the evolution over an ensemble of perturbations yields an effect analogous (although not equivalent) to the tracing out of unobserved degrees of freedom. Briefly, while the equivalence can be established for the average over an ensemble of noise realizations, it does not exist for individual members of the ensemble, which follow unitary evolution with a given noise. By contrast, a decohering system will lose purity after becoming entangled with the environment, even when the state of the environment is known beforehand (see Ref. [2] for a detailed discussion). We will find it convenient to consider a simple form of perturbation (even though results do not dependent strongly on it, provided we exclude situations where the perturbation changes substantially the nature of the Hamiltonian [19]). Let us assume that $\Delta(x, t) = V(x)J(t)$, where $V(x)$ is a function of the coordinates of our system and $J(t)$ is an external source. For this case, averaging over Δ consists of averaging over functions $J(t)$. We will assume that the probability density $P(J)$ is a Gaussian whose width defines the temporal correlation function for the sources:

$$P(J) = N\exp\left[-\frac{1}{2}\iint dtdt' J(t)\nu^{-1}(t, t')J(t')\right], \quad (5)$$

with $\nu(t, t') = \int DJP(J)J(t)J(t')$ the noise correlation function and N a normalization factor. Using this, we can show that the evolution operator for $\bar{\rho}(t)$ has a path integral representation with an influence functional [20] given by

$$F[x, x'] = \exp\left[-\frac{1}{2}\iint dtdt' V_-(t)\nu(t, t')V_-(t')\right], \quad (6)$$

where $V_-(t) = V(x(t)) - V(x'(t))$. In some simple but physically relevant cases it is possible to write a master equation for $\bar{\rho}(t)$. In fact, when the noise is white, i.e., $\nu(t, t') = 2D\delta(t - t')$, we can show that

$$\dot{\bar{\rho}} = \frac{1}{i\hbar}[H_0, \bar{\rho}] - D[V(x), [V(x), \bar{\rho}]]. \quad (7)$$

While the first term on the right-hand side generates unitary evolution, the second one is responsible for decoherence: It induces a tendency towards diagonalization in position basis and, in the Wigner representation, it gives rise to a diffusion term. For the simplest case of $V(x) = x$ the equation for the Wigner function reads

$$\dot{W}(x, p) = \{H_0, W\}_{\mathrm{MB}} + D\partial^2_{pp}W(x, p), \quad (8)$$

where $\{\cdots\}_{\mathrm{MB}}$ is the so-called Moyal bracket, responsible for unitary evolution [1].

Equations like (7) and (8) arise if we consider a quantum system interacting with a quantum environment formed by a set of harmonic oscillators [21]. In such a case the modulus of the influence functional generated by the environment is identical to (6) provided one chooses the spectral density and the initial state of the environment in such a way that its noise kernel is equal to the kernel $\nu(t, t')$ in (6). However, the influence functional is in general a complex number whose phase is responsible for dissipation (noise and dissipation kernels are connected as mandated by the fluctuation-dissipation theorem). In the physically relevant limit (usually associated with high temperatures) for decoherence studies aimed at understanding the quantum-classical correspondence, relaxation effects can be ignored [1]. Thus, in this limit, the evolution of the average state $\bar{\rho}$ is identical to that of a quantum system interacting with an environment.

A convenient way to visualize the transition from quantum to classical is provided by the Wigner function, whose oscillations are the signature of quantum interference. They should be suppressed by decoherence to make the quantum-classical correspondence possible. Indeed, when the Wigner function oscillates with a well-defined wave vector k_p along the momentum direction $[W(x, p, t) \simeq A(x, t)\cos(k_p p)]$, the decoherence term in (8) washes out oscillations exponentially fast with a rate $\Gamma_D = Dk_p^2$. We can see the behavior of a typical Wigner function for a chaotic system (a driven double well analyzed in [6]) with and without decoherence in Fig. 1. Taking into account our previous discussion, the echo $\bar{M}(t)$ is obtained by computing the overlap between the two Wigner functions displayed in the figures. The purity ς should be computed by taking the overlap of the decohered Wigner function with itself. Below, we will discuss the relation between these two quantities.

The master equation (8) can be used to obtain the time derivatives of the purity ς and the echo $\bar{M}$:

$$\dot{\varsigma} = 2D\int dxdp\bar{W}(x, p)\partial^2_{pp}\bar{W}(x, p), \quad (9)$$

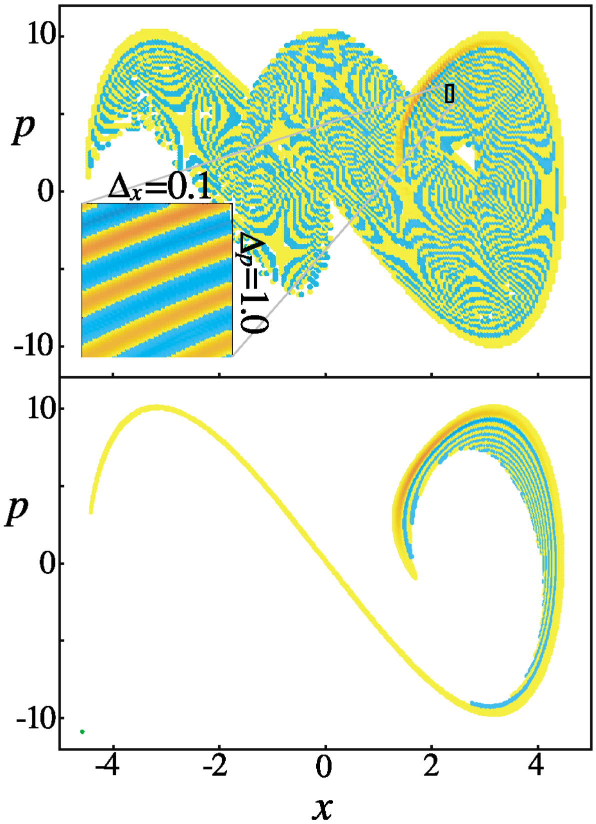

FIG. 1 (color). Wigner function of an initially Gaussian state evolved with a chaotic Hamiltonian without (top) and with (bottom) decoherence. The system is a particle moving in a driven double well potential (see [6]). A region of area $\hbar$ is shown in the top panel where sub-Planckian structure is evident [22]. The color scale is positive from yellow to red, shades of blue are negative, and white is zero. In the top panel we can appreciate the distinct regions A_O (A_C) where the Wigner function W_0 oscillates rapidly (is positive), used in Eq. (11).

$$\dot{\bar{M}} = D \int dx dp W_0(x, p) \partial^2_{pp} \bar{W}(x, p). \qquad (10)$$

Equation (9) has been used before to show the existence of a domain of exponential decay for the purity $\mathrm{Tr}\bar{\rho}^2$ [3,5,6,8] with a rate that, for classically chaotic systems, is independent of the diffusion D. The central piece of the argument is the following: After integrating by parts, Eq. (9) can be rewritten as $\dot{\varsigma}/\varsigma = -2D/\bar{\sigma}^2$, where $\bar{\sigma}$ characterizes the dominant wavelength in the spectrum of the Wigner function [i.e., $\bar{\sigma}^{-2} = \int(\partial_p \bar{W})^2/\int \bar{W}^2$]. Thus, the rate of change of the purity becomes independent of the diffusion constant when $\bar{\sigma}^2$ is proportional to D, which happens indeed as a consequence of the competition between two effects. The first one is the tendency of chaotic evolution to generate (exponentially fast, at a rate set by the Lyapunov exponent λ) small scale structure in the Wigner function. The second effect is due to diffusion, which tends to wash out small scales exponentially fast at a rate determined by the product Dk_p^2. These two effects reach a balance when $\bar{\sigma}^2 = 2D/\lambda$ [3], and then the purity ς decreases exponentially at a rate fixed by λ. For this behavior to take place D should be above a threshold [8], otherwise the critical width is not established (indeed, the implicit assumption is that the time scale for diffusion to wash out a k_p oscillation is shorter than the time scale for the oscillations to be regenerated by the dynamics). This simple scenario lets us understand why there is a regime where purity decreases exponentially with a Lyapunov rate.

The above argument can also be used to analyze the decay of the Loschmidt echo. In fact, Eqs. (9) and (10) just differ by a factor of 2 and by the presence of W_0 instead of $\bar{W}$ inside the integral. As before, we can transform the evolution equation of the echo into $\dot{\bar{M}}/\bar{M} = -D/\sigma^2$, $\sigma^{-2} = \int W_0 \partial^2_{pp} \bar{W}/\int W_0 \bar{W}$. When decoherence is effective and the dominant structure in $\bar{W}$ approaches the critical value, the smallest scales of the pure Wigner function W_0 continue contracting and developing smaller and smaller scales (sub-Planck scales are reached quickly in chaotic quantum systems [22]). In such a case, one obtains $\bar{\sigma}^2 = 2\sigma^2$. This is readily seen even in the crude approximation of $\bar{W} \sim \exp(-p^2/2\bar{\sigma}^2)$ and $W_0 \sim \exp(-p^2/2\sigma_0^2)$, with $\sigma_0 \sim \exp(-\lambda t)$ and $t \gg 1/\lambda$. Hence, when the purity starts decaying at the Lyapunov rate the echo does precisely the same.

Using the above ideas we now present a more illustrative picture of the time dependence of the echo $\bar{M}(t)$ and the purity $\varsigma(t)$. For the sake of simplicity we focus on the echo but the same reasoning applies to the purity. To compute the overlap $\bar{M} = \int dx dp W_0 \bar{W}$ we can split the phase space integral into two regions: the region A_C close to the classical unstable manifold of the initial state, where W_0 is positive, and the region A_O over which W_0 oscillates (see Fig. 1):

$$\bar{M}(t) = \int_{A_O} dx dp W_0 \bar{W} + \int_{A_C} dx dp W_0 \bar{W}. \qquad (11)$$

In the oscillatory region we can estimate the value of the integral assuming that there is a dominant wave vector k_p. In such a case, from Eq. (8) we assume $\bar{W} \simeq W_0 e^{-Dk_p^2 t}$. If more than one scale is present the result would be a sum of terms like this one. For the second integral, we can also use a crude estimate supposing that W_0 and $\bar{W}$ are constant over their respective effective support. In particular, $W_0 \sim 1/A_C$ since its integral over A_O cancels out. As $\bar{W}$ approaches the critical width $\bar{\sigma}$ along the stable manifold, the area of its effective support grows exponentially. Therefore, one gets that the second integral is $\int_{A_C} W_0 \bar{W} \sim W_0 A_C \bar{W} \sim \bar{W} \sim e^{-\lambda t}$. Thus, combining the two results we find that the expected behavior of the Loschmidt echo is

$$\bar{M}(t) = a \exp(-\lambda t) + b \exp(-Dk_p^2 t) \qquad (12)$$

for appropriate prefactors a and b. This result was previously derived for the Loschmidt echo using semiclassical techniques [13]. The first term gives the Lyapunov decay, while the second one describes the so-called Fermi golden rule regime (FGR) [15]. In this case the rate is proportional to the diffusion coefficient (which is itself proportional to the square of the strength of the perturbation). As mentioned above, a similar result is expected for the purity.

Our treatment is valid in a semiclassical regime where the evolution of the Wigner function is dominated by the classical Hamiltonian flow and the corresponding interference fringes generated when its phase space support folds. The virtue of this analysis, entirely based on properties of the evolution of $\bar{W}$ derived in the context of decoherence studies, is not only its simplicity but also the fact that it enables us to identify the regions of phase space that can be associated with each of the terms appearing in (12): The FGR contribution arises from the decay of the interference fringes while the Lyapunov contribution is associated with the behavior of $\bar{W}$ near the classical unstable manifold. Such a picture has also been noticed using semiclassical techniques [23].

It is also interesting to perform a better estimate of the integral over the region A_C. Assuming that the local Lyapunov exponent is constant along the unstable manifold, one can approximate the value of the integral using the corresponding result for the simplest system with an unstable fixed point: the inverted oscillator (IO) with Hamiltonian $H_0 = p^2/2m - m\lambda^2 x^2/2$. In such a case, the echo can be computed exactly and results in

$$\bar{M}_{\rm IO}(t) = [1 + r\,\mathrm{sh}(2\lambda t) + r^2(\mathrm{sh}^2(\lambda t) - \lambda^2 t^2)]^{-1/2}. \quad (13)$$

Here $r = \bar{\sigma}^2/4\sigma_i^2$, where σ_i is the momentum dispersion of the initial state. This exact result shows that for long times $[\lambda t \gg \ln(r)/2]$ the echo $M_{\rm IO}$ always decays as $\exp(-\lambda t)$. For short times a decay with a rate determined by diffusion is observed, but this transitory regime always leads to a decay dominated by the Lyapunov exponent. The initial transient is sensitive to the details of the noise statistics. For example, we can also evaluate the echo for a flat noise kernel [i.e., $\nu(t, t')$ independent of t and t']. For such a case the long time behavior of the echo is not changed but the initial transient displays a quadratic decay.

We expect the analogy between Loschmidt echo and decoherence not only to enable intuitive derivations like the one leading us to Eq. (12) but also to provide new insights into theoretically unexplained experimental features such as the Gaussian decay observed in [12]. Our results are also relevant for quantum computation as the Loschmidt echo is a measure of the fidelity with which a given algorithm is implemented. The Lyapunov decay of the fidelity could hinder the practical implementation of such computers, which would then have to deal with an exponential increase of error probability at a rate independent of the coupling to the environment and fixed solely by the (possibly chaotic) nature of the underlying physical system.

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